Randomness, Integrability and Universality
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Stationary half-space last passage percolation

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with D. Betea and A. Occelli
Comm. Math. Phys. 377 (2020), 421-467 (one-point)
Stoch. Process. Appl. 146 (2022), 207-263 (multi-point)

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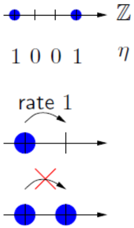
KPZ stationary models in full-space

- TASEP: Totally Asymmetric Simple Exclusion Process

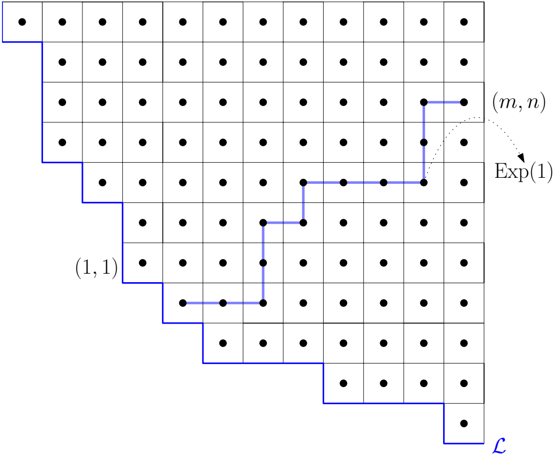
- **Configurations**
  $\eta = \{\eta_x\}_{x\in\mathbb{Z}}$, $\eta_x = \begin{cases} 1, & \text{if } x \text{ is occupied,} \\ 0, & \text{if } x \text{ is empty.} \end{cases}$



- **Dynamics**
  Independently, particles jump on the right site with rate 1, provided the right is empty.

⇒ Particles are ordered: position of particle n is $x_n(t)$ with $x_n(t) > x_{n+1}(t)$ for all n, t.

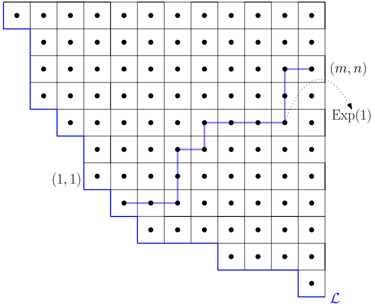
- Consider independent random variables $\{\omega_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ with $\omega_{i,j} \sim \mathrm{Exp}(1)$



- Consider independent random variables $\{\omega_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ with $\omega_{i,j} \sim \mathrm{Exp}(1)$
- The line-to-point LPP from a line $\mathcal{L}$ to the point (m,n) is given by

$$L_{m,n} = \max_{\pi:\mathcal{L}\to(m,n)} \sum_{(i,j)\in\pi} \omega_{i,j}$$

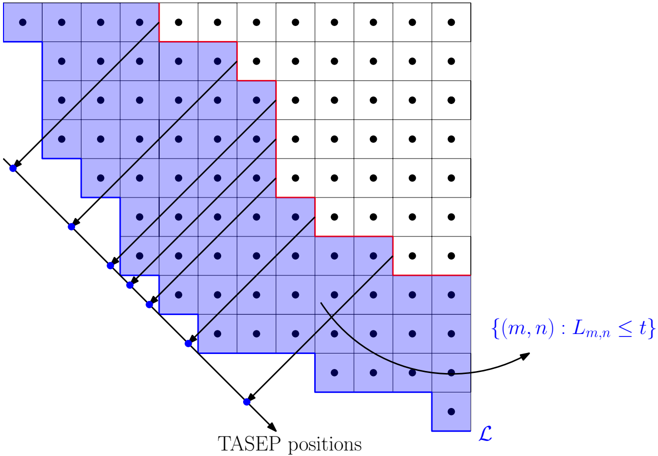
where the maximum is over up-right paths from $\mathcal{L}$ to (m,n), i.e. paths with increments in $\{(0,1),(1,0)\}$.



- The well-known connection between TASEP and LPP is

$$\mathbb{P}(L_{m,n} \leq t) = \mathbb{P}(x_n(t) \geq m - n).$$

where $\mathcal{L} = \{(x_k(0) + k, k), k \in \mathbb{Z} \text{ or } \mathbb{N}\}$.



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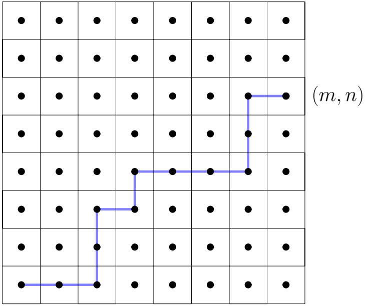
- Example: Step-initial condition $x_k(0) = -k + 1$, $k \geq 1$, $\mathcal{L} = \{(1, k), k \geq 1\}$, equivalent to reduce $\mathcal{L}$ to one point.

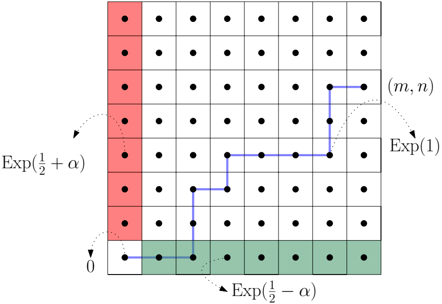


- Point-to-point LPP:

$$\omega_{i,j} = \begin{cases} \mathrm{Exp}(1), & i,j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Stationary LPP: fix $\alpha \in (-1/2, 1/2)$

$$\omega_{i,j} = \begin{cases} \mathrm{Exp}\left(\frac{1}{2}+\alpha\right) & i=0, j\geq 1, \\ \mathrm{Exp}\left(\frac{1}{2}-\alpha\right) & j=0, i\geq 1, \\ 0 & \text{if } i=j=0, \\ \mathrm{Exp}(1) & \text{otherwise.} \end{cases}$$



- Point-to-point LPP: GUE Tracy-Widom distribution

$$\lim_{t\to\infty} \mathbb{P}(L_{N,N} \leq 4N + s2^{4/3}N^{1/3}) = F_{\mathrm{GUE}}(s)$$

with

$$F_{\mathrm{GUE}}(s) = \det(\mathbb{1} - K_{\mathrm{Ai}})_{L^2(s,\infty)}$$

with $K_{\mathrm{Ai}}(x,y) = \int_{\mathbb{R}_+} d\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$ is the Airy kernel.

- Stationary initial condition: (stated for $\alpha = 0$)
  Baik-Rains distribution

$$\lim_{t\to\infty} \mathbb{P}(L^{\mathrm{stat}}_{N+w(2N)^{2/3},N-w(2N)^{2/3}} \leq 4N + s2^{4/3}N^{1/3}) = F_{\mathrm{BR},w}(s),$$

with $F_{\mathrm{BR},w}(s) = \frac{d}{ds}[F_{\mathrm{GUE}}(s+w^2)g(s,w)]$.

- w measures the distance from the characteristic line.

- The Baik-Rains distribution function is

$$F_{\mathrm{BR},w}(s) = \frac{d}{ds}[F_{\mathrm{GUE}}(s+w^2)g(s,w)].$$

- Let $\widehat{K}_{\mathrm{Ai}}(x,y) = K_{\mathrm{Ai}}(x+w^2, y+w^2)$, and

$$\mathcal{R} = s + e^{-\frac{2}{3}w^3}\int_s^\infty dx \int_0^\infty dy \mathrm{Ai}(x+y+w^2)\, e^{-w(x+y)},$$

$$\Psi(y) = e^{\frac{2}{3}w^3+wy} - \int_0^\infty dx \mathrm{Ai}(x+y+w^2)\, e^{-wx},$$

$$\Phi(x) = e^{-\frac{2}{3}w^3}\int_0^\infty d\lambda \int_s^\infty dy \mathrm{Ai}(x+w^2+\lambda)\mathrm{Ai}(y+w^2+\lambda)\, e^{-wy} - \int_0^\infty dy \mathrm{Ai}(y+x+w^2)\, e^{wy}.$$

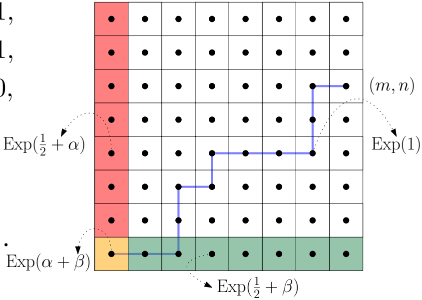
- Let P_s be the projection operator $P_s(x) = \mathbb{1}_{\{x>s\}}$, then the function g is given by

$$g(w,s) = \mathcal{R} - \big\langle (\mathbb{1} - P_s\widehat{K}_{\mathrm{Ai}}P_s)^{-1}P_s\Phi, P_s\Psi\big\rangle.$$

Step 1: An integrable model with a random shift τ.

- For $\alpha, \beta \in (-1/2, 1/2]$ with $\alpha + \beta > 0$:

$$\omega_{i,j} = \begin{cases} \mathrm{Exp}\left(\frac{1}{2}+\alpha\right) & i=0, j \geq 1, \\ \mathrm{Exp}\left(\frac{1}{2}+\beta\right) & j=0, i \geq 1, \\ \tau = \mathrm{Exp}\left(\alpha+\beta\right) & \text{if } i=j=0, \\ \mathrm{Exp}(1) & \text{otherwise.} \end{cases}$$



Using a Schur process:

$$\mathbb{P}(L^\tau_{m,n} \leq s) = \det(\mathbb{1} - K_{\alpha,\beta})_{L^2(s,\infty)}.$$

$$\boxed{L^{\mathrm{stat}}_{m,n} = \lim_{\beta\to-\alpha}(L^\tau_{m,n} - \tau)}$$

Step 2: Shift argument.

$$\mathbb{P}(L^\tau_{m,n} - \tau \leq s) = \left(1 + \frac{1}{\alpha+\beta}\frac{d}{ds}\right)\mathbb{P}(L^\tau_{m,n} \leq s).$$

Step 3: $K_{\alpha,\beta}$ is a rank-one perturbation:

$$K_{\alpha,\beta}(x,y) = \overline{K}(x,y) + (\alpha+\beta) f_\alpha(x) g_\beta(y)$$

gives

$$\det(\mathbb{1} - K_{\alpha,\beta}) = \det(\mathbb{1} - \overline{K})[1 - (\alpha+\beta)\langle(\mathbb{1}-\overline{K})^{-1} f_\alpha, g_\beta\rangle)].$$

Thus

$$\mathbb{P}(L_{m,n}^{\mathrm{stat}} \leq s) = \lim_{\beta\to-\alpha} \frac{d}{ds}\left[\det(\mathbb{1}-\overline{K})\left(\frac{1}{\alpha+\beta} - \langle(\mathbb{1}-\overline{K})^{-1} f_\alpha, g_\beta\rangle)\right)\right].$$

Step 4: Analytic continuation for $\alpha,\beta \in (-1/2, 1/2)$.

$$\frac{1}{\alpha+\beta} - \langle(\mathbb{1}-\overline{K})^{-1} f_\alpha, g_\beta\rangle) = \left[\frac{1}{\alpha+\beta} - \langle f_\alpha, g_\beta\rangle\right] - \langle(\mathbb{1}-\overline{K})^{-1}\overline{K} f_\alpha, g_\beta\rangle.$$

Step 5: Large time limit:

- $\overline{K}$ converges to $\widehat{K}_{\mathrm{Ai}}$,
- the term $\lim_{\beta\to-\alpha} \frac{1}{\alpha+\beta} - \langle f_\alpha, g_\beta\rangle$ converges to $\mathcal{R}$,
- $\overline{K} f_\alpha$ and $g_{-\alpha}$ converge to Φ and Ψ.

Determinantal systems: one-point distribution

- Polynuclear growth model Baik,Rains'00,Imamura,Sasamoto'04
- TASEP / last passage percolation Ferrari,Spohn'05

Determinantal systems: multi-point distributions

- TASEP Baik,Ferrari,Péché'09
- One-sided reflecting Brownian motion (low density limit of TASEP) Ferrari,Spohn,Weiss'15

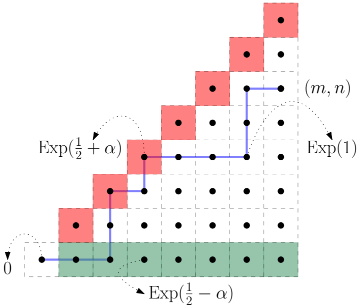
Integrable but not determinantal models (only one-point distribution)

- KPZ equation Imamura,Sasamoto'13
  Borodin,Corwin,Ferrari,Veto'14
- ASEP and stochastic six-vertex model Aggarwal'16
- q-TASEP and Semi-discrete directed polymer
  Imamura,Sasamoto'17

Half-space stationary models

- Fix $\alpha \in (-1/2, 1/2)$ and consider independent random variables $\{\omega_{i,j}\}_{(i,j)\in\mathcal{D}}$, $\mathcal{D} = \{(i,j) \in \mathbb{N}^2 | 1 \le j \le i\}$ and

$$\omega_{i,j} = \begin{cases} \mathrm{Exp}\left(\frac{1}{2}+\alpha\right) & i = j \ge 1, \\ \mathrm{Exp}\left(\frac{1}{2}-\alpha\right) & j = 0, i \ge 1, \\ 0 & \text{if } i = j = 0, \\ \mathrm{Exp}(1) & \text{otherwise.} \end{cases}$$



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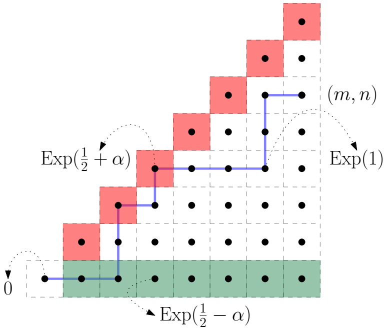
- A stationary half-space LPP time to the point (m,n) (for $n \leq m$), denoted $L^{\mathrm{stat}}_{m,n}$, is given by

$$L^{\mathrm{stat}}_{m,n} = \max_{\pi:(0,0)\to(m,n)} \sum_{(i,j)\in\pi} \omega_{i,j}$$

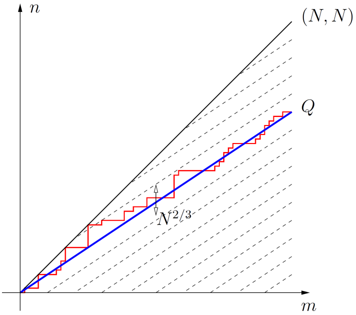
where the maximum is over up-right paths in $\mathcal{D}$ from $(1,1)$ to (m,n), i.e. paths with increments in $\{(0,1),(1,0)\}$.
- For TASEP, the boundary random variables are the injection waiting times at the origin.

- Why is this model called stationary?
- Increments $\{L^{\text{stat}}_{m+1,n} - L^{\text{stat}}_{m,n}, m \geq n\}$ are iid. $\text{Exp}(\frac{1}{2} - \alpha)$.
- Also $\{L^{\text{stat}}_{m,n} - L^{\text{stat}}_{m,n-1}, m \geq n\}$ are iid. $\text{Exp}(\frac{1}{2} + \alpha)$

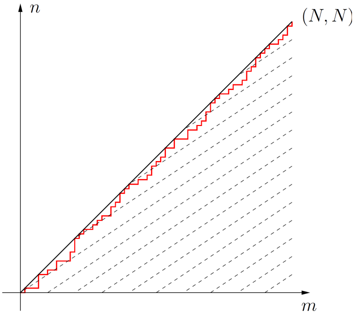
Balázs,Cator,Seppäläinen'06



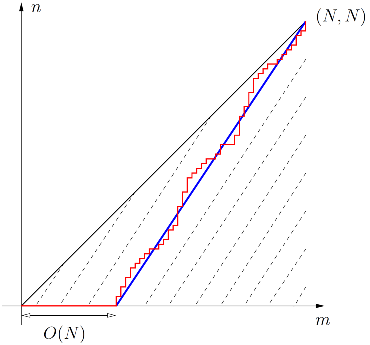
- Case $\alpha < 0$: large diagonal weights
- Characteristic lines have slopes $\left((\frac{1}{2}+\alpha)/(\frac{1}{2}-\alpha)\right)^2 < 1$
- End-point on characteristics from $(0,0)$: diagonal visited only $O(N^{2/3})$ around the origin: like full-space
- End-point (N,N): maximizer visits $O(N)$ times the diagonal: Gaussian fluctuations



$Q = N(1, (\frac{1}{2}+\alpha)^2/(\frac{1}{2}-\alpha)^2)$



- Case $\alpha > 0$: small diagonal weights
- Characteristic lines have slopes $\left((\frac{1}{2}+\alpha)/(\frac{1}{2}-\alpha)\right)^2 > 1$
- End-point (N,N): maximizer visits $O(N)$ times the first row: Gaussian fluctuations in $N^{1/2}$ scale



- Critical scaling:

$$\alpha = \delta 2^{-4/3} N^{-1/3}$$

and end-point $(N, N - \eta N)$ with

$$\eta = u 2^{5/3} N^{-1/3}.$$

- Law of large number gives:

$$L^{\text{stat}}_{N,N-\eta N} \simeq 4N - 4u(2N)^{2/3} + \delta(2u+\delta)2^{4/3}N^{1/3}.$$

Theorem

Let $\delta \in \mathbb{R}$, $u > 0$ be fixed. Set

$$\alpha = \delta 2^{-4/3} N^{-1/3}, \quad \eta N = u 2^{5/3} N^{2/3}.$$

Then

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{L^{\rm stat}_{N,N-\eta N} - (4N - 4u(2N)^{2/3})}{2^{4/3} N^{1/3}} \leq S \right) = F_{u,\delta}(S),$$

where $F_{u,\delta}(S) = \frac{d}{dS}\left\{ \mathrm{Pf}(J - \overline{\mathcal{A}}) G_{\delta,u}(S) \right\}$ *with* $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$

$$G_{\delta,u}(S) = e^{\delta,u}(S) - \left\langle -g_1^{\delta,u} \quad g_2^{\delta,u} \middle| (\mathbb{1} - J^{-1}\overline{\mathcal{A}})^{-1} \begin{pmatrix} -h_1^{\delta,u} \\ h_2^{\delta,u} \end{pmatrix} \right\rangle.$$

- The 2×2 matrix kernel $\overline{\mathcal{A}}$ is the one arising from the model with $\mathrm{Exp}(1)$ also for $j = 0$, instead of $\mathrm{Exp}(\frac{1}{2} - \alpha)$.
  Away from the diagonal: Imamura,Sasamoto'04
  General and rigorous case: Baik,Barraquand,Corwin,Suidan'18
- For moment computations the derivative is not a problem: denote $F_{u,\delta}(S) = \frac{d}{dS}T(S)$ and $\xi \sim F_{u,\delta}$, then:
  - by stationarity: $\mathbb{E}(\xi) = \delta(2u + \delta)$,
  - integrating by parts gives

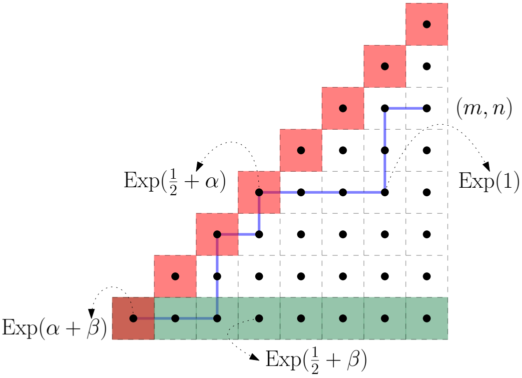
$$\mathbb{E}(\xi^\ell) = \ell(\ell-1)\int_{\mathbb{R}_+} dS S^{\ell-2}(T(S)-S) + \ell(\ell-1)\int_{\mathbb{R}_-} dS S^{\ell-2}T(S).$$

- The inverse of the operator is not a numerical issue either:

$$\mathrm{Pf}\,(J-K)\left\langle c \quad d \middle| (\mathbb{1}-J^{-1}K)^{-1}\begin{pmatrix} a \\ b \end{pmatrix}\right\rangle$$
$$= \mathrm{Pf}\,(J-K) - \mathrm{Pf}\left(J - K - \left| \begin{matrix} b \\ -a \end{matrix} \right\rangle \langle c \quad d| - \left| \begin{matrix} c \\ d \end{matrix} \right\rangle \langle -b \quad a| \right).$$

- Then use Bornemann's method to evaluate the Fredholm determinants (Pfaffians) Bornemann'08

Step 1: An integrable model. Consider the model



- The process $L_{N,1}, L_{N,2}, \ldots, L_{N,N}$ is the marginal of a Pfaffian Schur process. Baik,Barraquand,Corwin,Suidan'18

- For $\alpha+\beta>0$ and $\beta>0$ we a Fredholm Pfaffian expression on (s,∞)
$$\mathbb{P}(L_{N,N-n}\le s)=\mathrm{Pf}(J-K)$$
with

$$K_{11}(x,y)=-\oint\frac{dz}{2\pi\mathrm{i}}\oint\frac{dw}{2\pi\mathrm{i}}\frac{\Phi(x,z)}{\Phi(y,w)}\left[(\tfrac{1}{2}-z)(\tfrac{1}{2}+w)\right]^n\frac{(z+\beta)(w-\beta)}{(z-\beta)(w+\beta)}\frac{(z+\alpha)(w-\alpha)(z+w)}{4zw(z-w)},$$
$$K_{12}(x,y)=-\oint\frac{dz}{2\pi\mathrm{i}}\oint\frac{dw}{2\pi\mathrm{i}}\frac{\Phi(x,z)}{\Phi(y,w)}\left[\frac{\frac{1}{2}-z}{\frac{1}{2}-w}\right]^n\frac{z+\alpha}{w+\alpha}\frac{z+\beta}{z-\beta}\frac{w-\beta}{w+\beta}\frac{z+w}{2z(z-w)}$$
$$=-K_{21}(y,x),$$
$$K_{22}(x,y)=\oint\frac{dz}{2\pi\mathrm{i}}\oint\frac{dw}{2\pi\mathrm{i}}\frac{\Phi(x,z)}{\Phi(y,w)}\frac{1}{\left[(\frac{1}{2}+z)(\frac{1}{2}-w)\right]^n}\frac{1}{(z-\alpha)(w+\alpha)}\frac{z+\beta}{z-\beta}\frac{w-\beta}{w+\beta}\frac{z+w}{z-w}+\varepsilon(x,y),$$

with $\Phi(x,z)=e^{-xz}[(\frac{1}{2}+z)/(\frac{1}{2}-z)]^{N-1}$ and

$$\varepsilon(x,y)=-\operatorname{sgn}(x-y)\oint_{\Gamma_{1/2,\alpha}}\frac{dz}{2\pi\mathrm{i}}\frac{2ze^{-z|x-y|}}{(z^2-\alpha^2)\left(\frac{1}{4}-z^2\right)^n}.$$

Step 2: Shift argument. We want to get the limit of $\beta = -\alpha$ conditioned on $\omega_{0,0} = 0$.

- For $\alpha + \beta > 0$, we have

$$\mathbb{P}(L_{N,N-n} \leq s | \omega_{0,0} = 0) = \left(1 + \frac{1}{\alpha + \beta} \frac{d}{ds}\right) \mathbb{P}(L_{N,N-n} \leq s).$$

Step 3: Rank one decomposition.

- By deforming contours such that the expressions are analytic at $\alpha+\beta=0$ we get

$$K = \overline{K} + (\alpha+\beta)R$$

with R of the form

$$R = \begin{pmatrix} |g_1\rangle\langle f^\beta| - |f^\beta\rangle\langle g_1| & |f^\beta\rangle\langle g_2| \\ -|g_2\rangle\langle f^\beta| & 0 \end{pmatrix}$$

with $f^\beta(x) \sim e^{-\beta x}$.

- Thus we have

$$\mathbb{P}(L^{\mathrm{stat}}_{N,N-n} \le s) = \lim_{\beta\to-\alpha} \frac{d}{ds}\left[\mathrm{Pf}(J-\overline{K})\left(\tfrac{1}{\alpha+\beta} - \left\langle Y \left| (\mathbb{1}-\overline{G})^{-1} X \right.\right\rangle\right)\right]$$

with $X = \left| \begin{matrix} 0 \\ f^\beta \end{matrix} \right\rangle$ and $Y = \langle -g_1 \quad g_2|$ and $\overline{G} = J^{-1}\overline{K}$.

Step 4: Analytic continuation.

- Let $G = J^{-1}K$, then the idea is to use

$$\frac{1}{\alpha+\beta} - \langle Y \,|(\mathbb{1}-\overline{G})^{-1}X\rangle = \frac{1}{\alpha+\beta} - \langle Y\,|X\rangle - \langle Y\,|(\mathbb{1}-\overline{G})^{-1}\overline{G}X\rangle$$

- Problem: $\langle Y\,|\overline{G}X\rangle$ is a sum of 4 terms, some of which diverge for $\beta \leq 0$, due to the f^β term. A term-by-term limit $\beta \to -\alpha$ for $\alpha \geq 0$ is not possible.
- Solution: The diverging terms exactly cancels for any $\beta > 0$, namely we show that

$$\langle Y\,|(\mathbb{1}-\overline{G})^{-1}\overline{G}X\rangle = \left\langle Y\,\middle|(\mathbb{1}-\overline{G})^{-1}\widetilde{G}X\right\rangle$$

where $\widetilde{G}$ is without the problematic terms. The result is then analytic on $(\alpha,\beta) \in (-1/2,1/2)^2$.

Step 5: Large time asymptotics. Standard steep descent method.

Full-space vs. half-space stationary models

Full-space	Half-space
One-parameter family	Two-parameter family
Determinantal structure	Pfaffian structure
Simple analytic continuation	Tricky analytic continuation

- Is the full-space distribution a limit of half-space one?

- Taking $\delta \to -\infty$, the characteristic line has direction far away from the diagonal. Thus the maximizer of the LPP will touch less and less the diagonal away from a $O(N^{2/3})$-neighborhood of the origin, so one might expect to recover the Baik-Rains distribution.

Theorem

Let $S = s + \delta(2u + \delta)$ and $u = w - \delta$ (for $w = 0$ we are on the characteristic line). Then,

$$\lim_{u\to\infty} F_{u,\delta}(S) = F_{\mathrm{BR},w}(s).$$

Some remarks

- In arXiv:2012.10337 we extended the result to multi-point distributions
- In arXiv:2204.06782 we get some results on the time-time covariance close to the characteristic direction (compare with Alessandra Occelli's talk a few weeks ago).
- The general stationary process in TASEP has two parameters: one for the input rate and one for the density at infinity.

  Liggett'75

  This is reflected into the LPP setting as well (see (maybe) Barraquand's talk next week)

  Barraquand-Krajenbrink-Le Doussal'22;Barraquand-Corwin'22