Integrable quantum cluster algebras associated with affine root systems

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Outline

T- and Q-systems as discrete evolutions and cluster mutations

Quantization via cluster algebra structure

Proof of integrability via Macdonald theory
  Duality
  Fourier transform
  Time translation operator
T-system and Q-system for $\mathfrak{sl}_2$

Fusion relations for Heisenberg model transfer matrices, $T_V(\zeta)$, $V(\zeta) = \text{auxiliary space}$.

- Commuting family $\{T_{j,k} := T_{V(k\omega_1)}(q^j \zeta), j \in \mathbb{Z}, k \geq 0\}$, satisfy

  **T-system:**
  $$T_{j,k+1}T_{j,k-1} = T_{j+1,k}T_{j-1,k} - 1.$$  

- Asymptotics of TBA $\zeta \to \infty$: $T_{j,k} \to Q_k$, satisfy

  **Q-system:**
  $$Q_{k+1}Q_{k-1} = Q_k^2 - 1.$$  

Discrete evolution in "time" $k$

$$Q_{k+1} = (Q_k^2 - 1)Q_{k-1}^{-1}, \quad \text{Initial data } \{Q_0 = 1, Q_1\}$$

1. **Integrable:** There is a conserved quantity $H_1(Q_k, Q_{k+1})$, independent of $k$;

2. **Cluster algebra:** $\{Q_i\}$ are cluster variables in CA corresponding to the Kronecker quiver

   $$\circ \quad \Longrightarrow \quad \circ$$

3. **Canonical quantization:** Discrete, non-commutative integrable evolution equation.
T-system and Q-system for $\mathfrak{sl}_N$

Transfer matrices $\{T_{i,j,k} = T_{V(k\omega_i)}(q^j \zeta) : i \in [1, N - 1], j \in \mathbb{Z}, k \geq 0\}$.

**T-system:** $T_{i,j,k+1}T_{i,j,k-1} = T_{i,j+1,k}T_{i,j-1,k}-T_{i-1,j,k}T_{i+1,j,k}$, $T_{0,j,k} = 1 = T_{N+1,j,k}$.

Asymptotics $\zeta \to \infty$, $T_{i,j,k} \to Q_{i,k}$:

**Q-system:** $Q_{i,k+1}Q_{i,k-1} = Q_{i,k}^2 - Q_{i-1,k}Q_{i+1,k}$, $Q_{0,k} = 1 = Q_{N+1,k}$.

[Diagram of cluster algebras: Q-system quiver is T-system quiver wrapped around a cylinder of radius 2]
T-system and Q-system for $\mathfrak{sl}_N$

Transfer matrices $\{T_{i,j,k} = T_{V(k\omega_i)}(q^j \zeta) : i \in [1, N-1], j \in \mathbb{Z}, k \geq 0\}$.

**T-system:**

\[
T_{i,j,k+1} T_{i,j,k-1} = T_{i,j+1,k} T_{i,j-1,k} - T_{i-1,j,k} T_{i+1,j,k}, \quad T_{0,j,k} = 1 = T_{N+1,j,k}.
\]

Asymptotics $\zeta \to \infty$, $T_{i,j,k} \to Q_{i,k}$:

**Q-system:**

\[
Q_{i,k+1} Q_{i,k-1} = Q_{i,k}^2 - Q_{i-1,k} Q_{i+1,k}, \quad Q_{0,k} = 1 = Q_{N+1,k}.
\]

Discrete evolution equations in $k$:

1. **Integrable:** (Poisson-commuting) Hamiltonians $\{H_a\}_{a=1}^{N-1}$ independent of $k$;
2. **Cluster algebra:** $Q_{i,k}$ cluster variables, equations are mutations;
3. Canonical **Quantization** of CA: Discrete, non-commutative integrable evolution.
Other classical root systems

There is a T-systems and Q-systems for each affine root system:

Quantum Q-systems: $g \rightarrow (R, R^*)$.

<table>
<thead>
<tr>
<th>$g$</th>
<th>$A_N^{(1)}$</th>
<th>$B_N^{(1)}$</th>
<th>$C_N^{(1)}$</th>
<th>$D_N^{(1)}$</th>
<th>$A_{2N-1}^{(2)}$</th>
<th>$A_{2N}^{(2)}$</th>
<th>$D_{N+1}^{(2)}$</th>
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<tr>
<td>$R$</td>
<td>$A_N$</td>
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<tr>
<td>$R^*$</td>
<td>$A_N$</td>
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<td>$D_N$</td>
<td>$C_N$</td>
<td>$BC_N$</td>
<td>$B_N$</td>
</tr>
</tbody>
</table>
Q-systems\textsuperscript{2}

\[ Q_{a,k+1} Q_{a,k-1} = Q_{a,k}^2 - \prod_{b \neq a} Q_{b,k}^{-C_{ba}}, \quad 1 \leq a \leq r, \quad k \in \mathbb{Z} \]

\((C = \text{Cartan matrix of } R)\)

Except for

\[ B_{N}^{(1)} : \quad Q_{N-1,k+1} Q_{N-1,k-1} = Q_{N-1,k}^2 - Q_{N-2,k} Q_{N,2k}; \]
\[ Q_{N,k+1} Q_{N,k-1} = Q_{N,k}^2 - Q_{N-1,\left\lfloor \frac{k}{2} \right\rfloor} Q_{N,\left\lfloor \frac{k+1}{2} \right\rfloor}; \]

\[ C_{N}^{(1)} : \quad Q_{N-1,k+1} Q_{N-1,k-1} = Q_{N-1,k}^2 - Q_{N-2,k} Q_{N,\left\lfloor \frac{k}{2} \right\rfloor} Q_{N,\left\lfloor \frac{k+1}{2} \right\rfloor}; \]
\[ Q_{N,k+1} Q_{N,k-1} = Q_{N,k}^2 - Q_{N-1,2k}; \]

\[ A_{2n}^{(2)} : \quad Q_{N,k+1} Q_{N,k-1} = Q_{N,k}^2 - Q_{N-1,k} Q_{N,k}. \]

Mostly cluster algebra mutations

Our goals:

- Integrable structure for all \( g \), commuting Hamiltonians;
- Quantization \( Q_{i,k} \rightarrow Q_{i,k} \) non-commuting: Solutions \( Q_{i,k} \)? Hamiltonians?

Unifying framework: Macdonald-Koornwinder equations.

\textsuperscript{2}Kirillov-Reshtikhin, Kuniba, Nakanishi, Suzuki; Hatayama et al.
Quantization of Q-systems $Q_{a,k+1}Q_{a,k-1} = Q_{a,k}^2 - M_{a,k}$

Canonical quantization of the associated cluster algebra$^3$.

Commutations relations:

$$Q_{a,t_a k+i}Q_{b,t_b k+j} = q^{\Lambda_{ab} j - \Lambda_{ba} i} Q_{b,t_b k+j}Q_{a,t_a k+i}, \quad i, j = 0, 1.$$  

$$\Lambda_{a,b} = \omega^*_a \cdot \omega_b, \omega_a, \omega^*_a \text{ fundamental weights of } R, R^*$$  

$t_a = 2$ for short roots, $t_a = 1$ for long roots

Evolution equations:

$$q^{\Lambda_{aa}} Q_{a,k+1} Q_{a,k-1} = Q_{a,k}^2 - :M_{a,k}:,$$  

$a \in \Pi, k \in \mathbb{Z}$  

$\Longleftrightarrow$ normal ordered product

Example: $A_{N-1}^{(1)}$

$$Q_{a,k}Q_{b,k+i} = q^{\min(a,b)i} Q_{b,k+i}Q_{a,k}, \quad i = 0, 1;$$  

$$q^a Q_{a,k+1} Q_{a,k-1} = Q_{a,k}^2 - Q_{a-1,k} Q_{a+1,k}.$$  

$Q_{0,k} = 1, Q_{N+1,k} = 0$

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$^3$Berenstein-Zelevinsky, Gekhtman-Shapiro-Vainshtein
The program

For each affine root system:
1. Prove integrability of quantum Q-systems;
2. Find Hamiltonians (quantum relativistic Toda);
3. Find solutions of quantum Q-systems (q-difference operators in sDAHA).

The following program in type $A^{(1)}_{N-1}$ generalizes to the classical affine root systems $X^{(r)}_N, X = ABCD, r = 1, 2$.

Ingredients:
1. Duality in Macdonald (Koornwinder) theory
2. “Fourier transform”
3. $SL_2(\mathbb{Z})$-action on DAHA commutes with Toda Hamiltonians
4. $q$-Whittaker limit: Functional representation of quantum cluster variables
Macdonald’s equations and universal solutions

Macdonald’s commuting difference operators:

$$D_a(X) = \sum_{I \in [1,N]} \prod_{i \in I \setminus j \notin I} \frac{tx_i - x_j}{x_i - x_j} \Gamma_I, \quad \Gamma_I x_j = q^\delta_{j \in I} x_j \Gamma_I.$$  

Elements in functional representation of spherical DAHA.

Eigenvalue equation: $D_1(X)P_\lambda(X) = e_1(S)P_\lambda(X), \quad (a = 1, \ldots, N, s_i = q^\lambda_i t^{N-i})$

Unique monic, polynomial solution $P_\lambda(X; q, t)$ for $\lambda$ dominant integral.

Universal solutions: $P_\lambda(X) \rightsquigarrow P(X|S)$ in formal variables $s_i = t^{N-i} q^\lambda_i$

**Theorem:** Up to normalization, the eigenvalue equation

$$D_1(X)P(X|S) = e_1(S)P(X|S)$$

has a unique solution as a series in $\{X^{-\alpha_i} = x_{i+1}/x_i\}$ of the form

$$P(X|S) = q^{\mu \cdot \lambda} \sum_{\beta \in Q_+} c_\beta(S) X^{-\beta}, \quad x_i = t^{N-i} q^{\mu_i}.$$
Two normalizations for universal function $P(X|S)$:

1. **Macdonald polynomials:**

$$P^{(1)}(X|S) = q^\lambda \cdot \mu \sum_{\beta \in Q_+} c^{(1)}_{\beta}(S) X^{-\beta}, \quad c^{(1)}_0 = 1.$$  

When $\lambda \in P_+$ the series truncates:

$$t^{\lambda \cdot \rho} P^{(1)}(X|S)|_{\lambda \in P_+} = P_\lambda(X), \quad \rho_i = N - i.$$  

Any Macdonald polynomial is a specialization of the universal solution $P^{(1)}(X|S)$.

2. **Self-dual solutions:**

$$P^{(2)}(X|S) = q^\lambda \cdot \mu \sum_{\beta \in Q_+} c^{(2)}_{\beta}(S) X^{-\beta}, \quad c^{(2)}_0(S) = \Delta(S) = \prod_{\alpha \in R_+ \atop n > 0} \frac{1 - q^n S^{-\alpha}}{1 - t^{-1} q^n S^{-\alpha}}.$$  

**Theorem (Duality for the universal function)$^4$**

$$P^{(2)}(X|S) = P^{(2)}(S|X).$$  

When $\lambda, \mu \in P_+$: this is Macdonald’s duality:

$$\frac{P_\lambda(q^\mu t^\rho)}{P_\lambda(t^\rho)} = \frac{P_\mu(q^\lambda t^\rho)}{P_\mu(t^\rho)}.$$  

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$^4$ Cherednik, Noumi, Shiraishi
From duality to Pieri rules

Starting from the Macdonald eigenvalue equations

\[ D_a(X)P^{(2)}(X|S) = e_a(S)P^{(2)}(X|S) \]  \hspace{1cm} (1)

Rename the variables \( X \leftrightarrow S \),

\[ D_a(S)P^{(2)}(S|X) = e_a(X)P^{(2)}(S|X) \]

Use duality \( P^{(2)}(S|X) = P^{(2)}(X|S) \):

\[ D_a(S)P^{(2)}(X|S) = e_a(X)P^{(2)}(X|S). \]  \hspace{1cm} (2)

Pieri rule – Specialize to \( \lambda \in P_+ \):

\[ H_a(S)P_\lambda(X) = e_a(X)P_\lambda(X), \]

Pieri operators

\[ H_a(S) = t^{\rho \cdot \lambda} \Delta^{-1}(S) \ D_a(S) \ \Delta(S) t^{-\rho \cdot \lambda}. \]

**Theorem**

*Equation (2) has a unique solution, up to normalization, as a series in \( \{S^{-\alpha_i}\} \).*

The two series, in \( X^{-\alpha_i} \) and \( S^{-\alpha_i} \), must be equal!
The $q$-Whittaker limit $t \to \infty$

Symmetry of Macdonald polynomials $P_{\lambda}(X; q, t) = P_{\lambda}(X; q^{-1}, t^{-1})$
Duality and eigenvalue equations in the $q$-Whittaker limit $t \to \infty$

**Duality:** We lose symmetry: $(s_i = t^{N-i} q^{\lambda_i})$

\[
\Delta(S) = c_0^{(2)}(S) \to 1, \quad \Delta(X) \to \overline{\Delta}(X) = \prod_{n>0} (1 - q^n X^{-\alpha}).
\]

\[
P^{(2)}(X|S) = P^{(2)}(S|X) \quad \xrightarrow{t \to \infty} \quad \Pi(X|\Lambda) = \overline{\Delta}(X) K(\Lambda|X), \quad \Lambda_i = q^{\lambda_i}.
\]

In terms of series: \[
\sum_{\beta \in Q^+} c_\beta(\Lambda) X^{-\beta} = \overline{\Delta}(X) \sum_{\beta \in Q^+} \tilde{c}_\beta(X) \Lambda^{-\beta}, \quad c_0 = \tilde{c}_0 = 1.
\]

**Eigenvalue equations:**

\[
D_a(X) \Pi(X|\Lambda) = \Lambda^{\omega_a} \Pi(X|\Lambda), \quad D_a(X) = \sum_{I \in [1,N]} \prod_{I = a} \prod_{i \in I, j \notin I} \frac{x_i}{x_i - x_j} \Gamma_I
\]

The unique series solution $\Pi(X|\Lambda) = X^\lambda \sum_{\beta \in Q^+} c_\beta(\Lambda) X^{-\beta}$ with $c_0 = 1$ truncates when $\lambda \in P_+$ to a class-I $q$-Whittaker polynomial.
Pieri rules in the $q$-Whittaker limit

$$H_a(\Lambda)K(\Lambda|X) = e_a(X)K(\Lambda|X)$$

Commuting operators $H_a(\Lambda) =$ relativistic quantum Toda Hamiltonians$^5$:

$$H_1(\Lambda) = \sum_{i=0}^{N-1} (1 - \Lambda^{-\alpha_i})T_i, \quad \Lambda^{-\alpha_0} = 0, \quad T_i\Lambda_i = q\Lambda_iT_i.$$ 

$$H_a(\Lambda) = \sum_{I \subset [1,N]} \prod_{\substack{i \in I, \; i-1 \notin I \atop |I|=a}} (1 - \Lambda^{-\alpha_{i-1}})T_I.$$ 

Unique series solution

$$K(\Lambda|X) = X^\lambda \sum_{\beta \in \mathbb{Q}_+} \tilde{c}_\beta(X)\Lambda^{-\beta}$$

with $\tilde{c}_0 = 1$: fundamental $q$-Whittaker functions with weight $\mu$ where $X = q^{\mu+\rho}$. 

$^5$Etingof
$\mathbf{SL}_2(\mathbb{Z})$-action: Time translation operator

$\tau_+ \in \mathbf{SL}_2(\mathbb{Z})$ acts on the functional rep of DAHA via adjoint action of the Gaussian $\gamma$: 

$$
\gamma(X) = \prod_{a=1}^{N} e^{\frac{(\log x_a)^2}{2 \log q}}, \quad \text{Ad}_\gamma(x_i) = x_i, \quad \text{Ad}_{\gamma^{-1}}(\Gamma_i) = q^{1/2} x_i \Gamma_i.
$$

Define

$$
D_{a,k}(X) := q^{-\frac{ak}{2}} \gamma^{-k} D_a(X) \gamma^k.
$$

**Theorem (Di Francesco-K, 19)**

The difference operators $D_{a,k}$ satisfy the $A_{N-1}^{(1)}$ quantum Q-system.

1. $D_{a,i} D_{b,i+1} = q^{\min(ab)} D_{b,i+1} D_{a,i}$, (q-Commutation relations)
2. $q^a D_{a,k+1} D_{a,k-1} = D_{a,k}^2 - D_{a+1,k} D_{a-1,k}$, (Recursion relations)

The following proof is generalizable to other root systems.
“Fourier Transform:” From \( X \) to \( \Lambda \) variables

**Useful trick:** The Universal \( q \)-Whittaker functions are “complete”: If \( \{ f_i(X) \}_i \) are \( q \)-difference operators in \( X \) satisfying relations \( R \), with

\[
f_i(x)\Pi(X|\Lambda) = \hat{f}_i(\Lambda)\Pi(X|\Lambda),
\]

then \( \{ \hat{f}_i(\Lambda) \}_i \) satisfy relations \( R^{op} \) with opposite multiplication.

Define \( \hat{D}_{a,k} \): Solutions of the quantum \( Q^{op} \)-system with initial data:

1. \( \hat{D}_{a,0} = \hat{D}_a = \Lambda^{\omega_a} \);
2. \( \hat{D}_{a,1} = \Lambda^{\omega_a} T^{\omega_a} \), where \( T^{\omega_a} \Lambda^{\alpha_b} = q^{\delta_{ab}} \Lambda^{\alpha_b} T^{\omega_a} \);
3. \( \hat{D}_{a,k}, \ k \in \mathbb{Z} \) defined from \( Q^{op} \)-system

\[
q^a \hat{D}_{a,k-1} \hat{D}_{a,k+1} = \hat{D}_{a,k}^2 - \hat{D}_{a+1,k} \hat{D}_{a-1,k}.
\]

We already know \( D_{a,0}(X)\Pi(X|\Lambda) = \hat{D}_{a,0}(\Lambda)\Pi(X|\Lambda) \) (eigenvalue equation)
Theorem

1. There exists a unique q-difference operator \( g(\Lambda, T) \) such that

\[
g \hat{D}_{a,k}(\Lambda) g^{-1} = q^{a/2} \hat{D}_{a,k+1}(\Lambda)
\]

Explicitly:

\[
g(\Lambda) = \prod_{a=1}^{N} e^{\frac{(\log T_a)^2}{2 \log q}} \prod_{a=1}^{N-1} \prod_{n \geq 0} (1 - q^n \Lambda^{-\alpha_a})^{-1}.
\]

2. \( g(\Lambda) \) commutes with the quantum Toda Hamiltonians \( H_a(\Lambda) \).
3. \( g(\Lambda) \) is the Fourier transform of the Gaussian \( \gamma(X) \):

\[
\gamma(X) \Pi(X|\Lambda) = g(\Lambda) \Pi(X|\Lambda).
\]

Proof:
(1) Is by explicit calculation using \( Q^O \)-system; (2) by using explicit form of Hamiltonians.
Proof of (3): $\gamma(X)\Pi(X|\Lambda) = g(\Lambda)\Pi(X|\Lambda)$.

1. The Pieri rule $H_1(\Lambda)K(\Lambda|X) = e_1(X)K(\Lambda|X)$ has a unique solution.

2. Act on Pieri with $g(\Lambda)$, using $[g(\Lambda), H_1(\Lambda)] = 0$:

$$
  g(\Lambda)H_1(\Lambda)K(\Lambda|X) = H_1(\Lambda)g(\Lambda)K(\Lambda|X) = e_a(X)g(\Lambda)K(\Lambda|X),
$$

i.e. $g(\Lambda)K(\Lambda|X)$ is also a solution of the Pieri equation:

$$
  g(\Lambda)K(\Lambda|X) = \text{const } K(\Lambda|X).
$$

3. Use expression of $K(\Lambda|X)$ as a series in $\Lambda^{-\alpha_a}$, with leading term $x^\lambda$, and the relation

$$
  g(\Lambda)x^\lambda g(\Lambda)^{-1} = \gamma(x)(1 + \text{lower}) \implies \text{const} = \gamma(X). \quad \square
$$

**Corollary:**

1. FT of $\hat{D}_{a,k} = q^{-\frac{ak}{2}}g(\Lambda)\hat{D}_{a,0}g(\Lambda)^{-1}$ is $D_{a,k} = q^{-\frac{ak}{2}}\gamma(X)^{-1}D_{a,0}\gamma(X)$.

2. $D_{a,k}$ satisfy the quantum Q-system.

3. $[g(\Lambda), H_a(\Lambda)] = 0$: Conserved quantities of qQ-system are the Toda Hamiltonians.
Generalization to other root systems

1. Use Koornwinder operators + higher q-difference operators.
2. Depend on parameters \((a, b, c, d)\): Specialize for each \(g\):

<table>
<thead>
<tr>
<th>(g)</th>
<th>(g^*)</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(R)</th>
<th>(R^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_{1N}^{(1)})</td>
<td>(D_{1N}^{(1)})</td>
<td>1</td>
<td>−1</td>
<td>(q^{\frac{1}{2}})</td>
<td>−(q^{\frac{1}{2}})</td>
<td>(D_N)</td>
<td>(D_N)</td>
</tr>
<tr>
<td>(B_{1N}^{(1)})</td>
<td>(C_{1N}^{(1)})</td>
<td>(t)</td>
<td>−1</td>
<td>(q^{\frac{1}{2}})</td>
<td>−(q^{\frac{1}{2}})</td>
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<td>(C_{1N}^{(1)})</td>
<td>(B_{1N}^{(1)})</td>
<td>(t^{\frac{1}{2}})</td>
<td>−(t^{\frac{1}{2}})</td>
<td>(t^{\frac{1}{2}} q^{\frac{1}{2}})</td>
<td>−(t^{\frac{1}{2}} q^{\frac{1}{2}})</td>
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<td>(q^{\frac{1}{2}})</td>
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<td>(C_N)</td>
<td>(C_N)</td>
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<tr>
<td>(D_{N+1}^{(2)})</td>
<td>(D_{N+1}^{(2)})</td>
<td>(t)</td>
<td>−1</td>
<td>(t q^{\frac{1}{2}})</td>
<td>−(q^{\frac{1}{2}})</td>
<td>(B_N)</td>
<td>(B_N)</td>
</tr>
<tr>
<td>(A_{2N}^{(2)})</td>
<td>(A_{2N}^{(2)})</td>
<td>(t)</td>
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<td>(t^{\frac{1}{2}} q^{\frac{1}{2}})</td>
<td>−(t^{\frac{1}{2}} q^{\frac{1}{2}})</td>
<td>(B C_N)</td>
<td>(B C_N)</td>
</tr>
</tbody>
</table>

3. Koornwinder operators act on the space of \(BC\)-symmetric Laurent polynomials (symmetric under permutations of variables and inversion).
4. Find time translation operators in \(q\)-Whittaker limit for each \(g\).
Universal Koornwinder function: Eigenvalue equations

Koornwinder operator: \( D^{(a,b,c,d)}_1(X) = \sum_{\epsilon=\pm 1} \sum_{i=1}^{N} \Phi_{i,\epsilon}(X) \Gamma_{i}^\epsilon + \varphi(X), \)

\[ \Phi_{i,\epsilon}(X) = \prod_{\alpha=a,b,c,d} \frac{(1 - \alpha x_i^\epsilon)}{(1 - x_i^{2\epsilon})(1 - qx_i^{2\epsilon})} \prod_{j \neq i, \epsilon' = \pm 1} \frac{tx_i^\epsilon / x_j^{\epsilon'} - 1}{x_i^{\epsilon} / x_j^{\epsilon'} - 1} \text{ series in } \{X^{-\alpha_i}\}_{\alpha_i \in \Pi_{BN}} \]

Eigenvalue equation

\[ D^{(a,b,c,d)}_1(X) P^{(a,b,c,d)}(X|S) = \hat{e}_1(S) P^{(a,b,c,d)}(X|S), \quad \hat{e}_a(S) = \sum_{i=1}^{N} (s_i + s_i^{-1}) \]

has a unique solution, up to scalar \( c_0(S) \), of the form

\[ P^{(a,b,c,d)}(X|S) = q^{\lambda \cdot \mu} \sum_{\beta \in Q_+(B_N)} c_{\beta}(S) X^{-\beta}. \]
Duality

Two natural normalizations of Universal Koornwinder function:

1. \( c_0(S) = 1 \): If \( \lambda \in P_+(C_N) \), with \( s_i = \sqrt{\frac{abcd}{q}} t^{N-i} q^{\lambda_i} \):

\[
P(a,b,c,d)(X|S) \to P_{\lambda}^{(a,b,c,d)}(X)
\]

is symmetric Laurent polynomial under the Weyl group of \( C_N \) (Koornwinder polynomial).

2. Choosing

\[
c_0(S) = \Delta^{(a^*,b^*,c^*,d^*)}(S) = \prod_{i=1}^N \frac{\left( \frac{q}{x_i^2}; q \right)_{\infty}}{\prod_{\alpha=a,b,c,d} \left( \frac{q}{\alpha^* x_i}; q \right)_{\infty} \prod_{\epsilon=\pm 1} \frac{q x_i^\epsilon}{t x_i}; q)_{\infty}}
\]

\((a, b, c, d)^* = (\sqrt{\frac{abcd}{q}}, -\sqrt{\frac{qac}{bd}}, \sqrt{\frac{qac}{bd}}, \sqrt{\frac{qad}{bc}})\).

Duality for Koornwinder:

\[
P(X|S) = P^*(S|X).
\]

- Using eigenvalue equation + duality gives Pieri rules.
- Specialize, take \( q \)-Whittaker limit \( t \to \infty \): Commuting Hamiltonians.
- Compute \( g^g(\Lambda) \): FT of time translation operator for each \( g \), commute with Hamiltonians.

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\(^6\) Macdonald conjecture; Cherednik; van Diejen; Sahi; Di Francesco-K for Universal function
Summary

1. Solutions of $qQ$-systems are $\tau^+$-translated Macdonald-Koornwinder operators in $q$-Whittaker limit with specialized $(a, b, c, d)$.

2. Elements in the spherical DAHA: Cluster algebra structure in sDAHA.

3. Integrability: Hamiltonians = Pieri operators in $q$-Whittaker limit.

4. Time translation operators $g(\Lambda) = "Baxter\ Q"$ at specialized spectral parameter (c.f. Schrader-Shapiro).

5. Homework: Exceptional types.
Exchange matrices for the Q-systems

For $g = X^{(1)}_N$, the quiver/exchange matrix is skew-symmetric:

$$B = \begin{pmatrix}
C^t - C & -C^t \\
C & 0
\end{pmatrix}$$

with $C$ the Cartan matrix of $R$.

For $g = A^{(2)}_{2N-1}$ or $D^{(2)}_{N+1}$,

$$B = \begin{pmatrix}
0 & -C \\
C & 0
\end{pmatrix}.$$

For $g = A^{(2)}_{2N}$: Not a cluster algebra.