BBS Models

with Atsuo Kuniba et Grégoire Misguich

Motivation

- Find a simple model where to test ideas of GGE, GHD, G…
- Soliton decomposition of the Box-Ball System
- Pablo A. Ferrari, Chi Nguyen, Leonardo T. Rolla, Minmin Wang
- Study a model with direct relation to TBA and solitons.
-

Origin of the model

Journal of The Physical Society of Japan
Vol. 59, No. 10, October, 1990, pp. 3514–3519

A Soliton Cellular Automaton

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A cellular automaton (CA) of filter automata type is proposed. Any state of the CA consists only of solitary wave solutions. It is shown that the solitary waves interact with one another preserving their identities during a time evolution. It is also shown that the CA has infinitely many conserved quantities. Hence, this CA may be considered to be one of the simplest soliton systems.

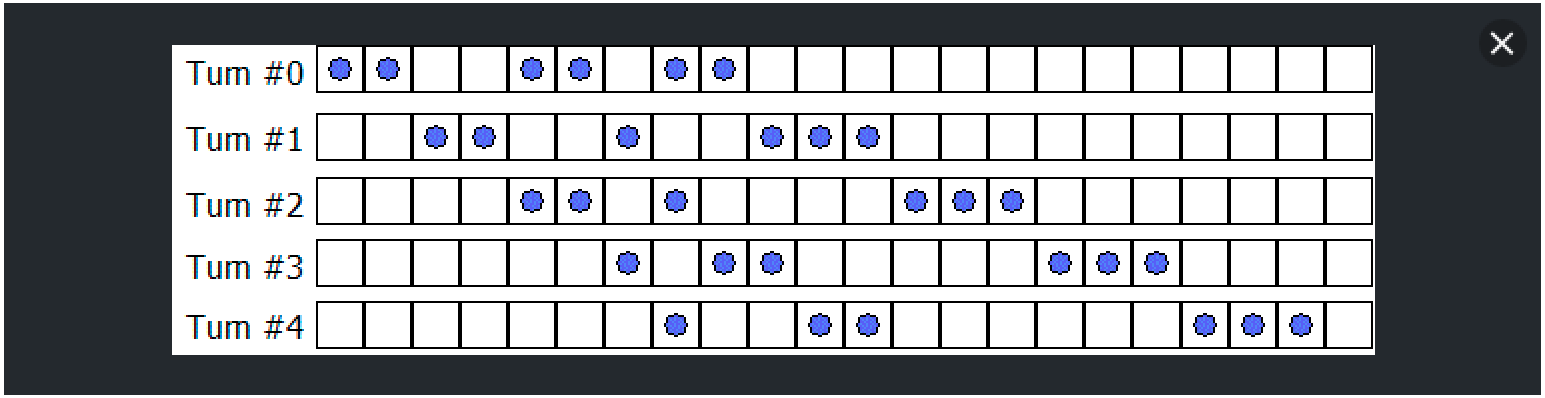
- Nuclear Physics 2000

Soliton Cellular Automata Associated With Crystal Bases

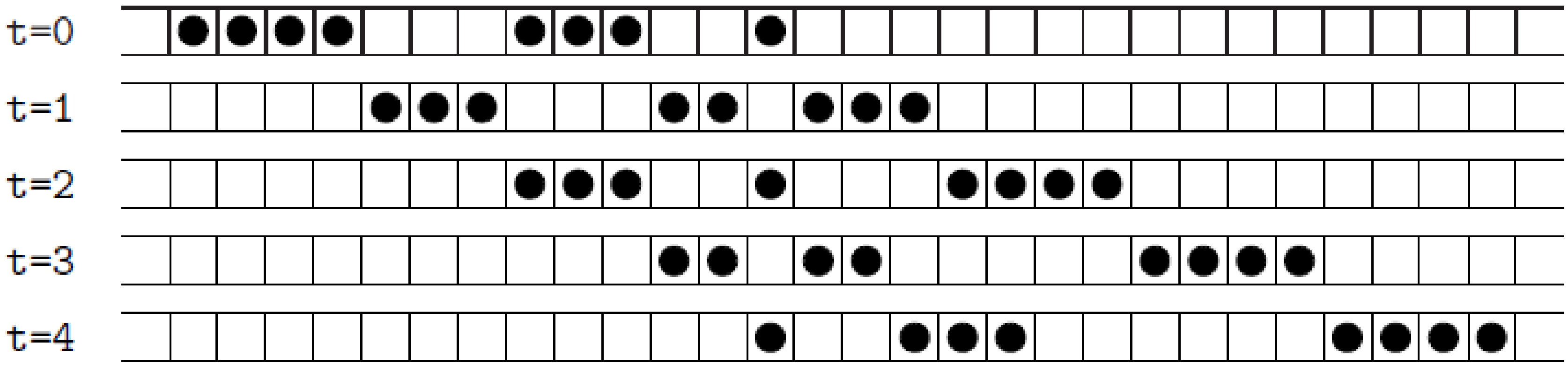
Goro Hatayama*, Atsuo Kuniba*, and Taichiro Takagi†

Hydrodynamics of BBS

Wat is BBS ?



Large Solitons overpass small solitons:



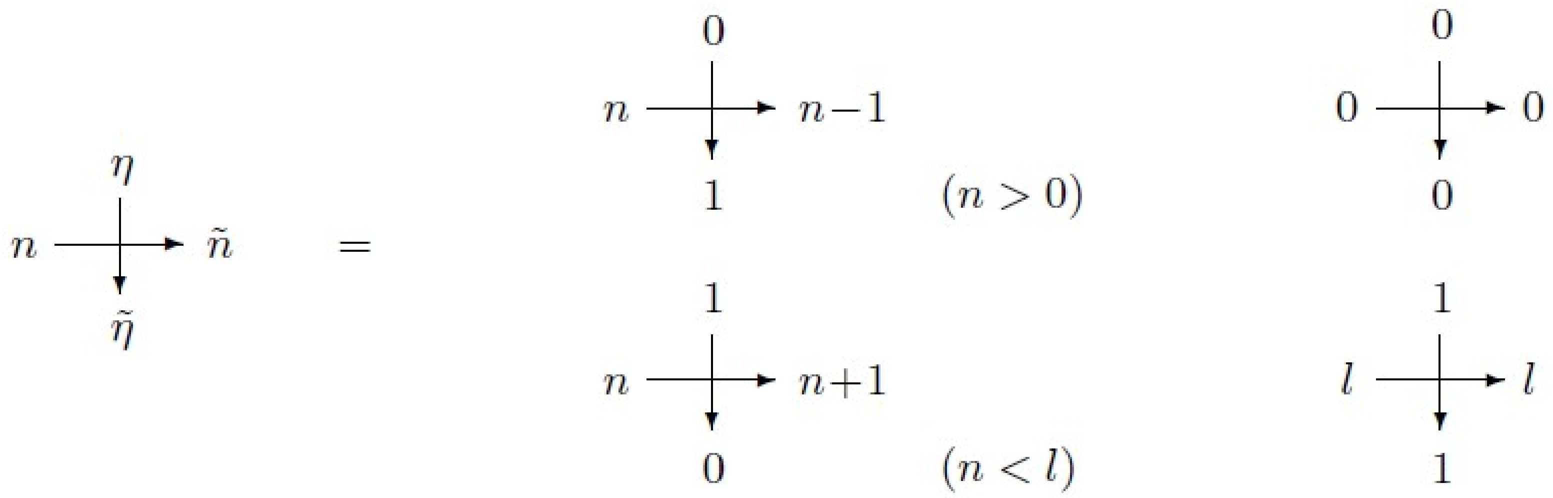
Bare velocity = size

Phase Shift

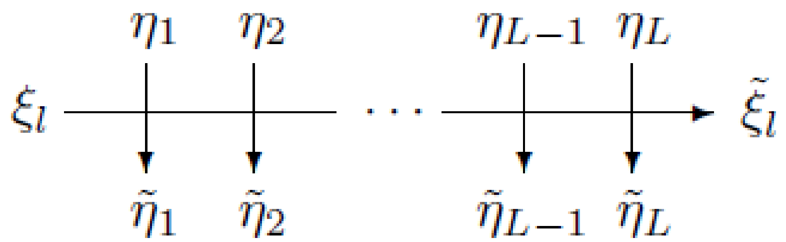
$$\phi(k,p) = 2\ min(k,p)$$

Where does it come from ? A1

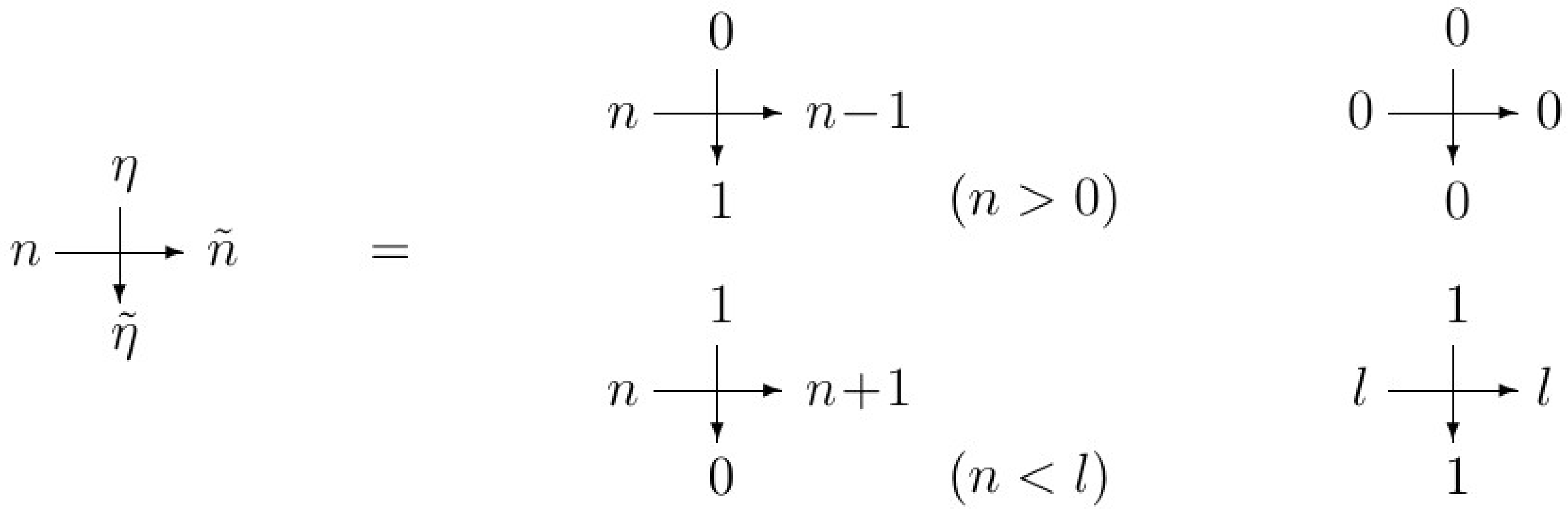
« Cristal » Vertex Model



« Cristal » Transfer Matrix

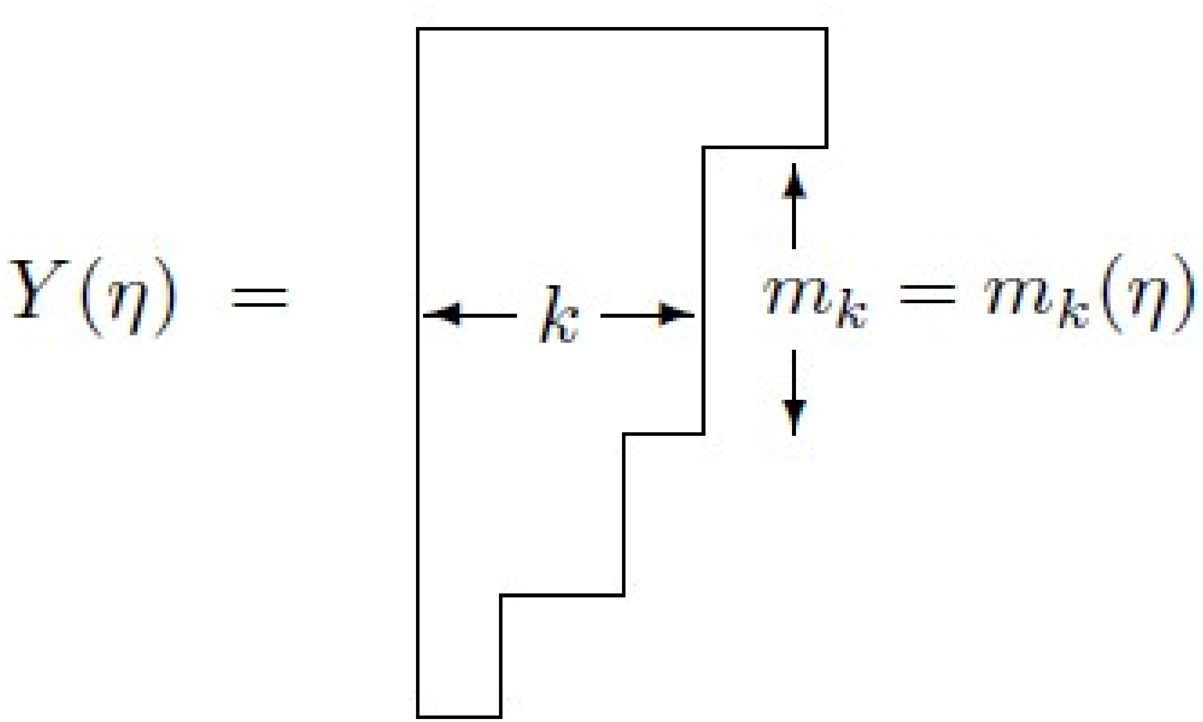


Commuting Combinatorial R



Young Diagram representation of soliton configurations

A2



$$m_k/N = \rho_k$$

ρ=soliton density

Thermodynamics

Bethe equations convert into hole density relation

$$\sigma_j = 1 - M_{ij}\rho_k$$

$$M_{ij} = 2\min(j,k)$$

Thermodynamics

$$\sigma_j = 1 - M_{ij}\rho_j = 1 - 2\epsilon_j$$

σ=hole density

ρ=density

$$Z = \sum_Y \prod_k e^{-\beta_k \epsilon_k} \binom{\rho_k + \sigma_k}{\rho_k}$$

Thermodynamics

- Minimize free energy :

$$\mathcal{F} = \beta_1 \varepsilon_1 + \cdots + \beta_s \varepsilon_s - \sum_{i=1}^{s} \big((\sigma_i + \rho_i) \log(\sigma_i + \rho_i) - \sigma_i \log \sigma_i - \rho_i \log \rho_i \big),$$

$$\sum_{j=1}^{s} \min(i,j) \beta_j = \log(1 + Y_i) - 2 \sum_{j=1}^{s} \min(i,j) \log(1 + Y_j^{-1}), \quad Y_i = \frac{\sigma_i}{\rho_i}.$$

$$\begin{aligned} Y_1^2 &= \mathrm{e}^{\beta_1}(1 + Y_2), \\ Y_i^2 &= \mathrm{e}^{\beta_i}(1 + Y_{i-1})(1 + Y_{i+1}) \quad (1 < i < s), \\ Y_s^2 &= \mathrm{e}^{\beta_s}(1 + Y_{s-1})(1 + Y_s). \end{aligned}$$

GGE

$$\mathcal{F} = -\sum_{i=1}^{s} \log(1 + Y_i^{-1}).$$

Thermodynamics

Distribute the balls independantly on each site of the lattice with probability z/(1+z) only one fugacity nonzero.

- Partition function sum on configurations z raised to the number of balls :

$$Z = (1+z)^N = \sum_Y z^{\sum_k k m_k} \prod_k \binom{m_k + p_k}{m_k}$$

TBA

$$\epsilon = \beta.h - T * \ln(1 + e^{-\epsilon})$$

$$T = (1 - M)^{-1}$$

Q system

- Change variable $Q_i = \sqrt{1 + Y_i}$

- Obtain Q-system (Schur functions):

$$Q_{i-1}Q_{i+1} + 1 = Q_i^2$$

$$Q_0 = 1 \qquad \frac{Q_{k+1}}{Q_k} \to \frac{1}{z}$$

Thermodynamics

$$\sigma_k = \frac{(1-z)(1+z^{k+1})}{(1+z)(1-z^{k+1})}$$

$$\rho_k = \frac{z^k(1-z)^3(1+z^{k+1})}{(1+z)(1-z^k)(1-z^{k+1})(1-z^{k+2})}$$

Speed of solitons in the medium

Speed of Soliton=

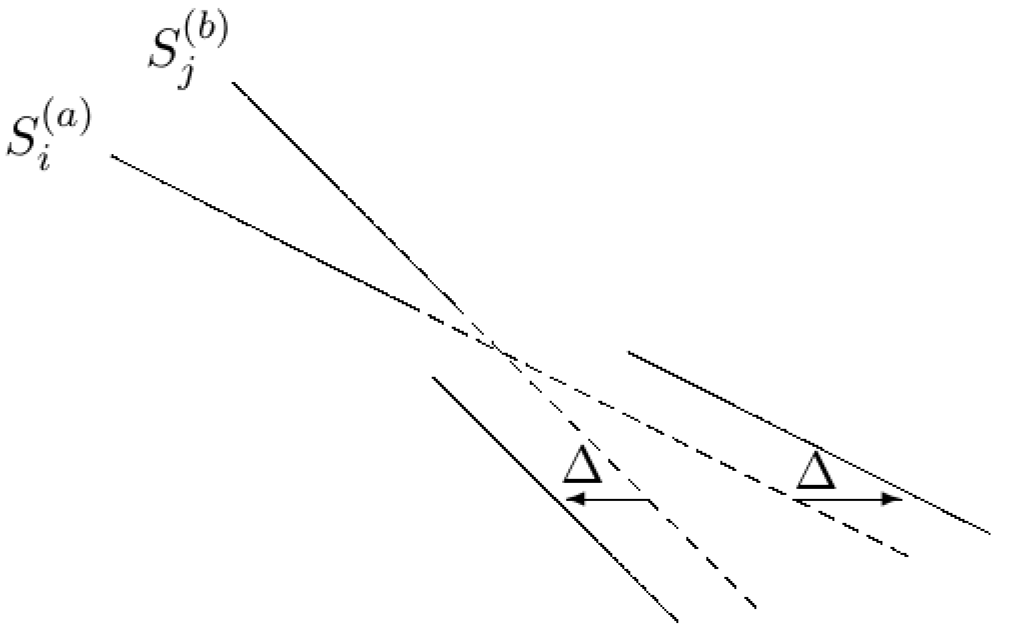
speed in vacuum

+

Nb of solitons of type p crossed per time unit

x relative shift

A3



Local determination of speed

$$V^{l}{}_{k} \;=\; min(k,l) + 2\sum_{p} min(k,p)(V^{l}{}_{k} - V^{l}{}_{p})\rho_{p}$$

Linear equation for the speeds.
Related to inverse of tropical matrix.

GHD formulation

$$A^{dr} = (1 + My)^{-1} A$$

$$y_k = \sigma_k / \rho_k$$

$$V^l = \frac{(\kappa^l)^{\mathrm{dr}}}{1^{\mathrm{dr}}}$$

V and densities are functions of filling fraction y

Normal modes, current

$$(j_\sigma)_k = \sigma_k v_k$$

$$\partial_t \sigma + \partial_x j_\sigma = 0.$$

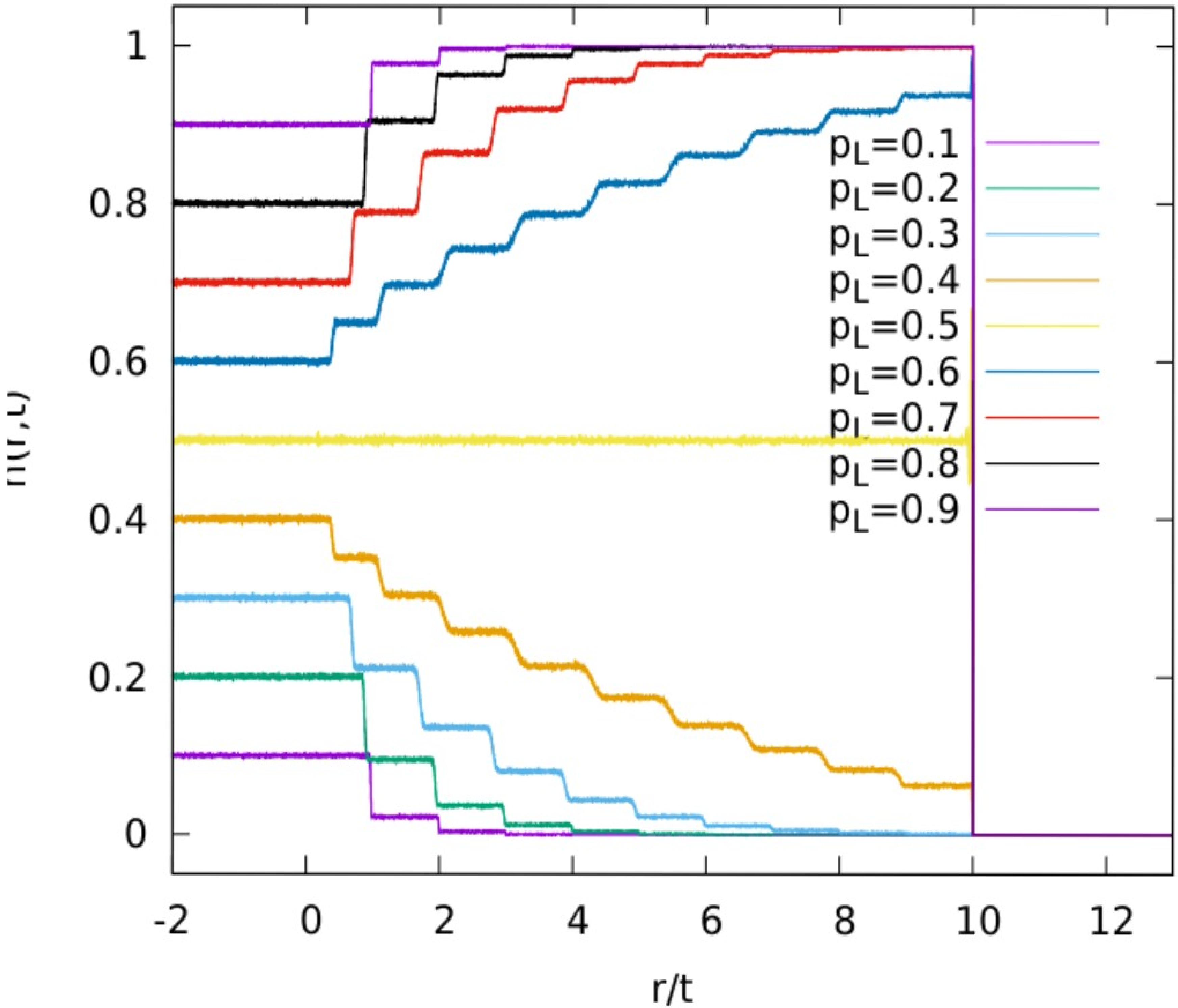
$$\partial_t y + \mathbf{v}\partial_x y = 0,$$

GHD

$$y_k = \sigma_k / \rho_k = e^{-\epsilon_k}$$

$$\langle \delta\epsilon_k \delta\epsilon_p \rangle = \delta_{kp} \frac{1 + \mathrm{e}^{\epsilon_k}}{\sigma_k}.$$

Yang-Yang



1
0.8
0.6
0.4
0.2
0
-2
0
2
4
6
8
10
12
r/t
p_L=0.1
p_L=0.2
p_L=0.3
p_L=0.4
p_L=0.5
p_L=0.6
p_L=0.7
p_L=0.8
p_L=0.9

Disappearance of solitons:

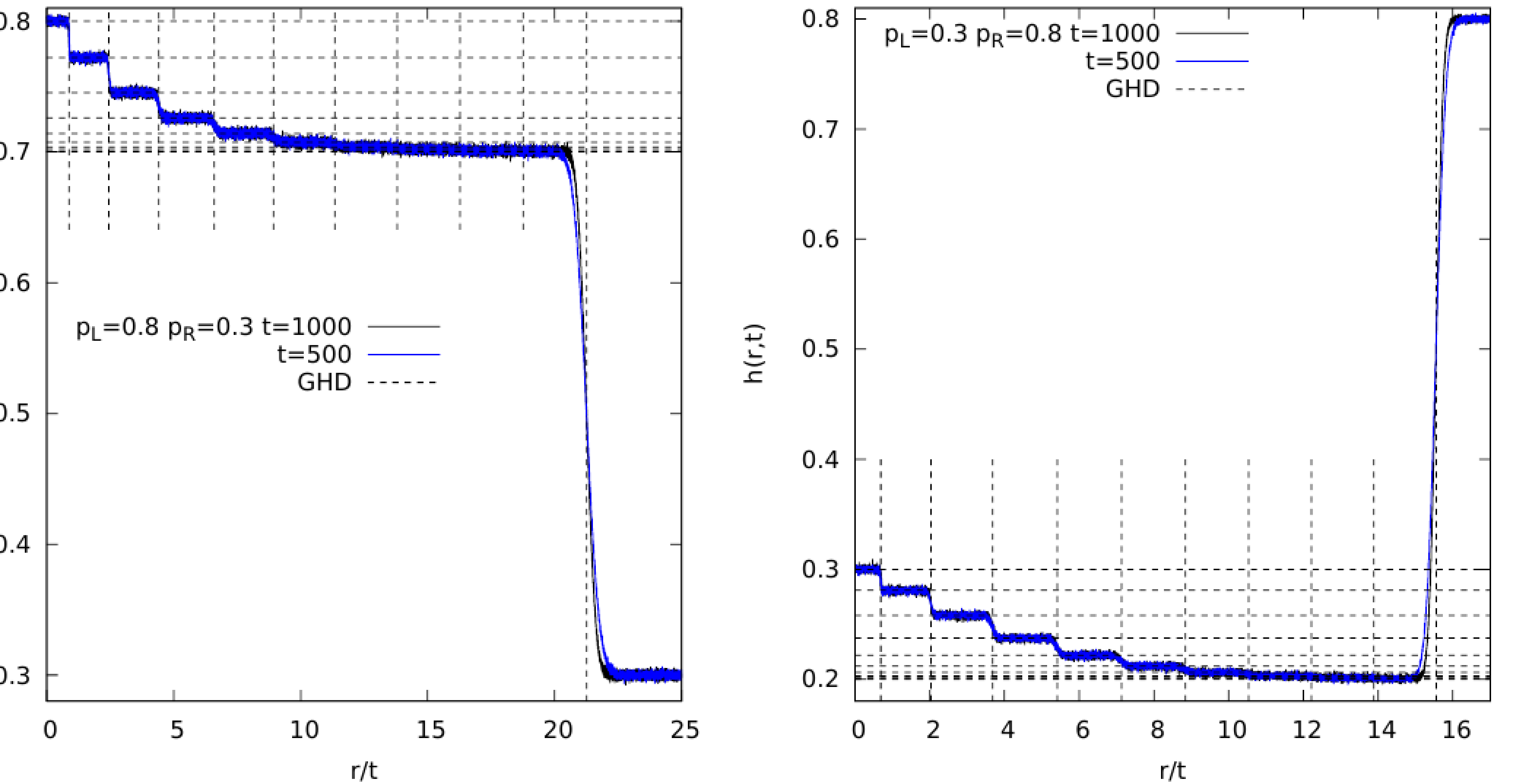
$x/t < V_1{}^{l}: \quad y_1, y_2, \cdots$ homogeneous

$V_1{}^{l} < x/t < V_2{}^{l}: \quad 0, y_2, \cdots$

....

$V_l{}^{l} < x/t: \quad 0, 0, \cdots$ empty

Ball density

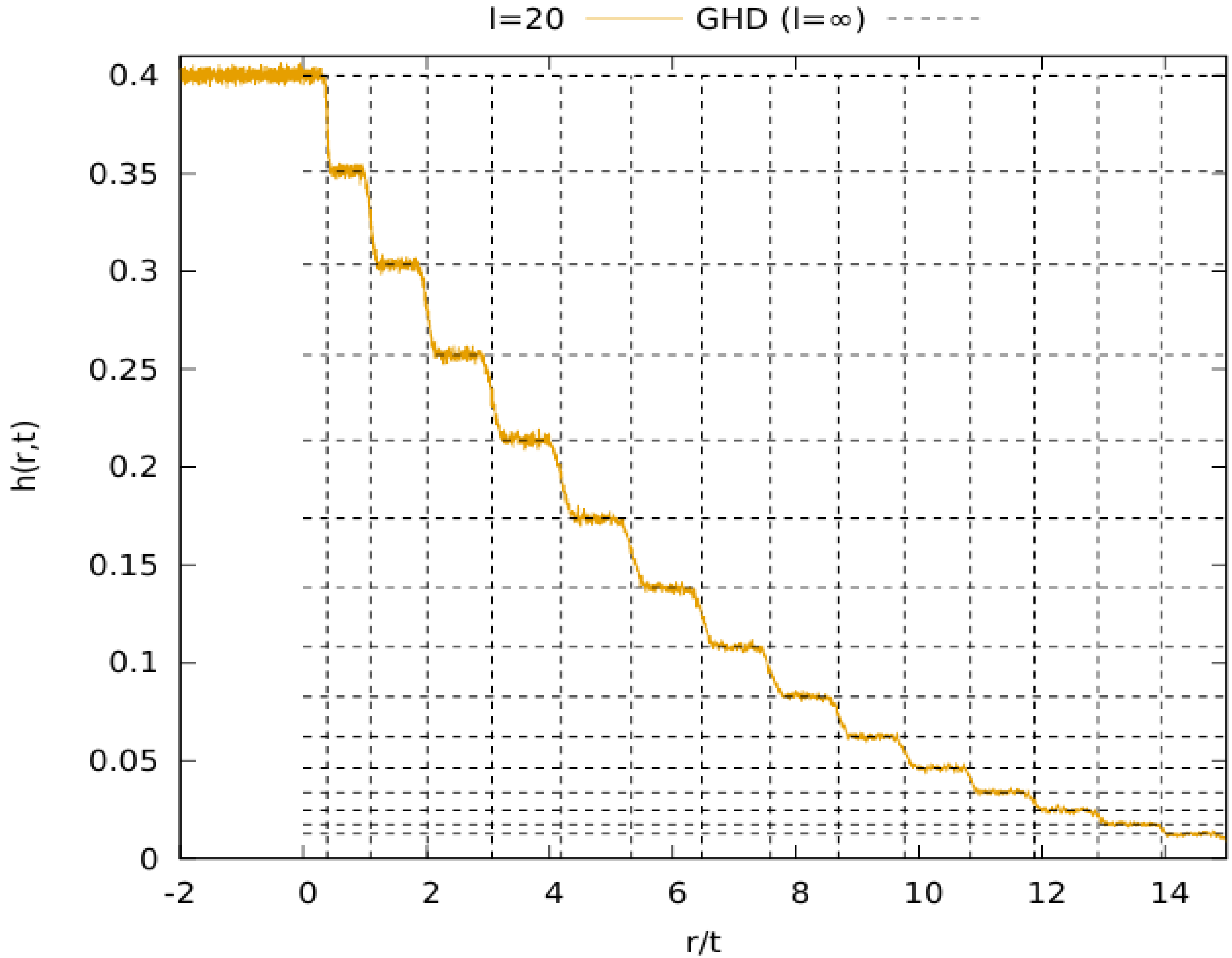


Many exact predictions

- Positions of speeds and heights of density plateaus empty on the right :

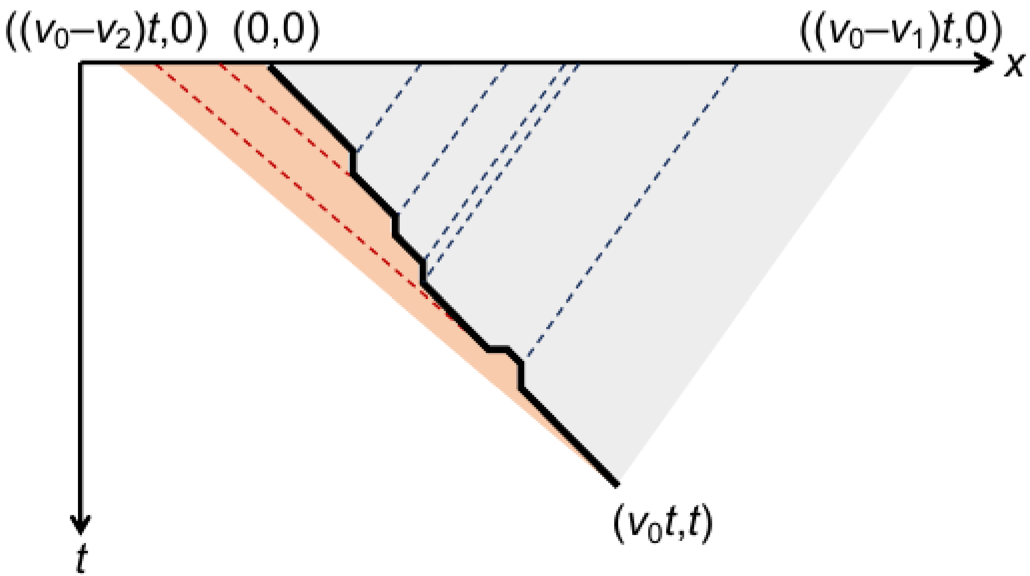
$$v_k = k\frac{1 - z^{k+1}}{1 + z^{k+1}}$$

$$h(k) = \sum_{j=1}^{\infty} j\rho_j(k) = \sum_{j=k+1}^{\infty} j n_j(k) r_j(k) = \frac{z^{k+1}([k+2] + k[1])}{[2k+3] + (2k+1)[1]z^{k+1}}.$$



l=20
GHD (l=∞)
h(r,t)
r/t
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-2
0
2
4
6
8
10
12
14

Diffusive corrections



J. De Nardis, D. Bernard and B. Doyon, Diffusion in generalized hydrodynamics and quasiparticle scattering, SciPost Phys. **6**, 049 (2019).

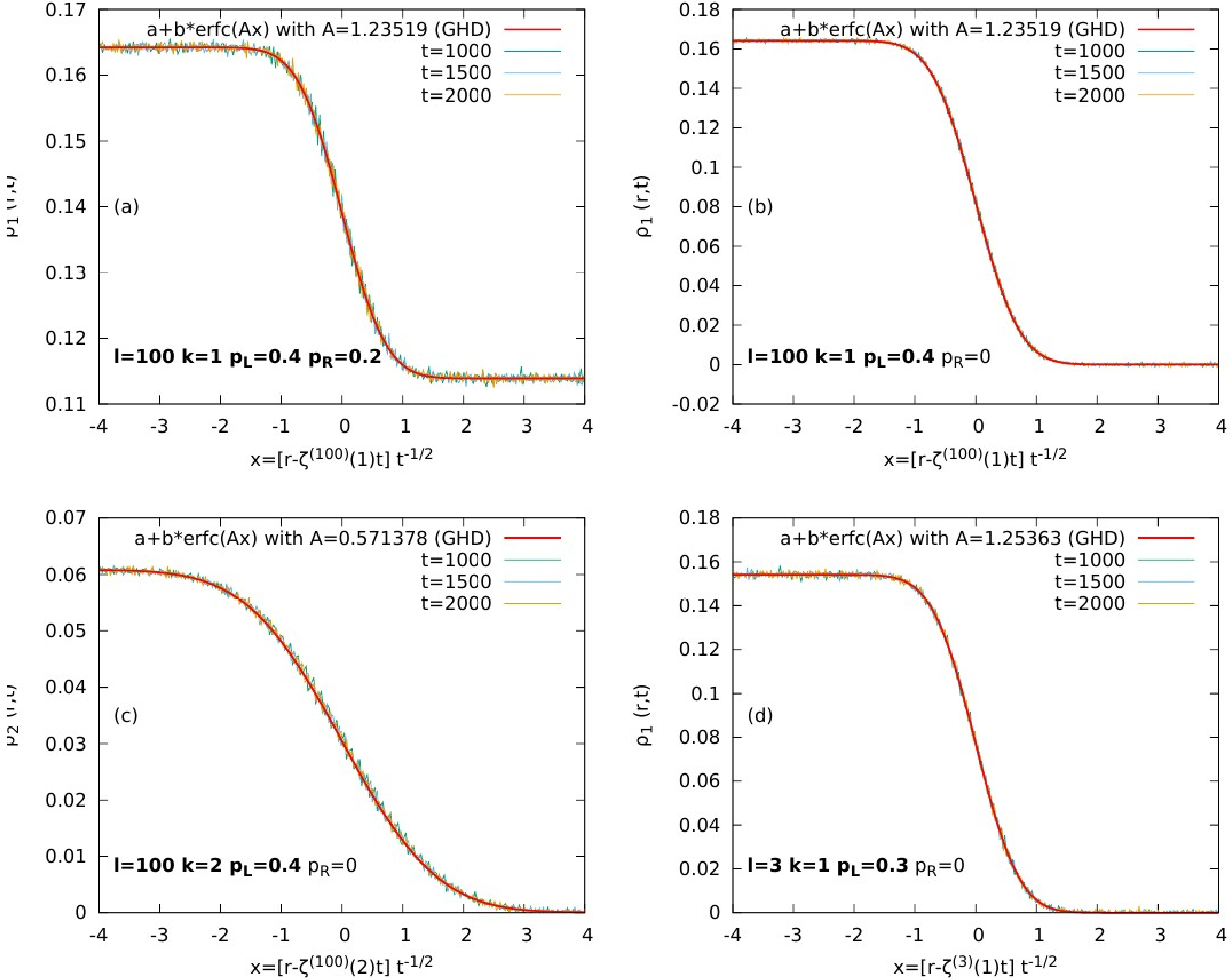
S. Gopalakrishnan, D. A. Huse, V. Khemani and R. Vasseur, Hydrodynamics of operator spreading and quasiparticle diffusion in interacting integrable systems, Phys. Rev. B **98**, 220303(R) (2018).

Diffusive corrections

$$t\langle(\delta\bar{v}_k)^2\rangle = \Sigma_k^2 = \sum_i \frac{\beta_{ki}^2}{\sigma_k^2}|v_k - v_i|\sigma_i y_i(1+y_i).$$

$$2\left(\Sigma_k^{(l)}\right)^2 = \frac{8k^2 z^{k+1}(1-z^{k+1})(1-z^{l-k})(1+z^{l+k+2})}{(1+z^{k+1})^3(1-z^{l+1})^2} \qquad (1 \le k \le l).$$

- Step width in the case empty to the right



a+b*erfc(Ax) with A=1.23519 (GHD)
t=1000
t=1500
t=2000
(a)
l=100 k=1 pL=0.4 pR=0.2
x=[r-ζ(100)(1)t] t-1/2
a+b*erfc(Ax) with A=1.23519 (GHD)
t=1000
t=1500
t=2000
ρ1 (r,t)
(b)
l=100 k=1 pL=0.4 pR=0
x=[r-ζ(100)(1)t] t-1/2
a+b*erfc(Ax) with A=0.571378 (GHD)
t=1000
t=1500
t=2000
(c)
l=100 k=2 pL=0.4 pR=0
x=[r-ζ(100)(2)t] t-1/2
a+b*erfc(Ax) with A=1.25363 (GHD)
t=1000
t=1500
t=2000
ρ1 (r,t)
(d)
l=3 k=1 pL=0.3 pR=0
x=[r-ζ(3)(1)t] t-1/2

Transport proprties

$$\langle QQ \rangle = p(1-p) = \sum_i \rho_i \sigma_i (\rho_i + \sigma_i)(v_i^{\infty})^2$$

Second cumulant

$$c_2 = \langle j(0) Q \rangle = p(1-p) = \sum_i \rho_i \sigma_i (\rho_i + \sigma_i)(v_i^l v_i^{\infty})^2$$

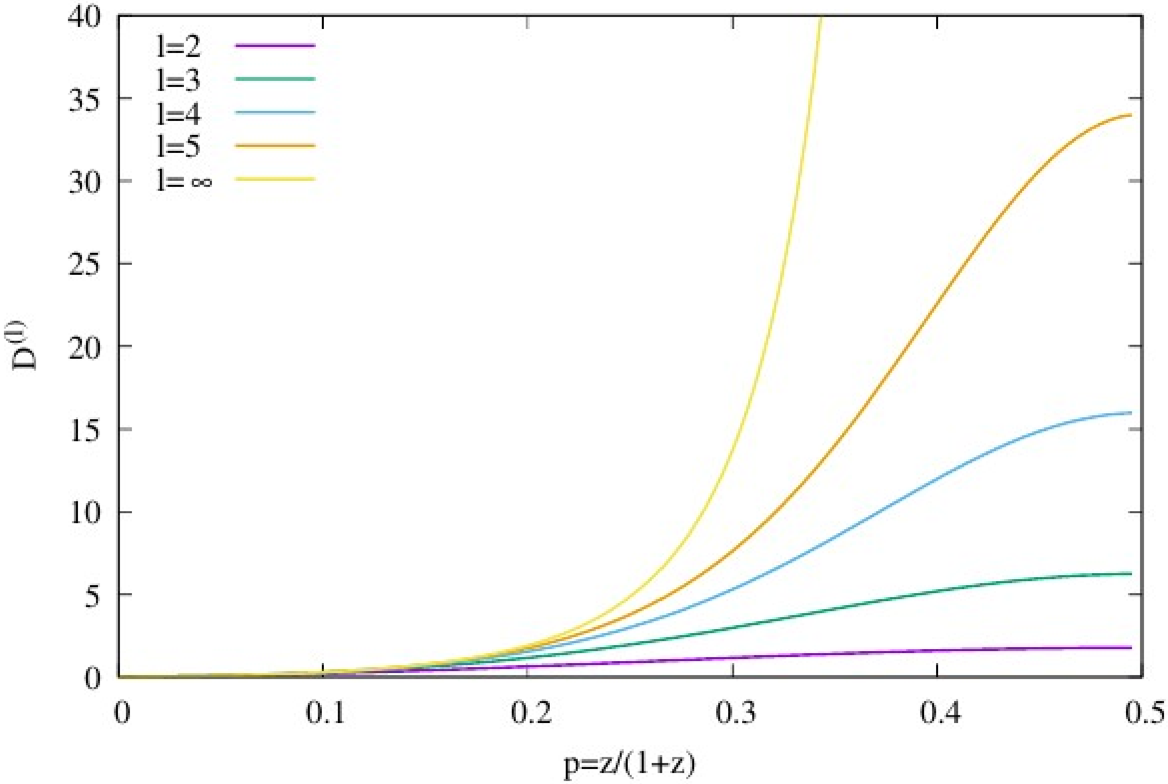
Drude

$$D = \langle J(0) J(t) \rangle = p(1-p) = \sum_i \rho_i \sigma_i (\rho_i + \sigma_i)((v_i^l)^2 v_i^{\infty})^2$$

Completely analogous to :

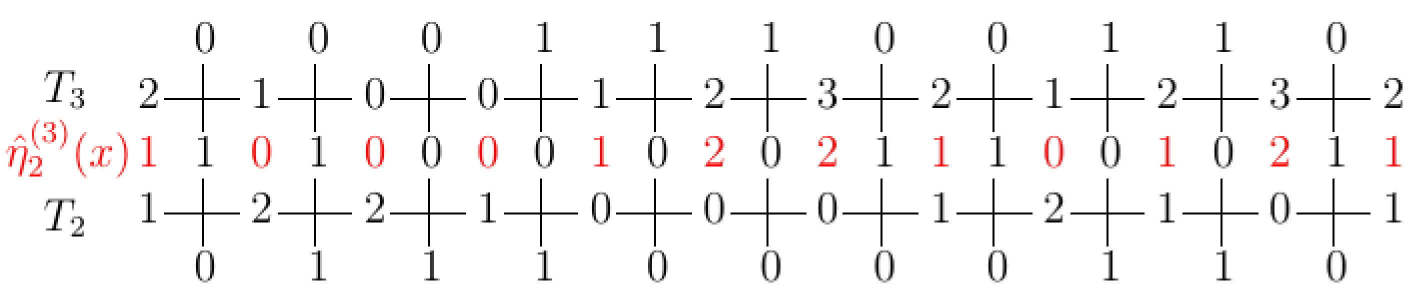
Doyon B and Spohn H 2017 Drude Weight for the Lieb-Liniger Bose Gas *SciPost Physics* **3** 039

Drude weight



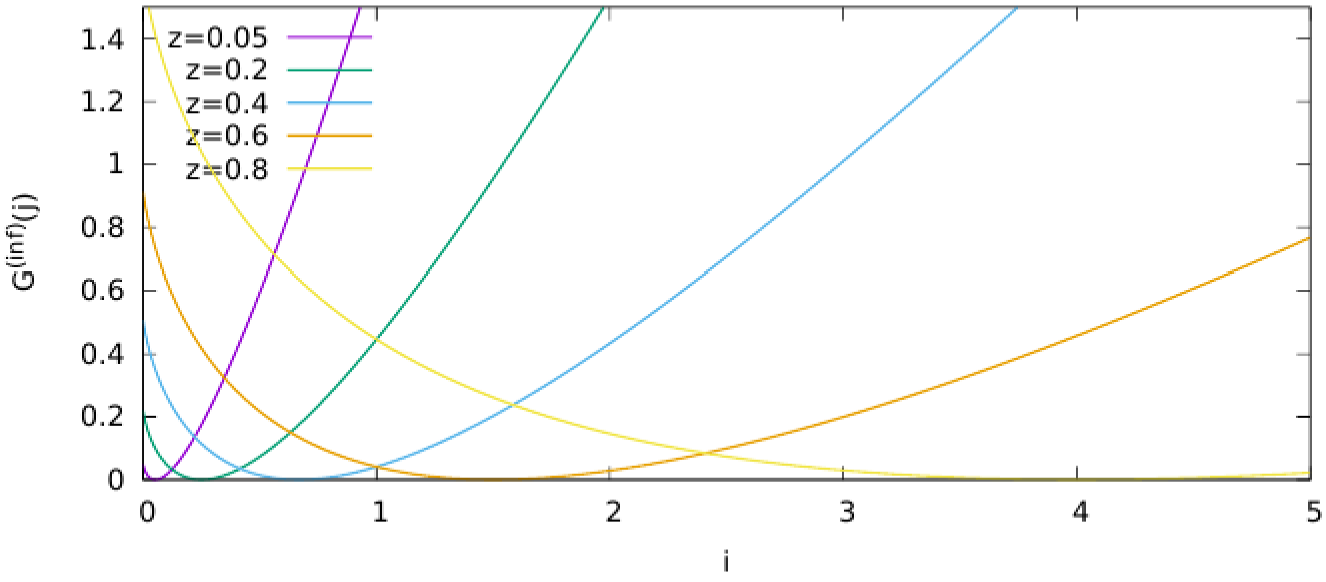
Generalized identity

$$
\begin{aligned}
C_{i,j}^{l,m} &= \sum_{k=1}^{\infty} \frac{\partial \eta_i^{(l)}}{\partial \bar{\epsilon}_k} \frac{\partial \eta_j^{(m)}}{\partial \bar{\epsilon}_k} \frac{1+e^{\bar{\epsilon}_k}}{\sigma_k} \\
&= \sum_{k \geq 1} \rho_k \sigma_k (\rho_k + \sigma_k) v_k^{(i)} v_k^{(j)} v_k^{(l)} v_k^{(m)}.
\end{aligned}
$$



Large deviation function

$$F^{(l)}(\lambda) = \ln\left(\frac{1-(ze^{\lambda})^{l+1}}{1-ze^{\lambda}}\right) - \ln\left(\frac{1-z^{l+1}}{1-z}\right).$$



Rate function

Outlook

- Simple integrable deterministic model gives amazing agreement with hydrodynimical predictions
- Related to interesting combinatorics such as Schensted’s row bumping algorithm (see Mateo’s talk)