Hermitian Schur measures

From quantum mechanics to new asymptotic statistics for random partitions

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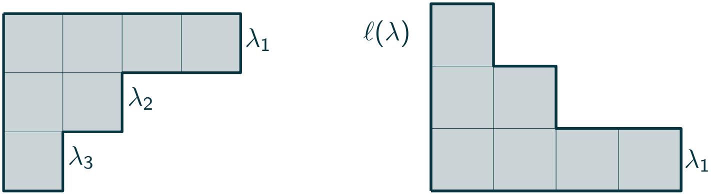
Joint work with Dan Betea (Université d'Angers)
and Jérémie Bouttier (CEA Saclay) [arXiv:2012.01995],
and some work in progress

Randomness, Integrability and Universality, Galileo Galilei Institute, 18 May 2022

Integer **partitions** $\lambda \vdash n$ are weakly decreasing sequences of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_{\ell(\lambda)} > 0)$, such that $\lambda_1 + \lambda_2 + \ldots + \lambda_{\ell(\lambda)} = n$. They are represented with Young diagrams.

e.g. for $\lambda = (4, 2, 1)$:



A natural measure on partitions of n arises from the uniform measure on permutations of $1, \ldots, n$.

Famously, the Robinson–Schensted algorithm maps each $\sigma \in S_n$ to a pair of standard Young tableaux (P, Q) of the same shape $\lambda \vdash n$. E.g.,

$$\sigma = (7, 5, 1, 6, 2, 3, 4)$$

$$\updownarrow$$

$$P = \begin{array}{|c|c|c|c|} \hline 7 & \multicolumn{3}{c}{} \\ \vee & \multicolumn{3}{c}{} \\ 5 < & 6 & \multicolumn{2}{c}{} \\ \vee & \vee & \multicolumn{2}{c}{} \\ 1 < & 2 < & 3 < & 4 \\ \hline \end{array}, \qquad Q = \begin{array}{|c|c|c|c|} \hline 3 & \multicolumn{3}{c}{} \\ \vee & \multicolumn{3}{c}{} \\ 2 < & 5 & \multicolumn{2}{c}{} \\ \vee & \vee & \multicolumn{2}{c}{} \\ 1 < & 4 < & 6 < & 7 \\ \hline \end{array}$$

λ_1 is the length of the longest increasing subsequence of σ.

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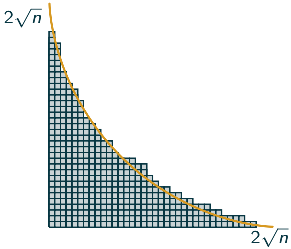
λ_1 is the length of the longest increasing subsequence of σ.

Where σ is sampled uniformly, this induces the **Plancherel measure**

$$\mathbb{P}_n(\lambda) = \mathbb{P}[\mathrm{Shape}(\mathrm{RS}(\sigma)) = \lambda] = \frac{f_\lambda^2}{n!}$$

where f_λ is the number of SYT of shape λ.



As $n \to \infty$, the Young diagram of a random partition distributed by $\mathbb{P}_n$ has a deterministic limit shape (Vershik & Kerov; Logan & Shepp, 1977). This is a simple solvable model in the "KPZ class".

Theorem (Baik–Deift–Johansson, 1998)

Let $\lambda \vdash n$ be a random partition under the Plancherel measure $\mathbb{P}_n$. Then

$$\lim_{n\to\infty} \mathbb{P}_n\left[\frac{\lambda_1 - 2n^{1/2}}{n^{1/6}} < s\right] = F_2^{TW}(s) := \det(1-\mathcal{A})_{[s,\infty)}$$

in law, where $F_2^{TW}(s)$ is the Tracy–Widom distribution of the largest eigenvalue of a GUE random matrix.

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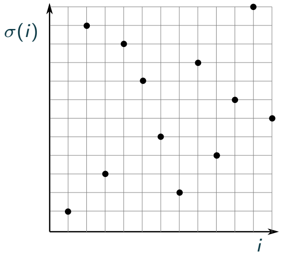

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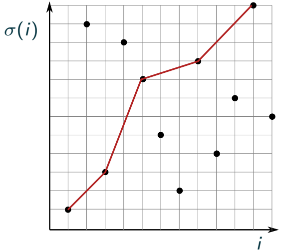
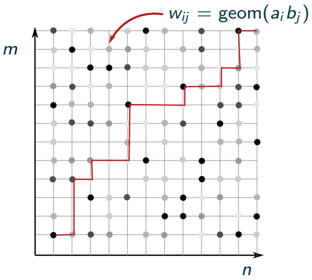


The statistic $L_{L.I.S.}(\sigma)$ or λ_1 has a geometric interpretation as the longest directed path in a uniform random medium.



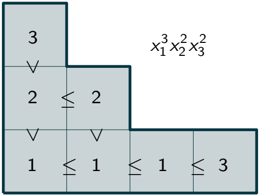
Via RSK, the Plancherel measure has a geometric generalisation corresponding to last passage percolation in more interesting random media, e.g.

$$\mathbb{P}(\lambda) = \frac{1}{Z} s_\lambda(a_1, a_2, \ldots, a_N) s_\lambda(b_1, b_2, \ldots, b_N)$$

where $a_i, b_i \in [0, 1)$ and s_λ is the Schur symmetric function (cf Guillaume's talk).

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det\limits_{1 \le i,j \le N} x_j^{\lambda_i - i + j}}{\det\limits_{1 \le i,j \le N} x_j^{i}}$$

$$s_\lambda(x_1, \ldots, x_N) = \sum_{T \text{ SSYT of } \lambda} \prod_{i=1}^{N} x_i^{\#\{i \text{ in } T\}}$$



$$s_{(4,2,1)}(x) = \sum_{i,j,k} (x_i^4 x_j^2 x_k + x_i^3 x_j^3 x_k + 2 x_i^3 x_j^2 x_k^2)$$

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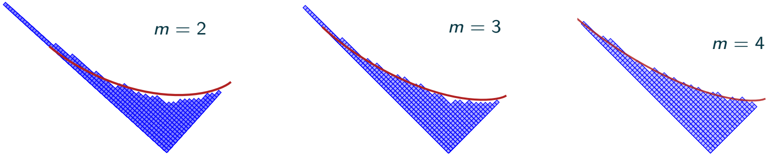

Theorem (Betea–Bouttier–W., 2020)

Let λ be a random partition of mean size θ^2 distributed by $\mathbb{P}_\theta^m$. Then*

$$\lim_{\theta \to \infty} \mathbb{P}_\theta^m \left[\frac{\lambda_1 - b\theta}{(d\theta)^{\frac{1}{2m+1}}} < s \right] = F(2m+1; s) := \det(1 - \mathcal{A}_{2m+1})_{[s,\infty)}$$

in law for constants b, d, where $\mathcal{A}_{2m+1}(x, y)$ is the order m generalized Airy kernel*.* ** to be defined.*

This edge behaviour was first observed for momenta of trapped fermions (Le Doussal, Majumdar & Schehr, 2018).

0. Partitions, Permutations, Plancherel, Schur

1. Lattice fermions and Hermitian Schur measures

2. Multicritical random partitions and unitary matrices

3. Splitting the Fermi sea

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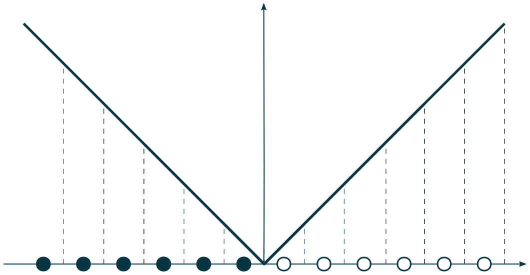
Partitions map to configurations of fermions on a 1D lattice at fixed charge, which can be read off tilted Young diagrams.



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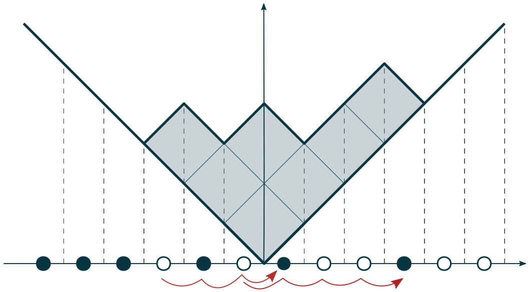

The empty partition corresponds to the domain wall state $|\emptyset\rangle$, a partition λ corresponds to the state

$$|\lambda\rangle := c^\dagger_{\lambda_1-\frac{1}{2}} c^\dagger_{\lambda_2-\frac{3}{2}} \cdots c^\dagger_{\lambda_\ell-\ell+\frac{1}{2}} c_{-\ell+\frac{1}{2}} \cdots c_{-\frac{3}{2}} c_{-\frac{1}{2}} |\emptyset\rangle \,.$$

These are eigenstates of a linear potential

$$\mathcal{H}_0 = \sum_k k : c^\dagger_k c_k :, \qquad \mathcal{H}_0 |\lambda\rangle = |\lambda| \, |\lambda\rangle \,, \qquad |\lambda| = \sum_i \lambda_i .$$

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We add kinetic hopping terms

$$a_r := \sum_k : c_k^\dagger c_{k+r} :$$

via the unitary operator

$$\mathcal{U} := e^{\sum_{r\geq 1}(t_r a_r^\dagger - \bar{t}_r a_r)}$$

for some finite sequence of complex parameters t_r (called Miwa times). Then

$$\mathcal{H} = \mathcal{U}\mathcal{H}_0\mathcal{U}^{-1} = \mathcal{H}_0 - \sum_{r\geq 1} r(\bar{t}_r a_r + t_r a_r^\dagger) + \sum_{r\geq 1} r^2 |t_r|^2$$

has the ground state

$$\mathcal{U}|\emptyset\rangle = e^{-\sum_{r\geq 1} r|t_r|^2/2} e^{\sum_{r\geq 1} t_r a_r^\dagger} |\emptyset\rangle = e^{-\sum_{r\geq 1} r|t_r|^2/2} \sum_\lambda s_\lambda[t_1, t_2, \ldots] \, |\lambda\rangle \, .$$

Here the specialisation is in the power sums $t_r = \sum_i x_i^r$.

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$$\mathcal{H} = \mathcal{U}\mathcal{H}_0\mathcal{U}^{-1} = \sum_{r\geq 1} a_r^\dagger a_r + \frac{a_0^2}{2} - \sum_{r\geq 1} r(\bar{t}_r a_r + t_r a_r^\dagger) + \sum_{r\geq 1} r^2 |t_r|^2$$

has the coherent ground state

$$\mathcal{U} |\emptyset\rangle = e^{-\sum_{r\geq 1} r|t_r|^2/2} e^{\sum_{r\geq 1} t_r a_r^\dagger} |\emptyset\rangle = e^{-\sum_{r\geq 1} r|t_r|^2/2} \sum_\lambda s_\lambda[t_1, t_2, \ldots] \, |\lambda\rangle \, .$$

Here the specialisation is of the power sums, via $t_r = \sum_i x_i^r$.

Imagine we could simultaneously measure the occupation number of every site on the lattice; the probability of observing $|\lambda\rangle$ is the Hermitian Schur measure (Okounkov, 2001)

$$\mathbb{P}(\lambda) = |\langle\lambda|\,\mathcal{U}\,|\emptyset\rangle|^2 = e^{-\sum_{r\geq 1} r|t_r|^2} s_\lambda[t_1, t_2, \ldots] s_\lambda[\bar{t_1}, \bar{t_2}, \ldots].$$

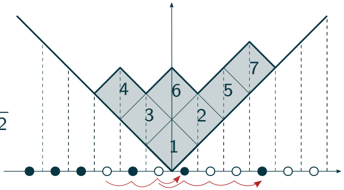
- $\mathbb{P}(\lambda)$ is a determinantal point process on the sets $\mathfrak{S}(\lambda) = \{\lambda_i - i + \frac{1}{2}, i \in \mathbb{Z} + \frac{1}{2}\}$, with

$$\mathbb{P}(k_1, \ldots, k_n \in \mathfrak{S}(\lambda)) = \det_{1\leq i,j\leq n} K(k_i, k_j)$$

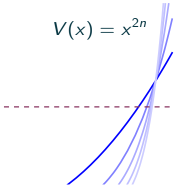
$$\sum_{k,\ell} \frac{z^k w^{-\ell}}{\sqrt{zw}} K(k,\ell) = \frac{e^{\sum_r (t_r z^r - \bar{t_r} z^{-r})}}{e^{\sum_r (t_r w^r - \bar{t_r} w^{-r})}}$$

- At $t_1 = \theta$, $t_{r>1} = 0$, we have the Poissonized Plancherel measure (Johansson; Borodin, Okounkov & Olshanski, 1999):

$$\mathbb{P}(\lambda) = |\,\langle\lambda|\, e^{-\theta^2/2} e^{\theta a_1}\,|\emptyset\rangle\,|^2 = f_\lambda^2 \frac{\theta^{2|\lambda|}}{e^{\theta^2}|\lambda|!^2}$$



Consider free fermions on a line in confining potentials $V(x) = x^{2m}$, $m = 1, 2, \dots$ (in first quantisation), and look at what happens on the edge (Le Doussal, Majumdar & Schehr, 2018).



- In position space: universal Airy fermions $H_{\text{edge}} \approx x - \dfrac{d^2}{dx^2}$.
- In momentum space: $H_{\text{edge}} \approx p + (-1)^m \frac{d^{2m}}{dp^{2m}}$.

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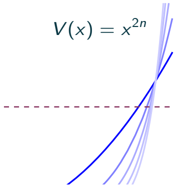

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The ground state correlator is

$$\mathcal{A}_{2m+1}(x, y) = \int_{-\infty}^{\infty} d\mu \, \mathrm{Ai}_{2m+1}(x+\mu)\, \mathrm{Ai}_{2m+1}(y+\mu),$$

$$\mathrm{Ai}_{2m+1}(p) = \int_{i\mathbb{R}+\delta} \frac{d\zeta}{2\pi i}\, e^{\frac{(-1)^{m+1}}{2m+1}\zeta^{2m+1} - p\zeta}$$

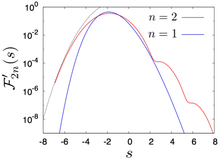


Figure from LDMS 2018.

Fluctuations are given by a Fredholm determinant

$$\mathbb{P}[p_{\max} - p_F < p_N s] = F(2m+1; s)$$
$$= \det(1 - \mathcal{A}_{2m+1})_{[s,\infty)}.$$

The Fredholm determinant

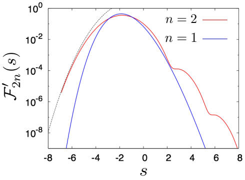


Figure from LDMS 2018.

$$F(2m+1;s) := \det(1-\mathcal{A}_{2m+1})_{[s,\infty)}$$
$$= \mathbb{P}[p_{\max} - p_F < p_N s]$$

generalises a connection between the Tracy–Widom distribution and classical integro-differential equations:

$$F(2m+1;s) = \exp\left[-\int_s^\infty (x-s) q_m^2((-1)^{m+1}x)\,dx\right]$$

where q_m is a solution of the mth equation of the Painlevé II hierarchy which coincide with Ai_{2m+1} at infinity (LDMS, 2018; Cafasso, Claeys & Girotti, 2019).

The same higher-order Painlevé equations are asymptotically satisfied by certain “multicritical” unitary matrix integrals (Periwal & Shevitz, 1990).

There is a finite temperature extension of $F(2m+1;s)$ and the Painlevé equations (Krajenbrink, 2020; Bothner, Cafasso & Tarricone, 2021).

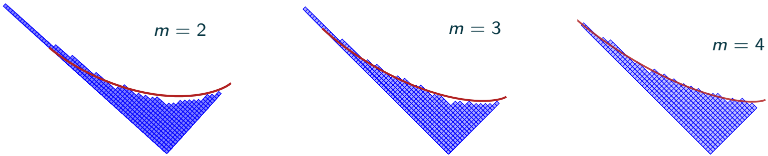
We can tune the Miwa times to have the same asymptotic edge fluctuations. E.g. take

$$\mathbb{P}_\theta^m(\lambda) = e^{-\theta^2 \sum_r \frac{\gamma_r^2}{r}} s_\lambda[\theta\gamma_1, \theta\gamma_2, \ldots]^2$$

with

$$\gamma_r = (-1)^r \binom{2m}{m-r} / \binom{2m}{m-1}, \; r = 1, \ldots, m$$

and all other $\gamma_r = 0$.



We can tune the Miwa times to have the same asymptotic edge behaviour. E.g. take

$$\mathbb{P}_\theta^{\mathrm{oe},m}(\lambda) = e^{-\theta^2 \sum_r \frac{\gamma_r^2}{r}} s_\lambda[\theta\gamma_1, \theta\gamma_2, \ldots]^2$$

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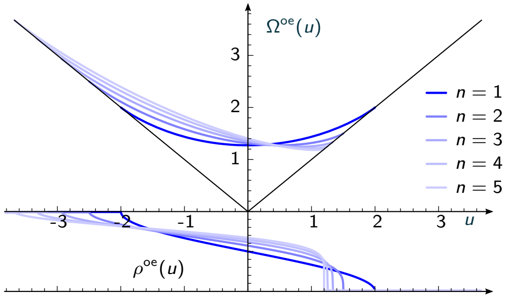

The $\theta \to \infty$ limit shape vanishes with a $1/2m$ exponent on the right. The $m = 2$ limit density also appeared for non-probabilistic time dependent free fermion processes (Bocini & Stéphan, 2020).

Or take

$$\mathbb{P}_\theta^{\circ,m}(\lambda) = e^{-\theta^2 \sum_r \frac{\gamma_r^2}{r}} s_\lambda[\theta\gamma_1, \theta\gamma_2, \ldots]^2$$

with

$$\gamma_{2r-1} = (-1)^r \binom{2m-1}{m-r} \Big/ \binom{2m-1}{m-1}, \; r = 1, \ldots, m$$

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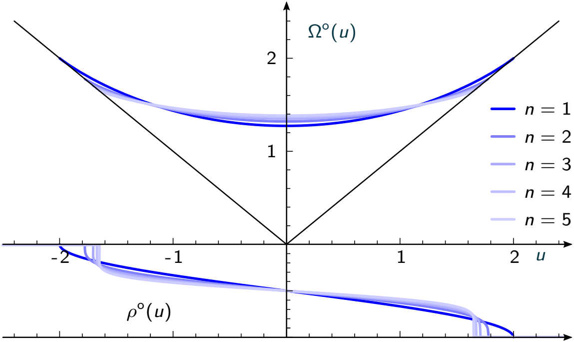

The $\theta \to \infty$ limit shape has two edges vanishing with a $1/2m$ exponent.

Define a class of Hermitian Schur measures with one parameter

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on integer partitions, where $\{\gamma_r\}$ satisfy

$$2\sum_r r^{2p}\gamma_r = \delta_{p,0} b + \delta_{p,m}(-1)^{m+1}(2m)!d\,, \qquad p = 0, 1, 2, \ldots, m$$

and $\sum_{r\geq 1} r\gamma_r \sin r\phi \geq 0, \phi \in [0, \pi]$ for some b, $d > 0$.

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in law for constants b, d, where $\mathcal{A}_{2m+1}(x, y)$ is the order m generalized Airy kernel.

This extends to finite temperature via cylindric partitions (Borodin, 2006; Betea & Bouttier, 2018)

Heuristically...

In the critical scaling regime $k = b\theta + (d\theta)^{\frac{1}{2n+1}}$, where $b := 2\sum_{r\geq 1} r\gamma_r > 0$ and $d := \frac{2(-1)^{n+1}}{(2n)!}\sum_{r\geq 1} r^{2n+1}\gamma_r$, the Hamiltonian associated with the underlying fermion model asymptotically coincides with that of fermion edge momenta in flat traps,

$$\frac{1}{\theta^{\frac{2n}{2n+1}}}\mathcal{H}_{\theta,n} \xrightarrow[\theta\to\infty]{} \mathcal{H}_{\infty,n} = \int_{\mathbb{R}} dx\ c^\dagger(x)\left[x + (-1)^n \frac{d^{2n}}{dx^{2n}}\right] c(x).$$

- Asymptotic analysis of wave functions was also used to study similar models (Kimura & Zahabi 2020).

Rigorously...

We compute asymptotics of the kernel for $\mathbb{P}_\theta^m$

$$K(k,\ell) = \frac{1}{(2\pi i)^2}\oint\!\!\oint \frac{dz}{z^{k+\frac{1}{2}}}\frac{dw}{w^{-\ell+\frac{1}{2}}}\frac{e^{\theta\sum_r \frac{\gamma_r}{r}(z^r - z^{-r})}}{e^{\theta\sum_r \frac{\gamma_r}{r}(w^r - w^{-r})}}$$

where w is integrated along $|w| = 1-\delta$ and z along $|z| = 1+\delta$. At order m multicriticality, there's an order $2m$ saddle point.

The distribution of λ_1 under $\mathbb{P}_\theta^m$ is equal to the partition function of a multicritical **unitary matrix model**.

Proposition (Betea–Bouttier–W.,2020)

For λ distributed by $\mathbb{P}_\theta^m$, we have:

$$e^{\sum_r r\theta^2\gamma_r^2} \cdot \mathbb{P}(\lambda_1 \leq \ell) = \int_{\mathcal{U}(\ell)} \mathcal{D}U \; e^{\theta \operatorname{tr} \sum_r (-1)^{r-1}\gamma_r(U^r + U^{*r})}$$

and the unitary matrix model with potential $V(U) = \theta \sum_r (-1)^{r-1}\gamma_r(U^r + U^{*r})$ has order $2m$ vanishing in its eigenvalue density.

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- Cauchy-Binet formula: $s_\lambda[\theta\gamma] = \det_{1\leq i,j \leq \ell(l)} h_{\lambda_i - i + j}$, so

$$\sum_{\lambda:\ell(\lambda)<\ell} s_\lambda[\theta\gamma] s_\lambda[\theta\gamma] = \det_{1\leq i,j\leq \ell} \sum_{k=0}^{\infty} h_{k-i} h_{k-j}$$

  This is a Toeplitz determinant, and $\sum_k h_k z^k = e^{\theta \sum_r \frac{\gamma_r}{r} z^r}$.

- Heine's identity:

$$\det_{1\leq i,j\leq \ell} [f_{j-i}] = \int d\theta_1 \int d\theta_2 \ldots \int d\theta_\ell \prod_{i<j} |e^{\theta_i} - e^{\theta_j}|^2 f(\theta_1) f(\theta_2) \cdots f(\theta_\ell)\,.$$

The distribution of λ_1 under $\mathbb{P}_\theta^m$ is equal to the partition function of a multicritical **unitary matrix model**.

Proposition (Betea–Bouttier–W.,2020)

For λ distributed by $\mathbb{P}_{n,\theta}^\gamma$, we have:

$$e^{\sum_r r\theta^2\gamma_r^2} \cdot \mathbb{P}(\lambda_1 \leq \ell) = \int_{\mathcal{U}(\ell)} \mathcal{D}U \, e^{\theta \operatorname{tr} \sum_r (-1)^{r-1}\gamma_r (U^r + U^{*r})}$$

and the unitary matrix model with potential $V(U) = \theta \sum_r (-1)^{r-1}\gamma_r(U^r + U^{*r})$ has order $2m$ vanishing in its eigenvalue density.

The unitary matrix integrals corresponding to $\mathbb{P}_\theta^{\mathrm{oe},m}$ were previously found by tuning to multicriticality; the mth order Painlevé equation was also found here (Periwal & Shevitz, 1990).

- $n = 1$ **case:** correspondence between the Poissonized Plancherel measure and the **Gross–Witten–Wadia** model (Johannson, 1998).
- In general: third order phase transition with scaling exponent $2 + \frac{1}{m}$.

The unitary matrix integral might give a combinatorial interpretation...

0. Partitions, Permutations, Plancherel, Schur

1. Lattice fermions and Hermitian Schur measures

2. Multicritical random partitions and unitary matrices

3. Splitting the Fermi sea

The second condition

$$\sum_{r\geq 1} r\gamma_r \sin r\phi \geq 0, \phi \in [0,\pi]$$

in our definition of multicritical Schur measures ensures that the ground state has a connected momentum spectrum.

In the corresponding unitary matrix model, this is a single cut assumption.

What if we lift it?

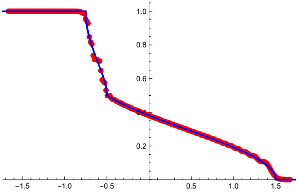

Figure from BS 2020.

- This situation has been studied for the Lieb–Liniger model (Fokkema, Eliëns & Caux 2014; Eliëns' PhD thesis)
- For free fermion models with a "split Fermi sea", limiting densities have been found (Bocini & Stéphan, 2020).

Take a Hermitian Schur measure $\mathbb{P}_\theta^{\mathrm{oe},m}(\lambda) = e^{-\theta^2 \sum_r \frac{\gamma_r^2}{r}} s_\lambda[\theta\gamma_1, \theta\gamma_2, \ldots]^2$ with

$$\gamma_1 = 1, \qquad \gamma_2 = -\frac{1}{4}$$

and all other $\gamma_r = 0$.

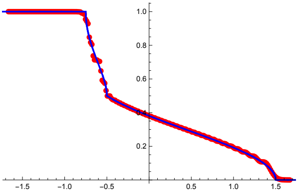

Figure from BS 2020.

The fluctuations around the right edge are in the generic $\theta^{1/3}$ scale, but the two-point function does not quite converge to the Airy kernel.

We have

$$\lim_{\theta\to\infty} \mathbb{P}\left[\frac{\lambda_1 - b\theta}{(d\theta)^{1/3}} < s\right] = \det\left(1 - \tilde{\mathcal{A}}\right)_{[s,\infty)}$$

$$\tilde{\mathcal{A}}(x,y) = \begin{cases} \mathcal{A}(x,x), & x = y \\ \frac{1}{2}\mathcal{A}(x,y), & x \neq y. \end{cases}$$

Further directions

- extension of the unitary matrix models to finite temperature
- multicriticality at non-integer n (cf Ambjorn, Budd & Makeenko, 2016)
- internal cusps in split fermi sea models
- combinatorial interpretations: is there a connection with surface enumeration? (cf Okounkov, 2000)

Thank you for your attention!