A case study: the tangent method applied to two-periodic Aztec diamonds

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Randomness, Integrability and Universality
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Spatial phase separation with emergence of limit shapes

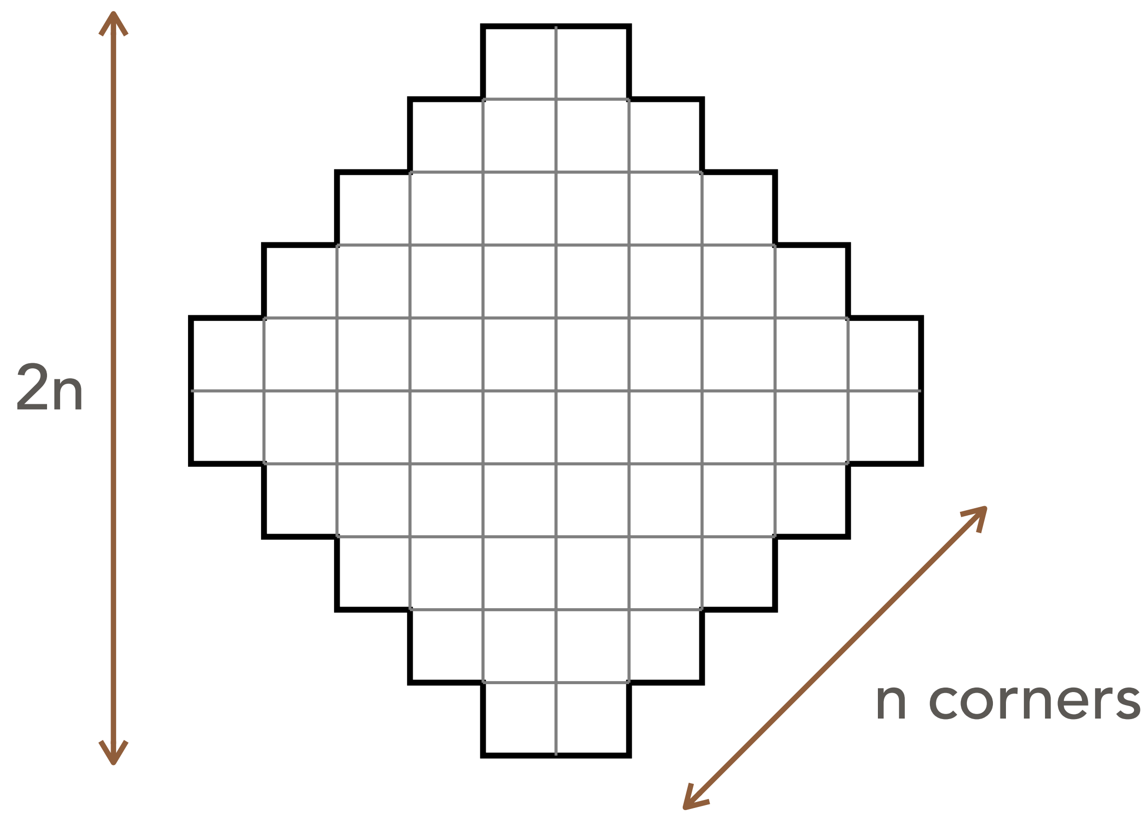
arctic phenomena :
frozen next to temperate regions

arctic curves :
interfaces

Historically, first observed in the **domino tiling of Aztec diamonds**
Cohn, Elkies and Propp (1996) & Jockusch, Propp and Shor (1998)

Since then, many other instances: tilings of hexagons, 6V model, alternating sign matrices, 2-periodic Aztec diamonds, Young tableaux, Aztec rectangles, 20V model(s), …

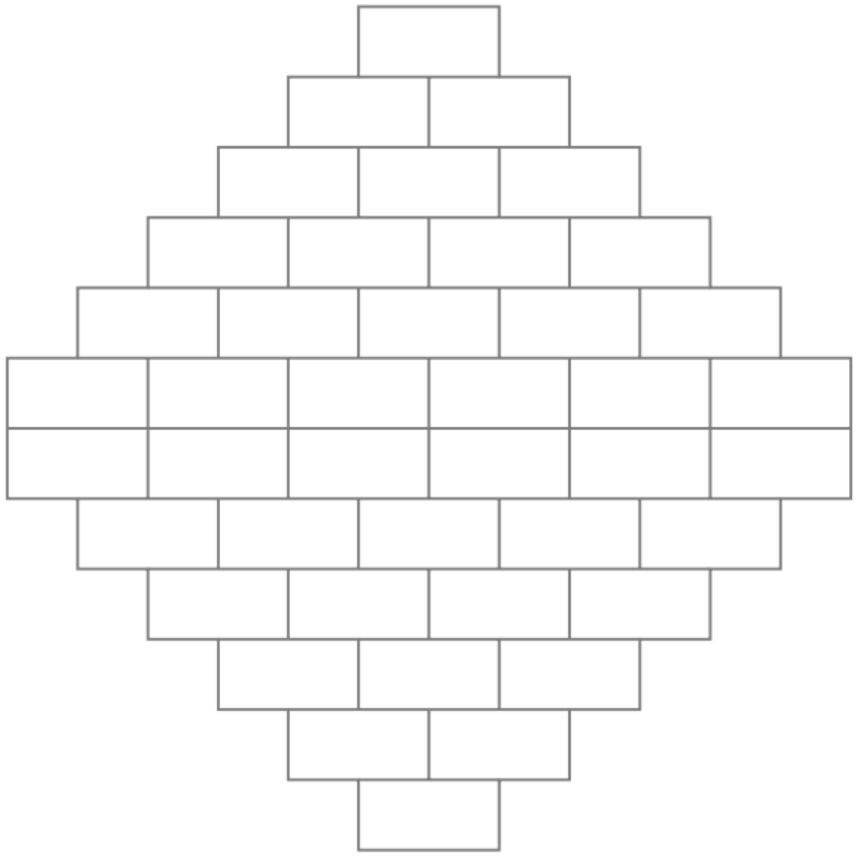
Aztec diamond of order n

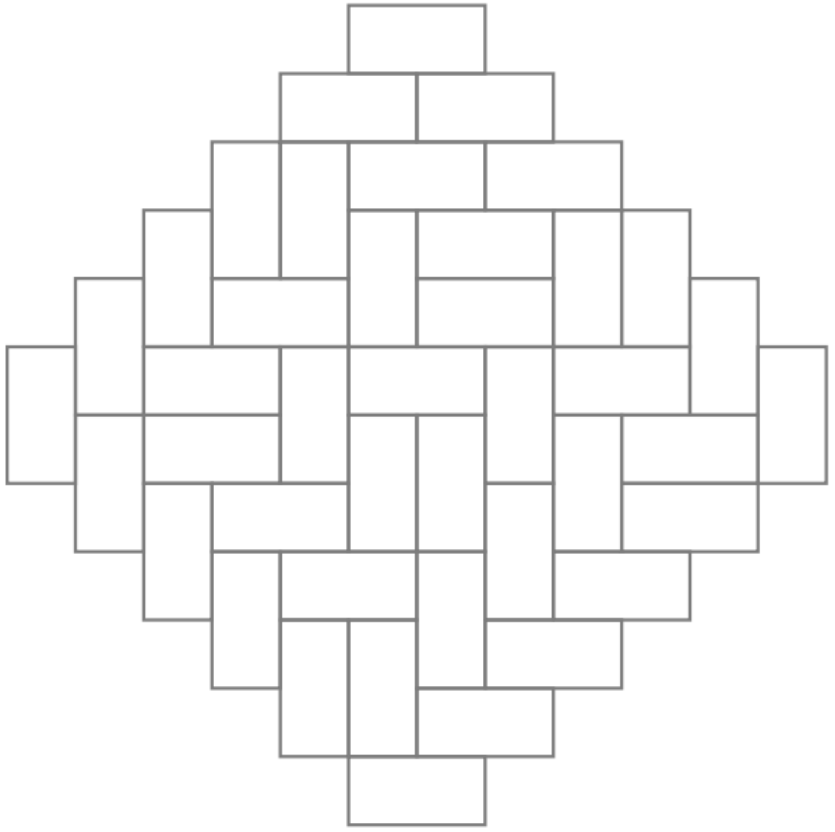


- finite domain in Z^2
- area $= 2n(n+1)$
- # dominos $= n(n+1)$

Then: consider tilings of AD_n by dominos (+ prob. measure, later)

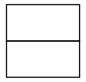
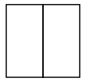
Aztec diamond of order 6



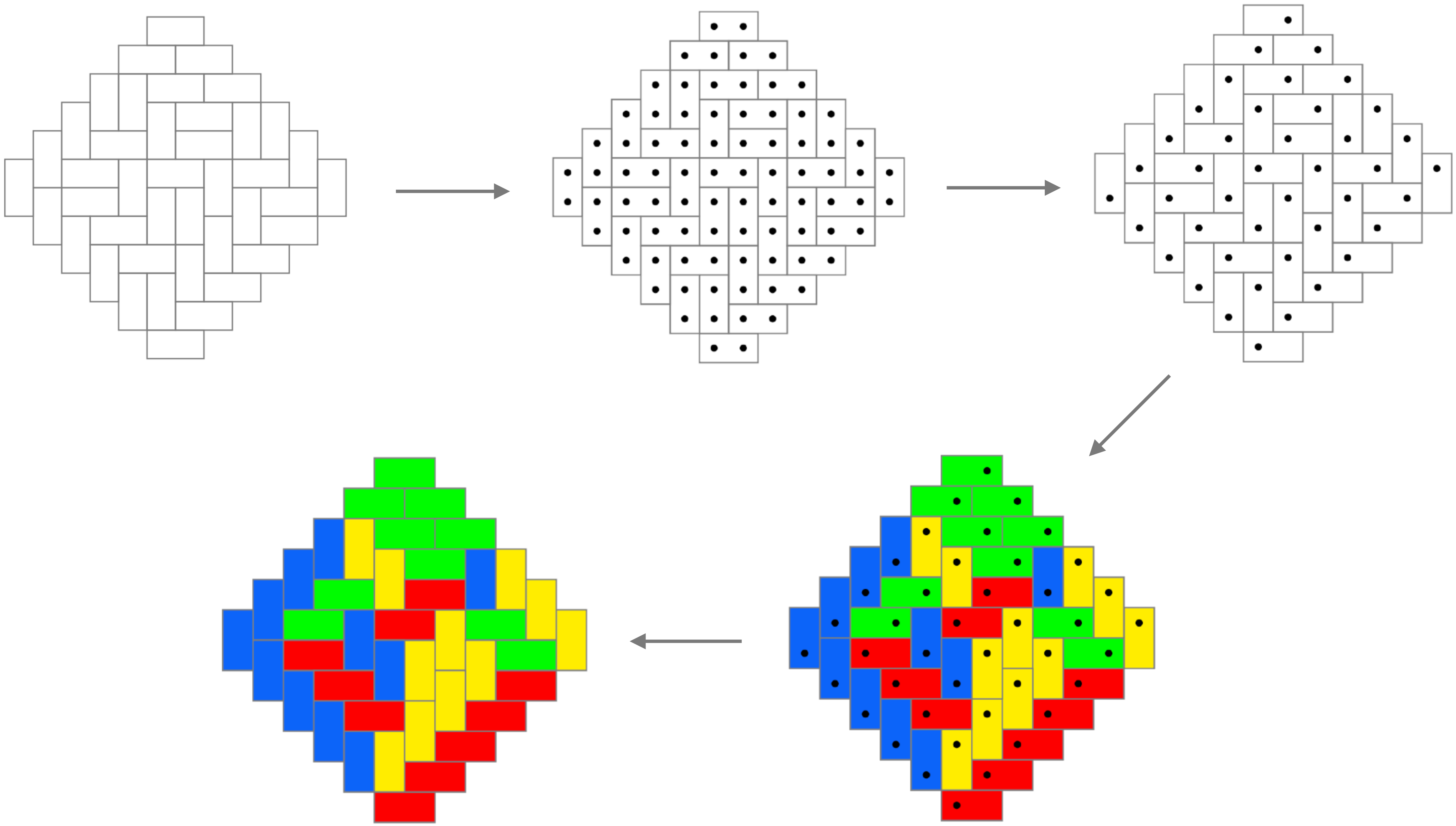


- # coverings $= 2^{21} = 2\,097\,152$, and $2^{n(n+1)/2}$ for general n

(ElkKupLarPro '92)

- they form a set on which the flip  $\longleftrightarrow$  is transitive

Coloring the tilings

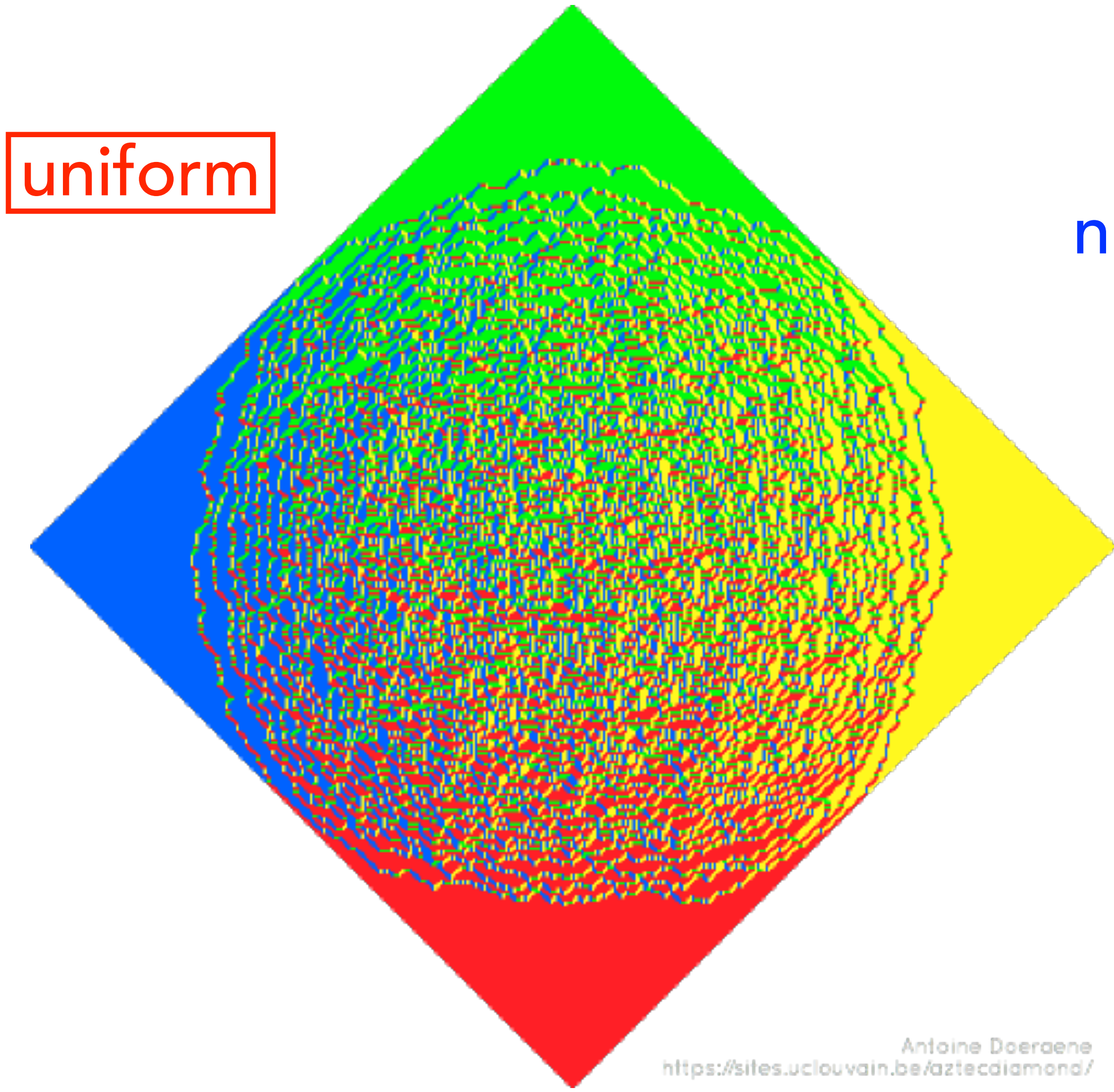


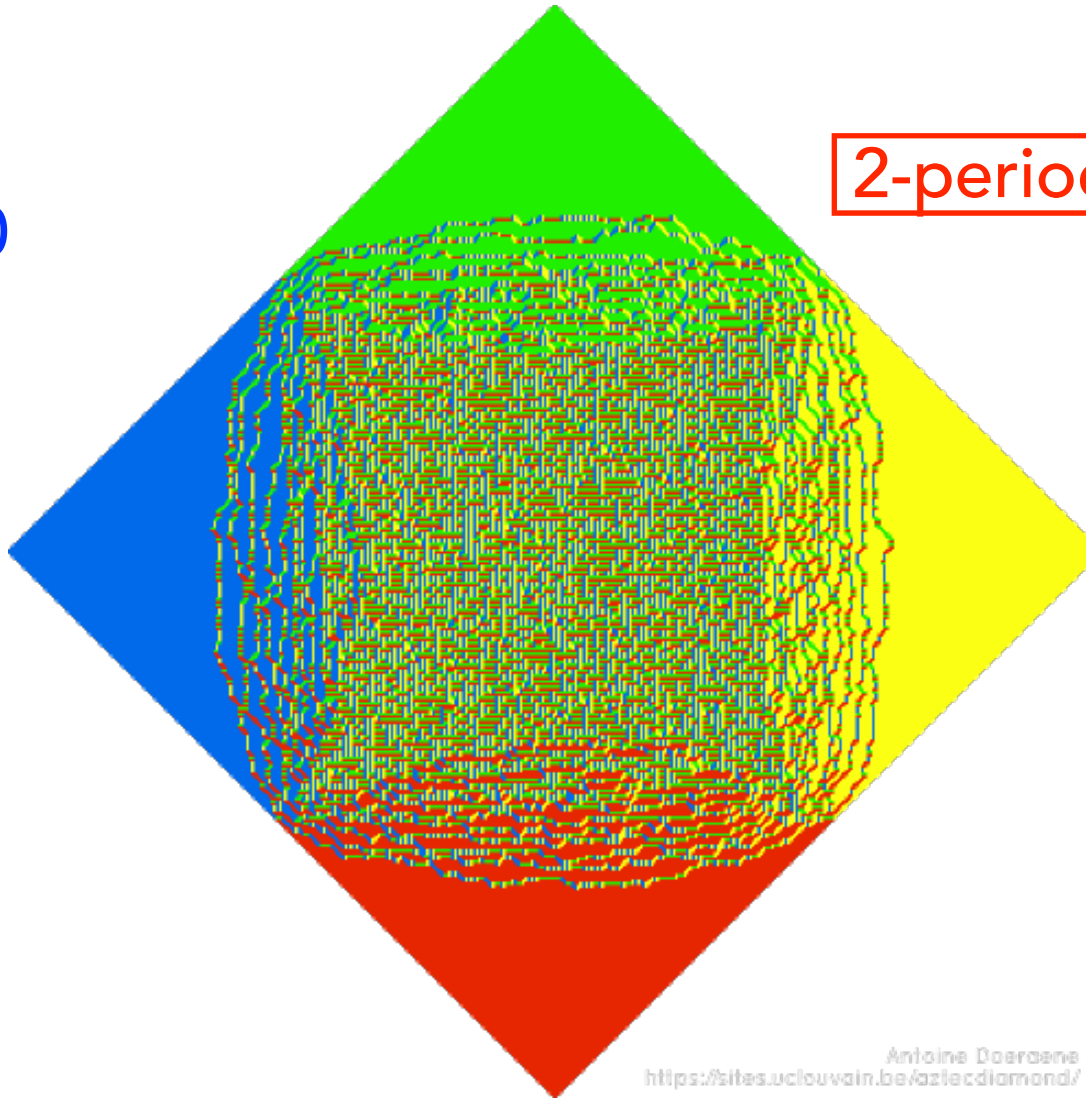
Typical behaviour at large size depends on prob. measure

uniform

n=200

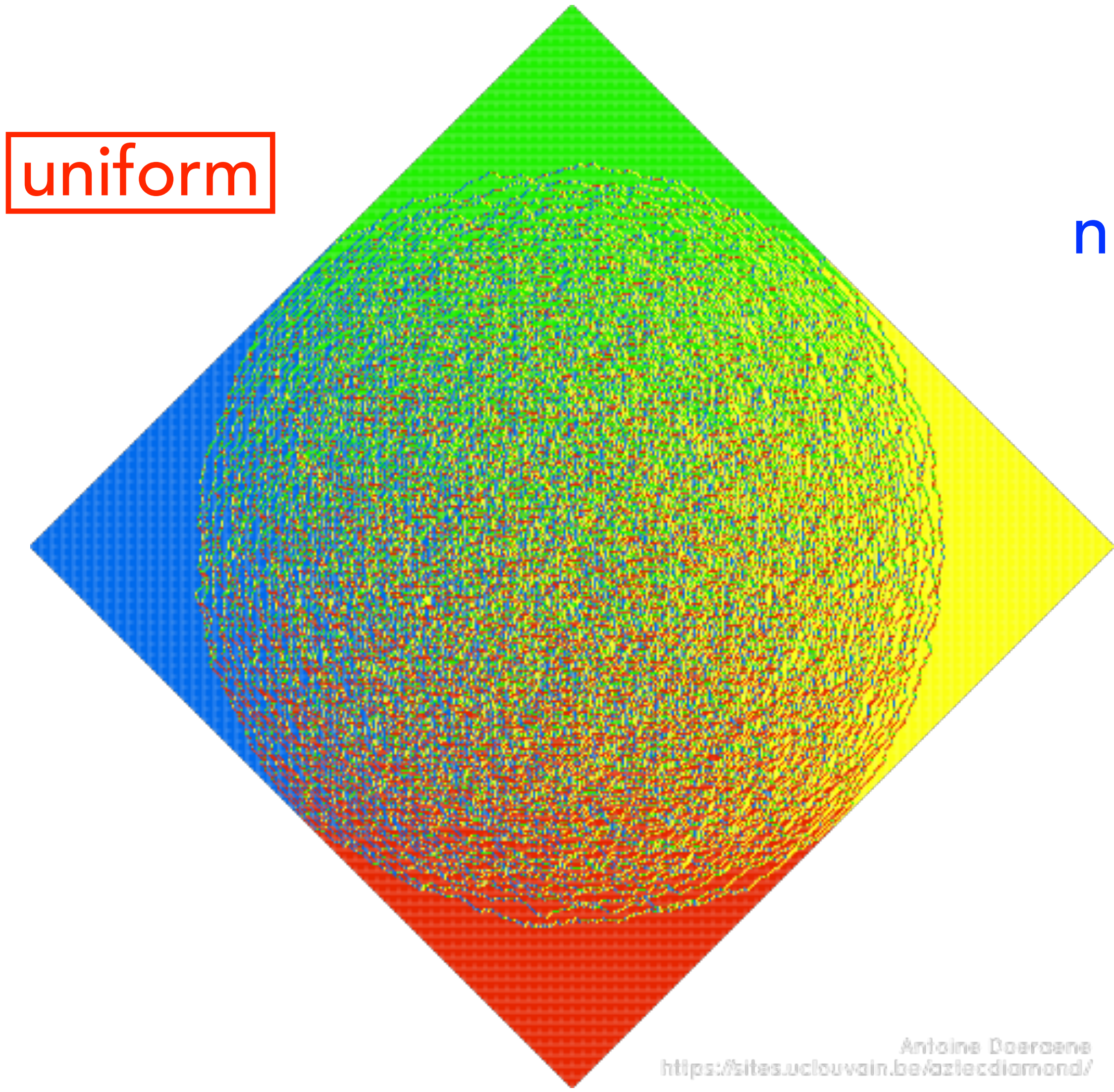
2-periodic





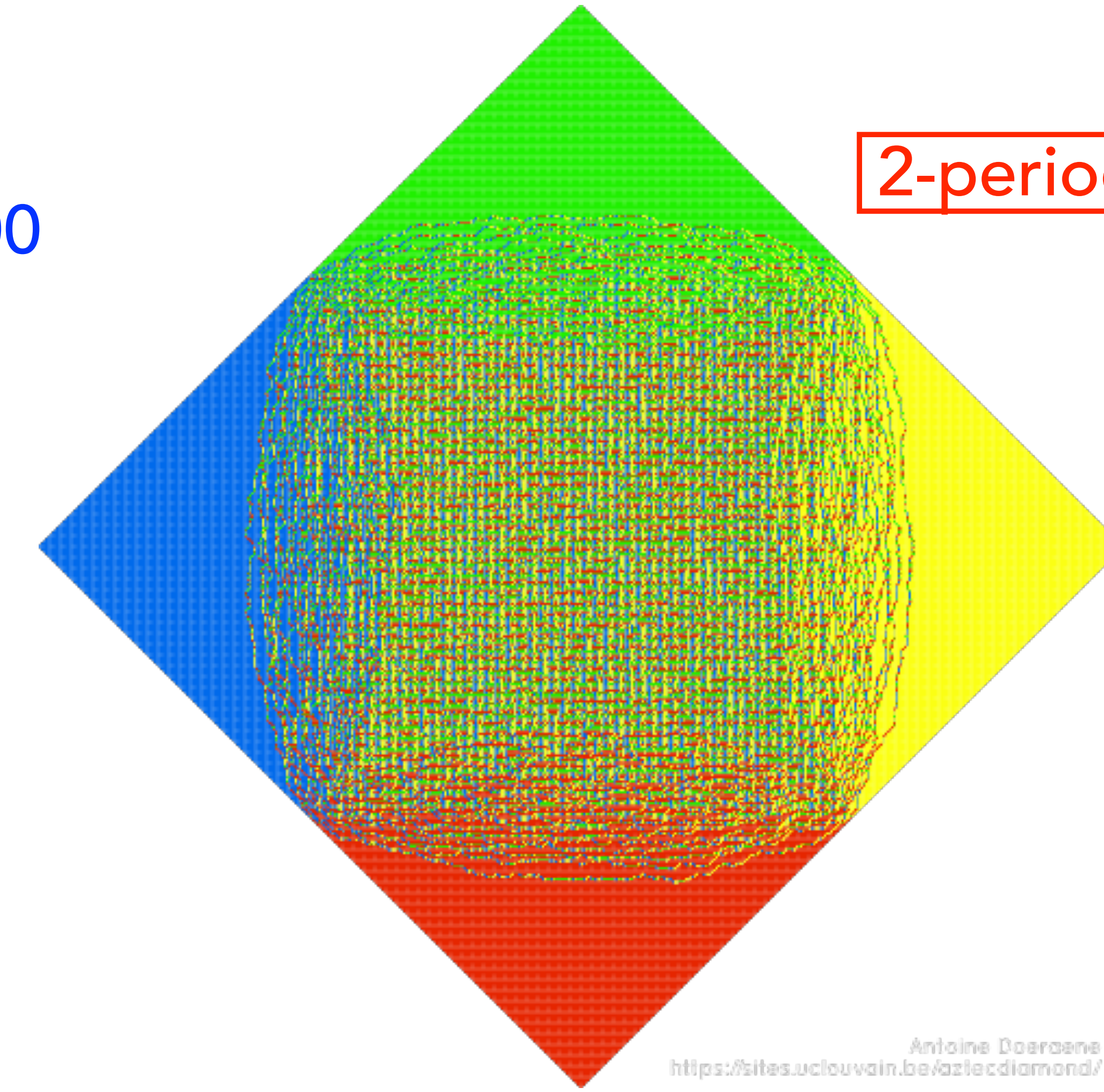
Limit shape at large size <u>depends on distribution</u>

uniform



n=400

2-periodic



- corners are frozen: only in N, in W, in S and in E
- central region contains randomness, and non-homogeneous
- interface(s) between distinct regions converge to non-random (arctic) curve(s)

How does one actually compute arctic curves ??

- Basic/naive idea: compute probability of relevant observable.

  In frozen regions, this probability saturates to 0 or 1, in the scaling limit. (In AzD, consider proba of )
  💪💪💪 The hard way, but yields more information.

- Alternative proposal by Colomo & Sportiello (2016).
  Use …

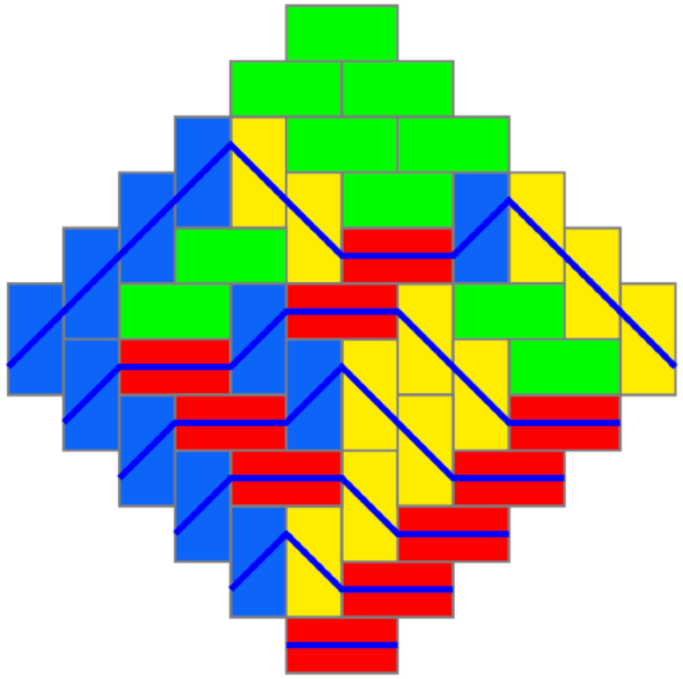
… the TANGENT METHOD

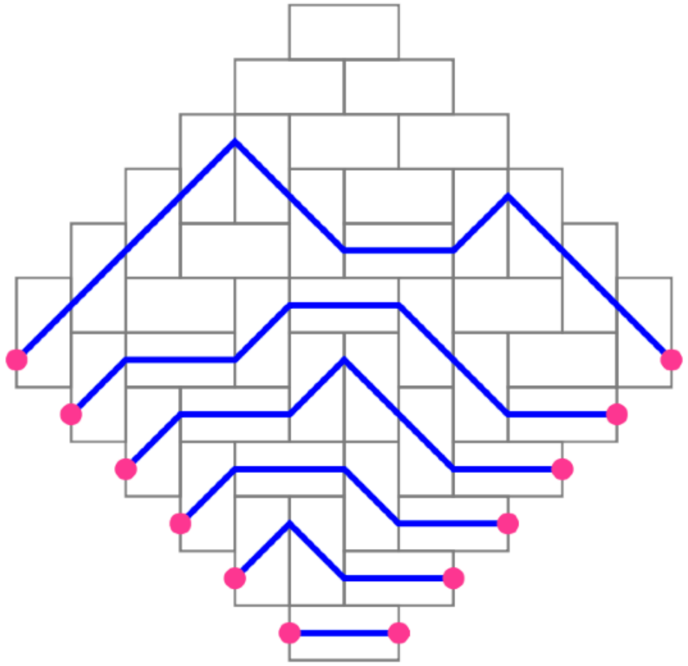
- Based on formulation in terms of non-intersecting lattice paths 😀
- Heuristic approach but successfully checked in many cases 😉
- Usually much easier than naive idea above 😃
- Relies on a reasonable but strong assumption, not under control 😕

Non-intersecting lattice path bijection

- depends on model considered, but available in all known cases ✓
- usually several ways to do that (really helps)
- paths are [non-intersecting random directed walks, with specific set S of elementary steps, starting and ending points are fixed.]

For AzD, draw elementary steps on 3 tiles :





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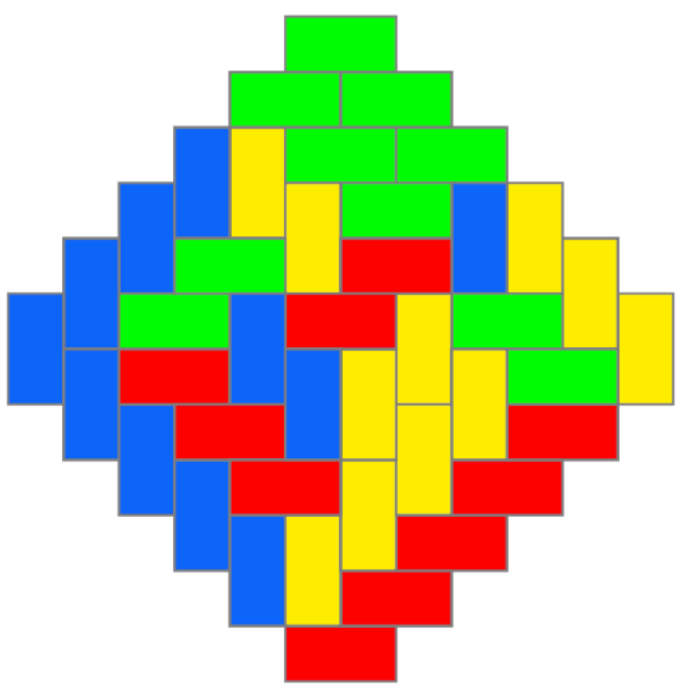
order n



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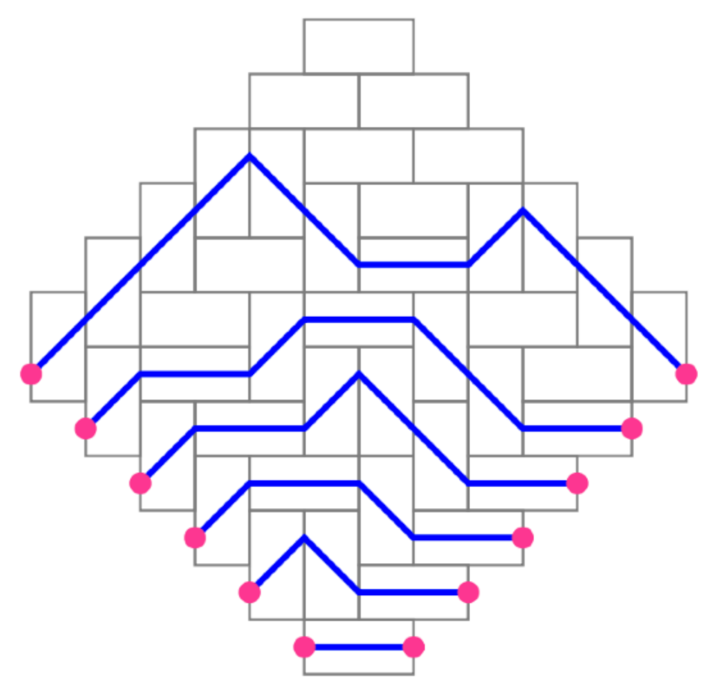

n paths

Non-intersecting lattice path bijection

The tilings of AD_n are in bijection with the set of n-uples of non-intersecting (Schröder) paths, starting and ending at fixed positions.
The lattice paths are made of elementary steps (1,1), (1,-1) and (2,0).

For AzD, draw elementary steps on 3 tiles :

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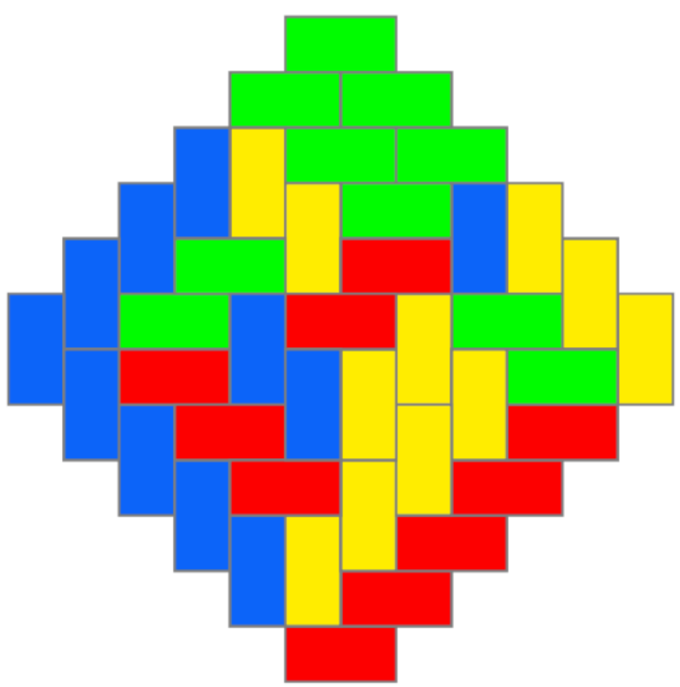

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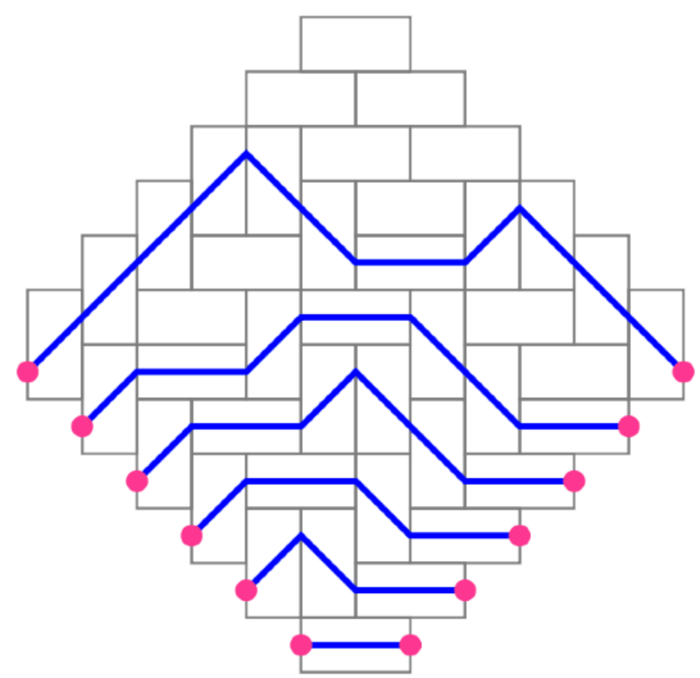
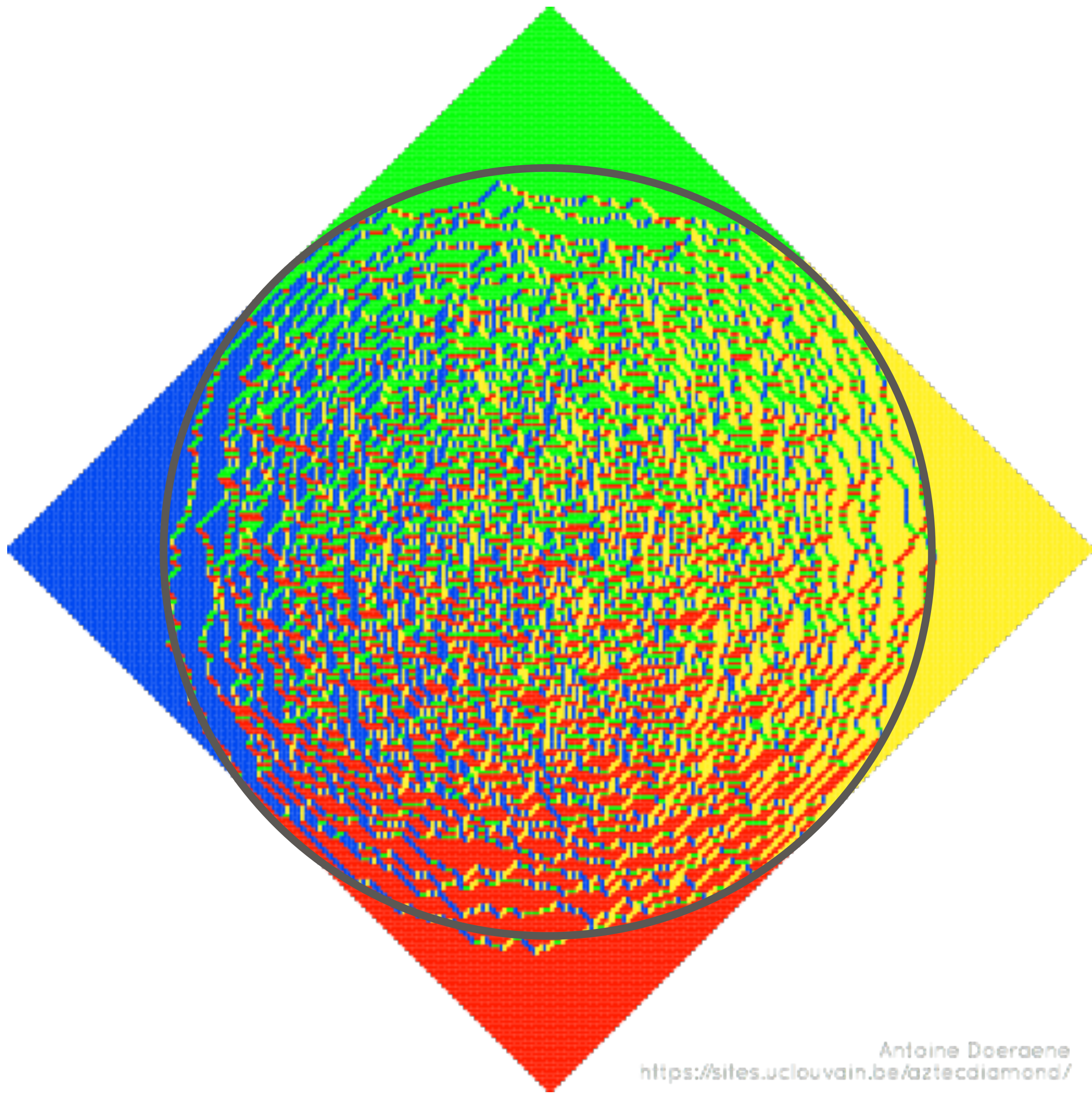
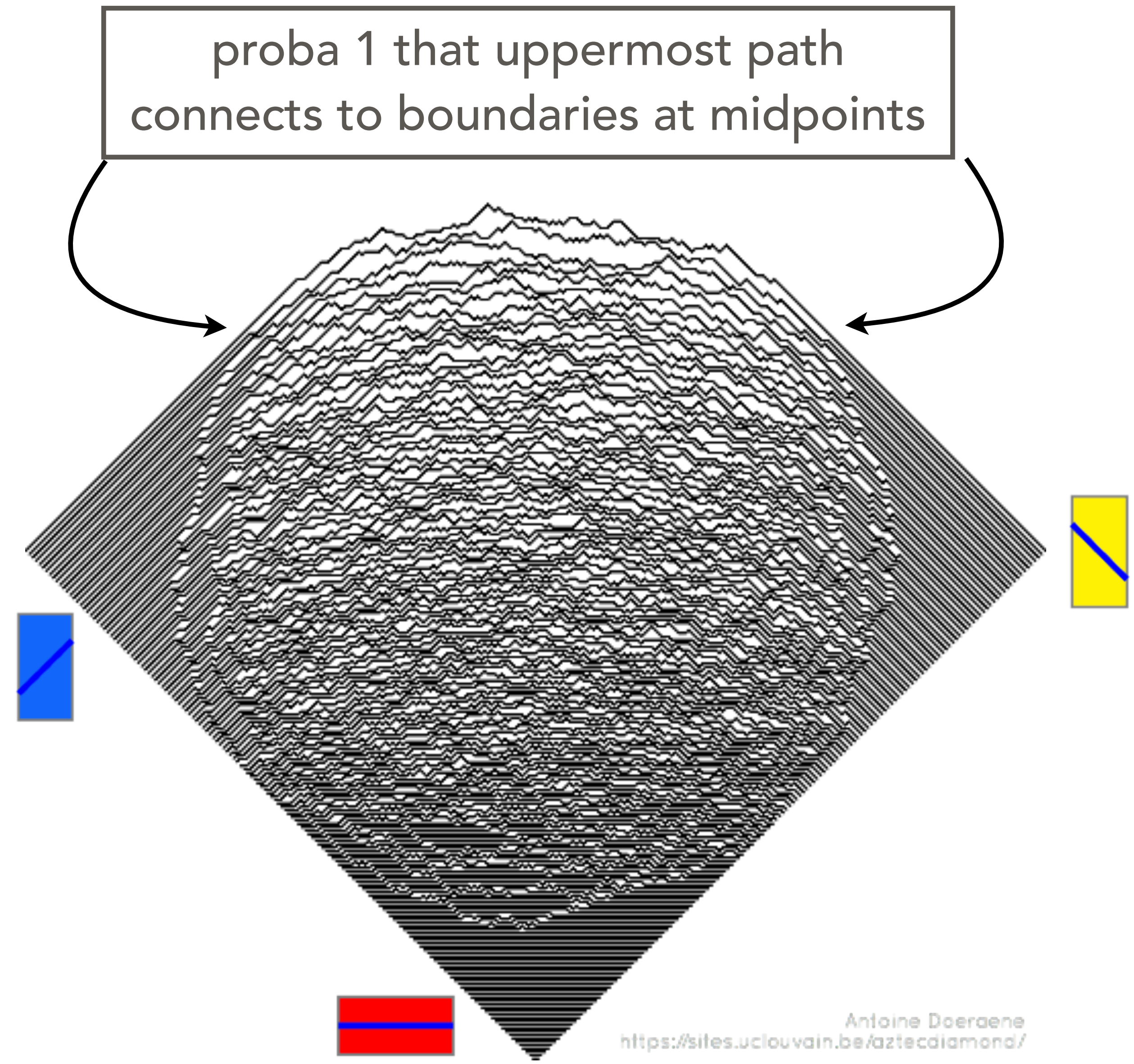


n paths

Arctic interface in terms of paths



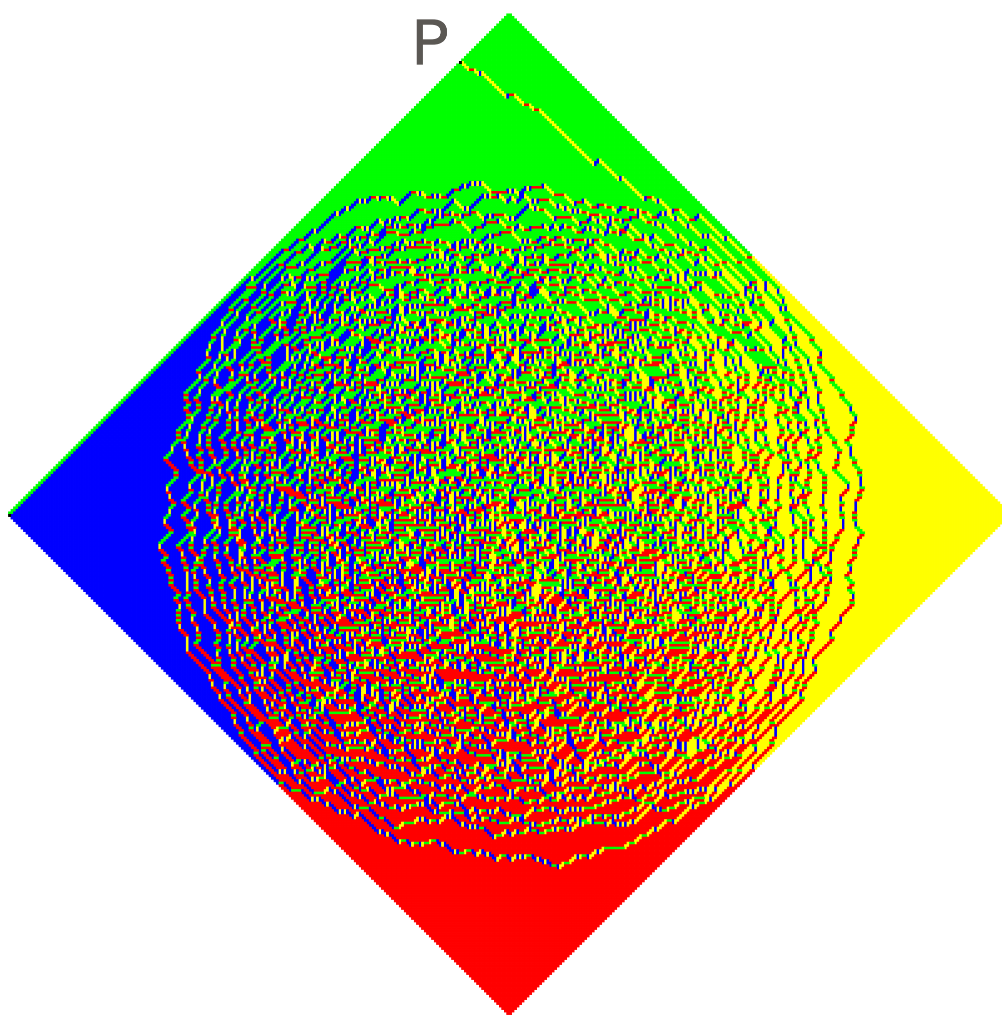


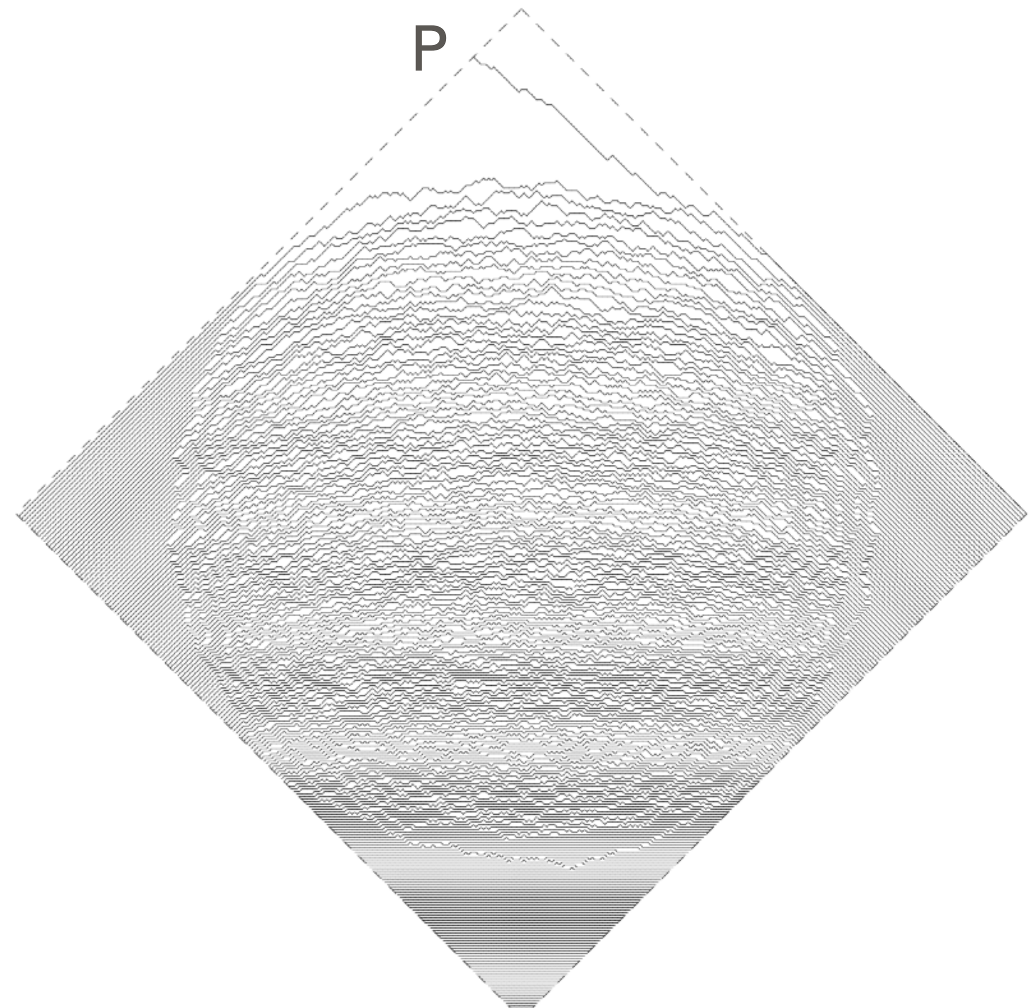
Regions where tiling is frozen ⟷ Regions maximally packed or void of paths

In scaling limit, the uppermost paths accumulate to form a sharp and deterministic interface between a void region and a region densely filled with paths.

Tangent method

Main idea: what if one conditions the uppermost path to exit boundary at P ??





Observations:

(1) the shape of the interface is preserved;

(2) the new uppermost path starts as a straight line from forced initial point, then approaches tangentially and merges in the compact cluster of paths forming the interface.

Tangent method

It builds on the two previous observations, become working hypotheses in the scaling limit:

H1	Perturbing the upper path has little influence on the other paths so that the arctic curve is not affected.
H2	The new upper path is a straight line until it touches the arctic curve, tangentially; from then on, it follows the arctic curve itself. ✓

Remarks:

- H1 is strong though reasonable; could turn out to be wrong if strong interactions between paths.
- H2 not as strong; looks surprising but somehow expected, for both linear part and tangency.

  Proved to hold in fairly general context ⟶ Debin, Granet & PhR (2019)

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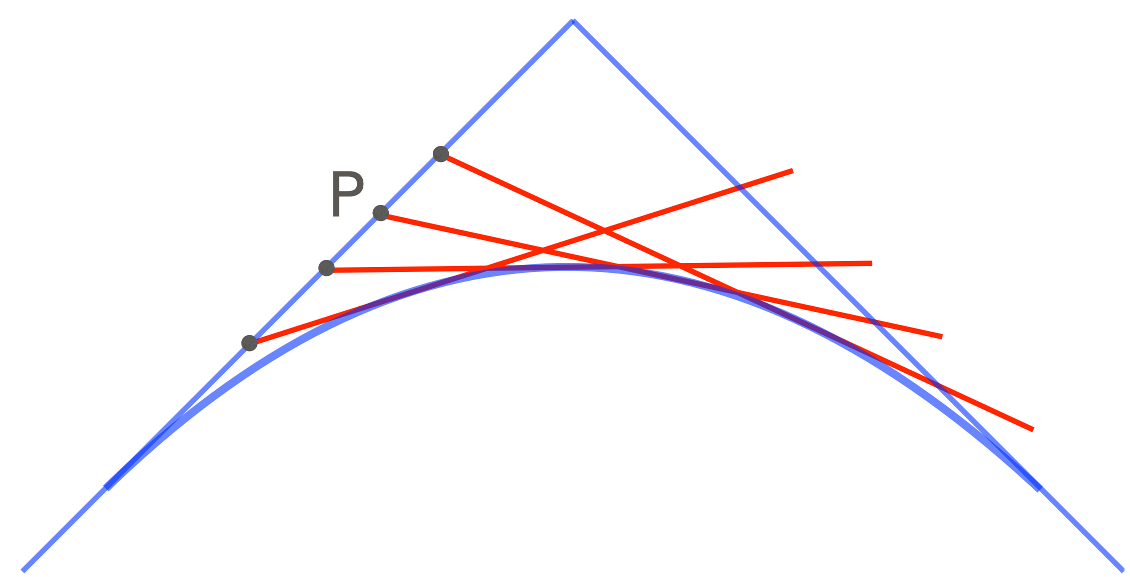
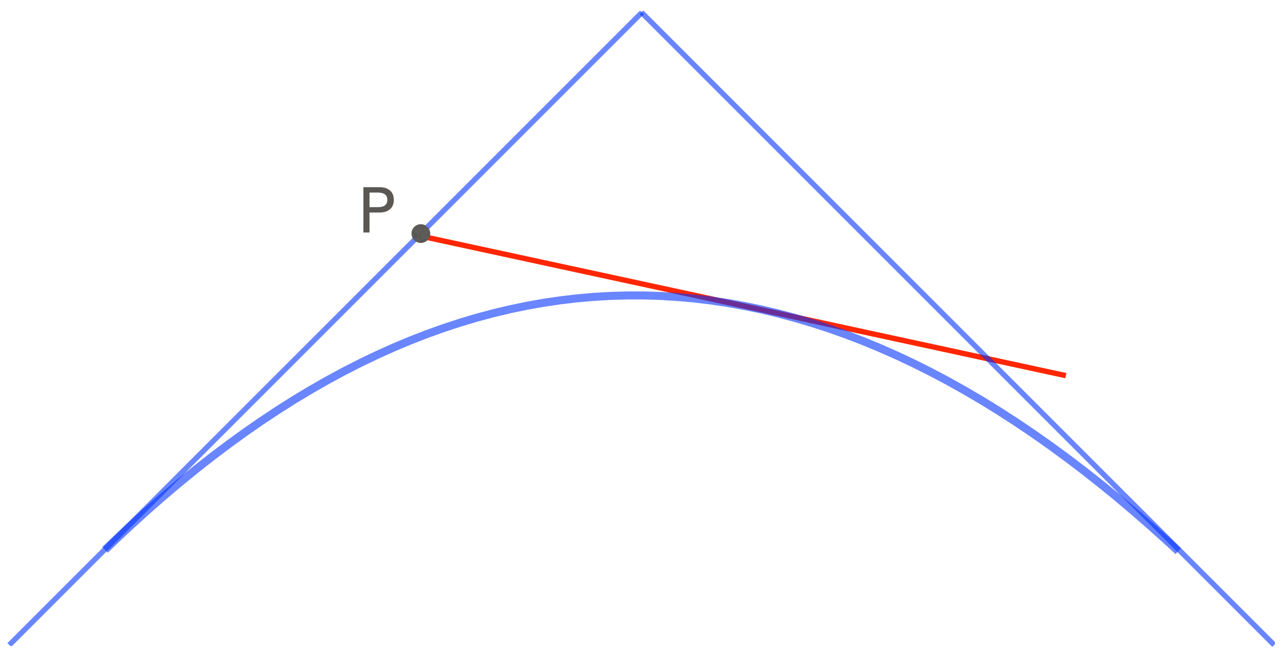
BUT :

Together H1 and H2 allow for an efficient way to compute arctic curves !



Tangent method at work

- Fix point P, at which upper path leaves left boundary
- ⇨ **Compute corresponding refined partition function**
- ⇨ **Extract the slope of the straight part in the scaling limit** (rescale all distances by n, and take limit)



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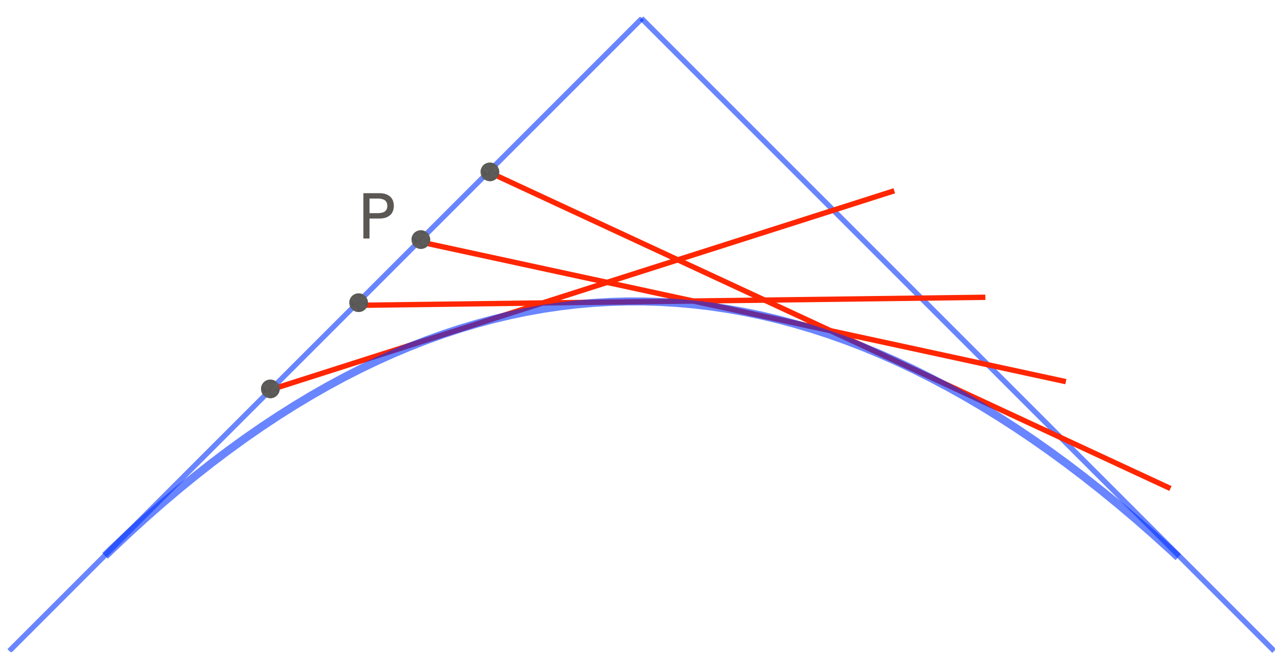
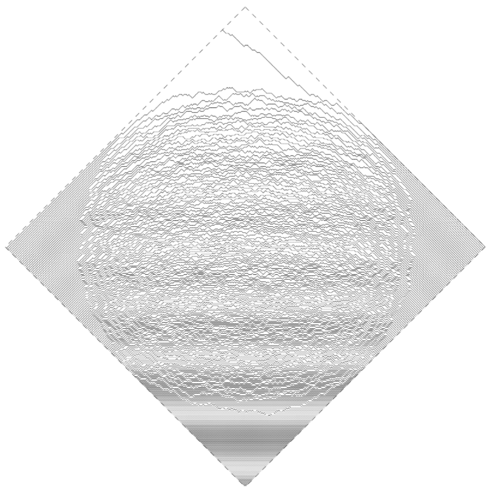
- Fix point P, at which upper path leaves left boundary
- ⇨ **Compute corresponding refined partition function**
- ⇨ **Extract the slope of the straight part in the scaling limit** (rescale all distances by n, and take limit)
- Vary starting point P over left boundary to obtain family of lines, all tangent to the arctic curve
- Retrieve the arctic curve as the enveloppe of the 1-parameter family of tangent lines



Problem 1: how to compute the refined partition function ?

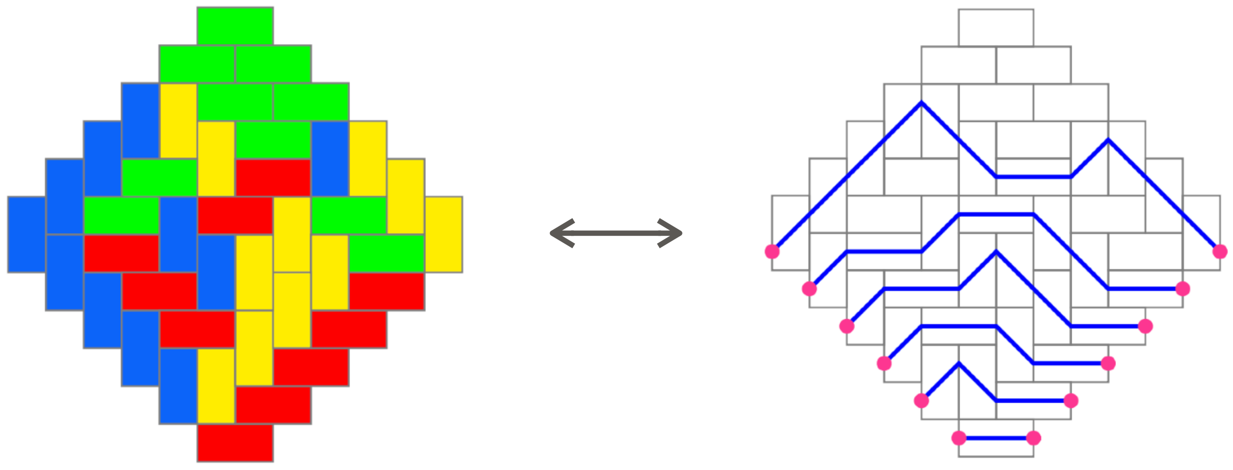
Problem 2: how to extract a slope from the partition function ?

1 - Refined partition function



To be computed : partition function for those AzD tilings s.t. the upper path leaves the left boundary at prescribed position.

From tiling-paths correspondence, equivalent to condition on fixed number of vertical dominos on WN boundary:



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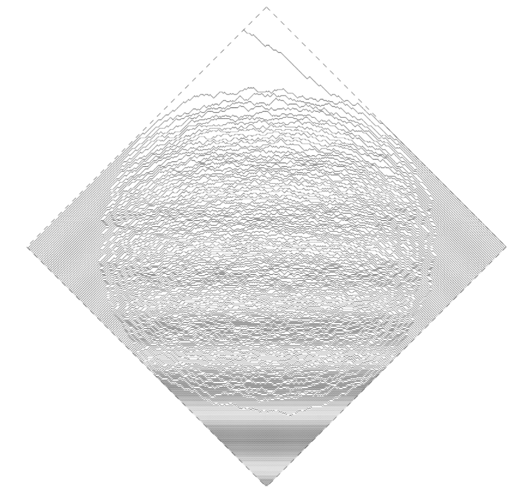

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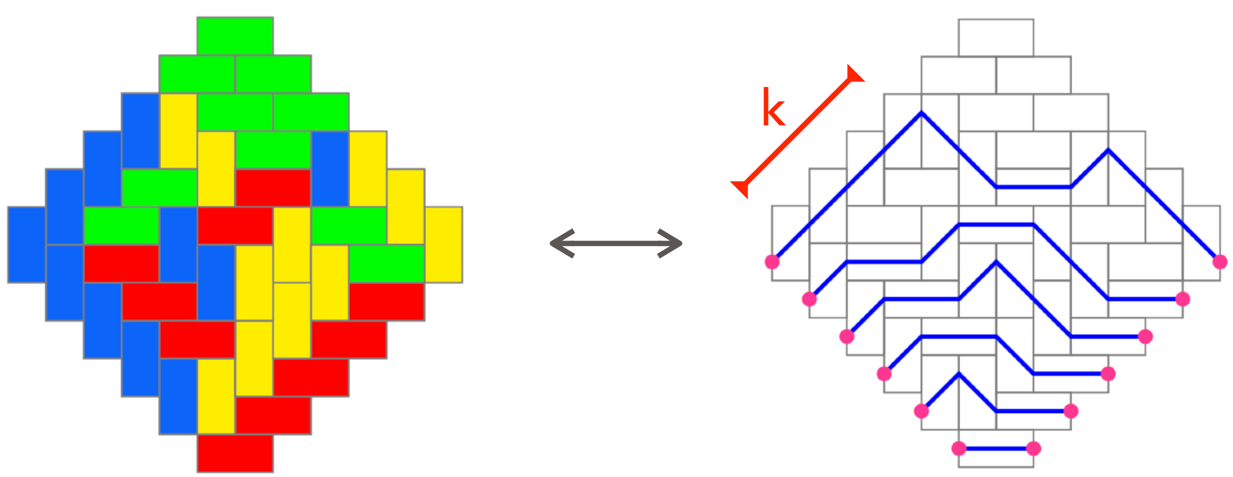


Let k be the number of vertical dominos; then (n-k) horizontal. (On figure, n=6, k=4)

Note: for tangent method, need $k \geq \frac{n}{2}$

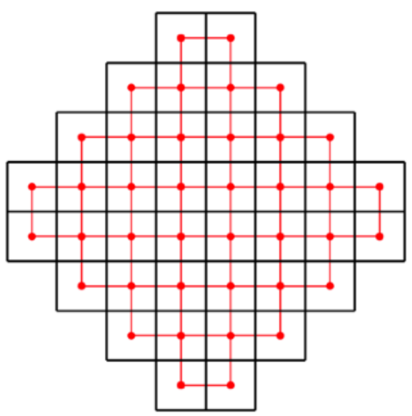
Let $Z_{n,k}$ be the refined partition function.

For standard AzD, not difficult: $Z_{n,k} = 2^{n(n-1)/2} \binom{n}{k}$.

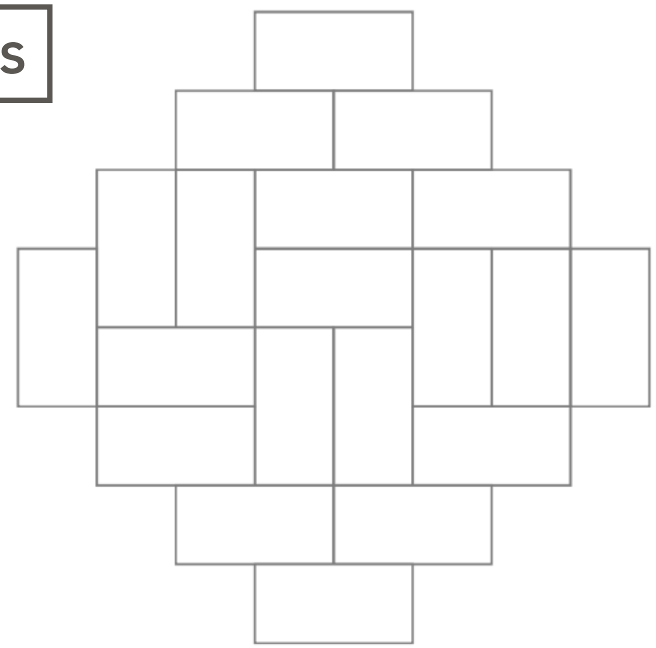
For 2-periodic AzD ???

2-periodic measure

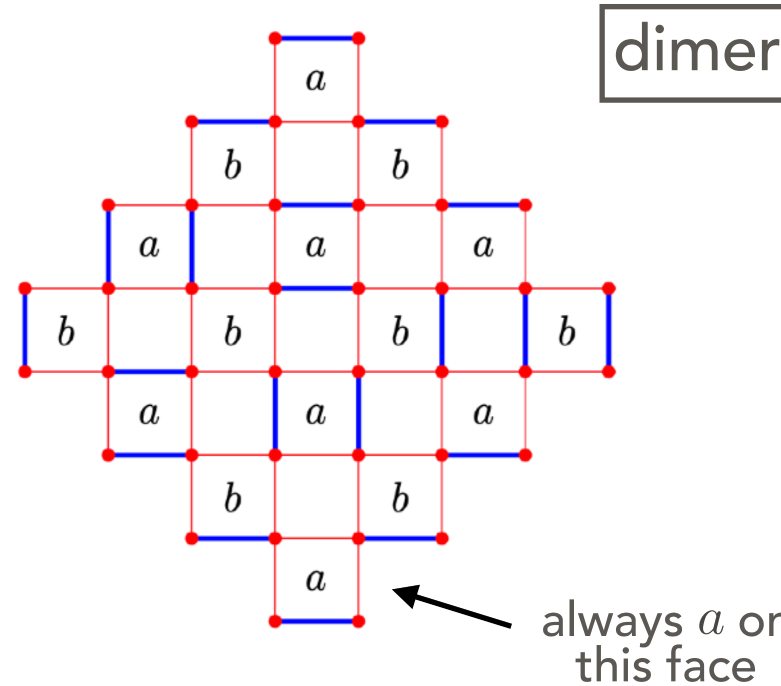
Is a <u>non-uniform measure</u> on the AzD tilings.
Better described in terms of perfect matchings of dual Aztec graph.



dominos



dimers



<u>2-periodic weighting</u>:

- a dimer is weighted by the weight of the face it is adjacent to
- weight of a perfect matching and corresponding tiling is product of dimer weights (above matching has weight $a^{12}\, b^{8}$)

1 - Standard partition function

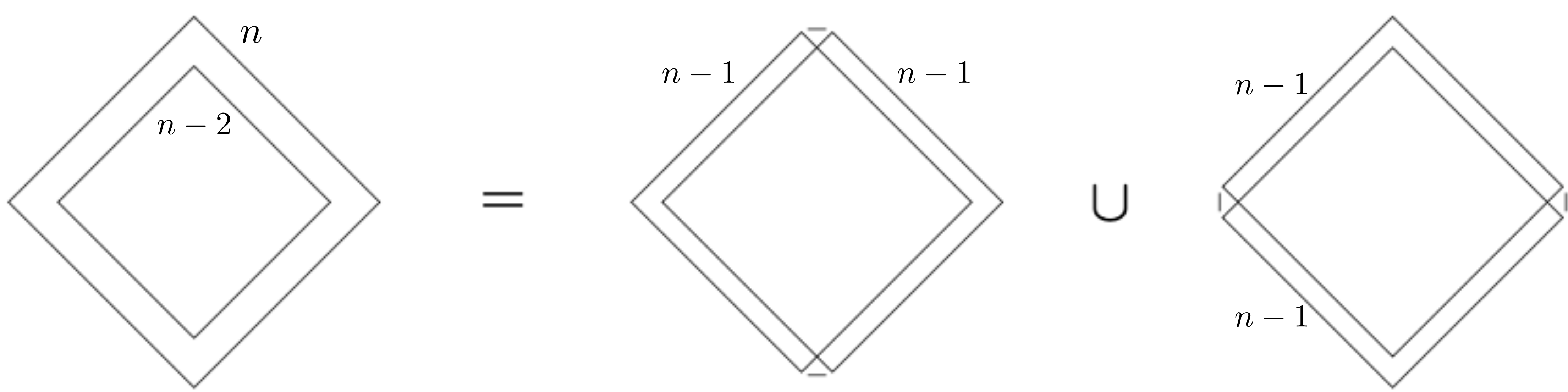
Follow Kuo, 2004 …

If Z_n denotes the partition function for <u>standard</u> AzD (their number), Kuo proved

$$\boxed{Z_n \, Z_{n-2} = Z_{n-1}^2 + Z_{n-1}^2}$$

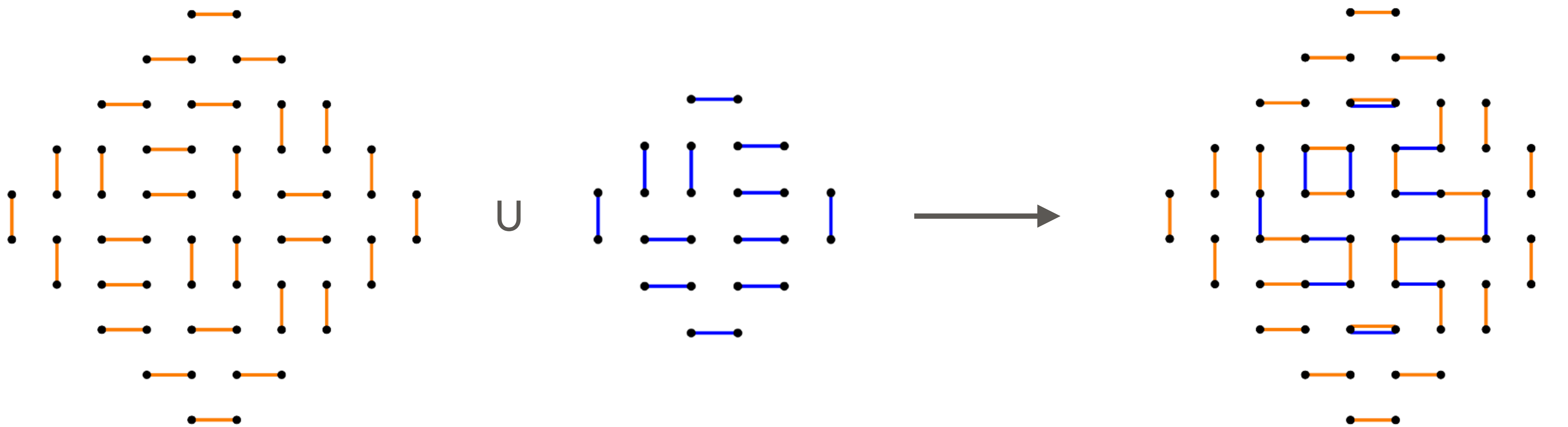
It readily implies $Z_n = 2^{n(n+1)/2}$.

The quadratic recurrence follows from the following graphical equation



1 - Standard partition function

Beautiful argument …



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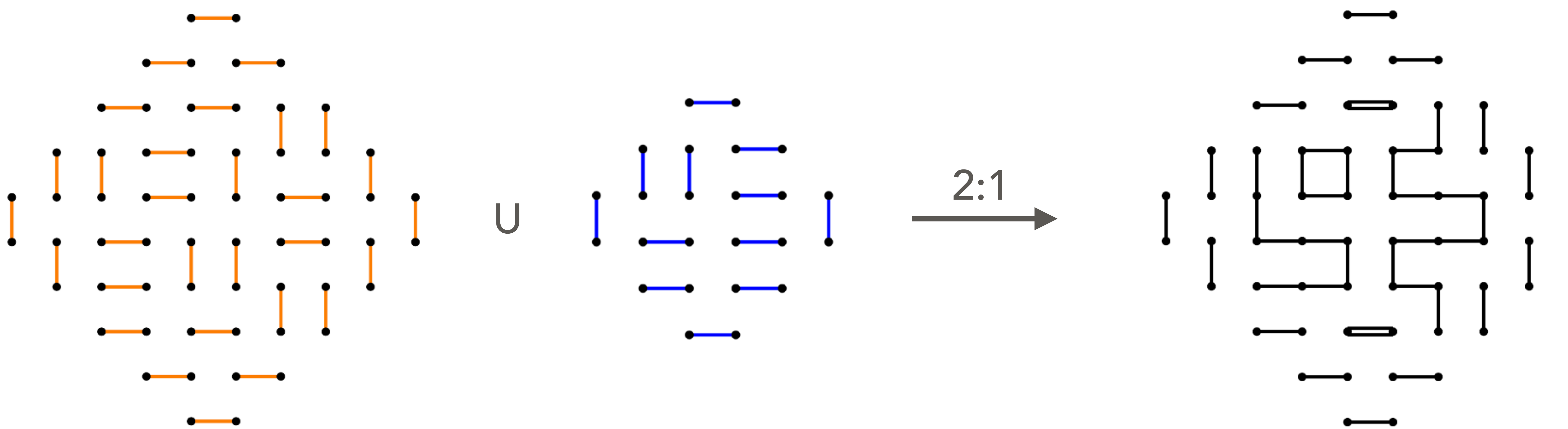

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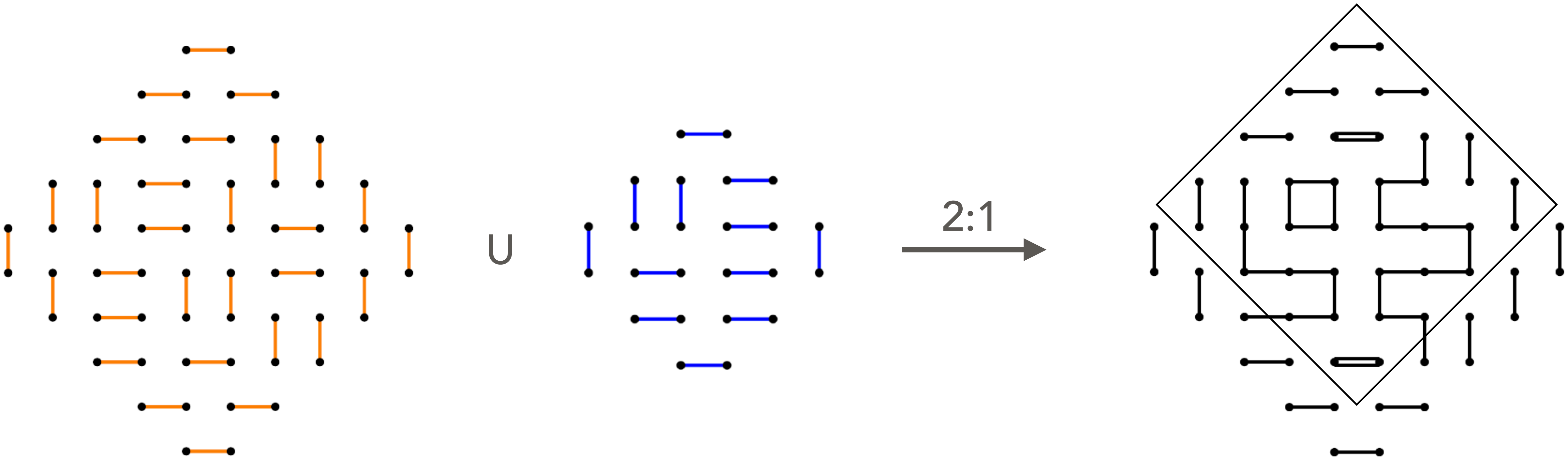

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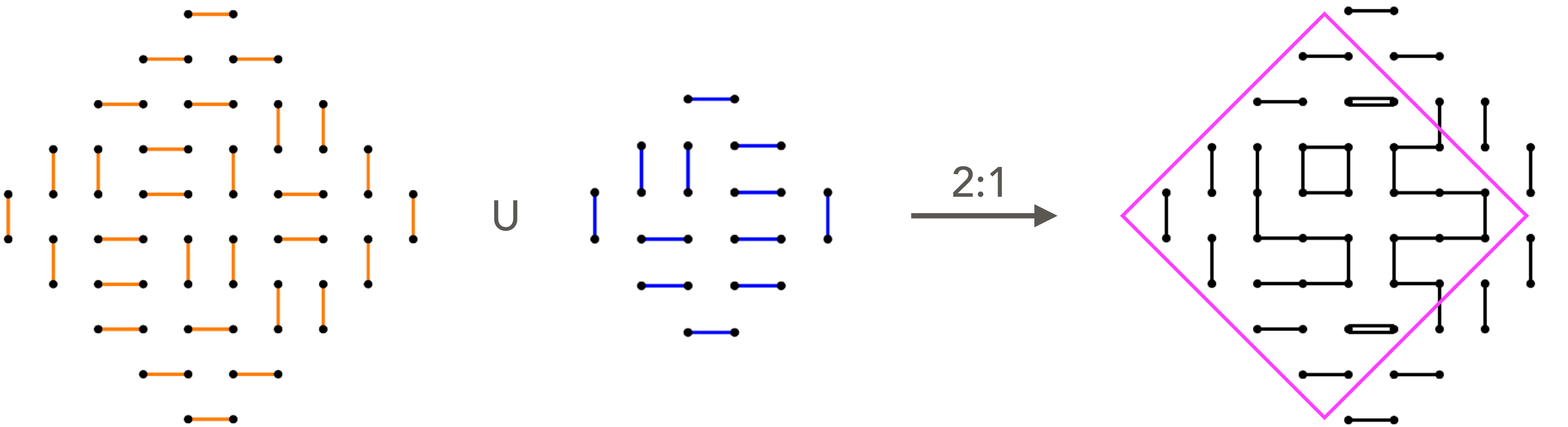

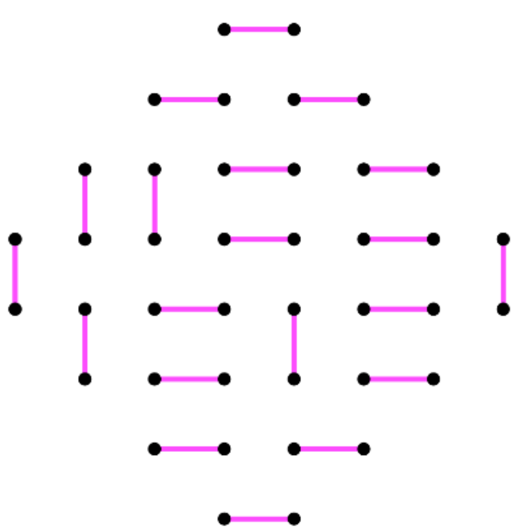

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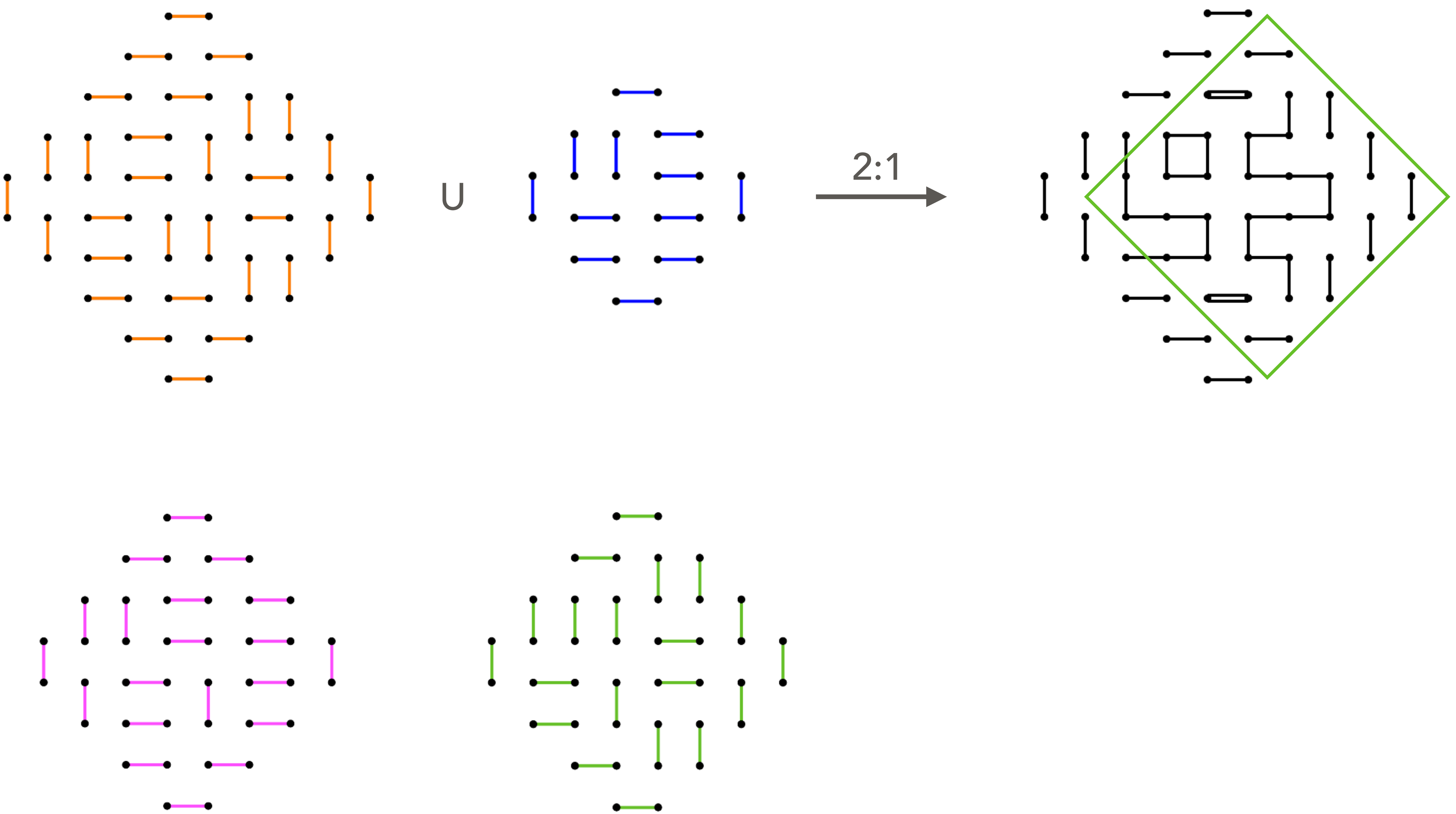

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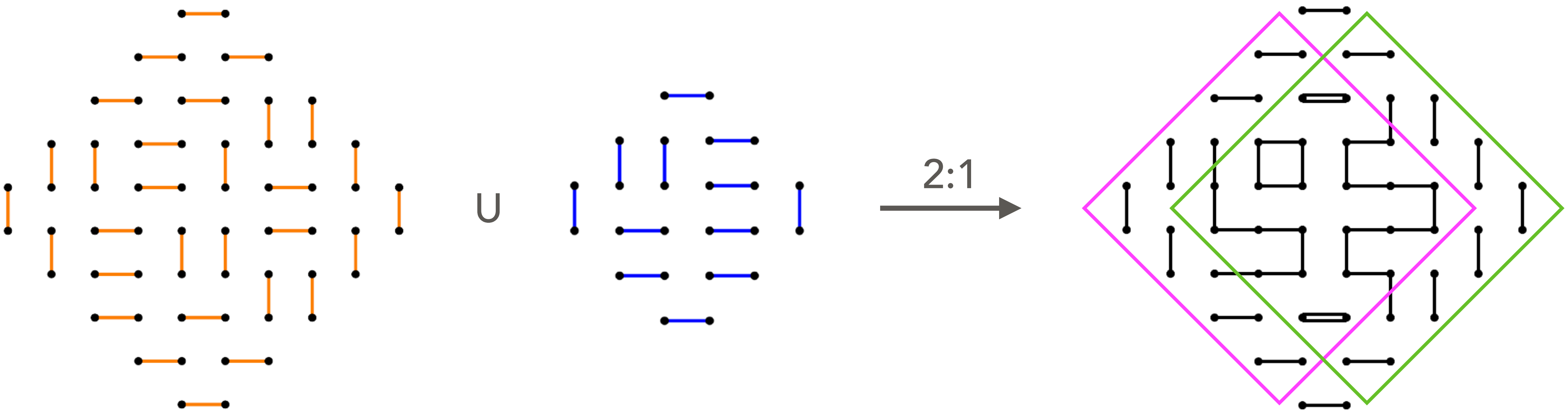
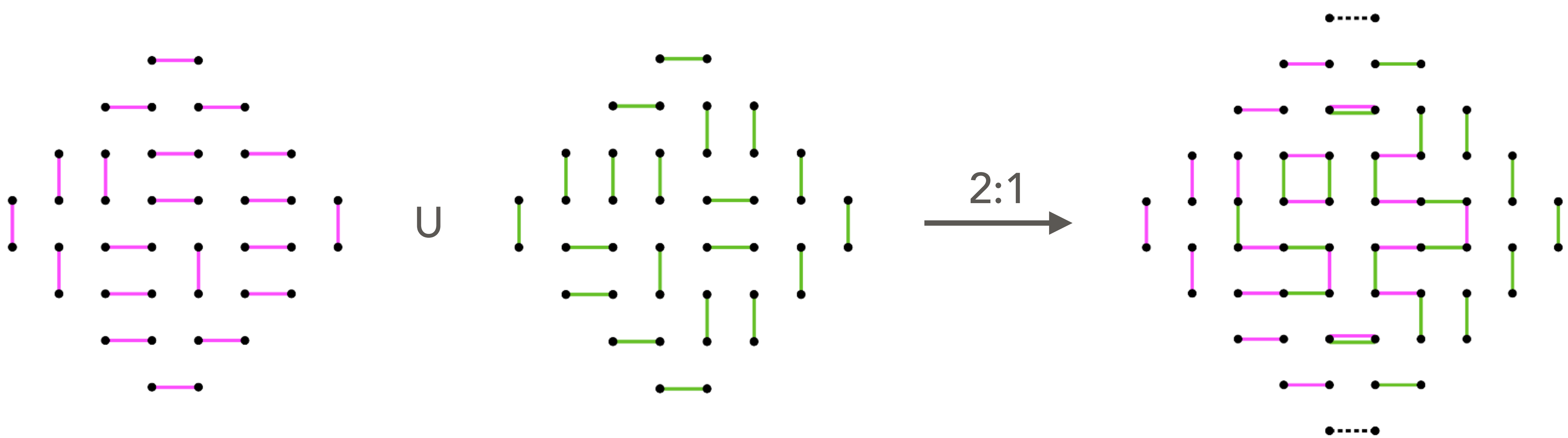




1 - Two-periodic partition function

Other beautiful observation, by Speyer (2007):

Kuo's recurrence extends to much more general weightings,
leads to so-called ***octahedron recurrence !***

For 2-periodic AzD, it implies for 2-periodic partition function $Z_n(a,b)$

$$Z_n(a,b)\, Z_{n-2}(a,b) = \begin{Bmatrix} b^2 \\ a^2 \end{Bmatrix} Z_{n-1}^2(a,b) + a^2 Z_{n-1}^2(b,a) \qquad \text{for } n \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$$

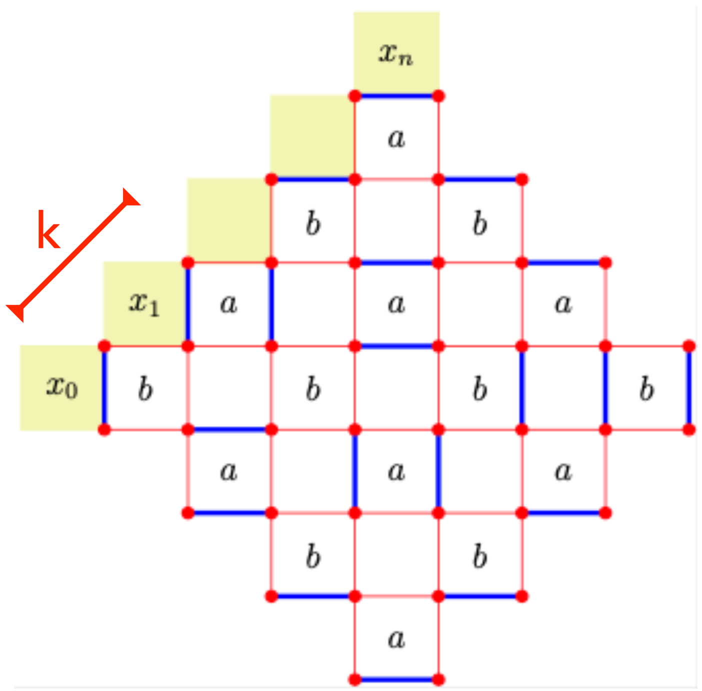
Yields a neat formula:

$$\boxed{Z_n(a,b) = (2ab)^{\lfloor \frac{(n+1)^2}{4} \rfloor} (a^2+b^2)^{\lfloor \frac{n^2}{4} \rfloor} \times \begin{cases} 1 & \text{if } n \neq 1 \bmod 4 \\ \frac{a}{b} & \text{if } n = 1 \bmod 4 \end{cases}}$$

(Di Francesco & Soto-Garrido, 2014)

1 - Refined partition function

The recurrence generalises upon inclusion of boundary face weights



If k vertical dimers along WN boundary, the yellow faces bring additional factor

$$x_0\, x_1 \ldots \hat{x}_k \ldots x_n$$

Thus

$$Z_{n,k}(a,b) = \frac{Z_n(a,b;x_0,\ldots,x_n)}{x_0 x_1 \ldots x_n}\Big|_{x_k^{-1}}$$

The generalized partition function satisfies

$$Z_n(a,b|x_0 \ldots x_n)\, Z_{n-2}(a,b) = x_0 \begin{Bmatrix} b^2 \\ a^2 \end{Bmatrix} Z_{n-1}(a,b|x_1 \ldots x_n)\, Z_{n-1}(a,b)$$

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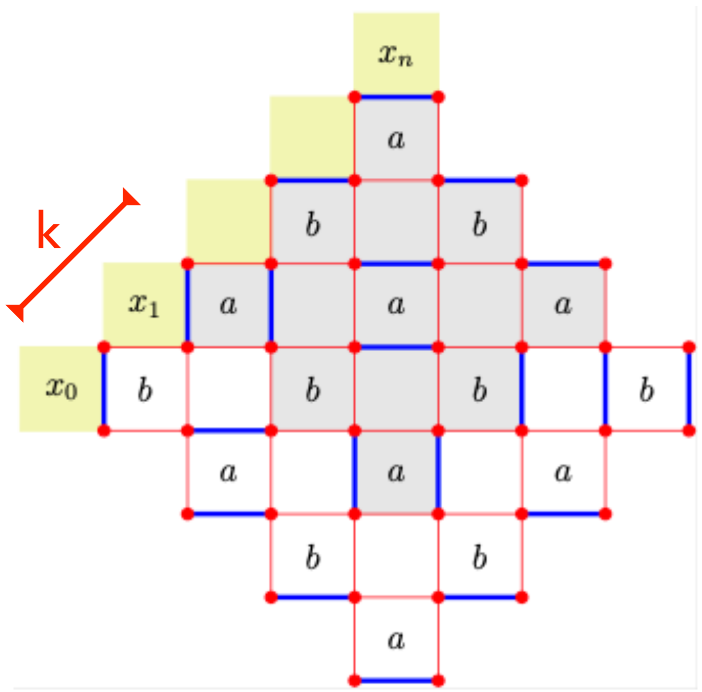

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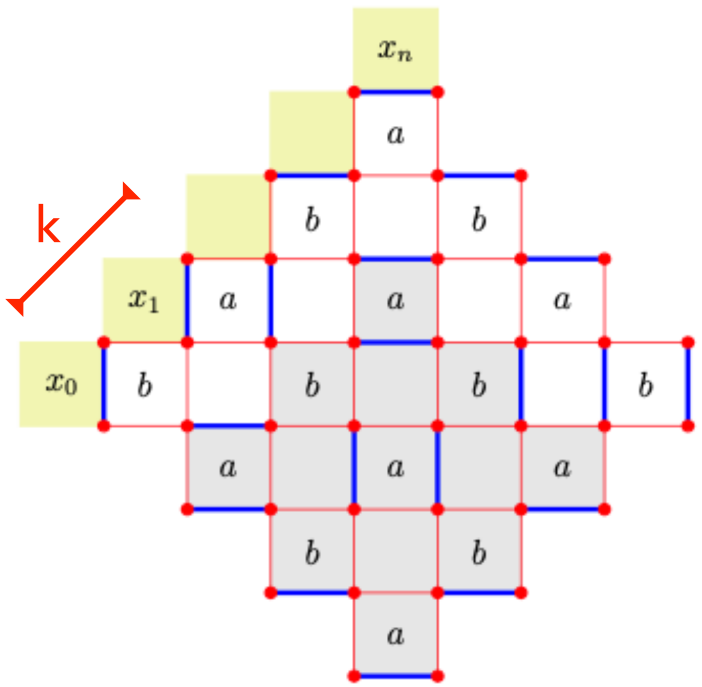

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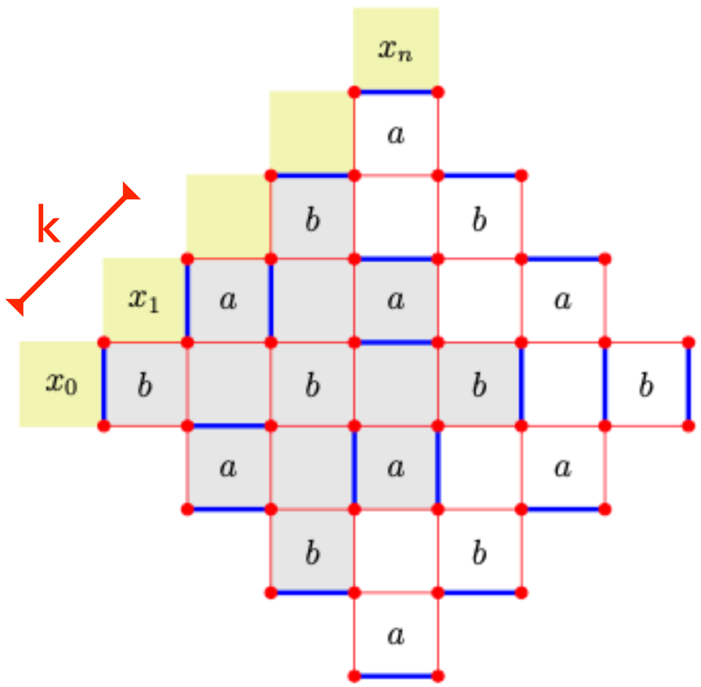

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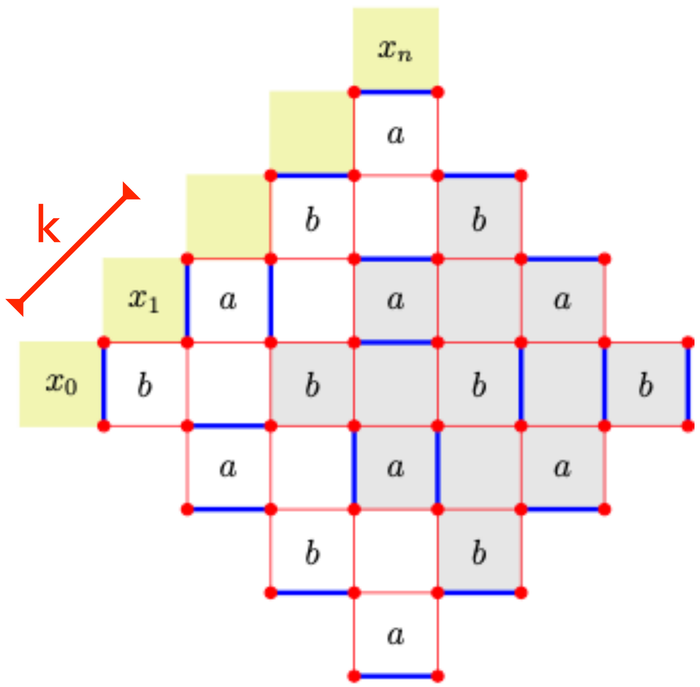


If k vertical dimers along WN boundary, the yellow faces bring additional factor

$$x_0\, x_1 \ldots \hat{x}_k \ldots x_n$$

Thus

$$Z_{n,k}(a,b) = \left.\frac{Z_n(a,b;x_0,\ldots,x_n)}{x_0 x_1 \ldots x_n}\right|_{x_k^{-1}}$$

The generalized partition function satisfies

$$Z_n(a,b|x_0\ldots x_n)\,Z_{n-2}(a,b) = x_0 \begin{Bmatrix} b^2 \\ a^2 \end{Bmatrix} Z_{n-1}(a,b|x_1\ldots x_n)\,Z_{n-1}(a,b)$$

$$+\,x_n\,a^2\,Z_{n-1}(b,a|x_0\ldots x_{n-1})\,Z_{n-1}(b,a) \qquad \text{for } n \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$$

1 - Refined partition function

Selecting the proper terms yields

$$Z_{n,k}(a,b)\, Z_{n-2}(a,b) = \begin{Bmatrix} b^2 \\ a^2 \end{Bmatrix} Z_{n-1,k-1}(a,b)\, Z_{n-1}(a,b) + a^2 Z_{n-1,k}(b,a)\, Z_{n-1}(b,a)$$

which becomes a linear recurrence for $S_{n,k}(a,b) \equiv \dfrac{Z_{n,k}(a,b)}{Z_{n-1}(a,b)}$!

Compute generating function, and use standard methods to extract asymptotic value of $S_{n,k}(a,b)$ for large k.

For $k = rn,\ 0 \le r \le 1$

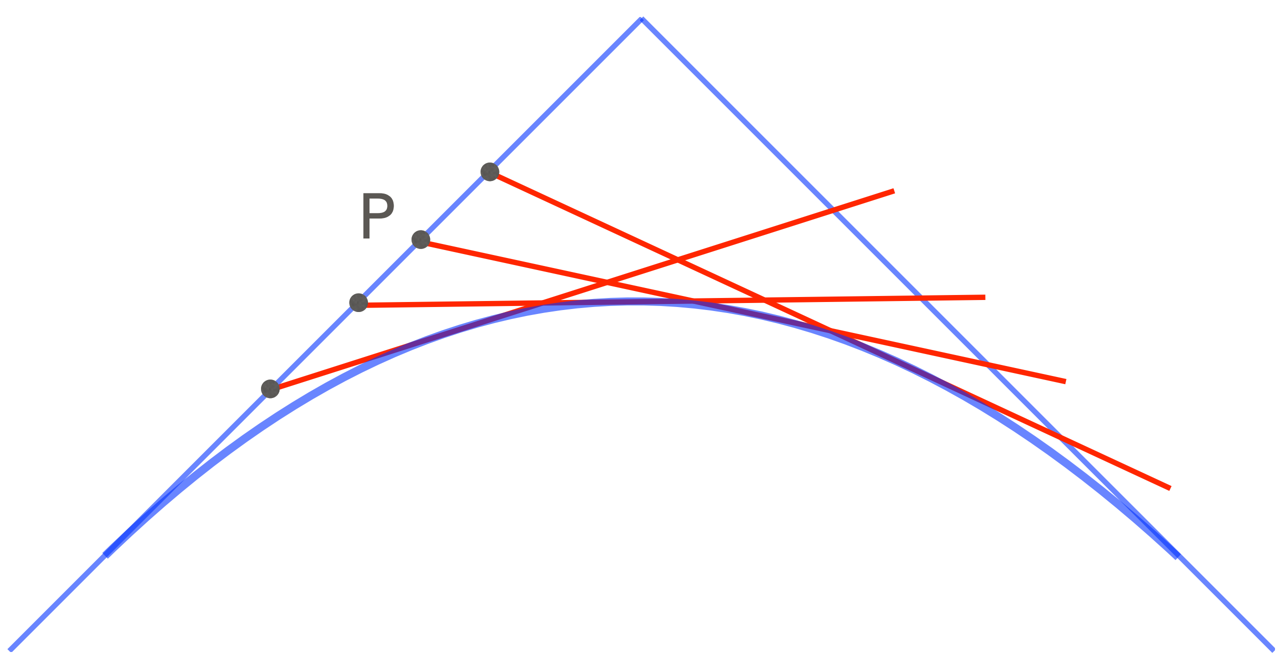
$$\lim_{n\to\infty} \frac{1}{n} \log \frac{Z_{n,rn}(a,b)}{Z_{n-1}(a,b)} = \log(ab) + F_1(r)$$

$(0 \le v \le +\infty)$

$$\frac{\mathrm{d}F_1(r)}{\mathrm{d}r} = -\log v(r) \quad \text{and} \quad r(v) = \frac{v}{1-v^2} \left\{ \frac{1+\beta}{2\sqrt{\beta}} \frac{1+v^2}{\sqrt{1+(\beta+\frac{1}{\beta})v^2+v^4}} - v \right\}$$

Tangent method at work

- Fix point P, at which upper path leaves left boundary
- ⇨ **Compute corresponding refined partition function**
- ⇨ **Extract the slope of the straight part in the scaling limit** (rescale all distances by n, and take limit)
- Vary starting point P over left boundary to obtain family of lines, all tangent to the arctic curve
- Retrieve the arctic curve as the enveloppe of the 1-parameter family of tangent lines

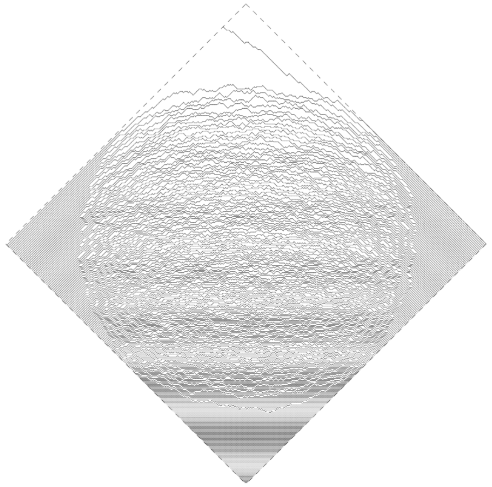


Problem 1: how to compute the refined partition function ? ✔

Problem 2: how to extract a slope from the partition function ?

2 - Computation of the slope

A priori, **not possible** to extract the slope from the partition function alone !
Need for further input …



Further input is provided by underlying assumptions of tangent method itself.

It assumes that

the uppermost path does not affect the way the other (n-1) paths behave in the scaling limit,

and suggests that

to dominant order, the refined partition function factorizes into a contribution of the bulk (n-1) paths and the contribution of the uppermost/isolated path:

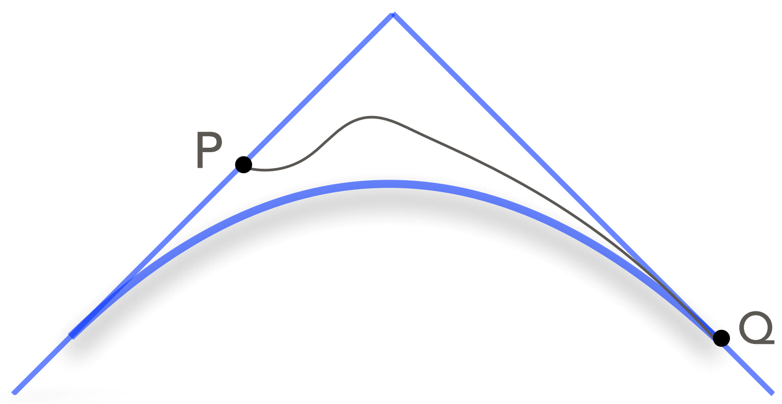
$$\boxed{Z_{n,k}(a,b) \sim Z_{n-1}(a,b)\, Z_1^{\text{up}}(a,b) \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{1}{n} \log \frac{Z_{n,rn}(a,b)}{Z_{n-1}(a,b)} = \lim_{n\to\infty} \frac{1}{n} \log Z_1^{\text{up}}}$$

(Debin & PhR, 2021)

2 - Computation of the slope

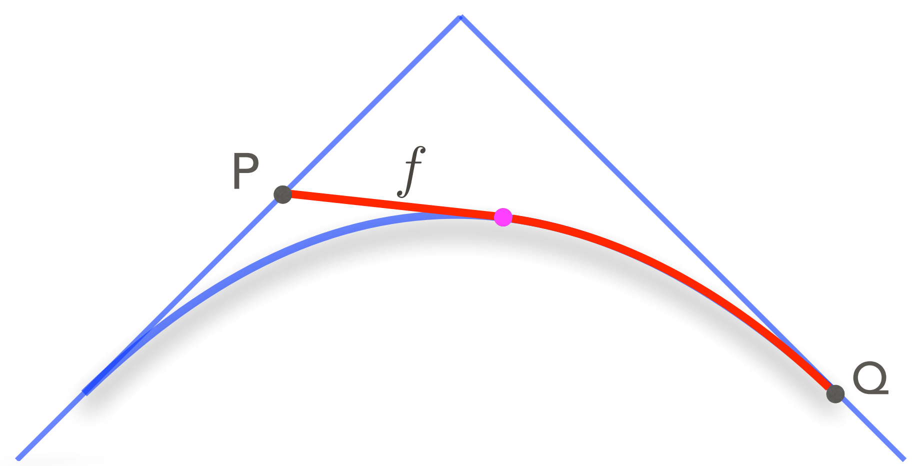
Next step is to compute $Z_1^{\rm up}$.

It is the partition function for a single path, travelling from (fixed) P to Q, and confined in a <u>deterministic</u> domain formed by the boundaries of the AzD and the arctic curve.



In scaling limit, partition function is exponentially dominated by lattice paths which condense on a deterministic trajectory f and is given asymptotically by

$$Z_1^{\rm up}[f] \simeq \exp\{nS[f]\}, \qquad S[f] = \int_P^Q {\rm d}x\, L\big(f'(x)\big)$$

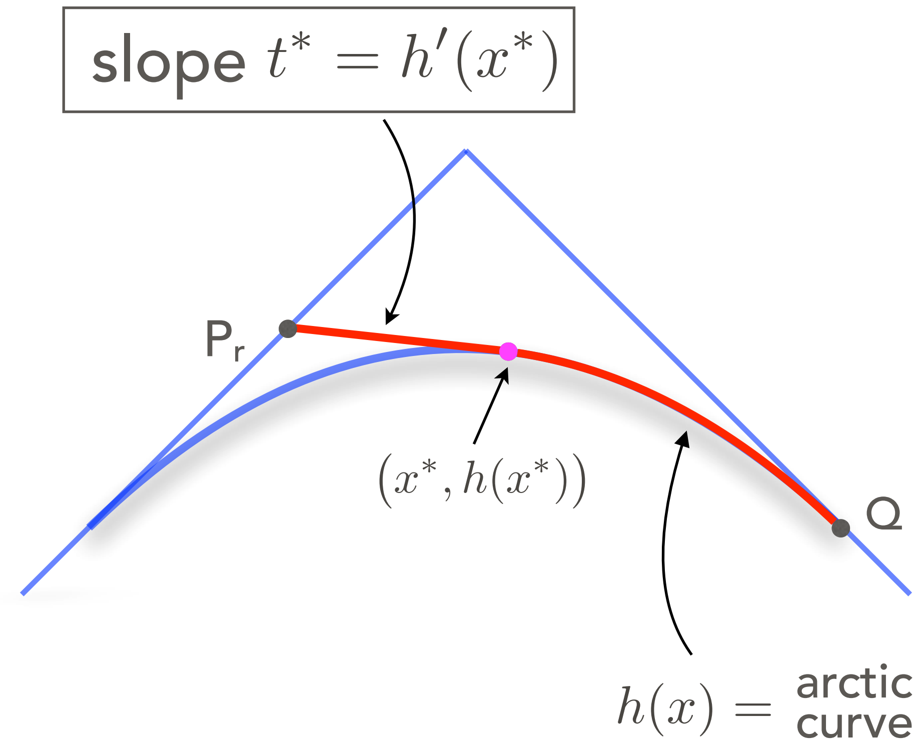


Moreover, f is the shortest path between P and Q entirely contained in the domain. In present case, f is the red curve (straight line + piece of arctic curve).

(Debin, Granet & PhR, 2019)

2 - Computation of the slope

Combining previous results,



$$\log(ab) + F_1(r) = S[f] = \int_{P_r}^{Q} \mathrm{d}x\, L\big(f'(x)\big)$$

$$= \int_{P_r}^{x^*} \mathrm{d}x\, L(t^*) + \int_{x^*}^{Q} \mathrm{d}x\, L\big(h'(x)\big)$$

$$= (x^* - P_r) L(t^*) + \int_{x^*}^{Q} \mathrm{d}x\, L\big(h'(x)\big)$$

Differentiate with respect to r to get

$$\boxed{F_1'(r) = \log\sqrt{ab} - L(t^*) - (1 - t^*) L'(t^*)}$$

i.e. an explicit relation between the slope and r .

$\longrightarrow$ if we can compute the function $L(t)$, <u>WE ARE DONE</u> 😀

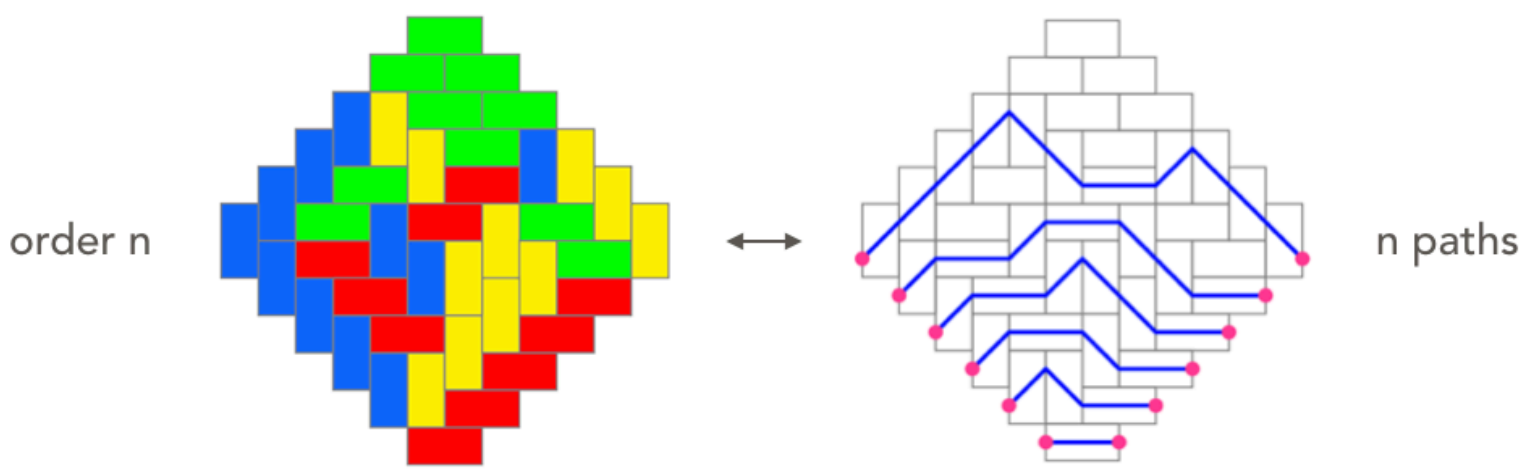
3 - Computation of function $L(t)$

Remember one of previous slides:

Non-intersecting lattice path bijection

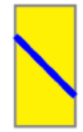
The tilings of AD_n are in bijection with the set of n-uples of non-intersecting (Schröder) paths, starting and ending at fixed positions.
The lattice paths are made of elementary steps (1,1), (1,-1) and (2,0).

For AzD, draw elementary steps on 3 tiles :

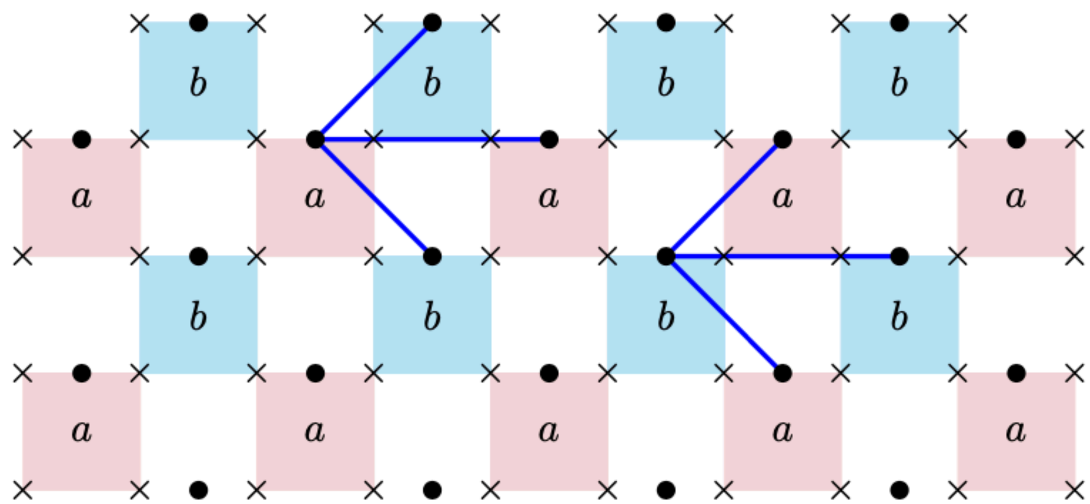


Must extend to a measure preserving bijection !

3 - Computation of function $L(t)$

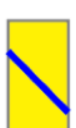
- The steps associated with     should be given weights a or b.

  These weights depend on where the steps are taken:



- Green dominos contribute to weight of tiling but have no associated step !
  Observe however: as many a/b-type red domino as a/b-type green domino.
  → simply square the weights of red dominos

- A path is made of elementary steps with position dependent weights

   and  get weight a or b  gets weight a^2 or b^2

3 - Computation of function $L(t)$

- Partition function $Z_{i,j}(a,b)$ for a single path = sum of weights of all paths $(0,0) \longrightarrow (i,j)$
  Write recurrence relations and form generating function $G(x,y) = \sum_{i,j} Z_{i,j}(a,b)\, x^i y^j$
  and get

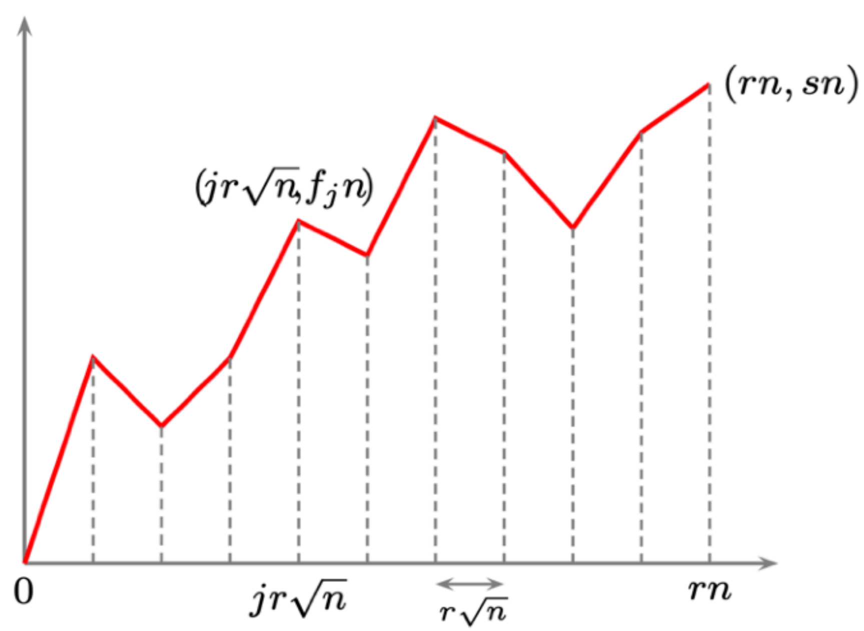
$$G(x,y) = \frac{1 - b^2x^2 + x(ay + b/y)}{(1-a^2x^2)(1-b^2x^2) - x^2(ay+b/y)(by+a/y)}$$

  Use standard methods to extract asymptotic value for large $i = rn,\ j = sn$

$$\boxed{Z_{rn,sn}(a,b) \simeq \exp\{nr\,L(t)\} \qquad t = \frac{s}{r}}$$

✔

- Allows to compute contribution of paths condensing on trajectory f



On each subinterval, we have $Z_{r\sqrt{n},(f_{j+1}-f_j)n} \simeq \exp\{r\sqrt{n}L(t_j)\}$

Then

$$Z[\{f_j\}] \simeq \prod_{j=1}^{\sqrt{n}} \exp\{r\sqrt{n}L(t_j)\} \longrightarrow Z[f(x)] \simeq \exp\left\{n \int \mathrm{d}x\, L\big(f'(x)\big)\right\}$$

$$= \exp\{nS[f]\}$$

FINAL RESULTS

The arctic curve for the 2-periodic AzD has the following parametrization

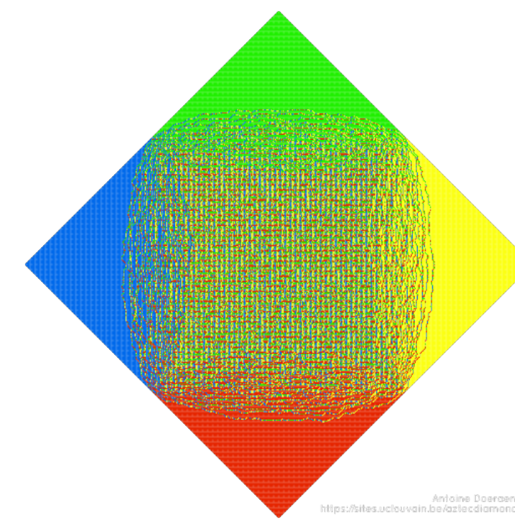
$$\begin{Bmatrix} X(v) \\ Y(v) \end{Bmatrix} = \frac{1}{(v^2+1)^4 + 4\frac{(\beta-1)^2}{\beta}\, v^4} \left\{ \frac{1}{2}(v^4-1)(v^2+1)^2 \mp 2\frac{\sqrt{\beta}}{\beta+1}\, v \left[v^4 + (\beta + \frac{1}{\beta})v^2 + 1 \right]^{\frac{3}{2}} \right\}$$

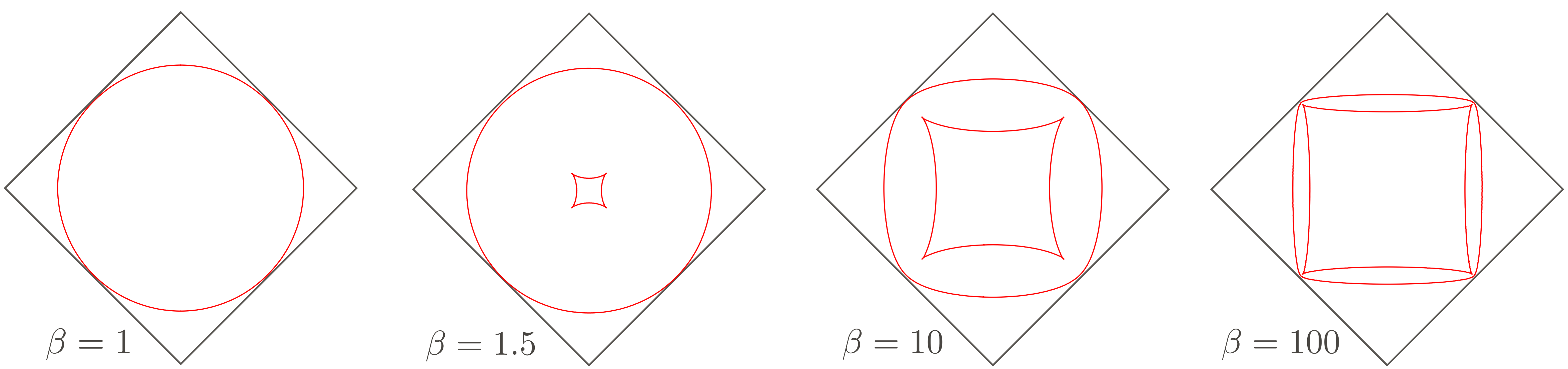
$$\left(\beta = \frac{a^2}{b^2}\right)$$

It satisfies 8th degree algebraic for $U = X + Y$ and $V = Y - X$

DF & S-G, '14
Ch & Jo, '16
Du & Ku, '21

$$\begin{aligned}
&(\beta+1)^6\,(U^8+V^8) - 4(\beta+1)^4(\beta^2-6\beta+1)\,U^2V^2(U^4+V^4) \\
&\quad + 2(\beta+1)^2(3\beta^4-20\beta^3+82\beta^2-20\beta+3)\,U^4V^4 - 4(\beta+1)^4(\beta^2-\beta+1)\,(U^6+V^6) \\
&\quad + 4(\beta+1)^2(\beta^4+17\beta^3-48\beta^2+17\beta+1)\,U^2V^2(U^2+V^2) \\
&\quad + 6(\beta^4-1)(\beta^2-1)\,(U^4+V^4) + 4(\beta-1)^2(\beta^4-22\beta^3-42\beta^2-22\beta+1)\,U^2V^2 \\
&\quad - 4(\beta-1)^4(\beta^2+\beta+1)\,(U^2+V^2) + (\beta-1)^6 = 0.
\end{aligned}$$





Selected references

- N. Elkies, G. Kuperberg, M. Larsen and J. Propp
  Alternating sign matrices and domino tilings I + II
  J. Alg. Combin. 1 (1992) 111-132 + 219-234 (arXiv:math/9201305 [math.CO])

- W. Jockusch, J. Propp and P. Shor
  Random domino tilings and the arctic circle theorem
  arXiv:math/9801068 [math.CO]

- P. Di Francesco and R. Soto-Garrido
  Arctic curves of the octahedron equation
  J. Phys. A: Math. Theor. 47 (2014) 285204 (arXiv:1402.4493 [math-ph])

- F. Colomo and A. Sportiello
  Arctic curves of the six-vertex model on generic domains: the tangent method
  J. Stat. Phys. 164 (2016) 1488-1523 (arXiv:1605.01388 [math-ph])

- B. Debin and P. Ruelle
  Factorization in the multirefined tangent method
  J. Stat. Mech. (2021) 103201 (arXiv:2105.02257 [math-ph]

- P. Ruelle
  Double tangent method for two-periodic Aztec diamonds
  Soon on arXiv (within a few days)

- Antoine Doeraene
  Aztec Diamond Generator
  Website https://sites.uclouvain.be/aztecdiamond/