Dimer model on minimal graphs: the elliptic case and beyond

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joint works with

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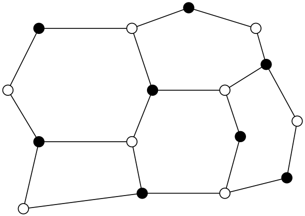
Outline

- Dimer model
- Dimer model and Harnack curves
- Minimal graphs and immersions
- Dimer model on minimal graphs
- Results

Dimer model: definition

- Planar, bipartite graph $\mathsf{G} = (\mathsf{V} = \mathsf{B} \cup \mathsf{W}, \mathsf{E})$.



- Dimer configuration M: subset of edges s.t. each vertex is incident to exactly one edge of $\mathsf{M} \rightsquigarrow \mathcal{M}(\mathsf{G})$.
- Positive weight function on edges: $\nu = (\nu_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}}$.
- Dimer Boltzmann measure (G finite):

$$\forall\, \mathsf{M} \in \mathcal{M}(\mathsf{G}), \quad \mathbb{P}_{\mathrm{dimer}}(\mathsf{M}) = \frac{\prod_{\mathsf{e} \in \mathsf{M}} \nu_{\mathsf{e}}}{Z_{\mathrm{dimer}}(\mathsf{G}, \nu)}.$$

where $Z_{\mathrm{dimer}}(\mathsf{G}, \nu)$ is the dimer partition function.

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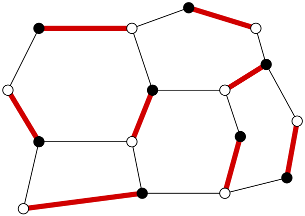

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- Positive weight function on edges: $\nu = (\nu_e)_{e \in E}$.
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$$\forall M \in \mathcal{M}(G), \quad \mathbb{P}_{\text{dimer}}(M) = \frac{\prod_{e \in M} \nu_e}{Z_{\text{dimer}}(G, \nu)}.$$

where $Z_{\text{dimer}}(G, \nu)$ is the dimer partition function.

Dimer model: Kasteleyn matrix

- Kasteleyn matrix (Percus-Kuperberg version)
  - Edge $wb \rightsquigarrow$ angle ϕ_{wb} s.t. for every face $w_1, b_1, \ldots, w_k, b_k$:

$$\sum_{j=1}^{k} (\phi_{w_j b_j} - \phi_{w_{j+1} b_j}) \equiv (k-1)\pi \mod 2\pi.$$

  - K is the corresponding twisted adjacency matrix.

$$\mathsf{K}_{w,b} = \begin{cases} \nu_{wb} e^{i\phi_{wb}} & \text{if } w \sim b \\ 0 & \text{otherwise.} \end{cases}$$

Dimer model: founding results

- Assume G finite.

Theorem ([Kasteleyn'61] [Kuperberg'98])

$$Z_{\mathrm{dimer}}(\mathsf{G},\nu) = |\det(\mathsf{K})|.$$

Theorem (Kenyon'97)

Let $\mathcal{E} = \{\mathsf{e}_1 = w_1 b_1, \dots, \mathsf{e}_n = w_n b_n\}$ *be a subset of edges of* G, *then:*

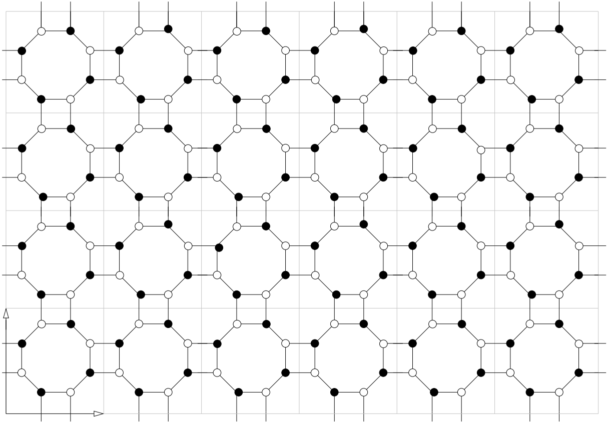
$$\mathbb{P}_{\mathrm{dimer}}(\mathsf{e}_1,\dots,\mathsf{e}_n) = \left|\Big(\prod_{j=1}^{n} \mathsf{K}_{w_j,b_j}\Big)\det(\mathsf{K}^{-1})_{\mathcal{E}}\right|,$$

where $(\mathsf{K}^{-1})_{\mathcal{E}}$ *is the sub-matrix of* K^{-1} *whose rows/columns are indexed by black/white vertices of* $\mathcal{E}$.

- G infinite: Boltzmann measure $\rightsquigarrow$ Gibbs measure
  - Periodic case [Cohn-Kenyon-Propp'01], [Ke.-Ok.-Sh.'06]
  - Non-periodic [dT'07], [Boutillier-dT'10], [B-dT-Raschel'19]

Dimer model: periodic case

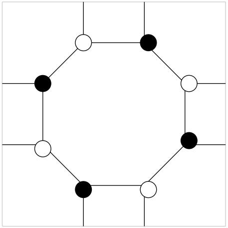
- Assume G is bipartite, infinite, $\mathbb{Z}^2$-periodic.



- Exhaustion of G by toroidal graphs: $(\mathsf{G}_n) = (\mathsf{G}/n\mathbb{Z}^2)$.

Dimer model: periodic case

- Fundamental domain: G_1



- Let K_1 be the Kasteleyn matrix of fundamental domain G_1.
- Multiply edge-weights by $\mathsf{z}, \mathsf{z}^{-1}, \mathsf{w}, \mathsf{w}^{-1} \rightarrow \mathsf{K}_1(\mathsf{z}, \mathsf{w})$.
- The characteristic polynomial is:

$$P(\mathsf{z}, \mathsf{w}) = \det \mathsf{K}_1(\mathsf{z}, \mathsf{w}).$$

Example: weight function $\nu \equiv 1$, $P(\mathsf{z}, \mathsf{w}) = 5 - \mathsf{z} - \frac{1}{\mathsf{z}} - \mathsf{w} - \frac{1}{\mathsf{w}}$.

Dimer model: periodic case

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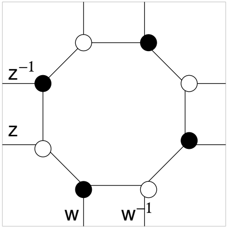


- Let K_1 be the Kasteleyn matrix of fundamental domain G_1.
- Multiply edge-weights by $z, z^{-1}, w, w^{-1} \to \mathsf{K}_1(z, w)$.
- The characteristic polynomial is:

$$P(z, w) = \det \mathsf{K}_1(z, w).$$

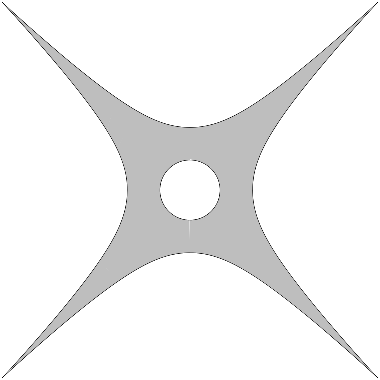
Example: weight function $\nu \equiv 1$, $P(z, w) = 5 - z - \frac{1}{z} - w - \frac{1}{w}$.

Dimer model: spectral curve

- The spectral curve:

$$\mathcal{C} = \{(\mathsf{z}, \mathsf{w}) \in (\mathbb{C}^*)^2 : \; P(\mathsf{z}, \mathsf{w}) = 0\}.$$

- Amoeba: image of $\mathcal{C}$ through the map $(\mathsf{z}, \mathsf{w}) \mapsto (\log|\mathsf{z}|, \log|\mathsf{w}|)$.



Amoeba of the square-octagon graph

Dimer model and Harnack curves

Theorems

- *Spectral curves of bipartite dimers* $\overset{\text{[Ke.-Ok.-Sh.'06] [Ke.-Ok.'06]}}{\longleftrightarrow}$ *Harnack curves with points on ovals.*

- *Spectral curves of isoradial, bipartite dimer models with critical weights [Kenyon '02]* $\overset{\text{[Kenyon-Okounkov'06]}}{\longleftrightarrow}$ *Harnack curves of genus 0.*

  Explicit ($\longleftarrow$) map.

- *Spectral curves of minimal, bipartite dimers* $\overset{\text{[Goncharov-Kenyon '13]}}{\longleftrightarrow}$ *Harnack curves with points on ovals.*

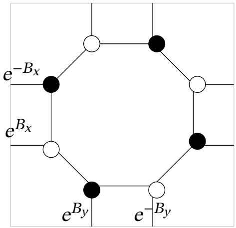
  Explicit ($\longrightarrow$) map

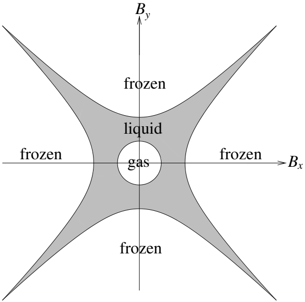
- *[Fock'15] Explicit ($\longleftarrow$) map for all algebraic curves.* *(no check on positivity).*

Gibbs measures for bipartite dimer models

Theorems (Kenyon-Okounkov-Sheffield'06)

- *The dimer model on a $\mathbb{Z}^2$-periodic, bipartite graph has a two-parameter family of ergodic Gibbs measures.*
- *The latter are obtained as weak limits of Boltzmann measures with magnetic field coordinates (B_x, B_y).*
- *The phase diagram is given by the amoeba of the spectral curve $\mathcal{C}$.*





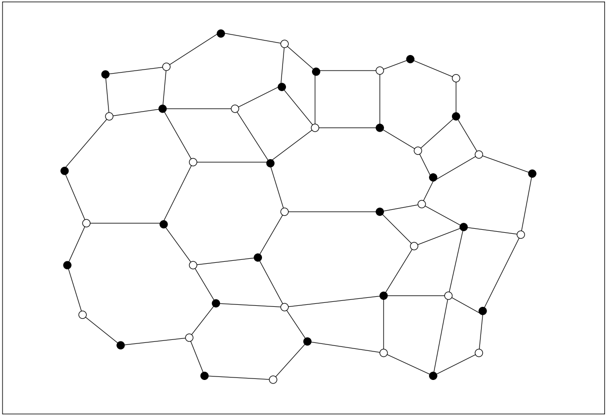
Goal of our work

- Find explicit ($\longleftarrow$) map for general genus Harnack curves.
- [Kenyon'02] proves "local" formula for the maximal entropy Gibbs measure in the case of the critical dimer model on isoradial graphs.

  $\rightsquigarrow$ Extension to the two-parameter family of Gibbs measures in the general genus case.
- Extension to the case of non-periodic graphs.

Quad-graph, train-tracks

- Infinite, planar, embedded graph G; embedded dual graph G^*.
- Corresponding quad-graph $\mathsf{G}^\diamond$, train-tracks.



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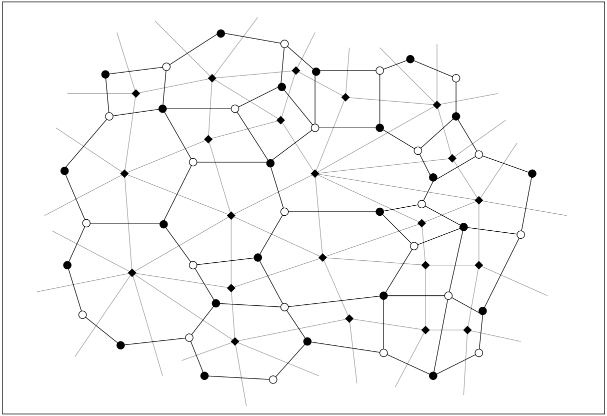

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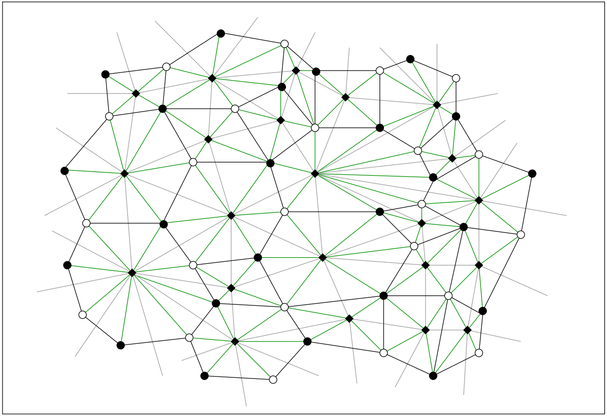

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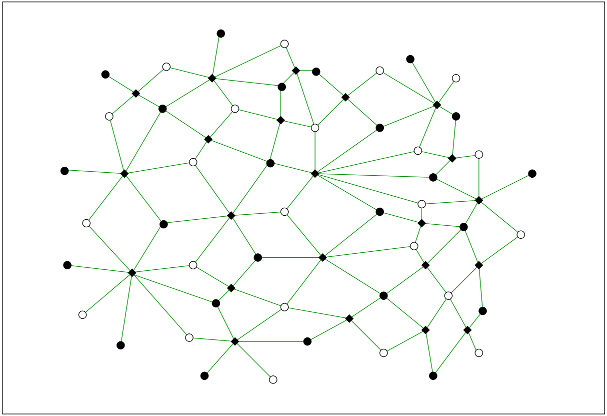

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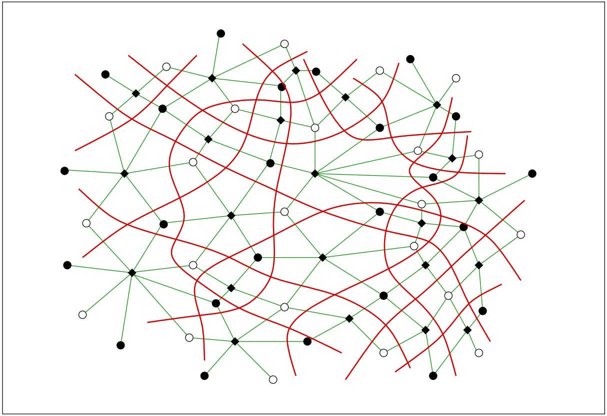

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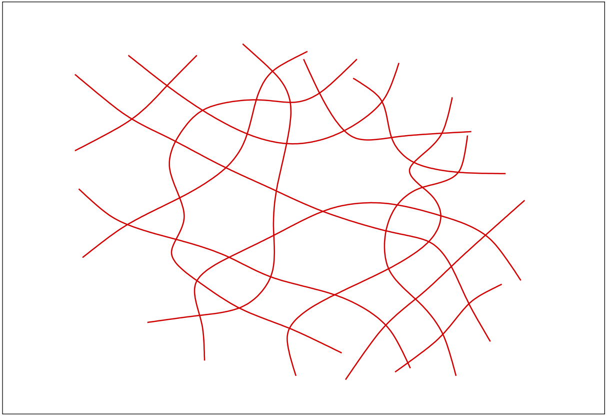
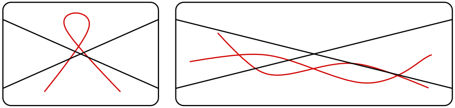


Isoradial graphs

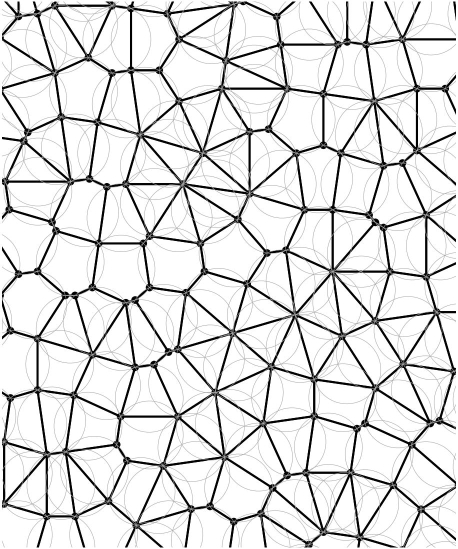
- An isoradial embedding of an infinite, planar graph G is an embedding such that all of its faces are inscribed in a circle of radius 1, and such that the center of the circles are in the interior of the faces [Duffin] [Mercat] [Kenyon].
- Equivalent to: the quad-graph $\mathsf{G}^\diamond$ is embedded so that of all its faces are rhombi.

Theorem (Kenyon-Schlencker'04)

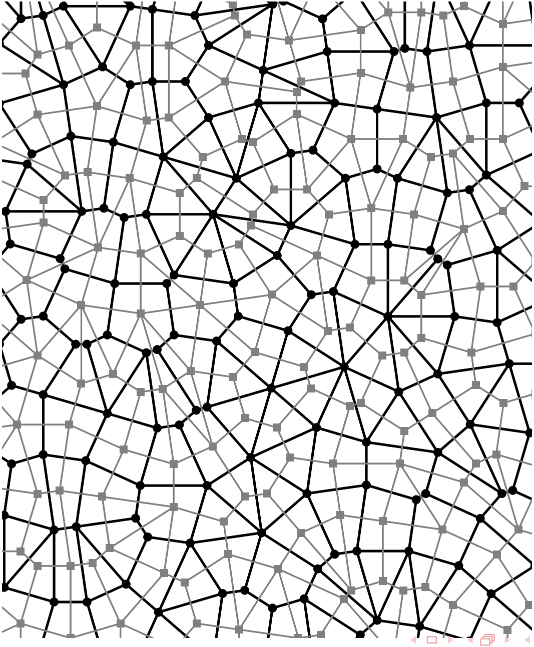
An infinite planar graph G *has an isoradial embedding iff*



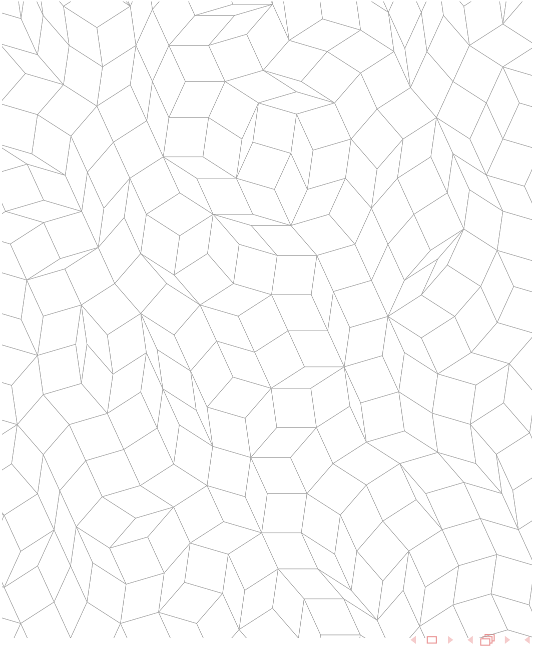
Isoradial embeddings



Isoradial embeddings

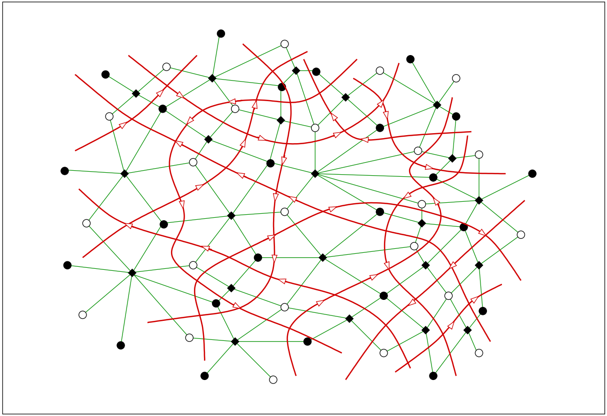


Isoradial embeddings



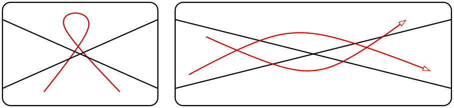
Minimal graphs

- If the graph G is bipartite, train-tracks are naturally oriented (white vertex of the left, black on the right) $\rightsquigarrow \vec{\mathcal{T}}$



Minimal graphs

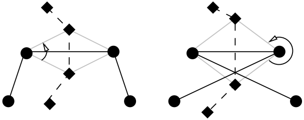
- If the graph G is bipartite, train-tracks are naturally oriented (white vertex of the left, black on the right) $\rightsquigarrow \vec{\mathcal{T}}$
- A bipartite, planar graph G is minimal if



[Thurston'04] [Gulotta'08] [Ishii-Ueda'11] [Goncharov-Kenyon'13]

Immersions of minimal graphs

- A minimal immersion of an infinite planar graph G is an immersion of the quadgraph $\mathsf{G}^\diamond$ such that:
  - all faces are rhombi (flat or reversed)



  - the immersion is flat: sum of rhombus angles around every vertex and every face is equal to 2π.

Theorem (Boutillier-Cimasoni-dT'19)

- *An infinite, planar, bipartite graph G has a minimal immersion iff it is minimal.*
- *The space of minimal immersions of G is an explicit subset of the angle maps $\{(\alpha) : \vec{\mathcal{T}} \to \mathbb{R}/\pi\mathbb{Z}\}$ (preserves cyclic order).*

Dimer version of Fock's weights

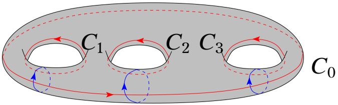
- Tool 1. Geometric data and theta functions.
  - Genus 1.
    - Parameter $q = e^{i\pi\tau}$, $\tau \in i\mathbb{R}$, $\Lambda(q) = \pi\mathbb{Z} + \pi\tau\mathbb{Z}$
    - $\mathbb{T}(q) = \mathbb{C}/\Lambda := \Sigma$
    - Jacobi's (first) theta function on $\mathbb{C}$

$$\theta(z) = 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)z.$$

    - Building block of meromorphic functions on Σ.
    - $\theta(z) \sim 2q^{\frac{1}{4}} \sin(z)$ as $q \to 0$.

Dimer version of Fock's weights

- Tool 1. Geometric data and theta functions.
  - Genus $g \geq 1$.
    - Maximal curve Σ of genus g. Riemann surface with σ, anti-holomorphic involution; Real locus: $g+1$ top. circles $C_0, C_1, \ldots, C_g$, fixed by σ.



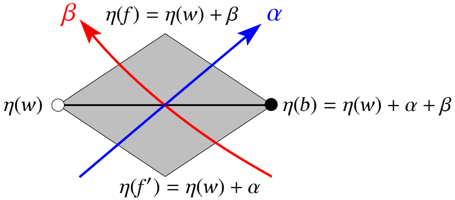
    - Jacobian variety: $\mathrm{Jac}(\Sigma) = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$
      Ω is pure imaginary period matrix constructed from Σ.
    - Theta function on $\mathbb{C}^g$

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(-i\pi\langle n, \Omega n\rangle + 2i\pi\langle z, n\rangle),$$

    - Abel map: $\Sigma \to \mathrm{Jac}(\Sigma) \rightsquigarrow$ theta function on Σ.
    - Prime form E on $\Sigma \times \Sigma$
      Building block of meromorphic functions on Σ.
  - Genus 1: $\Sigma \simeq \mathrm{Jac}(\Sigma)$ (easier!)

Dimer version of Fock's weights

- Tool 2. Another type of geometric data.
  - Minimal graph G.
  - Angle map $(\alpha) : \vec{\mathcal{T}} \to C_0$ preserving cyclic order.
- Tool 3. Discrete Abel map η
  - Function η on vertices of $\mathsf{G}^\diamond$: $\eta(f_0) = 0$ for given face f_0, then local rule



- Well chosen point $t \in \mathrm{Jac}(\Sigma)$: $t \in (\mathbb{R}/\mathbb{Z})^g$.

Dimer version of Fock's weights

- Fock's adjacency matrix

$$\mathsf{K}_{w,b} = \begin{cases} \dfrac{E(\beta-\alpha)}{\theta(t+\eta(f))\theta(t+\eta(f'))} & \text{if } w \sim b \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (B-C-dT)

If the following conditions hold:

- *Σ is a maximal-curve,*
- *angle map $(\alpha) : \vec{\mathfrak{T}} \to C_0$ preserves cyclic order,*
- *parameter $t \in \mathrm{Jac}(\Sigma)$ well chosen,*

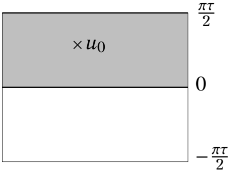
then, Fock's adjacency matrix is a Kasteleyn matrix for a dimer model on G (positive weights).

$\rightsquigarrow$ Good framework for doing probability.

Inverse(s) of Kasteleyn operator

Theorem (BCdT)

For any $u_0 \in$ upper half of Σ, the following local formula defines an inverse of the Kasteleyn operator K



$$\forall b, w \quad A^{u_0}_{b,w} := \frac{1}{2i\pi} \int_{C^{u_0}_{b,w}} g_{b,w}(u)$$

where $g_{b,w} = g_{b,x_1} g_{x_1,x_2} \dots g_{x_n,w}$ for $b, x_1, x_2, \dots, x_n, w$ path in $G^\diamond$

$$g_{f,w}(u) = \frac{\theta(u + t + \eta(w))}{E(u,\beta)} = g_{w,f}(u)^{-1}$$

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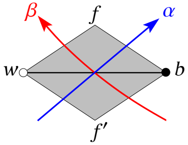

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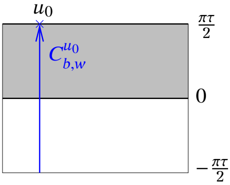

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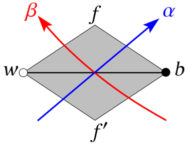

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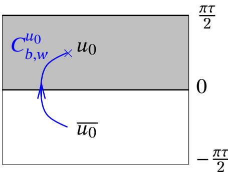

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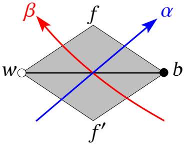
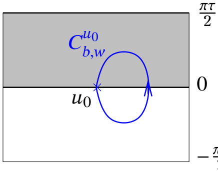


Inverse(s) of Kasteleyn operator

Theorem (BCdT)

For any $u_0 \in$ *upper half of* Σ, *the following local formula defines an inverse of the Kasteleyn operator* K



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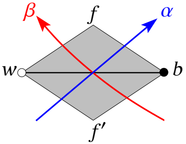


Idea of proof

- Show the identity $\mathsf{K}A^{u_0} = \mathrm{Id}$.
- Use Fay's trisecant identity:

$$\frac{\theta(s+u-\alpha-\beta)}{E(\alpha,u)E(\beta,u)}\frac{E(\alpha,\beta)}{\theta(s-\alpha)\theta(s-\beta)} =$$
$$\frac{\theta(s+u-\beta-\gamma)}{E(\beta,u)E(\gamma,u)}\frac{E(\gamma,\beta)}{\theta(s-\beta)\theta(s-\gamma)} - \frac{\theta(s+u-\alpha-\gamma)}{E(\alpha,u)E(\gamma,u)}\frac{E(\gamma,\alpha)}{\theta(s-\alpha)\theta(s-\gamma)}$$

- Show that the contours of integrations are such that one has 1's on the diagonal.

Gibbs measures and phase diagram

- Assume that the minimal graph G satisfies:

  (∗) any finite connected subgraph $\mathsf{G}_0 \subset \mathsf{G}$ is contained in a periodic minimal graph.

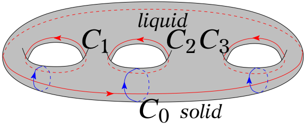
Theorem (BCdT)

For any u_0 in the upper half of Σ, there is a Gibbs measure $\mathbb{P}^{u_0}$ on $\mathcal{M}(\mathsf{G})$ such that for $\mathsf{e}_1 = w_1 b_1, \cdots, \mathsf{e}_k = w_k b_k$ distinct edges of G,

$$\mathbb{P}^{u_0}(\mathsf{e}_1, \ldots, \mathsf{e}_k) = \left(\prod_{i=1}^{k} \mathsf{K}_{w_i, b_i} \right) \det_{1 \le i,j \le k} \left[A^{u_0}_{b_i, w_j} \right].$$

Moreover, we have the phase diagram:

- $u_0 \in C_j, 1 \le j \le g, \Leftrightarrow$ *gaseous (expon. decay)*
- $u_0 \in C_0 \Leftrightarrow$ *frozen (no decay of correlations)*
- $u_0 \notin C_0 \cup \cdots \cup C_g \Leftrightarrow$ *liquid (polynomial decay)*



Remarks

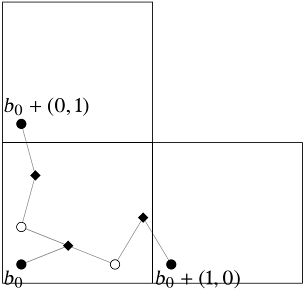
- Periodic case: explicit local expression for the two parameter family of Gibbs measures of [KOS'06].
- Non-periodic case: better understanding of possible phase diagram (upper half of the maximal curve Σ).

Explicit parameterization of the spectral curve

- Assume G is $\mathbb{Z}^2$-periodic. Define the map ψ,

$$\psi : \Sigma \to \mathbb{C}^2$$
$$u \mapsto \psi(u) = (\mathsf{z}(u), \mathsf{w}(u))$$

where $\mathsf{z}(u) = g_{b_0, b_0+(1,0)}(u)$, $\mathsf{w}(u) = g_{b_0, b_0+(0,1)}(u)$[1].

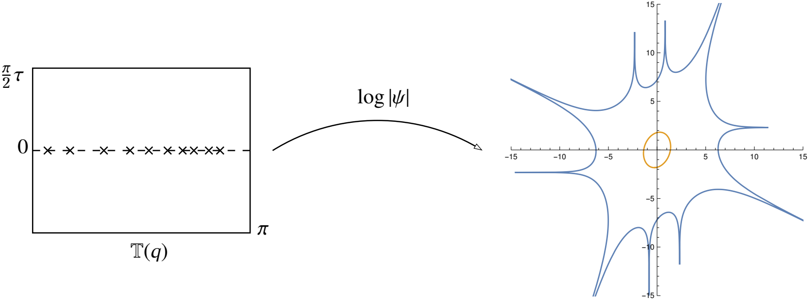


[1] with additional assumption to ensure periodicity

Explicit parameterization of the spectral curve

Proposition ([B-C-dT])

The map ψ is an explicit birational parameterization of the spectral curve $\mathcal{C}$, mapping $C_1, \ldots, C_g$ to the ovals of $\mathcal{C}$ and C_0 to the unbounded real component of $\mathcal{C}$, implying in particular that $\mathcal{C}$ has geometric genus g.



Dimer model and Harnack curves of genus g

Theorem ([B-C-dT])

Fix a Harnack curve with a standard divisor. Then there exists Σ, G, (α), t such that $\mathcal{C}$ is the corresponding spectral curve.

Connection to previous work

- Genus 0. (as limit of genus 1 case) [Kenyon'02].
- Genus 1. Two specific cases were handled before:
  - the bipartite graph arising from the Ising model [Boutillier-dT-Raschel'20]
  - the Z-Dirac operator [dT'18] $\rightsquigarrow$ massive discrete holomorphic functions.

Perspectives

- Prove the $(*)$ condition.
- Explore higher genus analogue of the massive Laplacian [George].
- Link with t-embeddings for dimers [Kenyon-Lam-Ramassamy-Russkikh].