Counting statistics of fermions in a trap and the Gaussian free field

Pierre Le Doussal
(LPT-ENS, Paris)

collaborations

- Yan Fyodorov (King's college)
- Naftali Smith (Ben Gurion U)
- Satya N. Majumdar (LPTMS, Orsay U)
- Gregory Schehr (LPTHE, Jussieu)

Counting statistics for noninteracting fermions in a d-dimensional potential

NS,PLD,SM,GS Phys. Rev. E 103, L030105 (2021)

Statistics of Extremes in Eigenvalue-Counting Staircases

YF,PLD Phys. Rev. Lett. 124, 210602 (2020)

- Xiangyu Cao

Full counting statistics for interacting trapped fermions

NS,PLD,SM,GS SciPost Phys. 11, 110 (2021)

Outline

- Non-interacting fermions d=1 in external (confining) potential V(x)

  mean density, bulk, edge

  counting statistics $\mathcal{N}_{[a,b]}$ number of fermions in [a,b]

variance: microscopic, macroscopic scales, edge

semi-classical calculation => Gaussian free field

application: variance sphere in higher dimension d>1 for central potential V(r)

for some special V(x)

↓

random matrix theory

$\beta = 2$

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- Extrema of counting staircases

circular unitary ensemble CUE

$\mathrm{C}\beta\mathrm{E}$

height field $\theta \to \mathcal{N}_{[\theta_A,\theta]} - \mathbb{E}(\mathcal{N}_{[\theta_A,\theta]})$

log-correlated field

pinned fBM H=0

stat-mech freezing

moments of partition sum = FCS of (interacting) fermions

Q: statistics of maximum

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- Interacting fermions in d=1

for some special V(x), interactions W(x,x') obtain full counting statistics

higher cumulants of $\mathcal{N}_{[a,b]}$

$\beta = 2$ entanglement entropy d=1 and d>1 in presence of V(x) and W=0

Counting statistics of non-interacting fermions in external potential V(x)

N. Smith, PLD, S. Majumdar, G. Schehr Phys. Rev. E 103, L030105 (2021)

Why the recent interest in non-interacting fermions ?

in a trap V(x): Fermi gas inhomogeneous and has an edge

at edge density vanishes:
strong quantum and thermal fluctuations

=> LDA, linear bosonization fail

Pauli exclusion principle
(effective repulsive interaction)

need other methods => universal edge correlations

bulk: inhomogeneous bosonisation

relations to RMT (ground state) and to KPZ equation (edge finite T)

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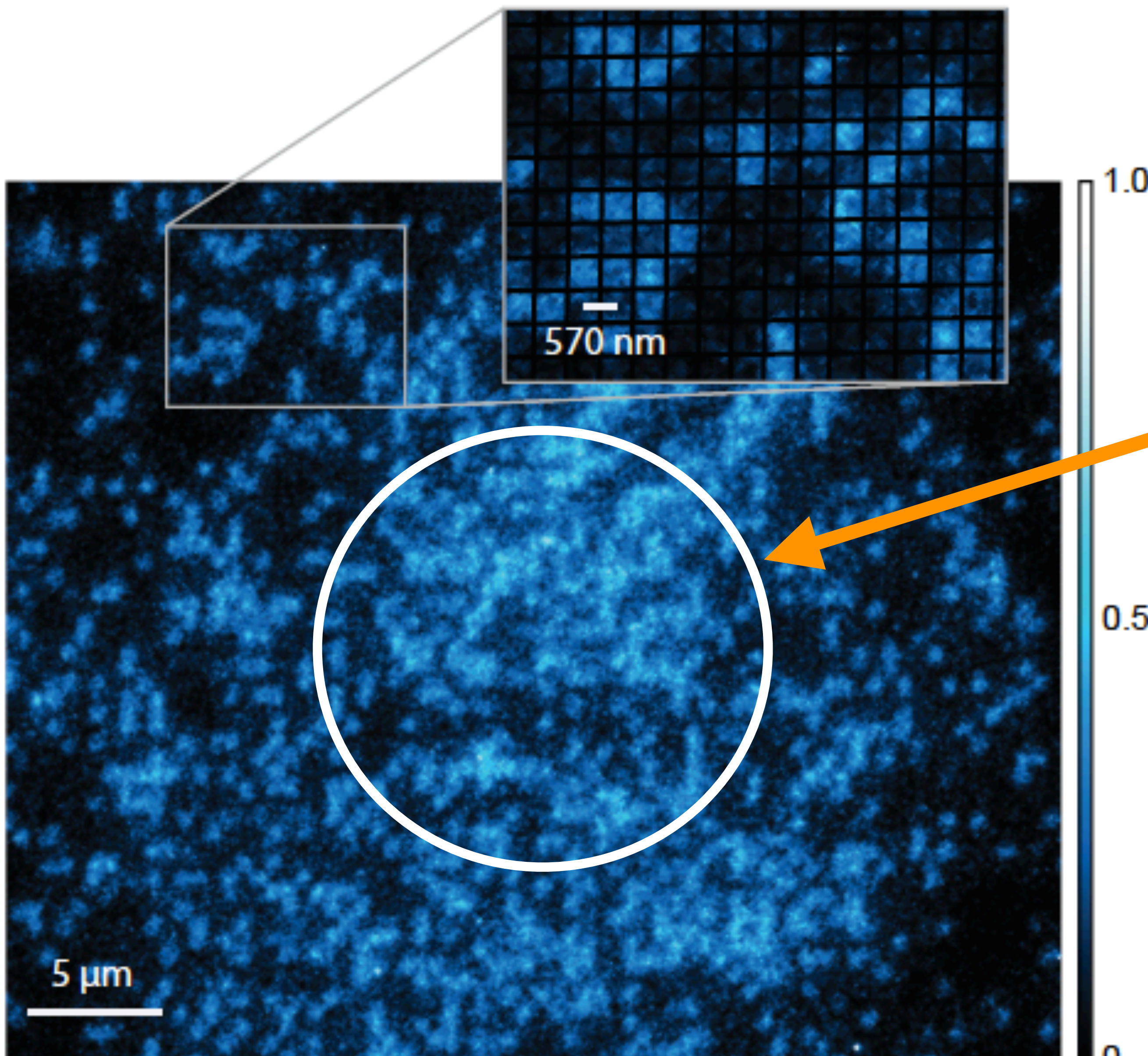
Experiments on ultra-cold atoms: confining potentials

- harmonic trap but can tune arbitrary shape potential
- can reach non-interacting limit for spinless fermions
- can reach low temperature, low entropy $\frac{S}{N} \simeq k_{\rm B} \frac{\pi^2}{3} \frac{T}{T_F}$ $\qquad \frac{T}{T_F} \approx 0.15$ 2D, ≈ 0.3 1D
- measure momentum distribution from time of flight
- can image position of each fermion in 2D or 1D (quantum microscope)

Fermionic quantum gas microscope

L. W. Cheuk *et al.*, Phys. Rev. Lett. **114** , 193001, (2015)
E. Haller *et al.*, Nature Physics **11**, 738 (2015).
M. F. Parsons *et al.*, Phys. Rev. Lett. **114**, 213002 (2015)
Omram et al. Phys. Rev. Lett. 115, 263001 (2015)

- Direct imaging of spatial fluctuations of the positions of fermions
- Counting statistics of fermions



domain D

How many fermions N_D are there?

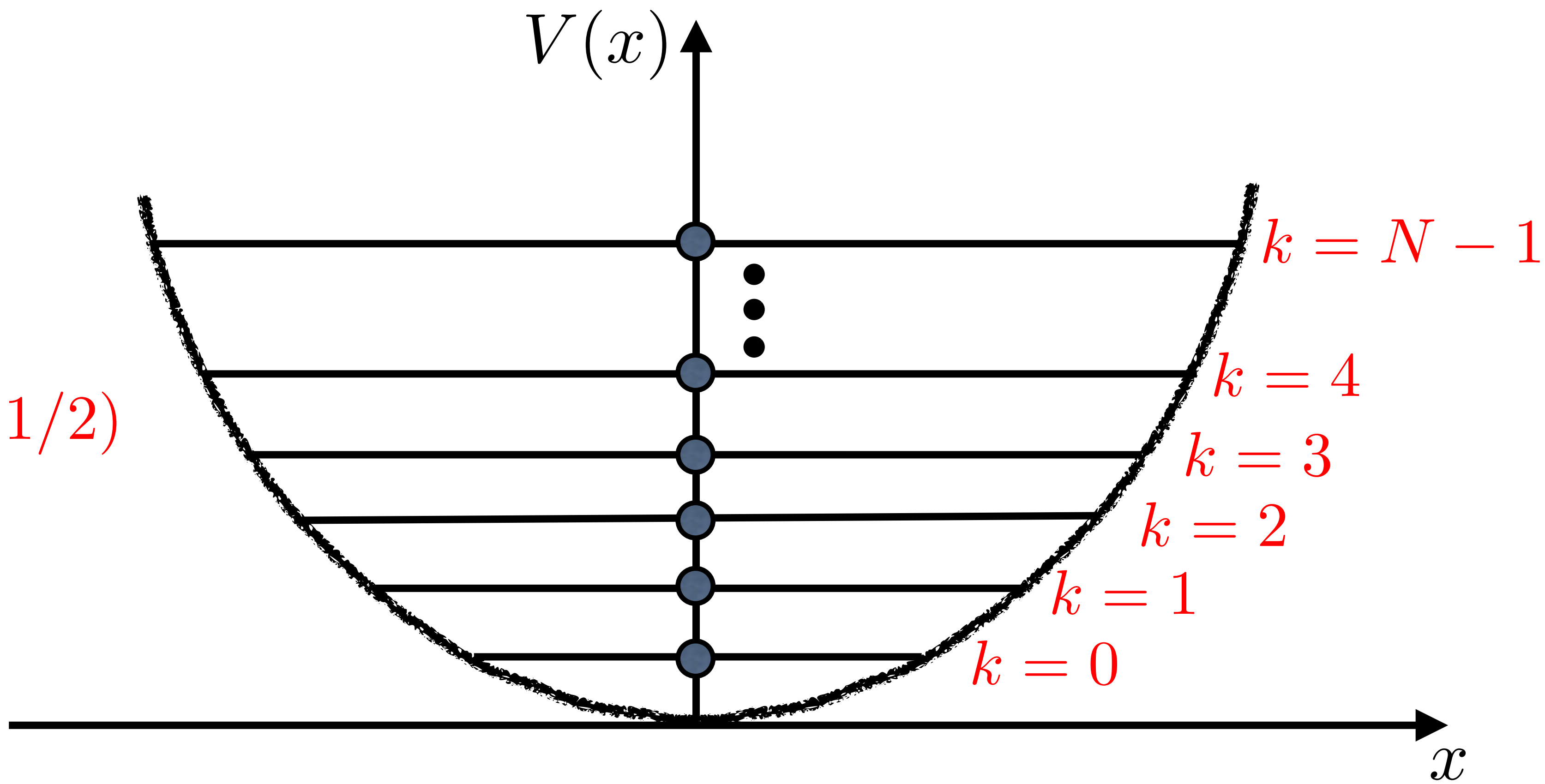
M. Greiner et al., PRL 2015

single-atom sensitivity, sub-micron resolution and more than 95% fidelity

N spinless noninteracting fermions in a 1d harmonic trap at T=0

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$$

$\epsilon_k = \hbar\omega(k + 1/2)$



- The N-particle GS wave function is given by a $N \times N$ Slater determinant

$$\Psi_0(x_1, x_2, \cdots, x_N) = \frac{1}{\sqrt{N!}}\det[\varphi_i(x_j)] \qquad \begin{array}{l} 0 \le i \le N-1 \\ 1 \le j \le N \end{array}$$

$$\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi}2^k k!}\right]^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_k(\alpha x)$$

Hermite polynomial

$\alpha = \sqrt{m\omega/\hbar}$

Ground state energy

$$E_0 = \sum_{k=0}^{N-1}\epsilon_k = \frac{N^2}{2}$$

Connection with RMT HO - GUE(N)

ground state wavefunction $\Psi_0(x_1, x_2, \cdots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)]$ $\quad 0 \le i \le N-1$, $\quad 1 \le j \le N$

$$\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi} 2^k k!}\right]^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_k(\alpha x)$$

$$\Longrightarrow \Psi_0(x_1, x_2, \cdots, x_N) \propto e^{-\frac{\alpha^2}{2}(x_1^2 + \cdots + x_N^2)} \det\left[H_i(\alpha x_j)\right]$$

Hermite polynomial of degree i

$$\prod_{1 \le i < j \le N} (x_j - x_i)$$

vandermonde

- Probability density function (PDF) of the positions $x_i's$

$$|\Psi_0(x_1, \cdots, x_N)|^2 = \frac{1}{z_N(\alpha)} \prod_{i<j} (x_i - x_j)^2 e^{-\alpha^2 \sum_{i=1}^N x_i^2}$$

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- Let M be a $N \times N$ random Hermitian matrix with Gaussian (complex) entries. The PDF of the (real) eigenvalues $\lambda_i' s$ is given by

$$P_{\rm joint}(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-\sum_{i=1}^N \lambda_i^2} \qquad \beta = 2$$

The positions of the fermions behave statistically like the eigenvalues of GUE random matrices

$$(\alpha\, x_1, \alpha\, x_2, \cdots \alpha\, x_N) \stackrel{d}{=} (\lambda_1, \lambda_2, \cdots, \lambda_N)$$

Spatial correlations of NI fermions in harmonic trap at T=0 can be obtained from known results in RMT

Eisler '13/Marino, Majumdar, G. S., Vivo '14/Calabrese, Le Doussal, Majumdar '15

Other potentials $V(x)$ related to matrix models

- Laguerre unitary ensemble can be realized with

$$V(x) = \frac{\alpha(\alpha-1)}{x^2} + \beta x^2 \ , \ x > 0$$

- More general Jacobi unitary ensemble can be realized with

$$V(x) = \begin{cases} \dfrac{\alpha^2 - 1/4}{8\sin^2(x/2)} + \dfrac{\beta^2 - 1/4}{8\cos^2(x/2)} \ , & x \in (0,\pi) \\ \\ +\infty \ , & x \notin (0,\pi) \end{cases}$$

Lacroix-A-Chez-Toine, Le Doussal, Majumdar, G. S., EPL '17

Correlations of NI fermions in general potential

$$\hat{H} = \frac{p^2}{2} + V(x) \qquad \psi_k(x) \qquad \{\epsilon_k\}_{k=1,2,\dots}$$

Slater determinant $\Psi_0(x_1,\dots,x_N) = \frac{1}{\sqrt{N!}} \det_{1\le i,j\le N} \psi_i(x_j)$

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$$\det_{1\le i,k\le N} \psi_k^*(x_i) \det_{1\le k,j\le N} \psi_k(x_j) = \det_{1\le i,j\le N} K_N(x_i,x_j)$$

$$|\Psi_0(x_1,\dots,x_N)|^2 = \frac{1}{N!} \det_{1\le i,j\le N} K_N(x_i,x_j)$$

Kernel $K_N(x,y) = \sum_{k=1}^{N} \psi_k^*(x)\psi_k(y)$

$$K_\mu(x,y) = \sum_{k\ge 1} \theta(\mu-\epsilon_k)\psi_k^*(x)\psi_k(y)$$

μ Fermi energy

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μ Fermi energy

all n-point correlations functions are determinants

$$R_n(x_1,\cdots,x_n) = \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} dx_{n+1} \cdots \int_{-\infty}^{\infty} dx_N |\psi_0(x_1,\cdots,x_n,x_{n+1},\cdots,x_N)|^2$$

DETERMINANTAL POINT PROCESS $\longleftrightarrow$ $$R_n(x_1,\cdots,x_n) = \det_{1\le i,j\le n} K_N(x_i,x_j)$$

kernel is self-reproducing

$$\int dy K_N(x,y)K_N(y,z) = K_N(x,z)$$

n=1 average density is

$$\rho(x) = \sum_{i=1}^{N} \langle \delta(x-x_i)\rangle_0 = K_N(x,x)$$

Counting statistics

1) mean number $\langle \mathcal{N}_D \rangle_0 = \int_D dx \rho(x)$ $\rho(x) = \sum_{i=1}^{N} \langle \delta(x - x_i) \rangle_0$

$N \gg 1$ Fermi energy μ is large

local Fermi momentum

in the bulk $V(x) < \mu$ $\xrightarrow[\text{semi-classics}]{\text{LDA}}$ $\rho(x) \simeq \frac{k_F(x)}{\pi}$ $k_F(x) = \sqrt{2(\mu - V(x))}$

LDA density vanishes at (and beyond) the edges $V(x_\pm) = \mu$ $N \simeq \frac{1}{\pi} \int dx \sqrt{2(\mu - V(x))_+}$

HO/GUE density/N -> semi-circle $\mu \simeq N$ $x_\pm = \pm\sqrt{2N}$

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2) variance

$$\boxed{\mathrm{Var}\mathcal{N}_D = \int_D dx \int_{\bar{D}} dy \, K_\mu(x,y)^2}$$

- microscopic scales $|x - y| \sim 1/k_F(x)$ $K_N(x,y)$ => sine kernel (universal)

Dyson, Mehta '62
GUE

$$\mathrm{Var}\,\mathcal{N}_{[a,b]} \simeq \frac{1}{\pi^2}\left[\log\left(\sqrt{2N - a^2}\,|b-a|\right) + c_2\right]$$

$\sqrt{N}|b-a| = O(1) \gg 1$

$c_2 = \gamma_E + 1 + \log 2$

- macroscopic scales ? $x^+ - x^-$

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- macroscopic scales ? $x^+ - x^-$ $\quad$ height field $h(x) = \mathcal{N}_{]-\infty,x]}$

height covariance $H(x,y) = \mathrm{Cov}[h(x), h(y)]$

$$\boxed{K_\mu(x,y)^2 = -\partial_x \partial_y H(x,y) + \delta(x-y)\rho(x)}$$

$$\mathrm{Var}\mathcal{N}_{[a,b]} = H(a,a) + H(b,b) - 2H(a,b)$$

$$\mathrm{Var}\mathcal{N}_{]-\infty,a]} = \mathrm{Var}\mathcal{N}_{[a,+\infty[} = H(a,a)$$

Kernel on macroscopic scales

$$\hat{H} = \frac{p^2}{2} + V(x) \qquad \psi_k(x) \quad \{\epsilon_k\}_{k=1,2,\ldots}$$

$$K_\mu(x,y) = \sum_{k=1}^{N} \psi_k^*(x)\psi_k(y)$$

semi-classical (WKB) form for eigenfunctions

$$\psi_k(x) \simeq \frac{C_k}{\left[2\left(\epsilon_k - V(x)\right)\right]^{1/4}} \sin\left(\phi_k(x) + \frac{\pi}{4}\right)$$

$$C_k^2 = \frac{2}{\pi}\frac{d\epsilon_k}{dk} \qquad \phi_k(x) = \int_{x^-}^{x} dz \sqrt{2(\epsilon_k - V(z))}$$

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$O(N)$

$$\phi_N(x) = \int_{x^-}^{x} dx\, k_F(x)$$

$$k_F(x) = \sqrt{2(\mu - V(x))}$$

two turning points

$$k_F(x_\pm) = 0$$

$$K_\mu(x,y) = \sum_{k=1}^{N} \psi_k^*(x)\psi_k(y)$$

relabel from Fermi level $\epsilon_N = \mu$

$$k = N - m$$

$$K_\mu(x,y) = \sum_{m=0}^{N-1} \psi_{N-m}^*(x)\psi_{N-m}(y)$$

$$\frac{dk_F(x)}{d\mu} = \frac{1}{\sqrt{2(\mu - V(x))}} = \frac{1}{k_F(x)}$$

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expand WKB phase around Fermi level

$$\phi_{N-m}(x) = \phi_N(x) - m\frac{d\phi_N(x)}{dN} + o(1)$$

$$= \underbrace{\phi_N(x)}_{O(N)} - \underbrace{m\theta_x}_{O(1)} + o(1)$$

$$\frac{d\phi_N}{dN} = \frac{d\phi_N}{d\mu} \Big/ \frac{dN}{d\mu}$$

$$\int_{x^-}^{x} \frac{dz}{k_F(z)} \Big/ \int_{x^-}^{x_+} \frac{dz}{\pi k_F(z)}$$

define

$$\boxed{\theta_x = \pi \frac{\int_{x^-}^{x} dz/k_F(z)}{\int_{x^-}^{x^+} dz/k_F(z)} \qquad \begin{cases} \theta_{x^-} = 0 \\ \theta_{x^+} = \pi \end{cases}}$$

$$\frac{d\theta_x}{dx} = \frac{d\mu}{dN}\frac{1}{k_F(x)}$$

Kernel on macroscopic scales

$$K_\mu(x,y) = \sum_{m=0}^{N-1} \psi^*_{N-m}(x)\psi_{N-m}(y)$$

approximate prefactors

$$C^2_{N-m} \simeq C^2_N \simeq \frac{2}{\pi}\frac{d\mu}{dN}$$

$$\frac{1}{(2(\epsilon_{N-m} - V(x))^{1/4}} \simeq \frac{1}{\sqrt{k_F(x)}}$$

insert WKB form

$$K_\mu(x,y) \simeq \frac{2d\mu/dN}{\pi\sqrt{k_F(x)k_F(y)}} \sum_{m\geq 0} \sin(\phi_N(x) - m\theta_x + \frac{\pi}{4}) \sin(\phi_N(y) - m\theta_y + \frac{\pi}{4})$$

perform geometric sum over m

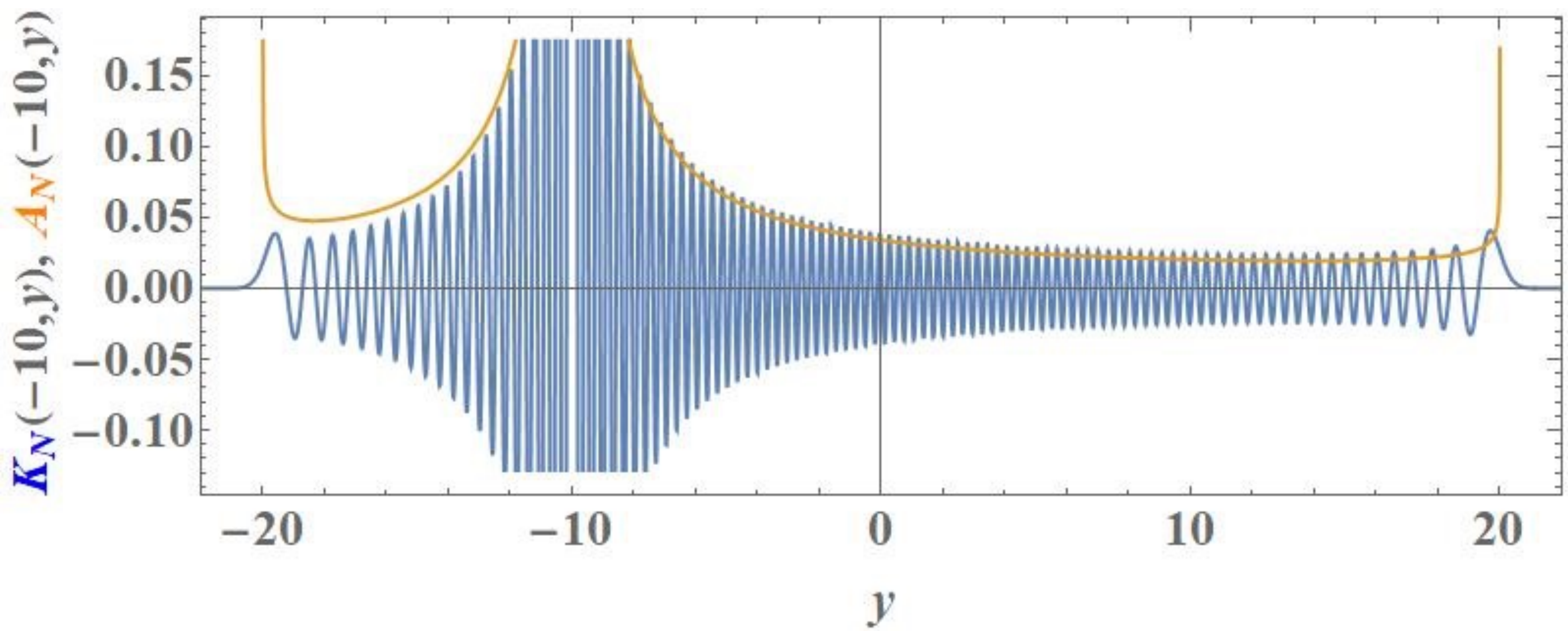
$$\simeq \frac{d\mu/dN}{2\pi\sqrt{k_F(x)k_F(y)}} \sum_{\sigma=\pm 1} \frac{\sin(\tilde{\phi}_N(x) - \sigma\tilde{\phi}_N(y))}{\sin\left(\left(\theta_x - \sigma\theta_y\right)/2\right)}$$

$$\tilde{\phi}_N(x) = \phi_N(x) + \frac{\theta_x}{2} - \frac{\pi}{4}$$

$O(N) \quad O(1)$

1d HO kernel

Harmonic oscillator



$K_N(x,y)$ and $A_N(x,y)$ vs. y for $N = 200$, $x = -10$
(edges at $y = \pm\sqrt{2N} = \pm 20$)

envelope

$$K_N(x,y) \simeq A_N(x,y) f_N(x,y)$$

$f_N(x,y)$ rapidly oscillating
with unit amplitude

Kernel on macroscopic scales

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$$K_\mu(x,y) \simeq \frac{2d\mu/dN}{\pi\sqrt{k_F(x)k_F(y)}} \sum_{m\geq 0} \sin(\phi_N(x) - m\theta_x + \frac{\pi}{4}) \sin(\phi_N(y) - m\theta_y + \frac{\pi}{4})$$

perform geometric sum over m

$$\simeq \frac{d\mu/dN}{2\pi\sqrt{k_F(x)k_F(y)}} \sum_{\sigma=\pm 1} \frac{\sin(\tilde\phi_N(x) - \sigma\tilde\phi_N(y))}{\sin\left((\theta_x - \sigma\theta_y)/2\right)}$$

$$\tilde\phi_N(x) = \phi_N(x) + \frac{\theta_x}{2} - \frac{\pi}{4}$$

$O(N) \quad O(1)$

$$K_\mu(x,y)^2 \simeq \frac{(d\mu/dN)^2}{2\pi^2 k_F(x)k_F(y)} \frac{1-\cos(\theta_x)\cos(\theta_y)}{(\cos\theta_x - \cos\theta_y)^2}$$

$$= \frac{1}{2\pi^2}\frac{d\theta_x}{dx}\frac{d\theta_y}{dy}\frac{1-\cos(\theta_x)\cos(\theta_y)}{(\cos\theta_x - \cos\theta_y)^2}$$

is a total derivative !

$$\sin(\tilde\phi_N(x) - \tilde\phi_N(y))^2 \to 1/2$$

$$\sin(\tilde\phi_N(x) - \tilde\phi_N(y))\sin(\tilde\phi_N(x) + \tilde\phi_N(y))$$

$$\to \cos(2\phi_N(x)) - \cos(2\phi_N(y)) \to 0$$

$$\sum_{\sigma=\pm 1}\frac{1}{\sin^2(\frac{\theta_x - \sigma\theta_y}{2})} = 4\frac{1-\cos(\theta_x)\cos(\theta_y)}{(\cos\theta_x - \cos\theta_y)^2}$$

Kernel on macroscopic scales

$$C_{N-m}^2 \simeq C_N^2 \simeq \frac{2}{\pi}\frac{d\mu}{dN}$$

approximate prefactors

$$K_\mu(x,y) = \sum_{m=0}^{N-1} \psi_{N-m}^*(x)\psi_{N-m}(y)$$

$$\frac{1}{(2(\epsilon_{N-m} - V(x))^{1/4}} \simeq \frac{1}{\sqrt{k_F(x)}}$$

insert WKB form

$$K_\mu(x,y) \simeq \frac{2d\mu/dN}{\pi\sqrt{k_F(x)k_F(y)}} \sum_{m\geq 0} \sin(\phi_N(x) - m\theta_x + \frac{\pi}{4})\sin(\phi_N(y) - m\theta_y + \frac{\pi}{4})$$

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$$\tilde\phi_N(x) = \phi_N(x) + \frac{\theta_x}{2} - \frac{\pi}{4}$$

$$O(N) \quad O(1)$$

$$\simeq \frac{d\mu/dN}{2\pi\sqrt{k_F(x)k_F(y)}} \sum_{\sigma=\pm1} \frac{\sin(\tilde\phi_N(x) - \sigma\tilde\phi_N(y))}{\sin\left((\theta_x - \sigma\theta_y)/2\right)}$$

$$\sin(\tilde\phi_N(x) - \tilde\phi_N(y))^2 \to 1/2$$

$$\sin(\tilde\phi_N(x) - \tilde\phi_N(y))\sin(\tilde\phi_N(x) + \tilde\phi_N(y))$$

$$\to \quad \cos(2\phi_N(x)) - \cos(2\phi_N(y)) \quad \to 0$$

$$K_\mu(x,y)^2 \simeq \frac{(d\mu/dN)^2}{2\pi^2 k_F(x)k_F(y)} \frac{1-\cos(\theta_x)\cos(\theta_y)}{(\cos\theta_x - \cos\theta_y)^2}$$

$$= \frac{1}{2\pi^2}\frac{d\theta_x}{dx}\frac{d\theta_y}{dy}\frac{1-\cos(\theta_x)\cos(\theta_y)}{(\cos\theta_x - \cos\theta_y)^2}$$

$$\sum_{\sigma=\pm1} \frac{1}{\sin^2(\frac{\theta_x - \sigma\theta_y}{2})} = 4\frac{1-\cos(\theta_x)\cos(\theta_y)}{(\cos\theta_x - \cos\theta_y)^2}$$

is a total derivative !

recall

$$K_\mu(x,y)^2 = -\partial_x\partial_y H(x,y) + \delta(x-y)\rho(x)$$

valid on mesoscopic/macroscopic scales in bulk

$$|x-y| \gg 1/k_F(x)$$

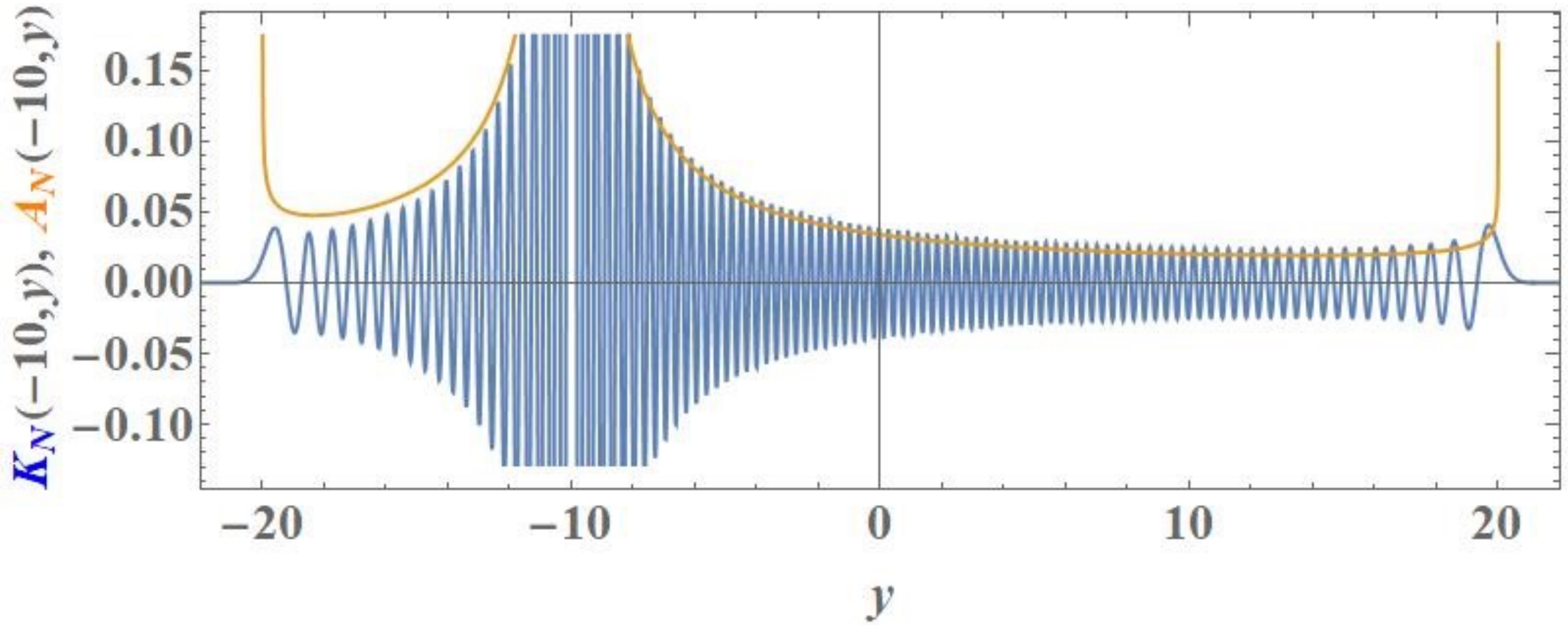
=> height covariance

$$\boxed{H(x,y) \simeq \frac{1}{2\pi^2}\left(\log\left|\sin\frac{\theta_x+\theta_y}{2}\right| - \log\left|\sin\frac{\theta_x-\theta_y}{2}\right|\right)}$$

assumed k_F(x) slowly varying

1d HO kernel

Harmonic oscillator



$K_N(x,y)$ and $A_N(x,y)$ vs. y for $N = 200$, $x = -10$
(edges at $y = \pm\sqrt{2N} = \pm 20$)

envelope

$$K_N(x,y) \simeq A_N(x,y) f_N(x,y)$$

$f_N(x,y)$ rapidly oscillating
with unit amplitude

- Using $N \gg 1$ Plancherel-Rotach asymptotics of Hermite polynomials [Forrester (2006)] (WKB approximation), one obtains (for $N \gg 1$):

$$A_N(x,y) = \frac{\sqrt{1-\frac{xy}{2N}}}{\pi|x-y|\left(1-\frac{x^2}{2N}\right)^{1/4}\left(1-\frac{y^2}{2N}\right)^{1/4}}$$

agrees with our general formula

$$\cos\theta_x = \frac{-x}{\sqrt{2\mu}}$$

$|x|, |y| < \sqrt{2N}$

GUE Brezin Zee (1993)
French Mello Pandey (1978) Beenakker (1994)

Recovering the sine kernel

$$\phi_N(x) = \int_{x^-}^{x} dx\, k_F(x)$$

towards microscopic scales $|x-y| \sim 1/k_F(x)$

$$\frac{d}{dx}\tilde{\phi}_N(x) \simeq k_F(x)$$

as y -> x only $\sigma = 1$ survives

$$K_\mu(x,y) \simeq \frac{d\mu/dN}{2\pi\sqrt{k_F(x)k_F(y)}} \sum_{\sigma=\pm 1} \frac{\sin(\tilde{\phi}_N(x) - \sigma\tilde{\phi}_N(y))}{\sin\left((\theta_x - \sigma\theta_y)/2\right)}$$

$$\tilde{\phi}_N(x) - \sigma\tilde{\phi}_N(y) \to (x-y)\frac{d\tilde{\phi}_N(x)}{dx}$$

$$\theta_x - \sigma\theta_y \to (x-y)\frac{d\theta_x}{dx}$$

$$\frac{d\theta_x}{dx} = \frac{d\mu}{dN}\frac{1}{k_F(x)}$$

$$\boxed{K_\mu(x,y) \simeq \frac{\sin\left(k_F(x)|x-y|\right)}{\pi|x-y|}}$$

$$\rho(x) = K_\mu(x,x) \simeq \frac{k_F(x)}{\pi}$$

$$\int_{-k_F}^{k_F} \frac{dk}{2\pi} e^{ik(x-y)}$$

Fermion height field for general V(x) and 2d Gaussian free field

$$h(x) = \mathcal{N}_{]-\infty,x]}$$

$$\theta_x = \pi \frac{\int_{x^-}^{x} dz/k_F(z)}{\int_{x^-}^{x^+} dz/k_F(z)} \quad \begin{cases} \theta_{x^-} = 0 \\ \theta_{x^+} = \pi \end{cases}$$

height covariance $\quad H(x,y) = \mathrm{Cov}[h(x), h(y)]$

$$H(x,y) \simeq \frac{1}{2\pi^2}\left(\log\left|\sin\frac{\theta_x+\theta_y}{2}\right| - \log\left|\sin\frac{\theta_x-\theta_y}{2}\right|\right)$$

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$$k_F(x) = \sqrt{2(\mu - V(x))}$$

$$H(x,y) = \frac{1}{\pi} C(z,w) \qquad C(z,w) = -\frac{1}{2\pi}\log\left|\frac{z-w}{z-\bar{w}}\right|$$

$$z = e^{i\theta_x}$$

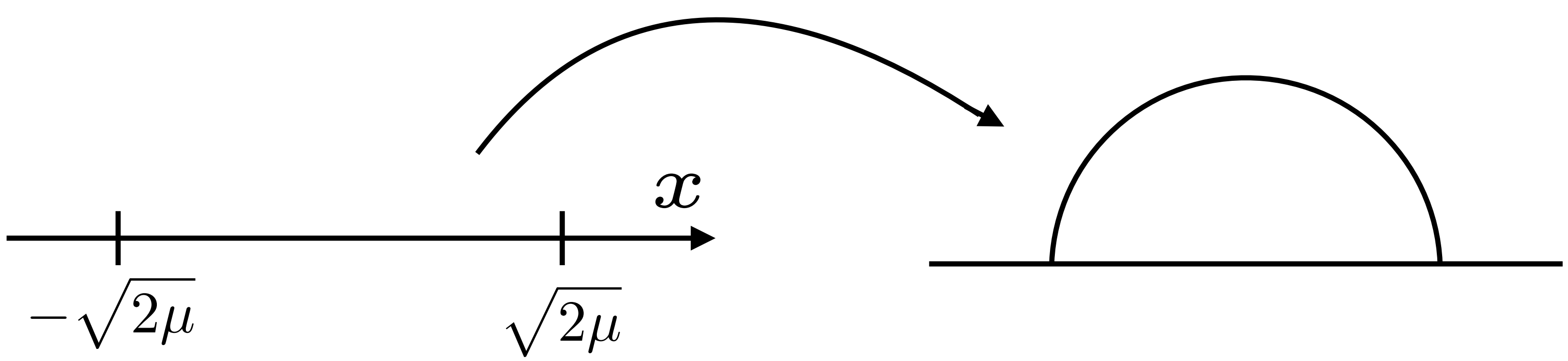
$$w = e^{i\theta_y}$$

2D Gaussian free field (GFF)

Gaussian process on 2d domain D
with covariance kernel
= Green function of Laplacian on D

here domain D is upper-half plane
Dirichlet BC on real axis

$$z = e^{i\theta_x}$$



map from interval to unit circle in complex upper-half plane

Fermion height field for HO/GUE and 2d Gaussian free field

Harmonic oscillator $V(x) = \frac{x^2}{2}$

GUE ensemble

$k_F(x) = \sqrt{2\mu - x^2}$

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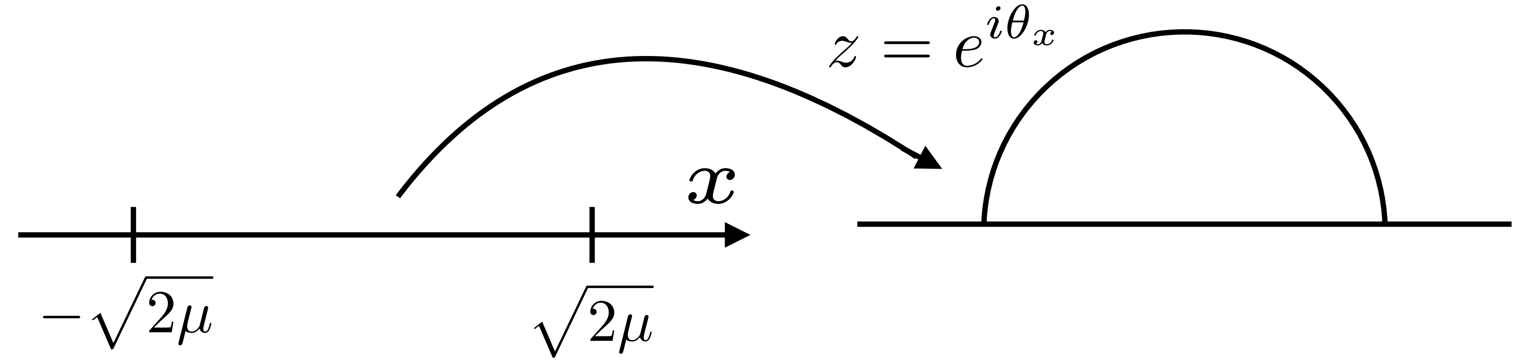
$$\boxed{\cos\theta_x = \frac{-x}{\sqrt{2\mu}}}$$



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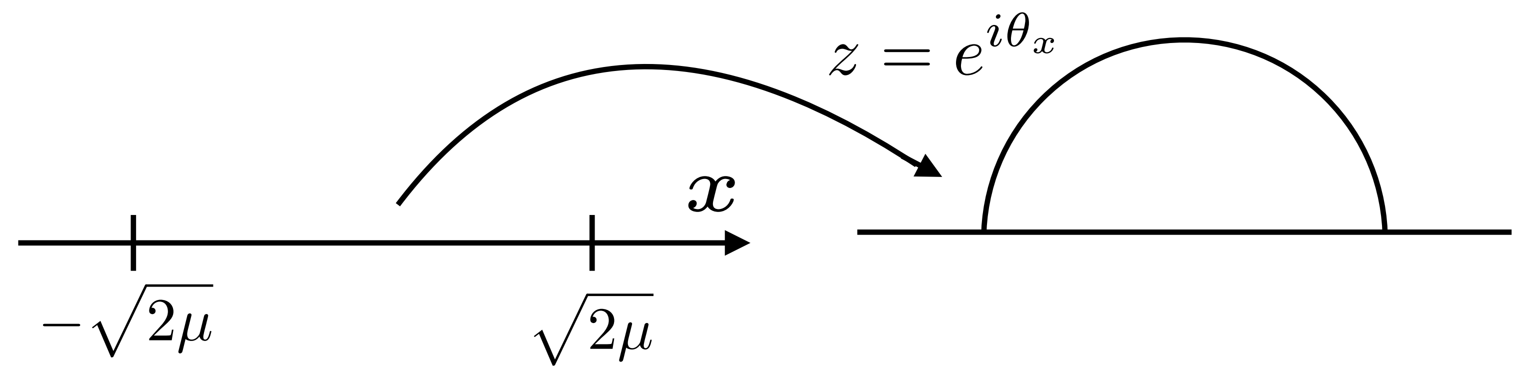
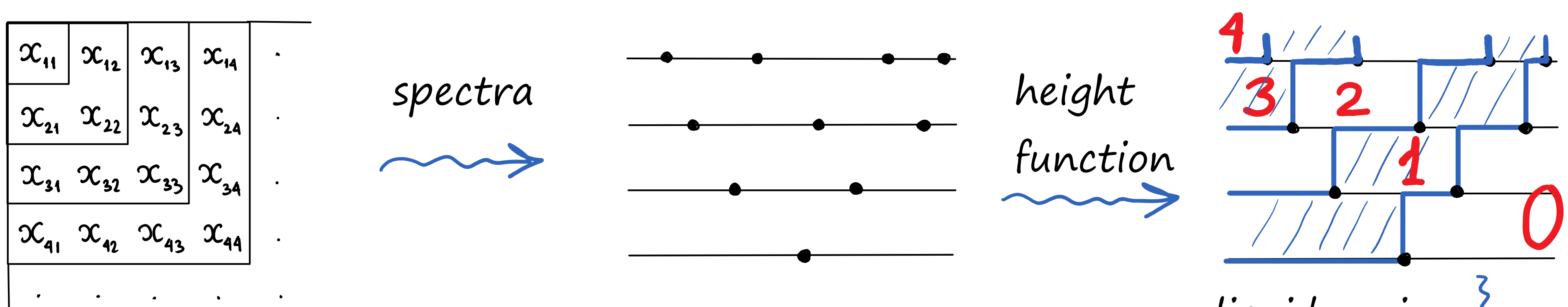


map from interval to unit circle in complex upper-half plane

Corners of random matrices and GFF

Alexei Borodin's talk



Theorem As $z \mapsto L^{-1}z$, $L \to \infty$,
Unscaled fluctuations $\Rightarrow$
Gaussian (massless)
Free Field on $\mathbb{H}$ with
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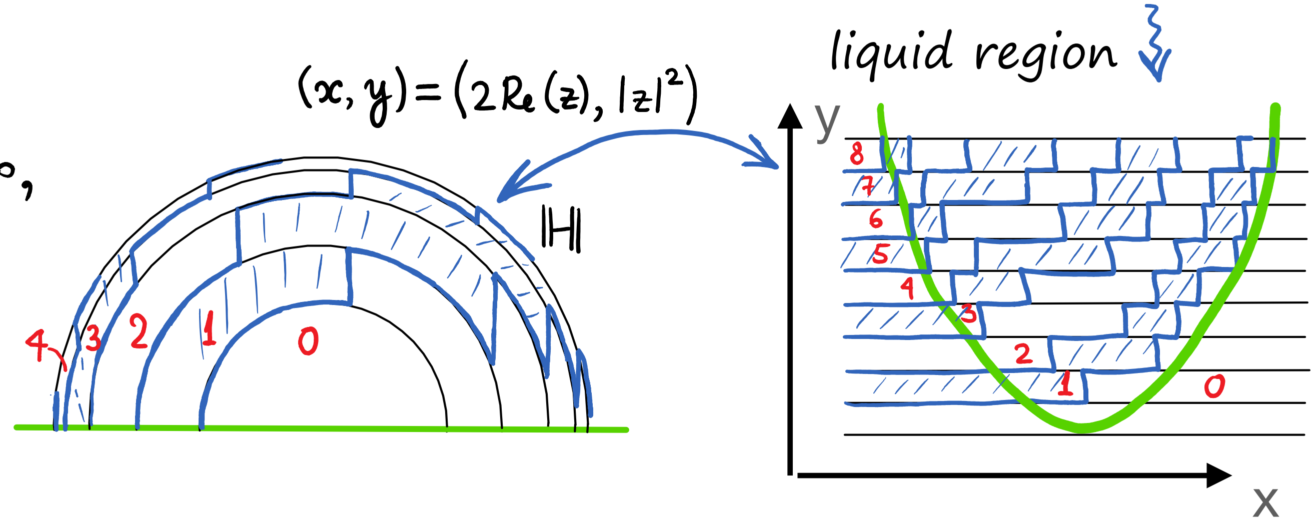

$(x, y) = (2\mathrm{Re}(z), |z|^2)$

- *Gaussianity follows from (more general) results of [Guionnet '02]*
- *GFF structure for GUE follows from [B-Ferrari, 2008], for GUE/GOE type Wigner matrices GFF fluctuations are proved in [B, 2010]*

Why GFF?

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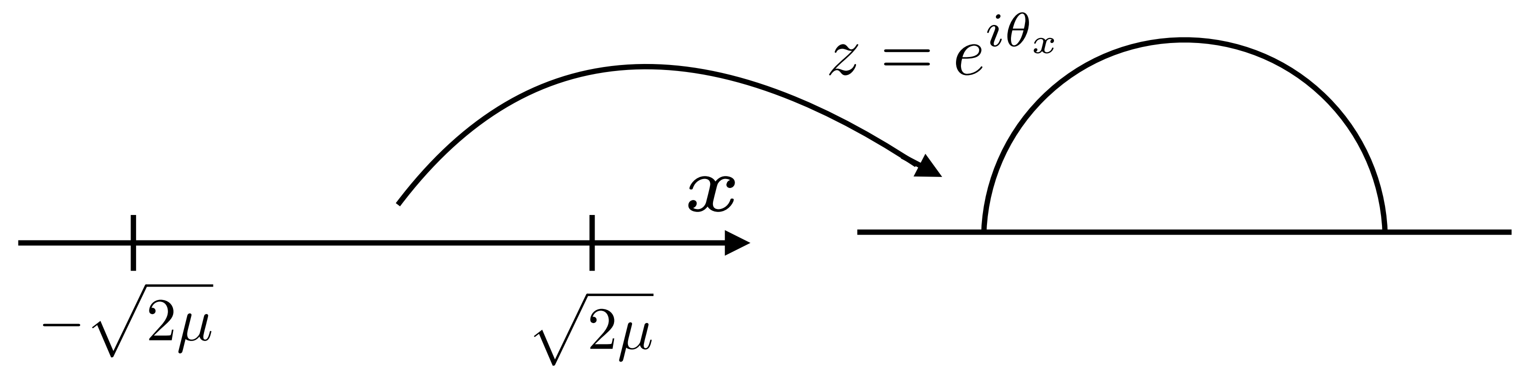

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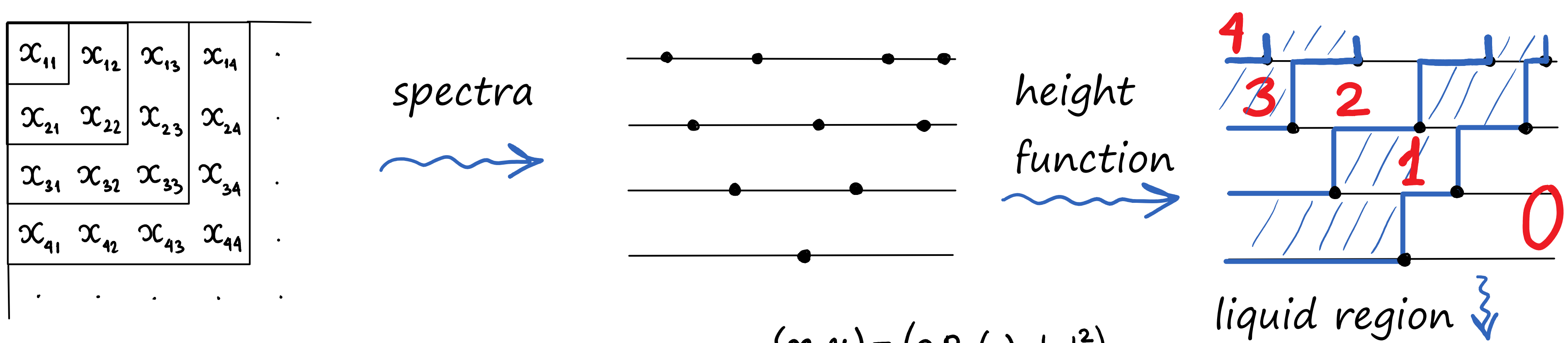
Inhomogeneous bosonisation

Dubail, Stephan,Viti,Calabrese (2017)

Alexei Borodin's talk

Corners of random matrices and GFF



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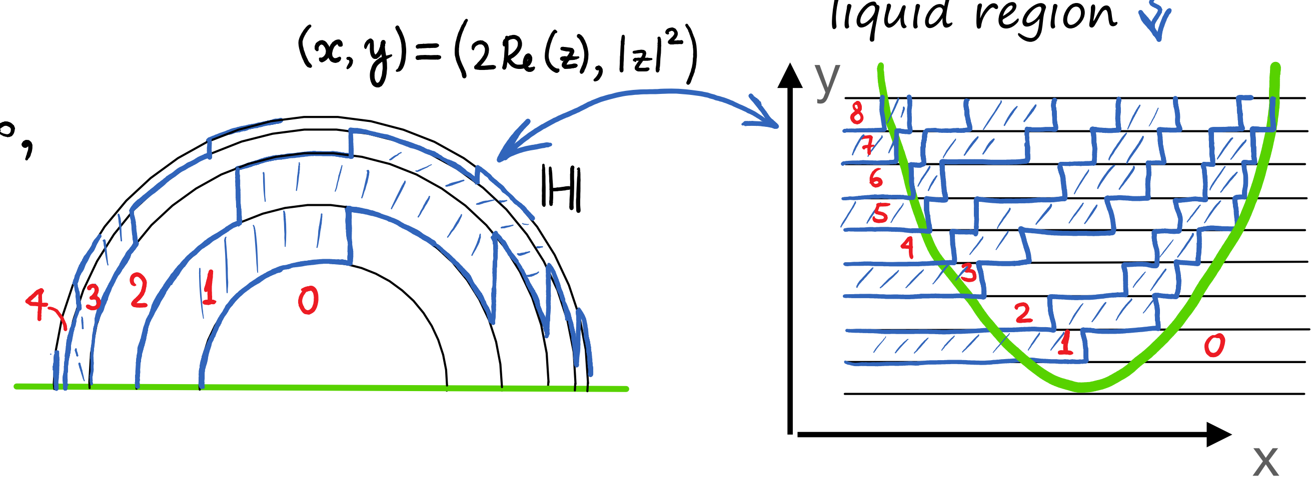

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Applications: number variance in interval in d=1

$$k_F(x) = \sqrt{2(\mu - V(x))}$$

$$\rho(x) = \frac{k_F(x)}{\pi}$$

$$\mathrm{Var}\mathcal{N}_{[a,b]} = H\left(a,a\right) + H\left(b,b\right) - 2H\left(a,b\right)$$

also needs H(a,a) =>
needs H(a,b) for |a-b| microscopic

$$\mathrm{Var}\mathcal{N}_{]-\infty,a]} = \mathrm{Var}\mathcal{N}_{[a,+\infty[} = H(a,a)$$

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solution on microscopic scales $|x-y| \sim 1/k_F(x)$
which matches macroscopic x->y

$$K_\mu(x,y)^2 = -\partial_x \partial_y H(x,y) + \delta(x-y)\rho(x)$$

insert sine-kernel

$$H(x,y) \simeq \frac{1}{2\pi^2}\left[U(k_F(x)|x-y|) + \log\frac{2k_F(x)\sin\theta_x}{d\theta_x/dx}\right]$$

neglect variations of $k_F(x)$

$$U''(z) = 2\sin^2 z/z^2$$

$$U(z \gg 1) = -\log z + o(1)$$

$$U'(0) = -\pi$$

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-> 1) number variance for microscopic interval

$$\pi^2 \mathrm{Var}\mathcal{N}_{[a,b]} \simeq U(0) - U(k_F(a)|a-b|)$$

$$\simeq \log k_F(a)|a-b| + c_2$$

$k_F(a)|a-b| \gg 1$

recovers Dyson-Mehta

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-> 2) number variance for macroscopic interval

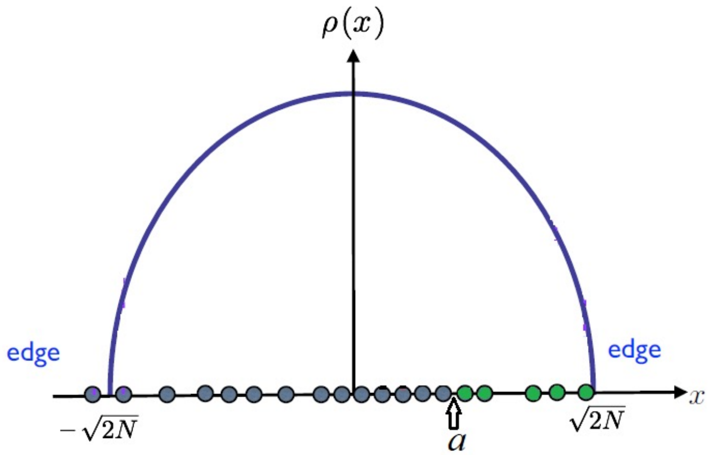
attention: a,b are in bulk !

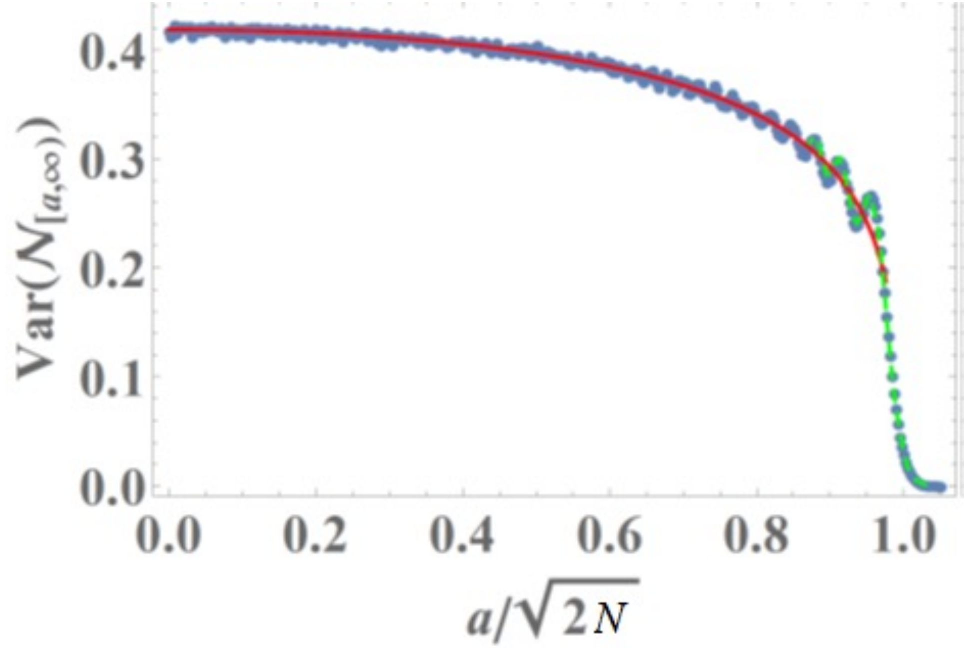
a,b near edge =>
edge regime is different !

$$H(a,a) = \mathrm{Var}\mathcal{N}_{[a,+\infty[} \simeq \frac{1}{2\pi^2}\left(\log \frac{2k_F(a)^2 \sin\theta_a}{d\mu/dN} + c_2\right)$$

$$\boxed{(2\pi^2)\mathrm{Var}\mathcal{N}_{[a,b]} = 2\log\left(2k_F(a)k_F(b)\int_{x^-}^{x^+} \frac{dz}{\pi k_F(z)}\right) + \log\left(\frac{\sin^2\frac{\theta_a-\theta_b}{2}}{\sin^2\frac{\theta_a+\theta_b}{2}}|\sin\theta_a \sin\theta_b|\right) + 2c_2 + o(1)}$$

1d HO results: $\mathrm{Var}\left(\mathcal{N}_{[a,\infty)}\right)$





Blue markers – numerics (N=100), red line – our prediction, green line – previously known edge prediction

$$\mathrm{Var}\left(\mathcal{N}_{[a,\infty)}\right) \simeq \frac{1}{2\pi^2}\left[\ln N + \frac{3}{2}\ln\left(1-\frac{a^2}{2N}\right) + \gamma + 1 + 3\ln 2\right]$$

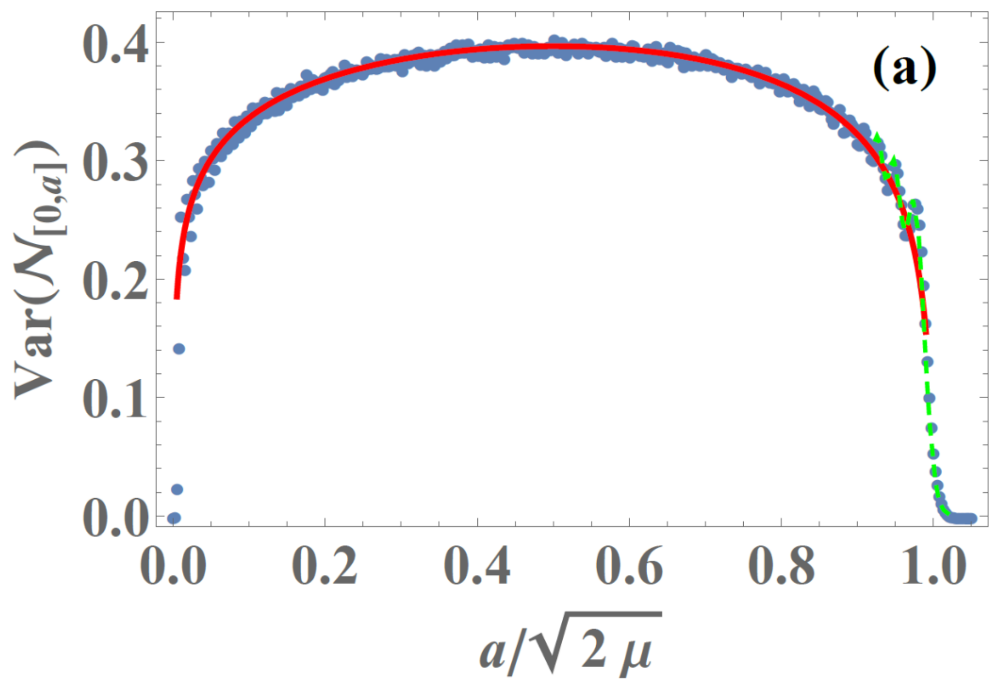
Subleading term is important at large (but not huge) N

$$\theta_x = \begin{cases} \arccos\left(\frac{-x}{\sqrt{2\mu}}\right) \quad , & V(x) = \frac{1}{2}x^2 \quad , \quad x^{\pm} = \pm\sqrt{2\mu} \\ \arccos\left(\frac{\mu - x^2}{\sqrt{\mu^2 - \alpha(\alpha-1)}}\right) \quad , & V(x) = \frac{x^2}{2} + \frac{\alpha(\alpha-1)}{2x^2} \quad , \quad (x^{\pm})^2 = \mu \pm \sqrt{\mu^2 - \alpha(\alpha-1)} \\ \arccos\left(\frac{\cos(\pi x/L) - A}{B}\right) \quad , & V(x) = \frac{\pi^2}{L^2}\left(\frac{\mathsf{a}^2 - \frac{1}{4}}{8\sin^2(\pi x/2L)} + \frac{\mathsf{b}^2 - \frac{1}{4}}{8\cos^2(\pi x/2L)}\right) \quad , \quad \cos(\pi x_{\pm}/L) = A \mp B \end{cases}$$

Particular examples ($W = 0$)

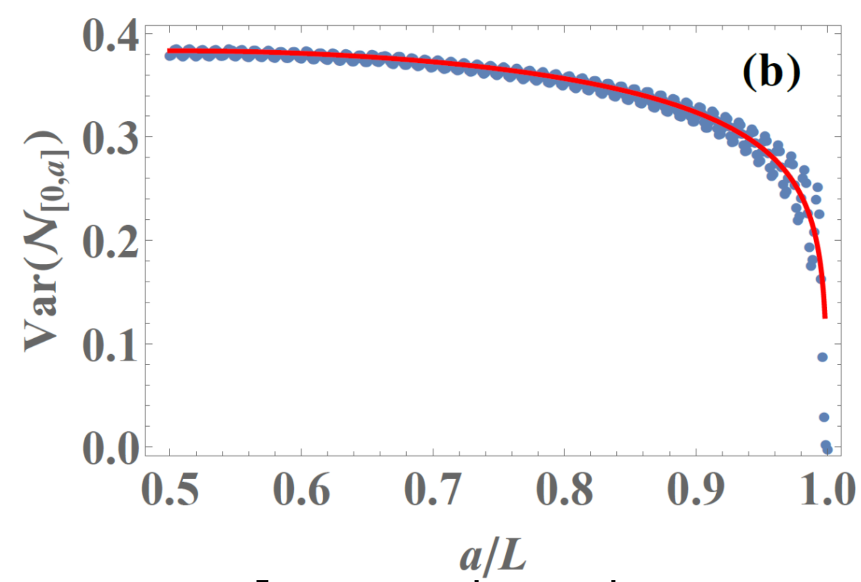
$$V(x) = \frac{x^2}{2} + \frac{\alpha(\alpha-1)}{2x^2}$$

(LUE random matrices)



$V(x) =$ particle in a box $[0, L]$

(JUE random matrices)



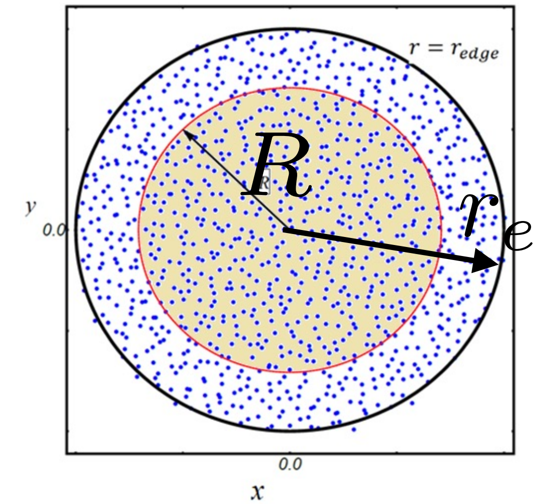
$$\mathrm{Var}(\mathcal{N}_{[0,a]}) \simeq \frac{1}{2\pi^2}\left[\ln N + \ln\left|\sin\frac{\pi a}{L}\right| + \gamma + 1 + 2\ln 2\right]$$

$$\mathrm{Var}(\mathcal{N}_{[0,a]}) \simeq \frac{1}{2\pi^2}\left[\ln\mu + \ln\left(4\tilde{a}\frac{\left(1 - \tilde{a}^2 - \frac{\lambda^2}{4\tilde{a}^2}\right)^{3/2}}{\sqrt{1-\lambda^2}}\right) + \gamma + 1 + \ln 2\right],$$

Where $\mu = 2N + \alpha + \frac{1}{2}$, $\tilde{a} = \frac{a}{\sqrt{2\mu}}$, $\lambda = \frac{\alpha}{\mu}$

Applications : d>1 central potential V(r)

$$H = -\frac{1}{2}\Delta_d + V(r)$$
$$r = |x|$$
$$\mu \sim N^{1/d}$$



$\mathcal{N}_R$ number of fermions in sphere of radius R centered at r=|x|=0

angular modes fluctuate independently

$\hat{\boldsymbol{L}}^2$ $\ell(\ell + d - 2)$

$\ell = 0, 1, ..$

$$\mathrm{Var}\,\mathcal{N}_R = \sum_{\ell \geq 0} g_d(\ell)\,\mathrm{Var}\,\mathcal{N}^{1d, V_\ell}_{[0,R]}$$

for all cumulants

$$g_d(\ell \gg 1) \simeq \frac{2\ell^{d-2}}{\Gamma(d-1)}$$

number of fermions in [0,R] for 1d problem

with Fermi energy μ

$$\hat{H}_\ell = -\frac{1}{2}\partial_r^2 + V_\ell(r)$$

$$V_\ell(r) = V(r) + \frac{\left(\ell + \frac{d-3}{2}\right)\left(\ell + \frac{d-1}{2}\right)}{2r^2}$$

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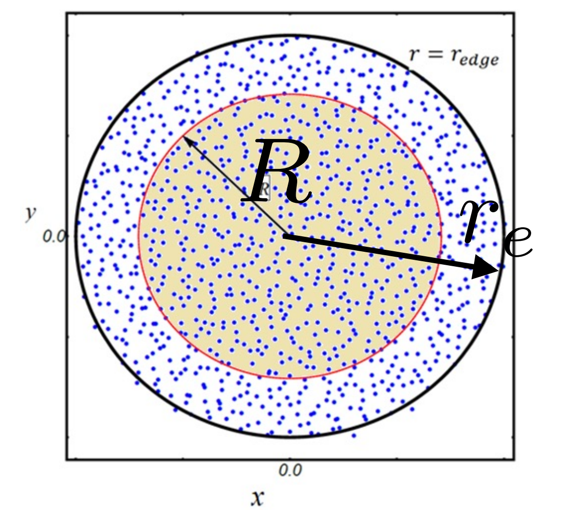

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1) free fermions V(r)=0

$$\mathrm{Var}\, \mathcal{N}_R \simeq \frac{1}{\pi^2 \Gamma(d)} (k_F R)^{d-1} \left[\log(k_F R) + b_d\right]$$

$$k_F = \sqrt{2\mu}$$

– Leading-order term was known
[Gioev and Klich, PRL 2006]
Widom

$$b_d = 2\log 2 - \frac{\gamma_E}{2} + 1 - \frac{3}{2}\psi^{(0)}\left(\frac{d+1}{2}\right)$$

we obtained exact result

$$\mathrm{Var}(\mathcal{N}_R) = \mathcal{U}_d(k_F R)$$

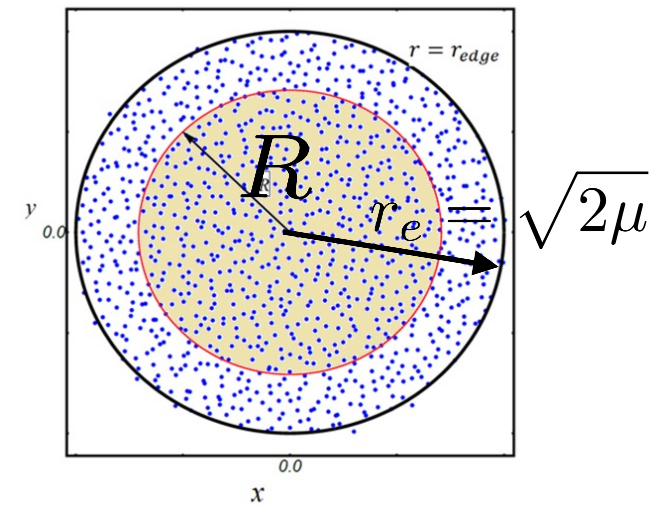
from d-dim version of sine-kernel

2) d>1 harmonic oscillator

$$H = -\frac{1}{2}\Delta_d + \frac{r^2}{2}$$

$\mathcal{N}_R$ number of fermions in sphere of radius R centered at r=|x|=0

$\mu \sim N^{1/d}$



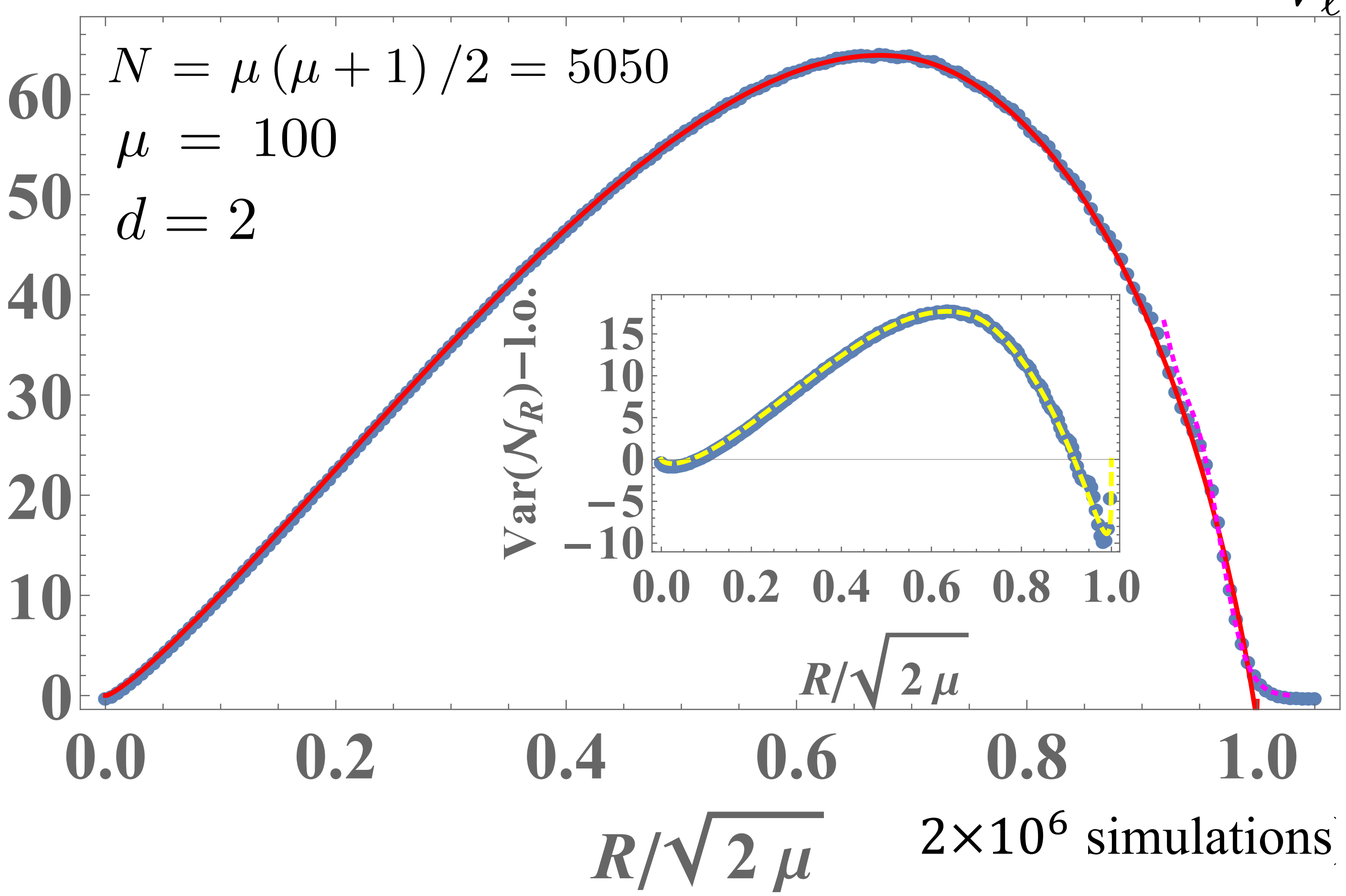
$$\mathrm{Var}\,\mathcal{N}_R = \sum_{\ell \geq 0} g_d(\ell)\,\mathrm{Var}\,\mathcal{N}^{1d,V_\ell}_{[0,R]}$$

number of fermions in [0,R] for 1d problem

$$\hat{H}_\ell = -\tfrac{1}{2}\partial_r^2 + V_\ell(r)$$

$$V_\ell(r) = \frac{r^2}{2} + \frac{(\ell + \frac{d-3}{2})(\ell + \frac{d-1}{2})}{2r^2}$$

<-> Laguerre-Wishart LUE ensemble



2×10^6 simulations

$$\boxed{\mathrm{Var}\mathcal{N}_R = \mu^{d-1}\left[A_d\left(\tilde{R}\right)\log\mu + B_d\left(\tilde{R}\right) + o(1)\right]}$$

$$\tilde{R} = R/\sqrt{2\,\mu}$$

$$A_d(\tilde{R}) = \frac{1}{\pi^2\Gamma(d)}\left(2\tilde{R}\sqrt{1-\tilde{R}^2}\right)^{d-1}$$

Inset: the sub-leading term $B_2(\tilde{R})$

$$2\pi^2 B_2(x) = 2x\sqrt{1-x^2}\left\{\log\left[\left(\frac{64x}{1-2x^2}\right)^2(1-x^2)^3\right] + 2\gamma_E - 2\right\} + \log\left(\frac{|1-2x\sqrt{1-x^2}|}{1+2x\sqrt{1-x^2}}\right)$$

Extrema of counting staircases

Y. Fyodorov, PLD Phys. Rev. Lett. 124, 210602 (2020)

Counting staircase (height funtion) for eigenvalues of $\text{Circular } \beta\text{-Ensemble } \text{CUE}_\beta(N)$

$\beta = 2$ random unitary matrix U(N)

eigenvalues e^{ix_j} $\quad -\pi < x_j \leq \pi \quad j = 1, \ldots, N$

$$P(x_1, \ldots, x_N) \sim \prod_{1 \leq j < k \leq N} |e^{ix_j} - e^{ix_k}|^\beta$$

R. Killip and I. Nenciu. (2004)

(Dumitriu-Edelman tridiag matrices)

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ground state for fermions on the circle

$$H = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{\beta(\beta-2)}{16 \sin^2 \frac{x_i - x_j}{2}}$$

Sutherland model

$\beta = 2$ free fermions
$\beta \neq 2$ interacting

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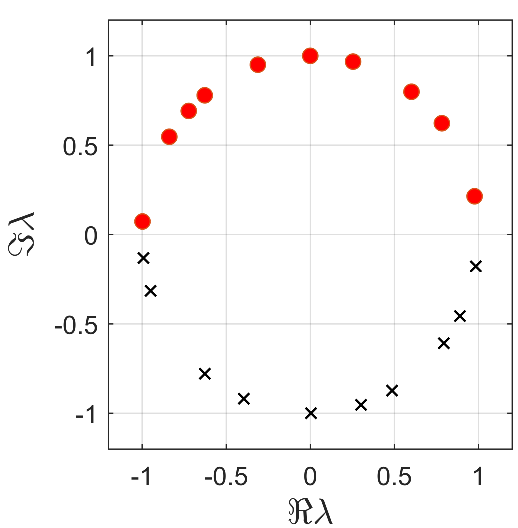

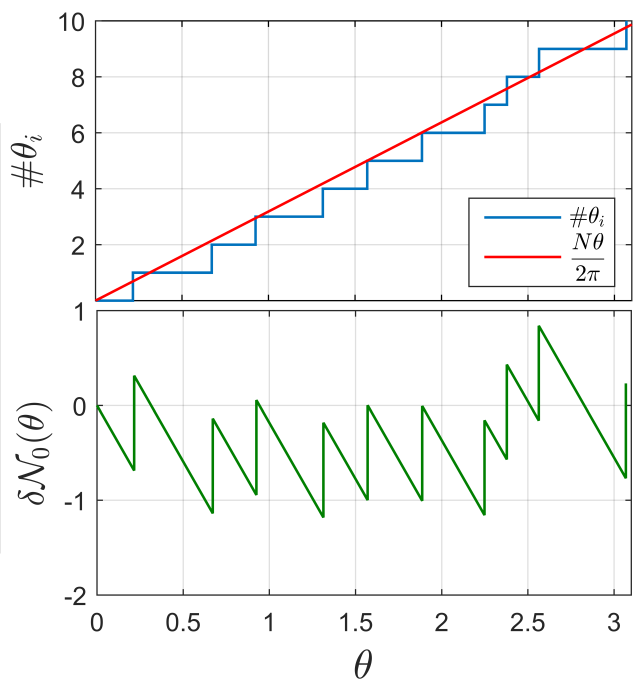

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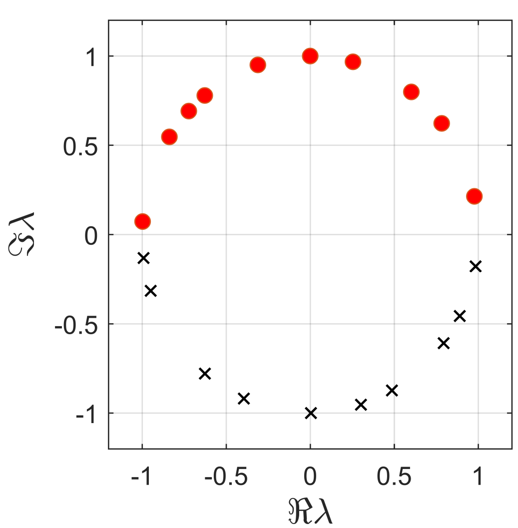
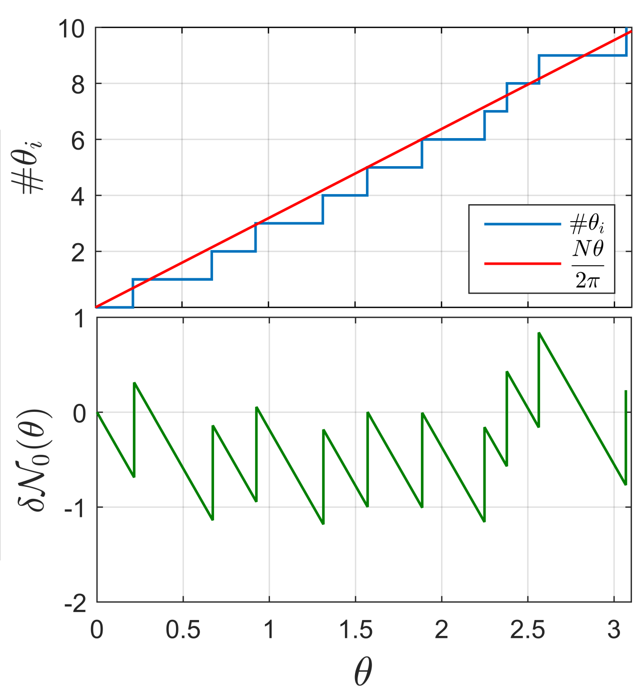
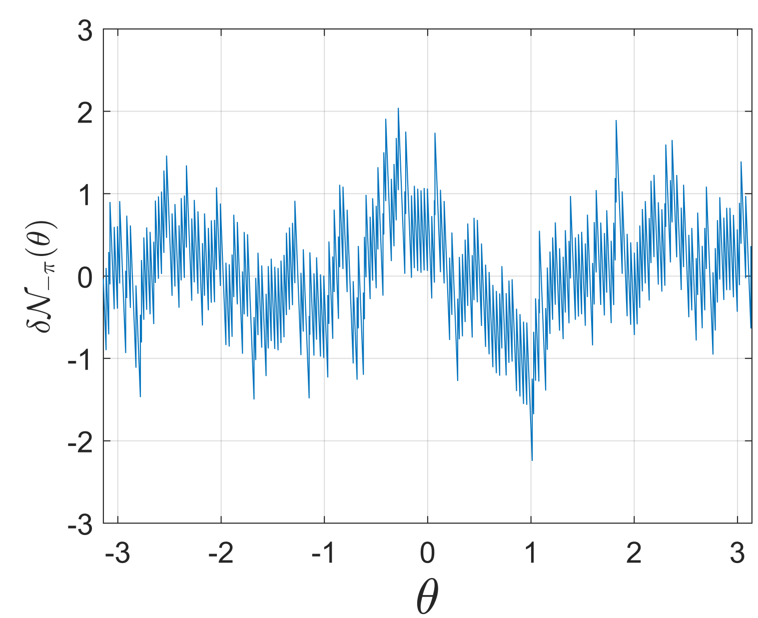
is a log-correlated field

some realization of GFF

$\mathbb{E} := \mathbb{E}_{\mathrm{C}\beta\mathrm{E}}$







Question: statistics of the maximum

$$\delta\mathcal{N}_m = \max_{x \in [x_A, x_B]} \delta\mathcal{N}_{x_A}(x)$$

$$\max_{x \in [x_A, x_B]} |\delta\mathcal{N}_{x_A}(x)|$$ Kolmogorov-Smirnov statistics

Question: find PDF of maximum

on some interval $[x_A, x_B]$

$$N \gg 1$$

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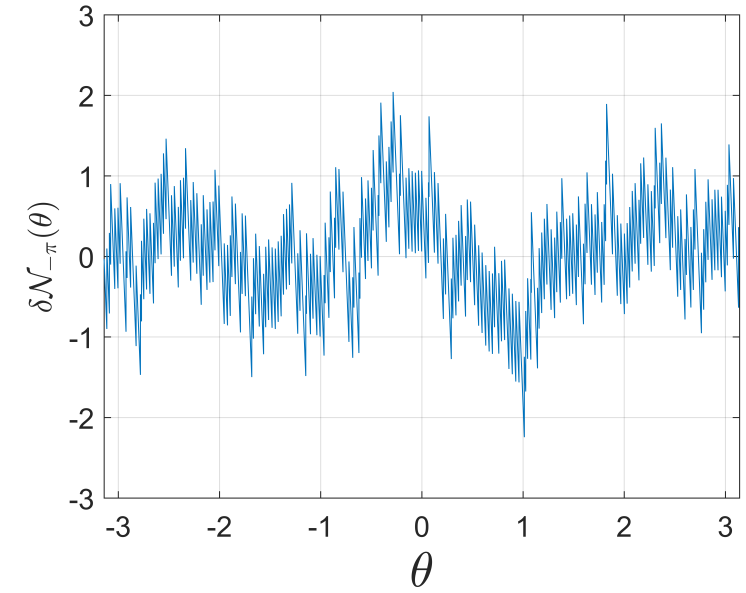

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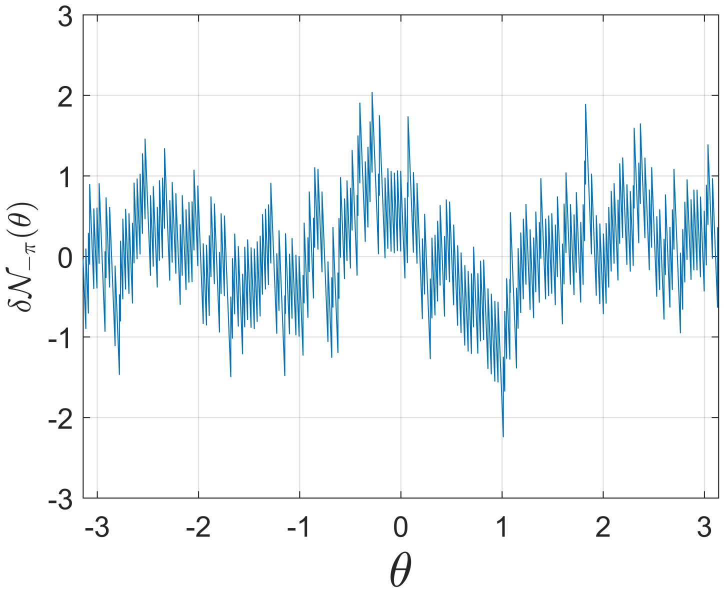

is a log-correlated field

some realization of GFF

$$\delta\mathcal{N}_A(x) = \frac{1}{\pi}\operatorname{Im}\log\xi_N(x) - (x \to x_A)$$

$$\xi_N(x) = \det(I - e^{-ix}U)$$

$$x_1 \neq x_2$$

$$(\operatorname{Im}\log\xi_N(x_1), \operatorname{Im}\log\xi_N(x_2)) \underset{N \gg 1}{\to} (W(x_1), W(x_2))$$

$$\operatorname{Cov}(W(x_1)W(x_2)) = \frac{-1}{2\beta}\log(4\sin^2\frac{x_1 - x_2}{2})$$

$\beta = 2$ Hughes, Keating, O'Connell. Commun. Math. Phys. (2001)

general β Chhaibi, Madaule, Najnudel. Duke Math. J. (2018)

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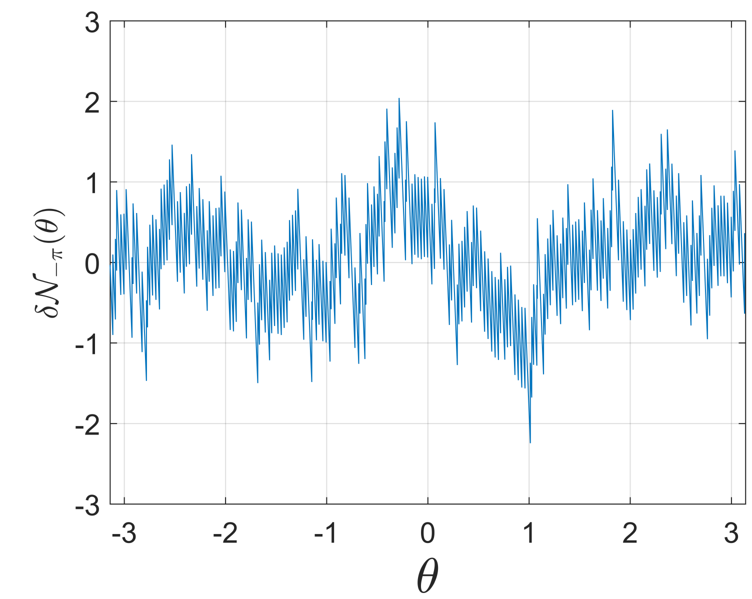

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fractional Brownian motion

with Hurst index $H = 0$

fBm0 Fyodorov, Khoruzhenko, N. J. Simm, Ann. Probab. (2016).

$$\mathrm{Cov}(W(x_1)W(x_2)) = \frac{-1}{2\beta}\log(4\sin^2\frac{x_1-x_2}{2})$$

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general β Chhaibi, Madaule, Najnudel. Duke Math. J. (2018)

$\delta\mathcal{N}_{x_A}(x = x_A) = 0$ pinned fBm0

$[x_A, x_B] =]-\pi, \pi]$ fBm0 bridge

$\frac{1}{N} \ll \ell \ll 1$ fBm0 on interval

$\ell = x_B - x_A$

Cao, Fyodorov, Le Doussal. Phys. Rev. E (2018)

Fyodorov, Le Doussal, J. Stat. Phys. (2016).

To find maximum: statistical mechanics

partition sum at inverse temperature b

$$Z_b = \frac{N}{2\pi} \int_{x_A}^{x_B} dy \, e^{2\pi b \sqrt{\frac{\beta}{2}} \delta\mathcal{N}_{x_A}(y)}$$

$$\mathcal{F} = \frac{1}{2\pi b \sqrt{\beta/2}} \log Z_b$$

$$\delta\mathcal{N}_m = \lim_{b \to +\infty} \mathcal{F}$$

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n replicas

$$\mathbb{E}[Z_b^n] = (\frac{N}{2\pi})^n \int_{x_A}^{x_B} dy_1 \dots dy_n \quad \mathbb{E}[e^{2\pi b\sqrt{\frac{\beta}{2}}\sum_{a=1}^n \delta\mathcal{N}_{[x_A,y_a]}}] \qquad \mathbb{E} := \mathbb{E}_{\mathrm{C}\beta\mathrm{E}}$$

$$e^{-b\sqrt{\frac{\beta}{2}}\sum_{a=1}^n N(y_a - x_A)} \times \mathbb{E}[\prod_{j=1}^N g(x_j)] \qquad g(x) = e^{2\pi b\sqrt{\frac{\beta}{2}}\sum_{a=1}^n \mathbb{1}_{[x_A,y_a]}(x)}$$

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symbol

$\beta = 2$

Toeplitz

$$= \det_{N\times N} g_{j-k}$$

smooth symbol $\Rightarrow$ Svegö theorem

$$\log g(x) = \sum_{p\in\mathbb{Z}} c_p e^{ipx}$$

$$= e^{Nc_0} e^{\sum_{p\geq 1} pc_p c_{-p} + o(1)}$$

singular symbol $\Rightarrow$ Fisher-Hartwig

jumps, power laws

(1969)

Deift, Its, Krasovsky (2011)

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rational β conjecture

Forrester, Frankel, J. Math. Phys. (2004).

Using Forrester-Frankel conjecture

YF,PLD Phys. Rev. Lett. 124, 210602 (2020)

we find for $\beta/2 = s/r$ s,r mutually prime

G(z) is Barnes function

$$G(z+1) = \Gamma(z) G(z)$$

$$G(1) = 1$$

1) for n=1

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FCS for number of fermions in interval

for Sutherland model

$$A_\beta(t) = r^{-t^2/2} \prod_{\nu=0}^{r-1} \prod_{p=0}^{s-1} \frac{G(1-\frac{p}{s}+\frac{\nu+it\sqrt{\frac{2}{\beta}}}{r})}{G\left(1-\frac{p}{s}+\frac{\nu}{r}\right)}$$

$\beta = 2$ $\quad A_2(t) = G(1+it)$

Ivanov, Abanov, Phys. Rev. B (2013)

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Ivanov, Abanov, Phys. Rev. B (2013)

$\beta = 4$ Stephan, Pollmann, Phys. Rev. B (2017)

2) any n $nb^2 < 1$

$$\mathbb{E}[Z_b^n] \underset{N \to +\infty}{\simeq} (\frac{N}{2\pi})^n N^{b^2(n+n^2)} |A_\beta(b)|^{2n} |A_\beta(bn)|^2$$

$$\times \int_{x_A}^{x_B} \prod_{a=1}^{n} dy_a |1 - e^{i(y_a - x_A)}|^{2nb^2} \prod_{1 \le a \le c \le n} |1 - e^{i(y_a - y_c)}|^{-2b^2}$$

Coulomb gas integrals for n replica

Maximum on the full circle

$$\mathbf{M}(n,a,b) = \prod_{j=0}^{n-1} \frac{\Gamma(1-2ab-jb^2)\Gamma(1-(j+1)b^2)}{\Gamma(1-ab-jb^2)^2\Gamma(1-b^2)}$$

$$\int_{-\pi}^{\pi} \prod_{a=1}^{n} dy_a |1-e^{i(y_a-x_A)}|^{2nb^2} \prod_{1\le a\le c\le n} |1-e^{i(y_a-y_c)}|^{-2b^2}$$

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Morris integral

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DSLT of the free energy
in high temperature phase b<1

$$t = -bn \qquad Q = b + \tfrac{1}{b}$$

$$\mathbb{E}\left(e^{-2\pi\sqrt{\frac{\beta}{2}}(\mathcal{F}-\mathcal{F}_1)t}\right) \simeq N^{-tQ+t^2} A_\beta(t)A_\beta(-t)\; \Gamma(1+tb)\frac{G_b(Q-2t)G_b(Q)^3}{G_b(Q-t)^3 G_b(Q+t)}$$

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Multiply both sides by $\Gamma(1+\frac{t}{b})$

=> self-dual $\quad b \to 1/b$

=> freezes at b=1

Freezing transition for log-correlated fields

Freezing transition in the REM Derrida => Log correlated field approximated as some REM Chamon et al.

Directed polymer on the Cayley-tree Derrida-Spohn (hierarchical log-correlated)

Carpentier, Le Doussal. Glass transition of a particle in a random potential, front selection in nonlinear renormalization group, and entropic phenomena in Liouville and sinh Gordon models. Phys. Rev. E (2001)

Fyodorov, Bouchaud. Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential. J. Phys. A: Math. Theor. (2008)

$g_b(y) = \mathbb{E}(e^{-e^{by} Z_b})$ freezes at b=1

$\mathcal{F}_b - b^{-1} G$ freezes $\mathcal{F}_{+\infty} \equiv \mathcal{F}_1 - G$ <=> $\mathbb{E}(e^{-2\pi\sqrt{\frac{\beta}{2}}\mathcal{F}t})\Gamma(1+\frac{t}{b}) = \mathbb{E}(e^{-2\pi\sqrt{\frac{\beta}{2}}\delta\mathcal{N}_m t})$

Freezing transition for log-correlated fields

Freezing transition in the REM Derrida => Log correlated field approximated as some REM Chamon et al.

Directed polymer on the Cayley-tree Derrida-Spohn (hierarchical log-correlated)

Carpentier, Le Doussal. Glass transition of a particle in a random potential, front selection in nonlinear renormalization group, and entropic phenomena in Liouville and sinh Gordon models. Phys. Rev. E (2001)

Fyodorov, Bouchaud. Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential. J. Phys. A: Math. Theor. (2008)

$$g_b(y) = \mathbb{E}(e^{-e^{by} Z_b})$$ freezes at b=1

$$\mathcal{F}_b - b^{-1} G$$ freezes $$\mathcal{F}_{+\infty} \equiv \mathcal{F}_1 - G$$ <=> $$\mathbb{E}(e^{-2\pi\sqrt{\frac{\beta}{2}}\mathcal{F}t})\Gamma(1+\frac{t}{b}) = \mathbb{E}(e^{-2\pi\sqrt{\frac{\beta}{2}}\delta\mathcal{N}_m t})$$

Fyodorov, Le Doussal, Rosso. Statistical Mechanics of Logarithmic REM: Duality, Freezing and Extreme Value Statistics of 1/f Noises Generated by Gaussian Free Fields. J. Stat. Mech. Theor. Exp. (2009)

FDC: all self-dual thermodynamic quantities freeze $$\mathcal{P}(x_m) = \lim_{b\to 1} \mathbb{E}(p_b(x))$$

Fyodorov, Le Doussal. Moments of the position of the maximum… J. Stat. Phys. (2016)

Cao et al. Liouville

Math:

1step RSB

Madaule, Rhodes, Vargas. Glassy phase and freezing of log-correlated Gaussian potentials. Ann. Appl. Probab. (2016)

Maximum on the full circle

$$\mathbf{M}(n,a,b) = \prod_{j=0}^{n-1} \frac{\Gamma(1-2ab-jb^2)\Gamma(1-(j+1)b^2)}{\Gamma(1-ab-jb^2)^2\Gamma(1-b^2)}$$

$$\int_{-\pi}^{\pi} \prod_{a=1}^{n} dy_a |1-e^{i(y_a-x_A)}|^{2nb^2} \prod_{1\le a\le c\le n} |1-e^{i(y_a-y_c)}|^{-2b^2}$$

$$= \mathbf{M}(n, a=-nb, b)$$

Morris integral

Generalized Barnes function $\Gamma(bx) = \frac{\widetilde{G}_b(x+b)}{\widetilde{G}_b(x)}$

$$G_b(z) = G_{1/b}(z)$$

$$G_1(z) = G(z)$$

$$\mathbb{E}[Z_b^n] \simeq (\frac{N}{2\pi})^n N^{b^2(n+n^2)} \quad |A_\beta(b)|^{2n}|A_\beta(bn)|^2 \ \mathbf{M}(n, a=-nb, b)$$

DSLT of the free energy

in high temperature phase b<1

$$t = -bn \qquad Q = b + \frac{1}{b}$$

$$\mathbb{E}\left(e^{-2\pi\sqrt{\frac{\beta}{2}}(\mathcal{F}-\mathcal{F}_1)t}\right) \simeq N^{-tQ+t^2} A_\beta(t)A_\beta(-t)\ \Gamma(1+tb)\frac{G_b(Q-2t)G_b(Q)^3}{G_b(Q-t)^3G_b(Q+t)}$$

Multiply both sides by $\Gamma(1+\frac{t}{b})$

=> self-dual $b \to 1/b$

=> freezes at b=1

$$\mathbb{E}(e^{-2\pi\sqrt{\frac{\beta}{2}}\delta\mathcal{N}_m t}) \simeq N^{-2t+t^2}\, e^{ct} A_\beta(t)A_\beta(-t)\,\frac{\Gamma(1+t)^2G(2-2t)}{G(2-t)^3G(2+t)}$$

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Similar result for the small interval

Selberg integral $\int_0^1 \prod_{i=1}^{n} [x_i^{-2ab} \mathrm{d}x_i] \prod_{i<j} |x_i - x_j|^{-2b^2}$

Summary of results for statistics of the maximum

$$\delta\mathcal{N}_m = \max_{x \in [x_A, x_B]} \delta\mathcal{N}_{x_A}(x)$$

$$\delta\mathcal{N}_{x_A}(x) = \mathcal{N}_{[x_A,x]} - \mathbb{E}(\mathcal{N}_{[x_A,x]})$$

$$2\pi\sqrt{\frac{\beta}{2}}\mathbb{E}(\delta\mathcal{N}_m) \simeq 2\log N - \frac{3}{2}\log\log N + c_\ell^{(\beta)}$$

fermionic, micro

fBm0, meso/macro

$$\mathbb{E}^c(\delta\mathcal{N}_m^2) \simeq \frac{2}{\beta(2\pi)^2}(2\log N + \tilde{C}_2^{(\beta)} + C_2(\ell))$$

$$\mathbb{E}^c(\delta\mathcal{N}_m^k) \simeq \frac{2^{k/2}}{\beta^{k/2}(2\pi)^k}(\tilde{C}_k^{(\beta)} + C_k(\ell))$$

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$$\tilde{C}_{2p}^{(\beta)} = (-2)^{1-p}\beta^p\sum_{\nu=0}^{\infty}\psi^{(2p-1)}(1+\frac{\beta\nu}{2}) = (-2)^{p+1}\frac{1}{\beta^p}\sum_{q=1}^{\infty}\psi^{(2p-1)}(\frac{2q}{\beta})$$

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maximum over the full circle

$$\ell = 2\pi$$

fBm0 bridge

$$C_k(2\pi) = (-1)^k\frac{d^k}{dt^k}|_{t=0}\log\left[\frac{\Gamma(1+t)^2G(2-2t)}{G(2-t)^3G(2+t)}\right]$$

$$\left\{\frac{\pi^2}{3}, -2\pi^2+8\zeta(3), \frac{14\pi^4}{15}-72\zeta(3)\right\}$$

maximum over a mesoscopic interval

$$\frac{1}{N} \ll \ell \ll 1$$

$$C_k(\ell) \simeq 2\log\ell\,\delta_{k,2}$$
$$+(-1)^k\frac{d^k}{dt^k}|_{t=0}\left[\frac{2\Gamma(1+t)^2G(2-2t)}{G(2+t)^2G(2-t)G(4-t)}\right]$$

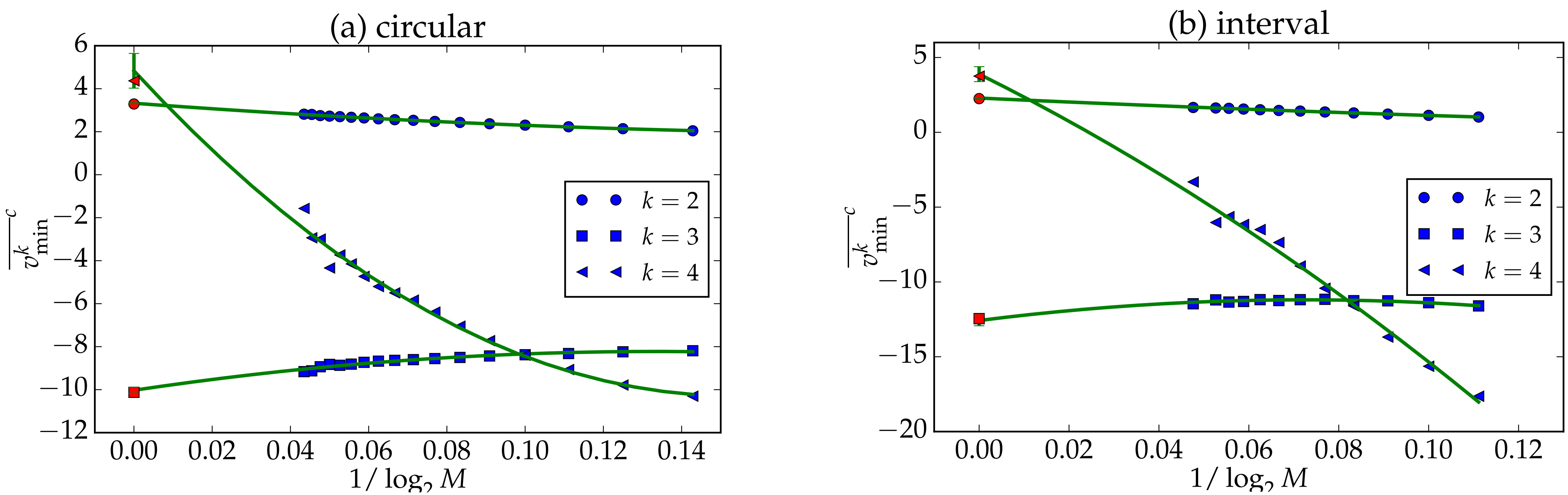
fBm0 interval

Cao, Fyodorov, Le Doussal. Phys. Rev. E (2018)

$$\left\{\frac{9}{4}, 8\zeta(3)-\frac{8\pi^2}{3}+\frac{17}{4}, -72\zeta(3)+\frac{4\pi^4}{5}+\frac{99}{8}\right\}$$

numerical checks for the fBm0 part

Cao, Fyodorov, Le Doussal. Phys. Rev. E (2018)



Full counting statistics for some models of interacting fermions in traps in d=1

N. Smith, PLD, S. Majumdar, G. Schehr, SciPost Phys. 11, 110 (2021)

Hamiltonian for spinless fermions d=1

- in external potential V(x)
- with two-body interaction W(x,y)

$$\mathcal{H}_N = \sum_{i=1}^{N} \left[\frac{p_i^2}{2} + V(x_i) \right] + \sum_{i<j} W(x_i, x_j)$$

find V, W so that ground state wavefunction has one and two-body form

x in real line, half-line, circle

$$|\Psi_0(\vec{x})|^2 = e^{-U(\vec{x})} \qquad U(\vec{x}) = \sum_i v(x_i) + \sum_{i<j} w(x_i, x_j)$$

example: Calogero-Sutherland model $\quad V(x) = \frac{x^2}{2} \quad W(x,y) = \frac{\beta(\beta-2)}{4(x-y)^2}$

$$v(x) = x^2 \qquad w(x,y) = -\beta \log|x-y|$$

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$$v(x) = x^2 \qquad w(x,y) = -\beta \log|x-y|$$

larger class: can be mapped to RMT ensemble

$$P(\vec{\lambda}) = \frac{e^{-F(\vec{\lambda})}}{Z_N} \qquad F(\vec{\lambda}) = \sum_{i=1}^{N} V_0(\lambda_i) - \beta \sum_{i<j} \log|\lambda_i - \lambda_j|$$

$$\lambda_i = \lambda(x_i)$$

$$v(x) = V_0(\lambda(x)) - \log|\lambda'(x)|$$
$$w(x,x') = -\beta \log|\lambda(x) - \lambda(x')|$$

Fermion Hamiltonian

$$\mathcal{H}_N = \sum_{i=1}^{N} \left[\frac{p_i^2}{2} + V(x_i) \right] + \sum_{i<j} W(x_i, x_j)$$

RMT eigenvalue JPDF

$$P(\vec{\lambda}) = \frac{e^{-F(\vec{\lambda})}}{Z_N} \qquad F(\vec{\lambda}) = \sum_{i=1}^{N} V_0(\lambda_i) - \beta \sum_{i<j} \log|\lambda_i - \lambda_j|$$

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Fermions' domain	Fermion potential $V(x)$	Fermion interaction $W(x,y)$	RMT ensemble	Matrix potential $V_0(\lambda)$	Map $\lambda(x)$
$x \in \mathbb{R}$	$x^2/2$	$\frac{\beta(\beta-2)}{4(x-y)^2}$	GβE	$\beta\lambda^2/2$	$\lambda = \sqrt{\frac{2}{\beta}}x$
$x \in [0, L]$	0	$\left(\frac{2\pi}{L}\right)^2 \frac{\beta(\beta-2)}{16\sin^2\frac{\pi(x-y)}{L}}$	CβE	0	$\lambda = e^{ix\frac{2\pi}{L}}$
$x \in \mathbb{R}^+$	$\frac{x^2}{2} + \frac{\gamma^2 - \frac{1}{4}}{2x^2}$	$\frac{\beta(\beta-2)}{4}\left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2}\right]$	WLβE	$\frac{\beta}{2}\lambda - \gamma\log\lambda$	$\lambda = \frac{2}{\beta}x^2$
$x \in [0, \pi]$	$\frac{1}{8}\left(\frac{\gamma_1^2 - \frac{1}{4}}{\sin^2\frac{x}{2}} + \frac{\gamma_2^2 - \frac{1}{4}}{\cos^2\frac{x}{2}}\right)$	$\frac{\beta(\beta-2)}{16}\left(\frac{1}{\sin^2\frac{x-y}{2}} + \frac{1}{\sin^2\frac{x+y}{2}}\right)$	JβE	$\log\frac{1}{\lambda^{\gamma_1}(1-\lambda)^{\gamma_2}}$	$\lambda = \frac{1-\cos x}{2}$

Fermion Hamiltonian

$$\mathcal{H}_N = \sum_{i=1}^{N}\left[\frac{p_i^2}{2} + V(x_i)\right] + \sum_{i<j} W(x_i, x_j)$$

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$$P(\vec{\lambda}) = \frac{e^{-F(\vec{\lambda})}}{Z_N} \qquad F(\vec{\lambda}) = \sum_{i=1}^{N} V_0(\lambda_i) - \beta \sum_{i<j} \log|\lambda_i - \lambda_j|$$

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Density from Coulomb gas any beta

$$\sigma(\lambda) = \frac{1}{N}\sum_i \langle \delta(\lambda - \lambda_i)\rangle$$

$$\longrightarrow \rho(x) = N\lambda'(x)\sigma(\lambda(x))$$

Calogero-Sutherland GβE

$$\rho(x) \simeq \frac{2}{\pi\beta}\sqrt{(N\beta - x^2)_+}$$

WLβE

$$\rho(x) \simeq \frac{2\,\theta(x)}{\pi\beta}\sqrt{(2N\beta - x^2)_+}$$

$$\rho(x) \simeq \frac{4x}{\beta}\sigma_{WL}\left(\frac{2x^2}{\beta N}; \frac{2\gamma}{\beta N}\right)$$

$$\sigma_{WL}(z;c) = \frac{\sqrt{(z-\zeta_-)(\zeta_+ - z)}}{2\pi z}$$

$$\zeta_\pm = (1 \pm \sqrt{1+c})^2$$

Full counting statistics

from formula for n=1 replica

Fyodorov, PLD PRL 124, 210602 (2020)

based on Forrester-Frankel conjecture

fermions on the circle (sutherland model)

$$\log \left\langle e^{2\pi\sqrt{\frac{\beta}{2}}t(\mathcal{N}_{[a,b]}-\langle\mathcal{N}_{[a,b]}\rangle)} \right\rangle = 2t^2 \log N + t^2 \log\left(4\sin^2\frac{|b-a|}{2}\right) + 2\log|A_\beta(t)|^2$$

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for $\mathrm{C}\beta\mathrm{E}$ but in fact for any model in the Table (conjecture)

even cumulants 4 and higher

$$\left\langle \left(\mathcal{N}_{[a,b]}^{(\beta)}\right)^{2p} \right\rangle^c = \frac{2}{(2\beta\pi^2)^p}\tilde{C}_{2p}^{(\beta)}$$

$$\tilde{C}_{2p}^{(\beta)} = (-2)^{p+1}\frac{1}{\beta^p}\sum_{q=1}^{\infty}\psi^{(2p-1)}\left(\frac{2q}{\beta}\right)$$

universality conjecture

higher cumulants come only from micro-scales

$$\tilde{C}_4^{(\beta=2)} = -12\zeta(3)\,, \quad \tilde{C}_4^{(\beta=1)} = \frac{\pi^4}{4} - 24\zeta(3)\,, \quad \tilde{C}_4^{(\beta=4)} = -24\zeta(3) - \frac{\pi^4}{4}$$

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second cumulant

$$\frac{\beta\pi^2}{2}\mathrm{Var}\mathcal{N}_{[a,b]}^{(\beta)} - c_\beta = \pi^2 \mathrm{Var}\mathcal{N}_{[a',b']}^{(\beta=2)} - c_2 + o(1)$$

$$a' = a\sqrt{2/\beta}$$
$$b' = b\sqrt{2/\beta}$$

in the bulk

$$c_\beta = \log 2 + \gamma_E + \sum_{\nu=0}^{+\infty}\left[\sum_{q=1}^{+\infty}\frac{\beta/2}{\left(\nu\frac{\beta}{2}+q\right)^2} - \frac{1}{1+\nu}\right]$$

$$= \gamma_E + \log\beta + \sum_{q=1}^{\infty}\left[\frac{2}{\beta}\psi^{(1)}\left(\frac{2q}{\beta}\right) - \frac{1}{q}\right]$$

$$c_1 = \log 2 + \gamma_E + 1 - \frac{\pi^2}{8}$$

$$c_2 = \log 2 + \gamma_E + 1$$

$$c_4 = 2\log 2 + \gamma_E + 1 + \frac{\pi^2}{8}$$

Calogero-Sutherland model

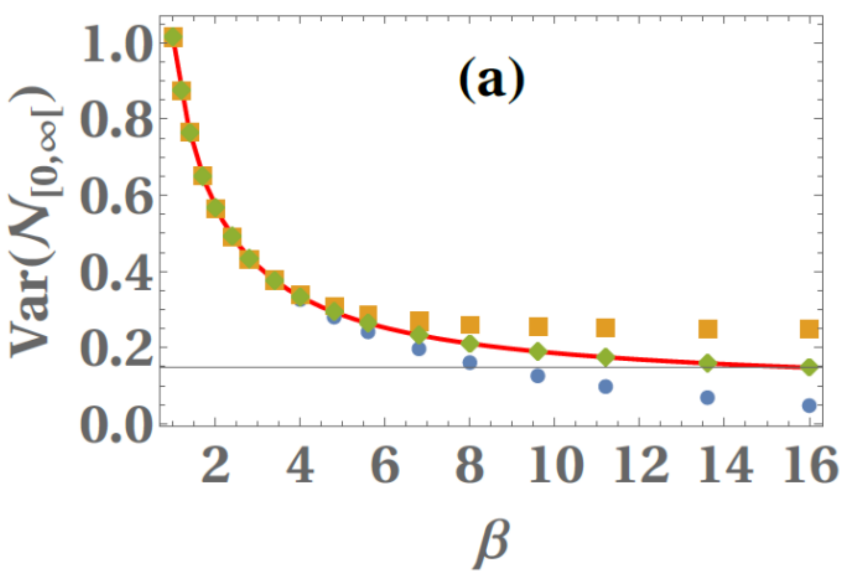
$$\frac{\beta\pi^2}{2}\mathrm{Var}\mathcal{N}_{[a,b]} = \log N + \frac{3}{4}\log\left[\left(1-\tilde{a}^2\right)\left(1-\tilde{b}^2\right)\right] + \log\left|\frac{4|\tilde{a}-\tilde{b}|}{1-\tilde{a}\tilde{b}+\sqrt{(1-\tilde{a}^2)(1-\tilde{b}^2)}}\right| + c_\beta + o(1)$$

$\tilde{a} = a/\sqrt{\beta N}$ in the bulk

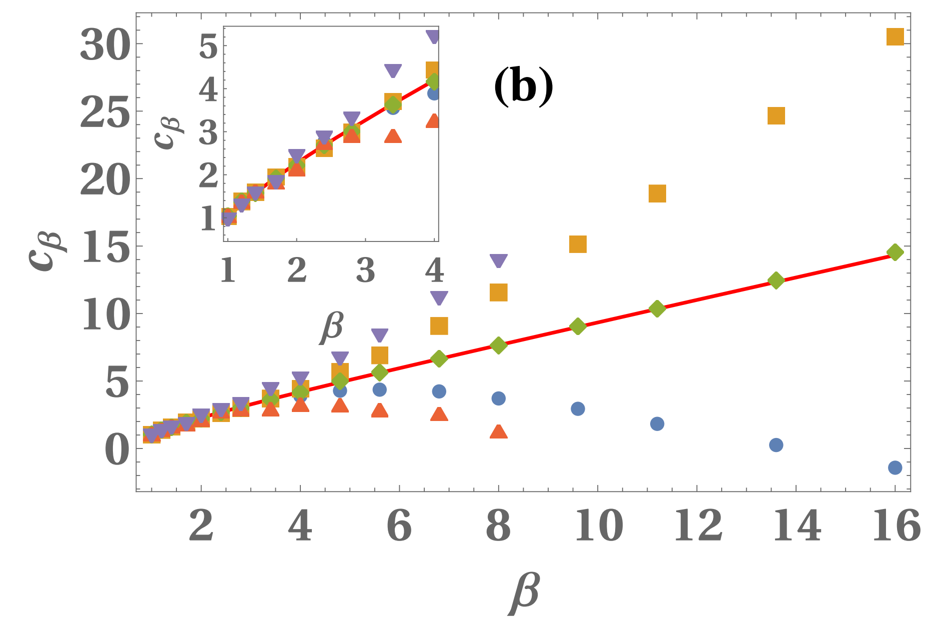
$\tilde{b} = b/\sqrt{\beta N}$ $\quad |\tilde{a}|, |\tilde{b}| < 1$

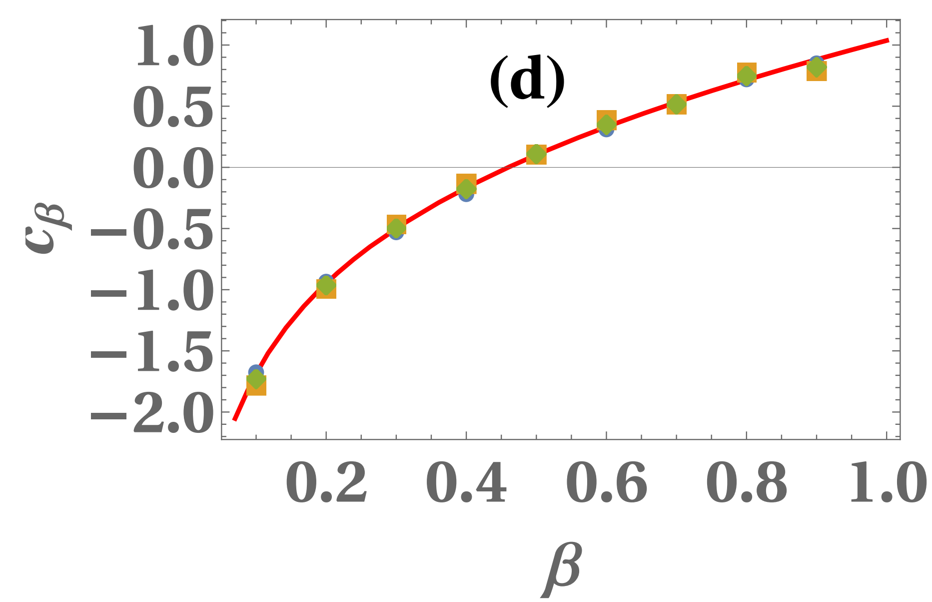
$\mathrm{Var}(\mathcal{N}_{[0,\infty)})$ - Agreement with numerics

- β dependence for N=2000, N=2001 (and their average)



$$\mathrm{Var}(\mathcal{N}_{[a,\infty)}) \simeq \frac{1}{\beta\pi^2}\left[\ln N + \frac{3}{2}\ln\left(1-\frac{a^2}{\beta N}\right) + 2\ln 2 + c_\beta\right]$$





Entanglement entropy $S_q(\mathcal{D})$ – our conjecture (W=0)

$S_q(\mathcal{D})$ - bipartite Rényi entanglement entropy between a domain $\mathcal{D}$ and the rest of the system $\overline{\mathcal{D}}$

$$S_q(\mathcal{D}) = \frac{1}{1-q} \ln \mathrm{Tr}\left(\rho_{\mathcal{D}}^q\right), \qquad \rho_{\mathcal{D}} = \mathrm{Tr}_{\overline{\mathcal{D}}}(\rho)$$

- For non-interacting fermions (W=0), $S_q(\mathcal{D})$ determined entirely by the distribution of $\mathcal{N}_{\mathcal{D}}$ [Calabrese, Mintchev, and Vicari, EPL 2012]

$$S_q(\mathcal{D}) = \sum_{n=1}^{\infty} s_n^{(q)} \left\langle \mathcal{N}_{\mathcal{D}}^{2n} \right\rangle_c$$

with known coefficients $s_n^{(q)}$.

- Our conjecture for the cumulants leads to the conjecture

$$S_q(\mathcal{D}) \simeq \frac{\pi^2}{6} \frac{q+1}{q} \mathrm{Var}(\mathcal{N}_{\mathcal{D}}) + E_q \times \begin{cases} 1, & d = 1, \mathcal{D} = [a,b], \\ \frac{1}{2}, & d = 1, \mathcal{D} = [a,\infty), \\ \frac{(k_F R)^{d-1}}{(d-1)!}, & d > 1, \mathcal{D} = B(0,R), \end{cases}$$

(E_q is known exactly)

- For some potentials (HO, free fermions,…) we proved this conjecture

Conclusion

d=1 height function, counting staircase

- non-interacting fermions in general V(x)

d=1 height covariance from 1st principle WKB semi-classics -> inhomogeneous GFF

Variance to leading and subleading order (V dependent)

For some V(x) connects to RMT ensembles -> FCS and entanglement entropy

FH/universality conjecture

Extension of these results for spherical domain in d>1

hole proba d>1

Gouraud,PLD,Schehr arXiv:2104.08574

Conclusion

d=1 height function, counting staircase

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- statistics extrema of height function

log correlated field contributions micro (fermionic stat) and macro (fBm0)

- interacting fermions some V(x), W(x,y) -> FCS Forrester-Frankel/universality conjectures

- Extension: variance in edge regime d>1

Conjectures numerical tests FCS any beta: matching bulk <-> edge results of

Bothner,Buckingham (2018) $\beta = 1, 2, 4$

Beyond: HO also valid for statistics momenta p, beyond HO? (TOF)

Tonks Girardeau, spin, dynamics, linear stat, large dev, any domain d>1,rotating LLL,...