Integrable quantum cluster algebras associated with affine root systems

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Outline

T-system and Q-system for $\mathfrak{sl}_2$

Fusion relations for Heisenberg model transfer matrices, $T_V(\zeta)$, $V(\zeta)$= auxiliary space.

- Commuting family $\{T_{j,k} := T_{V(k\omega_1)}(q^j\zeta), j \in \mathbb{Z}, k \geq 0\}$, satisfy

T-system: $$T_{j,k+1}T_{j,k-1} = T_{j+1,k}T_{j-1,k} - 1.$$

- Asymptotics of TBA $\zeta \to \infty$: $T_{j,k} \to \mathsf{Q}_k$, satisfy

Q-system: $$\mathsf{Q}_{k+1}\mathsf{Q}_{k-1} = \mathsf{Q}_k^2 - 1.$$

Discrete evolution in "time" k

$$\mathsf{Q}_{k+1} = (\mathsf{Q}_k^2 - 1)\mathsf{Q}_{k-1}^{-1}, \qquad \text{Initial data } \{\mathsf{Q}_0 = 1,\ \mathsf{Q}_1\}$$

1. **Integrable:** There is a conserved quantity $H_1(\mathsf{Q}_k, \mathsf{Q}_{k+1})$, independent of k;
2. **Cluster algebra:** $\{\mathsf{Q}_i\}$ are cluster variables in CA corresponding to the Kronecker quiver

$$\circ \Longrightarrow \circ$$

3. **Canonical quantization**: Discrete, non-commutative integrable evolution equation.

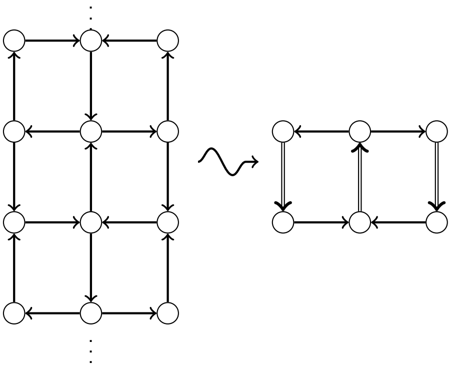
T-system and Q-system for $\mathfrak{sl}_N$

Transfer matrices $\{T_{i,j,k} = T_{V(k\omega_i)}(q^j\zeta) : i \in [1, N-1], j \in \mathbb{Z}, k \geq 0\}$.

T-system: $T_{i,j,k+1}T_{i,j,k-1} = T_{i,j+1,k}T_{i,j-1,k} - T_{i-1,j,k}T_{i+1,j,k}, \quad T_{0,j,k} = 1 = T_{N+1,j,k}.$

Asymptotics $\zeta \to \infty$, $T_{i,j,k} \to \mathsf{Q}_{i,k}$:

Q-system: $\mathsf{Q}_{i,k+1}\mathsf{Q}_{i,k-1} = \mathsf{Q}_{i,k}^2 - \mathsf{Q}_{i-1,k}\mathsf{Q}_{i+1,k}, \quad \mathsf{Q}_{0,k} = 1 = \mathsf{Q}_{N+1,k}.$



Cluster algebras: Q-system quiver is T-system quiver wrapped around a cylinder of radius 2

T-system and Q-system for $\mathfrak{sl}_N$

Transfer matrices $\{T_{i,j,k} = T_{V(k\omega_i)}(q^j\zeta) : i \in [1, N-1], j \in \mathbb{Z}, k \geq 0\}$.

T-system: $T_{i,j,k+1}T_{i,j,k-1} = T_{i,j+1,k}T_{i,j-1,k} - T_{i-1,j,k}T_{i+1,j,k}, \quad T_{0,j,k} = 1 = T_{N+1,j,k}.$

Asymptotics $\zeta \to \infty$, $T_{i,j,k} \to \mathsf{Q}_{i,k}$:

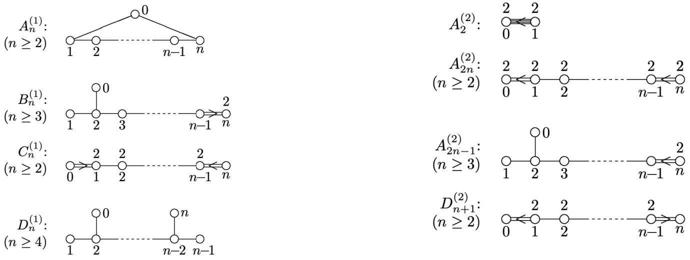
Q-system: $\mathsf{Q}_{i,k+1}\mathsf{Q}_{i,k-1} = \mathsf{Q}_{i,k}^2 - \mathsf{Q}_{i-1,k}\mathsf{Q}_{i+1,k}, \quad \mathsf{Q}_{0,k} = 1 = \mathsf{Q}_{N+1,k}.$

Discrete evolution equations in k:

1. **Integrable:** (Poisson-commuting) Hamiltonians $\{H_a\}_{a=1}^{N-1}$ independent of k;
2. **Cluster algebra:** $\mathsf{Q}_{i,k}$ cluster variables, equations are mutations;
3. Canonical **Quantization** of CA: Discrete, non-commutative integrable evolution.

Other classical root systems

There is a T-systems and Q-systems for each affine root system:



Quantum **Q-systems:** $\mathfrak{g} \to (R, R^*)$.

$\mathfrak{g}$	$A_N^{(1)}$	$B_N^{(1)}$	$C_N^{(1)}$	$D_N^{(1)}$	$A_{2N-1}^{(2)}$	$A_{2N}^{(2)}$	$D_{N+1}^{(2)}$
R	A_N	B_N	C_N	D_N	C_N	BC_N	B_N
R^*	A_N	C_N	B_N	D_N	C_N	BC_N	B_N

Q-systems[2]

$$\mathrm{Q}_{a,k+1}\mathrm{Q}_{a,k-1} = \mathrm{Q}_{a,k}^2 - \prod_{b\neq a} \mathrm{Q}_{b,k}^{-C_{ba}}, \qquad 1 \le a \le r, \quad k \in \mathbb{Z}$$

(C = Cartan matrix of R)

Except for

$$B_N^{(1)} \quad : \quad \mathrm{Q}_{N-1,k+1}\mathrm{Q}_{N-1,k-1} = \mathrm{Q}_{N-1,k}^2 - \mathrm{Q}_{N-2,k}\mathrm{Q}_{N,2k};$$

$$\mathrm{Q}_{N,k+1}\mathrm{Q}_{N,k-1} = \mathrm{Q}_{N,k}^2 - \mathrm{Q}_{N-1,\lfloor \frac{k}{2} \rfloor}\mathrm{Q}_{N-1,\lfloor \frac{k+1}{2} \rfloor};$$

$$C_N^{(1)} \quad : \quad \mathrm{Q}_{N-1,k+1}\mathrm{Q}_{N-1,k-1} = \mathrm{Q}_{N-1,k}^2 - \mathrm{Q}_{N-2,k}\mathrm{Q}_{N,\lfloor \frac{k}{2} \rfloor}\mathrm{Q}_{N,\lfloor \frac{k+1}{2} \rfloor};$$

$$\mathrm{Q}_{N,k+1}\mathrm{Q}_{N,k-1} = \mathrm{Q}_{N,k}^2 - \mathrm{Q}_{N-1,2k};$$

$$A_{2n}^{(2)} \quad : \quad \mathrm{Q}_{N,k+1}\mathrm{Q}_{N,k-1} = \mathrm{Q}_{N,k}^2 - \mathrm{Q}_{N-1,k}\mathrm{Q}_{N,k}.$$

Mostly cluster algebra mutations

Our goals:

- Integrable structure for all $\mathfrak{g}$, commuting Hamiltonians;
- Quantization $\mathrm{Q}_{i,k} \to \mathcal{Q}_{i,k}$ non-commuting: Solutions $\mathcal{Q}_{i,k}$? Hamiltonians?

Unifying framework: Macdonald-Koornwinder equations.

[2] Kirillov-Reshtikhin, Kuniba, Nakanishi, Suzuki; Hatayama et al.

Quantization of Q-systems $\mathrm{Q}_{a,k+1}\mathrm{Q}_{a,k-1} = \mathrm{Q}_{a,k}^2 - M_{a,k}$

Canonical quantization of the associated cluster algebra[3].

Commutations relations:

$$\mathcal{Q}_{a,t_a k+i}\mathcal{Q}_{b,t_b k+j} = q^{\Lambda_{ab}j-\Lambda_{ba}i}\mathcal{Q}_{b,t_b k+j}\mathcal{Q}_{a,t_a k+i}, \quad i,j=0,1.$$

$\Lambda_{a,b} = \omega_a^* \cdot \omega_b$, ω_a, ω_a^* fundamental weights of R, R^*
$t_a = 2$ for short roots, $t_a = 1$ for long roots

Evolution equations:

$$q^{\Lambda_{aa}}\mathcal{Q}_{a,k+1}\mathcal{Q}_{a,k-1} = \mathcal{Q}_{a,k}^2 - :\mathcal{M}_{a,k}:, \quad a\in\Pi, k\in\mathbb{Z}$$

$:\mathcal{M}_{a,k}:=$normal ordered product

Example: $A_{N-1}^{(1)}$

$$\begin{aligned} \mathcal{Q}_{a,k}\mathcal{Q}_{b,k+i} &= q^{\min(a,b)i}\mathcal{Q}_{b,k+i}\mathcal{Q}_{a,k}, & i=0,1; \\ q^a\mathcal{Q}_{a,k+1}\mathcal{Q}_{a,k-1} &= \mathcal{Q}_{a,k}^2 - \mathcal{Q}_{a-1,k}\mathcal{Q}_{a+1,k}. & \mathrm{Q}_{0,k}=1, \mathrm{Q}_{N+1,k}=0 \end{aligned}$$

[3] Berenstein-Zelevinsky, Gekhtman-Shapiro-Vainshtein

The program

For each affine root system:

1. Prove integrability of quantum Q-systems;
2. Find Hamiltonians (quantum relativistic Toda);
3. Find solutions of quantum Q-systems (q-difference operators in sDAHA).

The following program in type $A_{N-1}^{(1)}$ generalizes to the classical affine root systems $X_N^{(r)}$, $X = ABCD$, $r = 1, 2$.

Ingredients:

1. Duality in Macdonald (Koornwinder) theory
2. "Fourier transform"
3. $\mathrm{SL}_2(\mathbb{Z})$-action on DAHA commutes with Toda Hamiltonians
4. q-Whittaker limit: Functional representation of quantum cluster variables

Macdonald's equations and universal solutions

Macdonald's commuting difference operators:

$$\mathcal{D}_a(X) = \sum_{\substack{I \in [1,N] \\ |I|=a}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \Gamma_I, \qquad \Gamma_I \, x_j = q^{\delta_{j \in I}} x_j \Gamma_I.$$

elements in functional representation of spherical DAHA.

Eigenvalue equation: $\mathcal{D}_1(X) P_\lambda(X) = e_1(S) P_\lambda(X), \quad (a = 1, ..., N, s_i = q^{\lambda_i} t^{N-i})$

Unique monic, polynomial solution $P_\lambda(X; q, t)$ for λ dominant integral.

Universal solutions: $P_\lambda(X) \rightsquigarrow P(X|S)$ in formal variables $s_i = t^{N-i} q^{\lambda_i}$

Theorem: Up to normalization, the eigenvalue equation

$$\mathcal{D}_1(X) P(X|S) = e_1(S) P(X|S)$$

has a unique solution as a series in $\{X^{-\alpha_i} = x_{i+1}/x_i\}$ of the form

$$P(X|S) = q^{\mu \cdot \lambda} \sum_{\beta \in \mathsf{Q}_+} c_\beta(S) X^{-\beta}, \quad x_i = t^{N-i} q^{\mu_i}.$$

Two normalizations for universal function $P(X|S)$:

1. **Macdonald polynomials:**

$$P^{(1)}(X|S) = q^{\lambda\cdot\mu} \sum_{\beta\in\mathsf{Q}_+} c_\beta^{(1)}(S) X^{-\beta}, \quad c_0^{(1)} = 1.$$

   When $\lambda \in P_+$ the series truncates:

$$t^{\lambda\cdot\rho} P^{(1)}(X|S)|\Big|_{\lambda\in P_+} = P_\lambda(X), \qquad \rho_i = N - i.$$

   Any Macdonald polynomial is a specialization of the universal solution $P^{(1)}(X|S)$.

2. **Self-dual solutions:**

$$P^{(2)}(X|S) = q^{\lambda\cdot\mu} \sum_{\beta\in\mathsf{Q}_+} c_\beta^{(2)}(S) X^{-\beta}, \quad c_0^{(2)}(S) = \Delta(S) = \prod_{\substack{\alpha\in R_+ \\ n>0}} \frac{1 - q^n S^{-\alpha}}{1 - t^{-1} q^n S^{-\alpha}}$$

Theorem (Duality for the universal function[4])

$$P^{(2)}(X|S) = P^{(2)}(S|X).$$

When $\lambda, \mu \in P_+$: this is Macdonald's duality: $\frac{P_\lambda(q^\mu t^\rho)}{P_\lambda(t^\rho)} = \frac{P_\mu(q^\lambda t^\rho)}{P_\mu(t^\rho)}$.

[4] Cherednik, Noumi, Shiraishi

From duality to Pieri rules

Starting from the Macdonald eigenvalue equations

$$\mathcal{D}_a(X)P^{(2)}(X|S) = e_a(S)P^{(2)}(X|S) \tag{1}$$

Rename the variables $X \leftrightarrow S$,

$$\mathcal{D}_a(S)P^{(2)}(S|X) = e_a(X)P^{(2)}(S|X)$$

Use duality $P^{(2)}(S|X) = P^{(2)}(X|S)$:

$$\mathcal{D}_a(S)P^{(2)}(X|S) = e_a(X)P^{(2)}(X|S). \tag{2}$$

Pieri rule – Specialize to $\lambda \in P_+$: $\mathcal{H}_a(S)P_\lambda(X) = e_a(X)P_\lambda(X)$,

Pieri operators $\mathcal{H}_a(S) = t^{\rho\cdot\lambda}\Delta^{-1}(S)\ \mathcal{D}_a(S)\ \Delta(S)t^{-\rho\cdot\lambda}$.

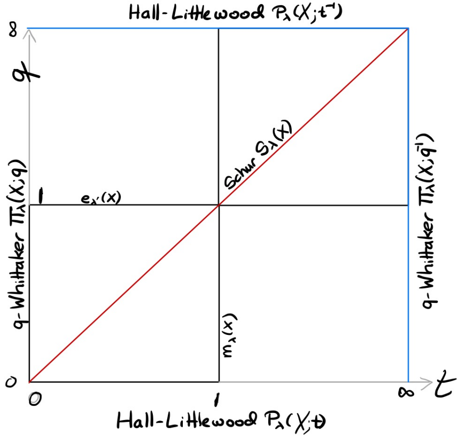
Theorem

Equation (2) has a unique solution, up to normalization, as a series in $\{S^{-\alpha_i}\}$.

The two series, in $X^{-\alpha_i}$ and $S^{-\alpha_i}$, must be equal!

The q-Whittaker limit $t \to \infty$

Symmetry of Macdonald polynomials $P_\lambda(X;q,t) = P_\lambda(X;q^{-1},t^{-1})$



Duality and eigenvalue equations in the q-Whittaker limit $t \to \infty$

Duality: We lose symmetry: $(s_i = t^{N-i} q^{\lambda_i})$

$$\Delta(S) = c_0^{(2)}(S) \to 1, \quad \Delta(X) \to \overline{\Delta}(X) = \prod_{\substack{\alpha \in \Pi \\ n > 0}} (1 - q^n X^{-\alpha}).$$

$$P^{(2)}(X|S) = P^{(2)}(S|X) \quad \underset{t \to \infty}{\longrightarrow} \quad \Pi(X|\Lambda) = \overline{\Delta}(X) K(\Lambda|X), \qquad \Lambda_i = q^{\lambda_i}.$$

In terms of series: $\sum_{\beta \in \mathsf{Q}_+} c_\beta(\Lambda) X^{-\beta} = \overline{\Delta}(X) \sum_{\beta \in \mathsf{Q}_+} \tilde{c}_\beta(X) \Lambda^{-\beta}, \qquad c_0 = \tilde{c}_0 = 1.$

Eigenvalue equations:

$$D_a(X)\Pi(X|\Lambda) = \Lambda^{\omega_a} \Pi(X|\Lambda), \qquad D_a(X) = \sum_{\substack{I \in [1,N] \\ |I| = a}} \prod_{\substack{i \in I \\ j \notin I}} \frac{x_i}{x_i - x_j} \Gamma_I$$

The unique series solution $\Pi(X|\Lambda) = X^\lambda \sum_{\beta \in \mathsf{Q}_+} c_\beta(\Lambda) X^{-\beta}$ with $c_0 = 1$ truncates when $\lambda \in P_+$ to a class-I q-Whittaker polynomial.

Pieri rules in the q-Whittaker limit

$$H_a(\Lambda)K(\Lambda|X) = e_a(X)K(\Lambda|X)$$

Commuting operators $H_a(\Lambda)$ = relativistic quantum Toda Hamiltonians[5]:

$$H_1(\Lambda) = \sum_{i=0}^{N-1}(1-\Lambda^{-\alpha_i})T_i, \qquad \Lambda^{-\alpha_0}=0, \quad T_i\Lambda_i = q\Lambda_i T_i.$$

$$H_a(\Lambda) = \sum_{\substack{I\subset[1,N]\\|I|=a}} \prod_{\substack{i\in I,\\ i-1\notin I}} (1-\Lambda^{-\alpha_{i-1}})T_I.$$

Unique series solution

$$K(\Lambda|X) = X^\lambda \sum_{\beta\in\mathsf{Q}_+} \tilde{c}_\beta(X)\Lambda^{-\beta}$$

with $\tilde{c}_0 = 1$: fundamental q-Whittaker functions with weight μ where $X = q^{\mu+\rho}$.

[5] Etingof

$\mathsf{SL}_2(\mathbb{Z})$-action: Time translation operator

$\tau_+ \in \mathsf{SL}_2(\mathbb{Z})$ acts on the functional rep of DAHA via adjoint action of the Gaussian γ:

$$\gamma(X) = \prod_{a=1}^{N} e^{\frac{(\log x_a)^2}{2\log q}}, \qquad \mathrm{Ad}_\gamma(x_i) = x_i, \quad \mathrm{Ad}_{\gamma^{-1}}(\Gamma_i) = q^{1/2} x_i \Gamma_i.$$

Define

$$D_{a,k}(X) := q^{-\frac{ak}{2}} \gamma^{-k} D_a(X) \gamma^k.$$

Theorem (Di Francesco-K, 19)

The difference operators $D_{a,k}$ satisfy the $A_{N-1}^{(1)}$ quantum Q-system.

(1) $\quad D_{a,i} D_{b,i+1} = q^{\min(ab)} D_{b,i+1} D_{a,i}, \quad$ *(q-Commutation relations)*

(2) $\quad q^a D_{a,k+1} D_{a,k-1} = D_{a,k}^2 - D_{a+1,k} D_{a-1,k}, \quad$ *(Recursion relations)*

The following proof is generalizable to other root systems.

"Fourier Transform:" From X to Λ variables

Useful trick: The Universal q-Whittaker functions are "complete":
If $\{f_i(X)\}_i$ are q-difference operators in X satisfying relations R, with

$$f_i(x)\Pi(X|\Lambda) = \hat{f}_i(\Lambda)\Pi(X|\Lambda),$$

then $\{\hat{f}_i(\Lambda)\}_i$ satisfy relations R^{op} with opposite multiplication.

Define $\widehat{D}_{a,k}$: Solutions of the quantum Q^{op}-system with initial data:

1. $\widehat{D}_{a,0} = \widehat{D}_a = \Lambda^{\omega_a}$;
2. $\widehat{D}_{a,1} = \Lambda^{\omega_a} T^{\omega_a}$, where $T^{\omega_a}\Lambda^{\alpha_b} = q^{\delta_{ab}}\Lambda^{\alpha_b}T^{\omega_a}$;
3. $\widehat{D}_{a,k}$, $k \in \mathbb{Z}$ defined from Q^{op}-system

$$q^a \widehat{D}_{a,k-1}\widehat{D}_{a,k+1} = \widehat{D}_{a,k}^2 - \widehat{D}_{a+1,k}\widehat{D}_{a-1,k}.$$

We already know $D_{a,0}(X)\Pi(X|\Lambda) = \widehat{D}_{a,0}(\Lambda)\Pi(X|\Lambda)$ (eigenvalue equation)

Fourier transform of the Gaussian: Time translation operator

Theorem

1. *There exists a unique q-difference operator* $g(\Lambda, T)$ *such that*

$$g\widehat{D}_{a,k}(\Lambda)g^{-1} = q^{a/2}\widehat{D}_{a,k+1}(\Lambda)$$

Explicitly:

$$g(\Lambda) = \prod_{a=1}^{N} e^{\frac{(\log T_a)^2}{2\log q}} \prod_{a=1}^{N-1}\prod_{n\geq 0}(1-q^n\Lambda^{-\alpha_a})^{-1}.$$

2. $g(\Lambda)$ *commutes with the quantum Toda Hamiltonians* $H_a(\Lambda)$*.*
3. $g(\Lambda)$ *is the Fourier transform of the Gaussian* $\gamma(X)$*:*

$$\gamma(X)\Pi(X|\Lambda) = g(\Lambda)\Pi(X|\Lambda).$$

Proof:
(1) Is by explicit calculation using Q^{op}-system; (2) by using explicit form of Hamiltonians.

Proof of (3): $\gamma(X)\Pi(X|\Lambda) = g(\Lambda)\Pi(X|\Lambda)$.

1. The Pieri rule $H_1(\Lambda)K(\Lambda|X) = e_1(X)K(\Lambda|X)$ has a unique solution.
2. Act on Pieri with $g(\Lambda)$, using $[g(\Lambda), H_1(\Lambda)] = 0$:

$$g(\Lambda)H_1(\Lambda)K(\Lambda|X) = H_1(\Lambda)g(\Lambda)K(\Lambda|X) = e_a(X)g(\Lambda)K(\Lambda|X),$$

   i.e. $g(\Lambda)K(\Lambda|X)$ is also a solution of the Pieri equation:

$$g(\Lambda)K(\Lambda|X) = \text{const } K(\Lambda|X).$$

3. Use expression of $K(\Lambda|X)$ as a series in $\Lambda^{-\alpha_a}$, with leading term x^λ, and the relation

$$g(\Lambda)x^\lambda g(\Lambda)^{-1} = \gamma(x)(1 + \text{lower}) \Longrightarrow \text{const} = \gamma(X). \qquad \square$$

Corollary:

1. FT of $\widehat{D}_{a,k} = q^{-\frac{ak}{2}} g(\Lambda)\widehat{D}_{a,0}g(\Lambda)^{-1}$ is $D_{a,k} = q^{-\frac{ak}{2}}\gamma(X)^{-1}D_{a,0}\gamma(X)$.
2. $D_{a,k}$ satisfy the quantum Q-system.
3. $[g(\Lambda), H_a(\Lambda)] = 0$: Conserved quantities of qQ-system are the Toda Hamiltonians.

Generalization to other root systems

1. Use Koornwinder operators + higher q-difference operators.
2. Depend on parameters (a, b, c, d): Specialize for each $\mathfrak{g}$:

$\mathfrak{g}$	$\mathfrak{g}^*$	a	b	c	d	R	R^*
$D_N^{(1)}$	$D_N^{(1)}$	1	-1	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	D_N	D_N
$B_N^{(1)}$	$C_N^{(1)}$	t	-1	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	B_N	C_N
$C_N^{(1)}$	$B_N^{(1)}$	$t^{\frac{1}{2}}$	$-t^{\frac{1}{2}}$	$t^{\frac{1}{2}}\, q^{\frac{1}{2}}$	$-t^{\frac{1}{2}}\, q^{\frac{1}{2}}$	C_N	B_N
$A_{2N-1}^{(2)}$	$A_{2N-1}^{(2)}$	$t^{\frac{1}{2}}$	$-t^{\frac{1}{2}}$	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	C_N	C_N
$D_{N+1}^{(2)}$	$D_{N+1}^{(2)}$	t	-1	$t\, q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	B_N	B_N
$A_{2N}^{(2)}$	$A_{2N}^{(2)}$	t	-1	$t^{\frac{1}{2}}\, q^{\frac{1}{2}}$	$-t^{\frac{1}{2}}\, q^{\frac{1}{2}}$	BC_N	BC_N

3. Koornwinder operators act on the space of BC-symmetric Laurent polynomials (symmetric under permutations of variables and inversion).
4. Find time translation operators in q-Whittaker limit for each $\mathfrak{g}$.

Universal Koornwinder function: Eigenvalue equations

Koornwinder operator: $D_1^{(a,b,c,d)}(X) = \sum_{\epsilon=\pm 1} \sum_{i=1}^{N} \Phi_{i,\epsilon}(X)\Gamma_i^{\epsilon} + \varphi(X),$

$$\Phi_{i,\epsilon}(X) = \frac{\prod_{\alpha=a,b,c,d}(1-\alpha x_i^{\epsilon})}{(1-x_i^{2\epsilon})(1-qx_i^{2\epsilon})} \prod_{\substack{j\neq i \\ \epsilon'=\pm 1}} \frac{tx_i^{\epsilon}/x_j^{\epsilon'}-1}{x_i^{\epsilon}/x_j^{\epsilon'}-1} \text{ series in } \{X^{-\alpha_i}\}_{\alpha_i \in \Pi_{B_N}}$$

Eigenvalue equation

$$D_1^{(a,b,c,d)}(X)P^{(a,b,c,d)}(X|S) = \widehat{e}_1(S)P^{(a,b,c,d)}(X|S), \quad \widehat{e}_a(S) = \sum_{i=1}^{N}(s_i + s_i^{-1})$$

has a unique solution, up to scalar $c_0(S)$, of the form

$$P^{(a,b,c,d)}(X|S) = q^{\lambda\cdot\mu} \sum_{\beta\in\mathsf{Q}_+(B_N)} c_\beta(S)X^{-\beta}.$$

Duality

Two natural normalizations of Universal Koornwinder function:

1. $c_0(S) = 1$: If $\lambda \in P_+(C_N)$, with $s_i = \sqrt{\frac{abcd}{q}} t^{N-i} q^{\lambda_i}$:
   $P^{(a,b,c,d)}(X|S) \to P_\lambda^{(a,b,c,d)}(X)$ symmetric Laurent polynomial under the Weyl group of C_N (**Koornwinder polynomial**).
2. Choosing

$$c_0(S) = \Delta^{(a^*,b^*,c^*,d^*)}(S) = \prod_{i=1}^{N} \frac{(\frac{q}{x_i^2};q)_\infty}{\prod_{\alpha=a,b,c,d} (\frac{q}{\alpha^* x_i};q)_\infty} \prod_{\substack{i<j \\ \epsilon=\pm 1}} \frac{(\frac{q x_j^\epsilon}{x_i};q)_\infty}{(\frac{q x_j^\epsilon}{t x_i};q)_\infty}$$

$(a,b,c,d)^* = (\sqrt{\frac{abcd}{q}}, -\sqrt{\frac{qac}{bd}}, \sqrt{\frac{qac}{bd}}, \sqrt{\frac{qad}{bc}})$.

Duality for Koornwinder:[6]

$$P(X|S) = P^*(S|X).$$

- Using eigenvalue equation + duality gives Pieri rules.
- Specialize, take q-Whittaker limit $t \to \infty$: Commuting Hamiltonians.
- Compute $g^{\mathfrak{g}}(\Lambda)$: FT of time translation operator for each $\mathfrak{g}$, commute with Hamiltonians.

[6] Macdonald conjecture; Cherednik; van Diejen; Sahi; Di Francesco-K for Universal function

Summary

1. Solutions of qQ-systems are τ_+-translated Macdonald-Koornwinder operators in q-Whittaker limit with specialized (a, b, c, d).

2. Elements in the spherical DAHA: Cluster algebra structure in sDAHA.

3. Integrability: Hamiltonians = Pieri operators in q-Whittaker limit.

4. Time translation operators $g(\Lambda) =$ "Baxter Q" at specialized spectral parameter (c.f. Schrader-Shapiro).

5. Homework: Exceptional types.

Exchange matrices for the Q-systems

For $\mathfrak{g} = X_N^{(1)}$, the quiver/exchange matrix is skew-symmetric:

$$B = \left[\begin{array}{c|c} C^t - C & -C^t \\ \hline C & 0 \end{array}\right]$$

with C the Cartan matrix of R.

For $\mathfrak{g} = A_{2N-1}^{(2)}$ or $D_{N+1}^{(2)}$,

$$B = \left[\begin{array}{c|c} 0 & -C \\ \hline C & 0 \end{array}\right].$$

For $\mathfrak{g} = A_{2N}^{(2)}$: Not a cluster algebra.

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