Dimer model on minimal graphs:  
the elliptic case and beyond

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Outline

• Dimer model
• Dimer model and Harnack curves
• Minimal graphs and immersions
• Dimer model on minimal graphs
• Results
**Dimer model: definition**

- Planar, bipartite graph $G = (V = B \cup W, E)$.

- Dimer configuration $M$: subset of edges s.t. each vertex is incident to exactly one edge of $M \rightsquigarrow M(G)$.

- Positive weight function on edges: $\nu = (\nu_e)_{e \in E}$.

- Dimer Boltzmann measure ($G$ finite):

  \[ \forall M \in \mathcal{M}(G), \quad P_{\text{dimer}}(M) = \frac{\prod_{e \in M} \nu_e}{Z_{\text{dimer}}(G, \nu)}. \]

  where $Z_{\text{dimer}}(G, \nu)$ is the dimer partition function.
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Dimer model: Kasteleyn matrix

- Kasteleyn matrix (Percus-Kuperberg version)
  - Edge $wb \rightsquigarrow$ angle $\phi_{wb}$ s.t. for every face $w_1, b_1, \ldots, w_k, b_k$:
    \[
    \sum_{j=1}^{k} (\phi_{w_j b_j} - \phi_{w_{j+1} b_j}) \equiv (k - 1)\pi \mod 2\pi.
    \]
  - $K$ is the corresponding twisted adjacency matrix.
  \[
  K_{w,b} = \begin{cases} 
  \nu_{wb} e^{i\phi_{wb}} & \text{if } w \sim b \\
  0 & \text{otherwise.}
  \end{cases}
  \]
Dimer model: founding results

▶ Assume $G$ finite.

**Theorem (Kasteleyn’61, Kuperberg’98)**

$$Z_{\text{dimer}}(G, \nu) = |\det(K)|.$$ 

**Theorem (Kenyon’97)**

Let $\mathcal{E} = \{e_1 = w_1b_1, \ldots, e_n = w_nb_n\}$ be a subset of edges of $G$, then:

$$P_{\text{dimer}}(e_1, \ldots, e_n) = \left| \left( \prod_{j=1}^{n} K_{w_j,b_j} \right) \det(K^{-1})_\mathcal{E} \right|,$$

where $(K^{-1})_\mathcal{E}$ is the sub-matrix of $K^{-1}$ whose rows/columns are indexed by black/white vertices of $\mathcal{E}$.

▶ $G$ infinite: Boltzmann measure $\leadsto$ Gibbs measure

- Periodic case [Cohn-Kenyon-Propp’01], [Ke.-Ok.-Sh.’06]
- Non-periodic [dT’07], [Boutillier-dT’10], [B-dT-Raschel’19]
Dimer model: periodic case

- Assume $G$ is bipartite, infinite, $\mathbb{Z}^2$-periodic.

- Exhaustion of $G$ by toroidal graphs: $(G_n) = (G/n\mathbb{Z}^2)$. 
DIMER MODEL: PERIODIC CASE

- Fundamental domain: $G_1$

Let $K_1$ be the Kasteleyn matrix of fundamental domain $G_1$.

- Multiply edge-weights by $z, z^{-1}, w, w^{-1} \rightarrow K_1(z, w)$.

- The characteristic polynomial is:

$$P(z, w) = \det K_1(z, w).$$

Example: weight function $\nu \equiv 1$, $P(z, w) = 5 - z - \frac{1}{z} - w - \frac{1}{w}$. 
Dimer model: periodic case

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**Dimer model: spectral curve**

- **The spectral curve:**

  \[ \mathcal{C} = \{(z, w) \in (\mathbb{C}^*)^2 : P(z, w) = 0\}. \]

- **Amoeba:** image of \( \mathcal{C} \) through the map \((z, w) \mapsto (\log|z|, \log|w|)\).

  ![Amoeba of the square-octagon graph](image)
**Dimer model and Harnack curves**

**Theorems**

▶ **Spectral curves of bipartite dimers**

\cite{Ke-Ok-Sh '06} \cite{Ke-Ok '06}

\[\leftrightarrow\] Harnack curves with points on ovals.

▶ **Spectral curves of isoradial, bipartite dimer models with critical weights**

\cite{Kenyon '02} \cite{Kenyon-Okounkov '06}

\[\leftrightarrow\] Harnack curves of genus 0.

Explicit (\(\leftarrow\)) map.

▶ **Spectral curves of minimal, bipartite dimers**

\cite{Goncharov-Kenyon '13}

\[\leftrightarrow\] Harnack curves with points on ovals.

Explicit (\(\rightarrow\)) map

▶ \cite{Fock '15} **Explicit (\(\leftarrow\)) map for all algebraic curves.**

(no check on positivity).
GIBBS MEASURES FOR BIPARTITE DIMER MODELS

Theorems (Kenyon-Okounkov-Sheffield’06)

- The dimer model on a $\mathbb{Z}^2$-periodic, bipartite graph has a two-parameter family of ergodic Gibbs measures.
- The latter are obtained as weak limits of Boltzmann measures with magnetic field coordinates $(B_x, B_y)$.
- The phase diagram is given by the amoeba of the spectral curve $\mathcal{C}$.

![Diagram of the phase diagram and a hexagonal lattice]
Goal of our work

- Find explicit (←→) map for general genus Harnack curves.

- [Kenyon’02] proves “local” formula for the maximal entropy Gibbs measure in the case of the critical dimer model on isoradial graphs.

  Extension to the two-parameter family of Gibbs measures in the general genus case.

- Extension to the case of non-periodic graphs.
**Quad-graph, train-tracks**

- Infinite, planar, embedded graph $G$; embedded dual graph $G^*$.  
- Corresponding *quad-graph* $G^\diamond$, *train-tracks*.  

![Quad-graph, train-tracks diagram](image)
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- Corresponding quad-graph $G^\circ$, train-tracks.
**Isoradial graphs**

- An *isoradial embedding* of an infinite, planar graph $G$ is an embedding such that all of its faces are inscribed in a circle of radius 1, and such that the center of the circles are in the interior of the faces [Duffin] [Mercat] [Kenyon].

- Equivalent to: the quad-graph $G^\diamond$ is embedded so that of all its faces are rhombi.

**Theorem (Kenyon-Schlencker’04)**

*An infinite planar graph $G$ has an isoradial embedding iff*
Isoradial embeddings
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If the graph $G$ is bipartite, train-tracks are naturally oriented (white vertex of the left, black on the right) $\rightsquigarrow \vec{T}$.
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A bipartite, planar graph $G$ is minimal if

[Thurston’04] [Gulotta’08] [Ishii-Ueda’11] [Goncharov-Kenyon’13]
**Immersions of minimal graphs**

- A **minimal immersion** of an infinite planar graph $G$ is an immersion of the quadgraph $G^\diamond$ such that:
  - all faces are rhombi (flat or reversed)
  - the immersion is **flat**: sum of rhombus angles around every vertex and every face is equal to $2\pi$.

**Theorem (Boutillier-Cimasoni-dT’19)**

- An infinite, planar, bipartite graph $G$ has a minimal immersion iff it is minimal.
- The space of minimal immersions of $G$ is an explicit subset of the angle maps $\{(\alpha) : \mathcal{T} \to \mathbb{R}/\pi\mathbb{Z}\}$ (preserves cyclic order).
Tool 1. Geometric data and theta functions.

- Genus 1.
  - Parameter $q = e^{i\pi \tau}$, $\tau \in i\mathbb{R}$, $\Lambda(q) = \pi \mathbb{Z} + \pi \tau \mathbb{Z}$
  - $\mathbb{T}(q) = \mathbb{C}/\Lambda := \Sigma$
  - Jacobi’s (first) theta function on $\mathbb{C}$

  $$\theta(z) = 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n + 1)z.$$  

  - Building block of meromorphic functions on $\Sigma$.
  - $\theta(z) \sim 2q^{\frac{1}{4}} \sin(z)$ as $q \to 0$.  

**Dimer version of Fock’s weights**

- **Tool 1. Geometric data and theta functions.**
  - Genus $g \geq 1$.
    - Maximal curve $\Sigma$ of genus $g$. Riemann surface with $\sigma$, anti-holomorphic involution; Real locus: $g + 1$ top. circles $C_0, C_1, \ldots, C_g$, fixed by $\sigma$.
    - Jacobian variety: $\text{Jac}(\Sigma) = \mathbb{C}^g / (\mathbb{Z}^g + \Omega\mathbb{Z}^g)$
      $\Omega$ is pure imaginary period matrix constructed from $\Sigma$.
    - Theta function on $\mathbb{C}^g$
      $$\theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(-i\pi \langle n, \Omega n \rangle + 2i\pi \langle z, n \rangle),$$
    - Abel map: $\Sigma \rightarrow \text{Jac}(\Sigma) \leadsto$ theta function on $\Sigma$.
    - Prime form $E$ on $\Sigma \times \Sigma$
      Building block of meromorphic functions on $\Sigma$.
  - Genus 1: $\Sigma \simeq \text{Jac}(\Sigma)$ (easier!)
Dimer version of Fock’s weights

- Tool 2. Another type of geometric data.
  - Minimal graph $G$.
  - Angle map $(\alpha) : \tilde{\mathcal{F}} \rightarrow C_0$ preserving cyclic order.

- Tool 3. Discrete Abel map $\eta$
  - Function $\eta$ on vertices of $G^\circ$: $\eta(f_0) = 0$ for given face $f_0$, then local rule

- Well chosen point $t \in \text{Jac}(\Sigma)$: $t \in (\mathbb{R}/\mathbb{Z})^g$. 
Dimer version of Fock’s weights

▶ Fock’s adjacency matrix

\[
K_{w,b} = \begin{cases} 
\frac{E(\beta - \alpha)}{\theta(t + \eta(f))\theta(t + \eta(f'))} & \text{if } w \sim b \\
0 & \text{otherwise.}
\end{cases}
\]

Theorem (B-C-dT)

If the following conditions hold:

· \(\Sigma\) is a maximal-curve,

· angle map \((\alpha) : \vec{T} \to C_0\) preserves cyclic order,

· parameter \(t \in \text{Jac}(\Sigma)\) well chosen,

then, Fock’s adjacency matrix is a Kasteleyn matrix for a dimer model on \(\mathbb{G}\) (positive weights).

\(\rightsquigarrow\) Good framework for doing probability.
**Inverse(s) of Kasteleyn operator**

**Theorem (BCdT)**

For any \( u_0 \in \text{upper half of } \Sigma \), the following local formula defines an inverse of the Kasteleyn operator \( K \)

\[
\forall b, w \quad A_{b,w}^{u_0} := \frac{1}{2i\pi} \int_{C_{b,w}^{u_0}} g_{b,w}(u)
\]

where \( g_{b,w} = g_{b,x_1}g_{x_1,x_2} \ldots g_{x_n,w} \) for \( b, x_1, x_2, \ldots, x_n, w \) path in \( G^\diamond \)

\[
g_{f,w}(u) = \frac{\theta(u + t + \eta(w))}{E(u, \beta)} = g_{w,f}(u)^{-1}
\]

\[
g_{b,f}(u) = \frac{\theta(u - t - \eta(b))}{E(u, \alpha)} = g_{f,b}(u)^{-1}
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Inverse(s) of Kasteleyn operator

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Idea of proof

▶ Show the identity $KA^{u_0} = \text{Id}$.
▶ Use Fay’s trisecant identity:

$$
\frac{\theta(s + u - \alpha - \beta)}{E(\alpha, u)E(\beta, u)} \frac{E(\alpha, \beta)}{\theta(s - \alpha)\theta(s - \beta)} = \frac{\theta(s + u - \beta - \gamma)}{E(\beta, u)E(\gamma, u)} \frac{E(\gamma, \beta)}{\theta(s - \beta)\theta(s - \gamma)} - \frac{\theta(s + u - \alpha - \gamma)}{E(\alpha, u)E(\gamma, u)} \frac{E(\gamma, \alpha)}{\theta(s - \alpha)\theta(s - \gamma)}
$$

▶ Show that the contours of integrations are such that one has 1’s on the diagonal.
Gibbs measures and phase diagram

Assume that the minimal graph $G$ satisfies:

(*) any finite connected subgraph $G_0 \subset G$ is contained in a periodic minimal graph.

**Theorem (BCdT)**

For any $u_0$ in the upper half of $\Sigma$, there is a Gibbs measure $\mathbb{P}^{u_0}$ on $\mathcal{M}(G)$ such that for $e_1 = w_1b_1, \ldots, e_k = w_kb_k$ distinct edges of $G$,

$$\mathbb{P}^{u_0}(e_1, \ldots, e_k) = \left( \prod_{i=1}^{k} K_{w_i, b_i} \right) \det_{1 \leq i, j \leq k} \left[ A_{b_i, w_j}^{u_0} \right].$$

Moreover, we have the phase diagram:

- $u_0 \in C_j, 1 \leq j \leq g \iff$ **gaseous** (expon. decay)
- $u_0 \in C_0 \iff$ **frozen** (no decay of correlations)
- $u_0 \notin C_0 \cup \cdots \cup C_g \iff$ **liquid** (polynomial decay)
Remarks

▶ Periodic case: explicit local expression for the two parameter family of Gibbs measures of [KOS’06].

▶ Non-periodic case: better understanding of possible phase diagram (upper half of the maximal curve $\Sigma$).
Explicit parameterization of the spectral curve

Assume $G$ is $\mathbb{Z}^2$-periodic. Define the map $\psi$,

$$\psi : \Sigma \to \mathbb{C}^2$$

$$u \mapsto \psi(u) = (z(u), w(u))$$

where $z(u) = g_{b_0, b_0 + (1,0)}(u)$, $w(u) = g_{b_0, b_0 + (0,1)}(u)$\(^1\).

\(^1\)with additional assumption to ensure periodicity.
Explicit parameterization of the spectral curve

Proposition ([B-C-dT])

The map $\psi$ is an explicit birational parameterization of the spectral curve $C$, mapping $C_1, \ldots, C_g$ to the ovals of $C$ and $C_0$ to the unbounded real component of $C$, implying in particular that $C$ has geometric genus $g$. 

![Diagram showing the parameter space $\mathbb{T}(q)$ and the map $\log|\psi|$]
Dimer model and Harnack curves of genus $g$

**Theorem ([B-C-dT])**

Fix a Harnack curve with a standard divisor. Then there exists $\Sigma$, $G$, $(\alpha)$, $t$ such that $C$ is the corresponding spectral curve.
Connection to previous work

- Genus 0. (as limit of genus 1 case) [Kenyon’02].
- Genus 1. Two specific cases were handled before:
  - the bipartite graph arising from the Ising model [Boutillier-dT-Raschel’20]
  - the Z-Dirac operator [dT’18] \(\mapsto\) massive discrete holomorphic functions.
 Perspectives

- Prove the (*) condition.
- Explore higher genus analogue of the massive Laplacian [George].
- Link with t-embeddings for dimers [Kenyon-Lam-Ramassamy-Russkikh].