Geometrical web models

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Galileo Galilei Institute, Florence

Randomness, Integrability and Universality

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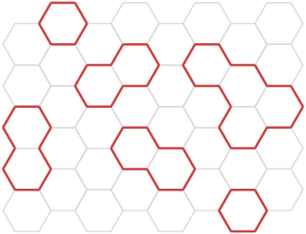
From loops to webs

Loop models: what, why, how?

- Self-avoiding (open or closed) simple curves in two dimensions
- Polymers, level lines, domain walls, electron gases
- Lattice: Integrability, knot theory, cellular algebras, category theory
- Continuum limit: CFT, CLE, SLE

Definition and features

- Fix lattice of nodes and links
- Place bonds on some links so as to form set of loops
- Weight x per bond (+ maybe further local weights) and N per loop
- For $|N| \leq 2$, dense and dilute critical points $x_{\rm c}^{\pm}$
- Continuum limit of compactified free bosonic field (Coulomb gas) [Nienhuis, Di Francesco-Saleur-Zuber, Duplantier, Cardy ...]



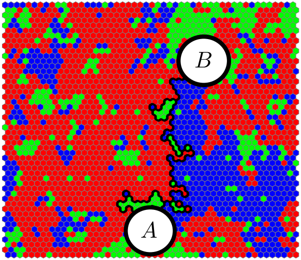
Generalisation to webs

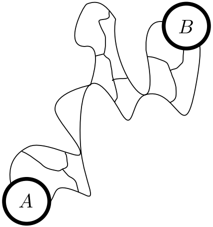
- Allow for branchings and bifurcations (with weights)
- Topological rules give weight to each connected web component
- Properties and possible critical behaviour?

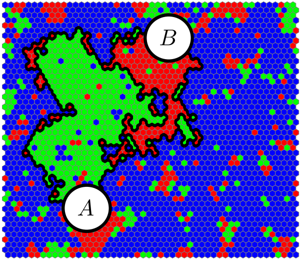
Motivations for webs

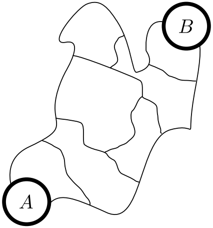
- Domain walls in spin systems [Dubail-JJ-Saleur, Picco-Santachiara]
- Network models for topological phases [Kitaev, Levin-Wen, Fendley]
- Spiders in invariance theory [Kuperberg, Kim, Cautis-Kamnitzer-Morrison]

Thin and thick domain walls ($Q = 3$ Potts model)







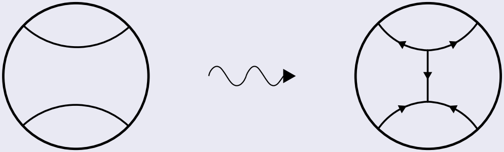


Questions (physics)

- How to define a "good" model of webs on the lattice?
- Fractal dimension of such domain walls (bulk / boundary)?
- Fractal dimension of an entire web component?
- Topological weight of web versus chromatic polynomial in $Q = 3$?
- Web model away from this special point?

Questions (mathematics)

- Algebraic construction accounting for bifurcations?
- Loop model has $U_{-q}(\mathfrak{sl}_2)$ symmetry, can we get $U_{-q}(\mathfrak{sl}_n)$?

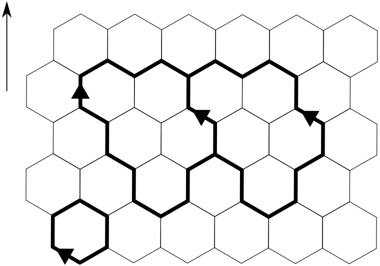


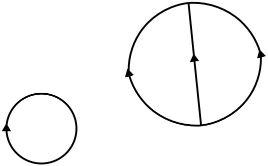
Web model from Kuperberg A_2 spider ($U_{-q}(\mathfrak{sl}_3)$ case)

Lattice considerations

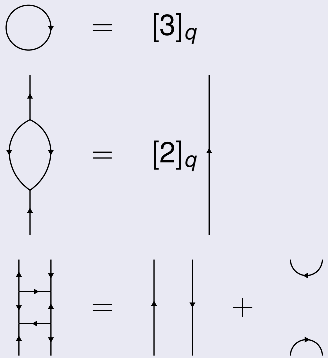
- Hexagonal (honeycomb) lattice $\mathbb{H}$ with nodes and links
- Configuration c by drawing bonds on some links, with constraints:
  - Nodes have valence 0, 2 or 3: closed web with 3-valent vertices
  - Each bond is oriented. Orientations conserved at 2-valent nodes
  - Vertices are sources or sinks (all bonds point in or out)

Each configuration can be seen as an abstract graph (vertices/edges). It is closed, planar, trivalent, bipartite. Fix an orientation (= 'up').





Rules for 'reducing' a configuration [Kuperberg]



(1)

(2)

(3)

- Rotated and arrow-reversed diagrams not shown.
- A web component always has ≥ 1 polygon of degree 0, 2 or 4.
- The three rules thus evaluate any web to a number (its weight)

Define q-deformed numbers: $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$

Defining the web model

- Sum over configurations $c \in K$ on $\mathbb{H}$
- Local weights: x_1 (up bond), x_2 (down bond), y (sink), z (source)
- Partition function:

$$Z_{\mathrm{K}} = \sum_{c \in K} x_1^{N_1} x_2^{N_2} (yz)^{N_V} w_{\mathrm{K}}(c)$$

with N_1 up-bonds, N_2 down-bonds, and N_V vertex pairs

$\mathbb{Z}_3$ spin model

Definition

- Spins $\sigma_i \in \mathbb{Z}_3 := \{0, 1, 2\}$ defined on triangular lattice $\mathbb{T} = \mathbb{H}^*$.
- Weight of link $(ij) \in \mathbb{T}$ defined as $x_{\sigma_j - \sigma_i}$, with j to the right of i.
- Normalise $x_0 = 1$. Weight x_1 or x_2 for a piece of domain wall.

Note: vertex is a sink (source) if spins follow cyclically $0 \to 1 \to 2 \to 0$ upon turning anticlockwise (clockwise).

Partition function

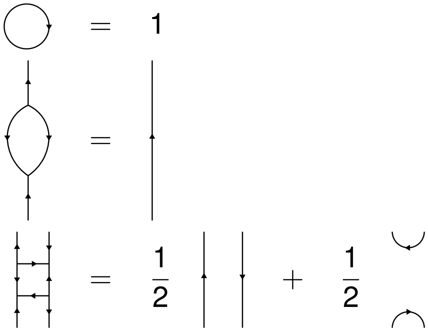
$$Z_{\mathrm{spin}} = 3 \sum_{c \in K} x_1^{N_1} x_2^{N_2}$$

- Equivalent to web model if $w'_{\mathrm{K}}(c) := (yz)^{N_V} w_{\mathrm{K}}(c) = 1$ for any c.

Equivalence at a special point:

$$q = e^{i\frac{\pi}{4}} \,,$$
$$yz = 2^{-\frac{1}{2}} \,.$$

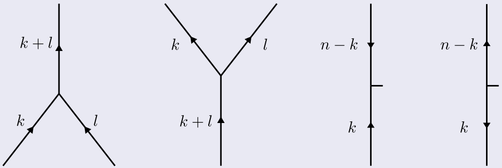
Proof: Absorb y and z into the vertices. Use $[3]_q = 1$ and $[2]_q = \sqrt{2}$. Then the rules become probabilistic:



Generalisation to $U_{-q}(\mathfrak{sl}_n)$ symmetry

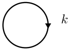
Based on spider defined by [Cautis-Kamnitzer-Morrison]

- Webs are still closed, oriented, planar, trivalent graphs. But not always bipartite as before.
- Edges carry an integer flow $i \in [\![1, n-1]\!]$.
- Generators conserve flow, or change by n due to 'tags':

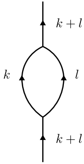


- Flow labels fundamental representations of $U_{-q}(\mathfrak{sl}_n)$. Orientation distinguishes between dual or not.

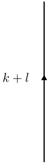
Rules (mirrored and the arrow-reversed versions omitted):

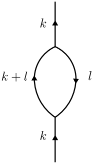


$= \begin{bmatrix} n \\ k \end{bmatrix}_q$

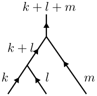


$= \begin{bmatrix} k+l \\ k \end{bmatrix}_q$





$= \begin{bmatrix} n-k \\ l \end{bmatrix}_q$



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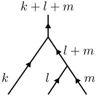
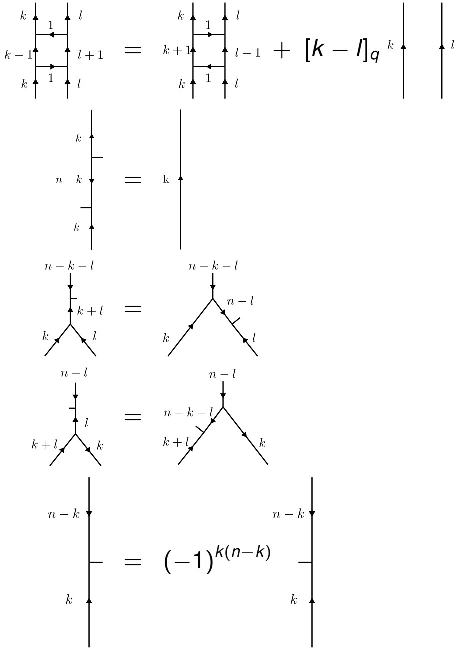



k l 1 k − 1 l + 1 1 k l
=
k l 1 k + 1 l − 1 1 k l
+ [k − l]_q k l
k n − k k = k
n − k − l k + l k l = n − k − l n − l k l
n − l l k + l k = n − l n − k − l k + l k
n − k k = (−1)^{k(n−k)} n − k k

Short summary of results

- Case $n = 3$ gives back the Kuperberg web model.
- Case $n = 2$ gives the well-known Nienhuis loop model.
- Special point $q = e^{i\frac{\pi}{n+1}}$ equivalent to $\mathbb{Z}_n$ spin model.

Outlook this far

- $\mathbb{Z}_n$ spin models known to be critical (with appropriate weights) [Fateev-Zamolodchikov]
- Therefore expect the special point to be critical for any n.
- Web models likely have larger critical manifold (vary q and x, y, z).

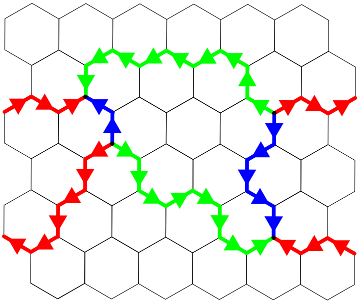
To investigate criticality we wish a local formulation

- Analogous to vertex models for Potts and O(N) models.
- The locality enables us to define a transfer matrix.
  - Good for numerical study and makes contact with integrability.
  - Non-local TM also possible for loops, but seems difficult for webs.
- Vertex model defines equivalent ($n-1$ component) height model.
  - Starting point for Coulomb gas construction and CFT identification.

Local reformulation for $U_{-q}(\mathfrak{sl}_3)$ web model

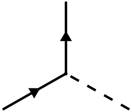
Basic idea

- Decorate bonds by extra degrees of freedom ($n = 3$ colours).
- They allow to redistribute the web weight locally.
- Summing over colours gives back the undecorated model.
- Each link can now be in 7 different states.

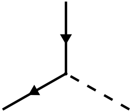


Reminder for $n = 2$ loop case

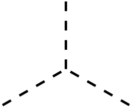
- Write $N = q + q^{-1} = [2]_q$.
- Orient each loop in two ways (clockwise, anticlockwise).
- Give $q^{-\frac{\theta}{2\pi}}$ to a left-turn through angle θ.



$= xq^{-\frac{1}{6}}$,



$= xq^{\frac{1}{6}}$,



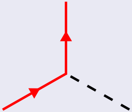
$= 1$

Remark

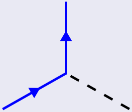
Better to think of these two ‘orientations’ as colourings. The analogue for $n = 3$ is the three colours. The orientations distinguish (for $n \geq 3$) fundamental and dual fundamental, but for $n = 2$ the two coincide!

Basic idea for $n = 3$

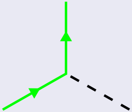
- Three colours RBG.
- Weight $q^2 + 1 + q^{-2} = [3]_q$ for sum over (say) clockwise loop. Opposite phases for an anticlockwise loop (same sum). Set $x_1 = x_2$ for convenience.



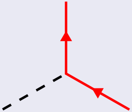
$= xq^{-\frac{1}{3}}$,



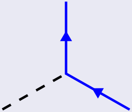
$= x$,



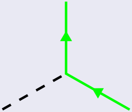
$= xq^{\frac{1}{3}}$



$= xq^{\frac{1}{3}}$,

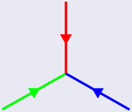


$= x$,

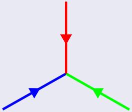


$= xq^{-\frac{1}{3}}$

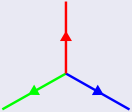
The ‘tricky’ part involving vertices



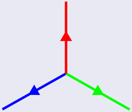
$$= zx^{\frac{3}{2}} q^{-\frac{1}{6}} \,,$$



$$= zx^{\frac{3}{2}} q^{\frac{1}{6}}$$

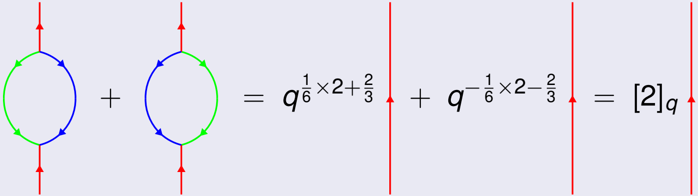


$$= yx^{\frac{3}{2}} q^{\frac{1}{6}} \,,$$

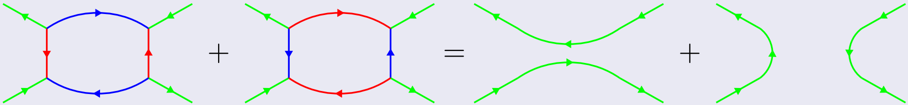


$$= yx^{\frac{3}{2}} q^{-\frac{1}{6}}$$

Proof for the ‘digon’ rule (2)

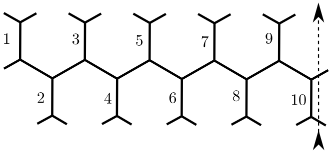


Proof for the ‘square’ rule (3)



Other colours / arrangements of external legs work similarly.

Defining the transfer matrix



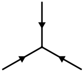
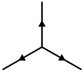
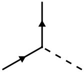
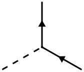
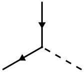
Built of pieces $t_{(1)} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and $t_{(2)} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, so that

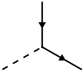
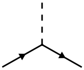
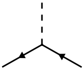
$$T = \left(\prod_{k=0}^{L-1} t_{2k+1} \right) \left(\prod_{k=1}^{L-1} t_{2k} \right)$$

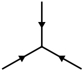
with $t = t_{(2)} t_{(1)}$. Write t_i, with i specifying the position.

Technically T is an intertwiner of the quantum group action.

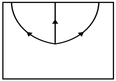
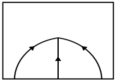
- Let $\{v_1, v_2, v_3\}$ be a basis of the first fundamental V_1 of $U_{-q}(\mathfrak{sl}_3)$.
- Let $\{w_1, w_2, w_3\}$ be a basis of the dual V_1^*, so that $w_i(v_j) = \delta_{ij}$.
- Relate $\{v_1, v_2, v_3, w_1, w_2, w_3, 1\}$ to the basis $\{|{\color{red}\uparrow}\rangle, |{\color{blue}\uparrow}\rangle, |{\color{green}\uparrow}\rangle, |{\color{red}\downarrow}\rangle, |{\color{blue}\downarrow}\rangle, |{\color{green}\downarrow}\rangle, |\ \rangle\}$ of coloured arrows.
  Amounts to drawing each link vertically and providing the corresponding powers of q.
- Draw the diagrams of all transitions in $t_{(1)}$ and $t_{(2)}$. For instance:

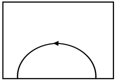
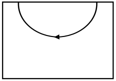
$t_{(1)} = z x_1 x_2^{\frac{1}{2}}$  $+\, y x_1^{\frac{1}{2}} x_2$  $+\, x_1$  $+\, x_1$  $+\, x_2$ 

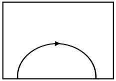
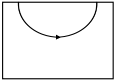
$+\, x_2$  $+\, x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$  $+\, x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$  $+$ 

- Let us have a look at just the first term! 

- Express each diagram in terms of the elementary blocks (maps)

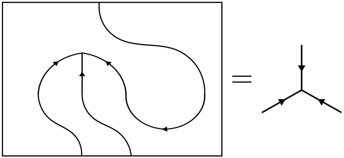
$v =$  $w =$ 

$\mathrm{ev} =$  $\mathrm{coev} =$ 

$\widetilde{\mathrm{ev}} =$  $\widetilde{\mathrm{coev}} =$ 

Their expressions follow from quantum group considerations.

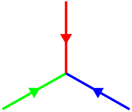
- The first term is the composition of coev and w:

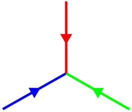


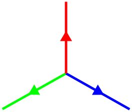
- In the bases $\{|\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle\}$ of $V_1 \otimes V_1$ and $\{|\downarrow\rangle, |\downarrow\rangle, |\downarrow\rangle\}$ of V_1^*, we finally get

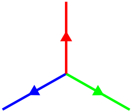
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{6}} & 0 & q^{-\frac{1}{6}} & 0 \\ 0 & 0 & q^{\frac{1}{6}} & 0 & 0 & 0 & q^{\frac{1}{6}} & 0 & 0 \\ 0 & q^{\frac{1}{6}} & 0 & q^{-\frac{1}{6}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Looks familiar?
- Hint:


$= zx^{\frac{3}{2}}q^{-\frac{1}{6}}$,


$= zx^{\frac{3}{2}}q^{\frac{1}{6}}$


$= yx^{\frac{3}{2}}q^{\frac{1}{6}}$,

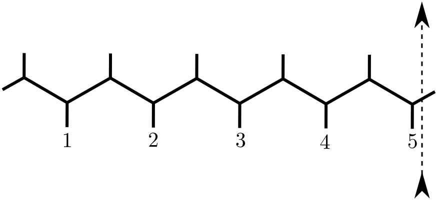

$= yx^{\frac{3}{2}}q^{-\frac{1}{6}}$

Summary of this technical part

- The diagrams are intertwiners of $U_{-q}(\mathfrak{sl}_3)$.
- We can compute all elements of T in this way.
- We are now ready to diagonalise T numerically.

Phase diagram of the web model

- More efficient to use the geometry

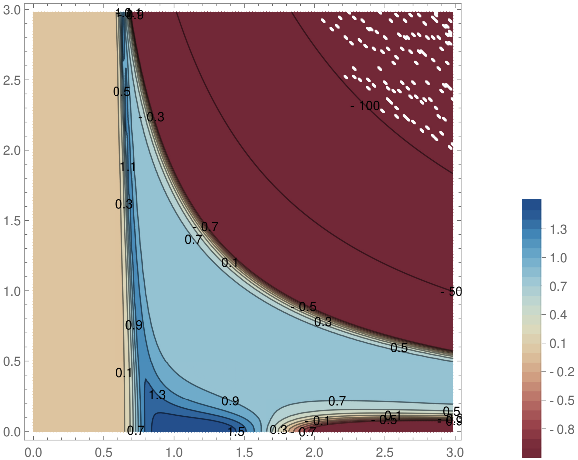


- Connection to the (effective) central charge of CFT:

$$f_L = -\frac{2}{\sqrt{3}L} \log(\Lambda_{\max}) \,,$$

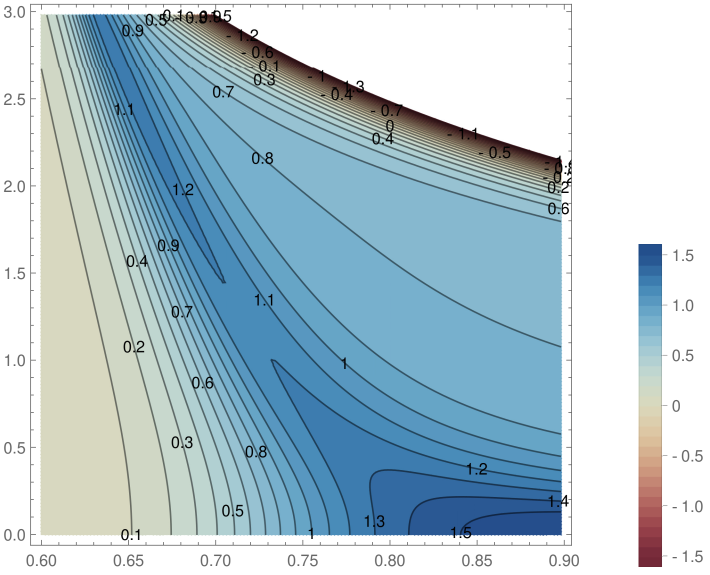
$$f_L = f_\infty - \frac{\pi c_{\rm eff}}{6L^2} + o\left(\frac{1}{L^2}\right) \,.$$

$c_{\rm eff}$ for $q = e^{i\pi/5}$ in the $(\sqrt{x}, y)$ plane



- Based on sizes $L = 5$ and $L = 6$.
- Coulomb gas prediction: dilute $c = \frac{4}{5}$ and dense $c = \frac{6}{5}$ phases.

Zoom of the interesting region



Coulomb gas predictions

Set $q = \mathrm{e}^{i\gamma}$ with $\gamma \in [0, \pi]$.

CG of two bosons compactified on the root lattice of sl_3

Coupling constant $g = 1 \pm \frac{\gamma}{\pi}$ in dilute $(+)$ or dense $(-)$ phase.
Central charge $c = 2 - 24\frac{(g-1)^2}{g}$.

Example I: $\gamma = \frac{\pi}{5}$ as in numerical figures

Coupling constant $g = \frac{6}{5}$ (dilute) or $g = \frac{4}{5}$ (dense).
Central charge $c = \frac{6}{5}$ (dilute) or $c = \frac{4}{5}$ (dense).

Example II: $\gamma = \frac{\pi}{4}$ as at special point

Coupling constant $g = \frac{5}{4}$ (dilute) or $g = \frac{3}{4}$ (dense).
Central charge $c = \frac{4}{5}$ (dilute) or $c = 0$ (dense).
Corresponds to $Q = 3$ Potts model at $T = T_{\mathrm{c}}$ or $T = \infty$.

Summary

- Web models generalise the $U_{-q}(\mathfrak{sl}_2)$ loop model to $U_{-q}(\mathfrak{sl}_n)$.
- Geometrical content with applications to $\mathbb{Z}_n$ spin interfaces.
- Dense and dilute critical points for $q = e^{i\gamma}$ and $\gamma \in [0, \pi]$.

In the pipeline

- Coulomb gas description and fractal dimension of defects
- Statistical models for other spiders
- Detailed representation theoretical study

Further possibilities

- Link to integrable models
- SLE-like description of branching curves?