Dynamical universality classes: Results and open questions

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1. One-dimensional particle systems with short-range interactions
2. Nonlinear fluctuating hydrodynamics
3. Fibonacci universality classes
4. Numerical and exact results
5. Main results and open problems

1. One-dimensional particle systems with short-range interactions

- Unusual static properties $\neq$ mean field, higher dimensions

- Anomalous transport in nonequilibrium steady states

- Phase separation [Lahiri, Barma, Ramaswamy (2000)]

- Non-diffusive scale-invariant critical dynamics

- Superdiffusive spatio-temporal scaling [KPZ (1985), Dhar (1987), Gwa and Spohn (1992)]

- Dynamical universality classes with dynamical exponent $z < 2$

- Nongaussian universal scaling functions

Some generic model systems with local conservation laws

- Anharmonic chains (Conserved energy, momentum, ...)

- Lattice gas models with M conserved species of particles

$\Longrightarrow$ M stationary currents as functions of conserved densities $n_k^\alpha(t)$

Goal: Universality classes for dynamical structure functions

Stationary correlations of fluctuation fields $u_k^\alpha(t) = n_k^\alpha(t) - \rho^\alpha$:

$$S_k^{\alpha\beta}(t) = \langle\, n_k^\alpha(t) n_0^\beta(0)\,\rangle - \rho^\alpha \rho^\beta = \langle\, u_k^\alpha(t) u_0^\beta(0)\,\rangle$$

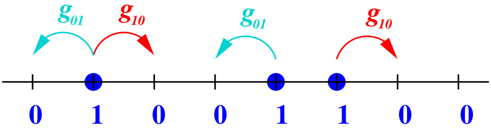
- Static compressibility matrix $K^{\alpha\beta} = \sum_k S_k^{\alpha\beta}(t)$
- Current Jacobian $J^{\alpha\beta}$ from $\sum_k k\dot{S}_k^{\alpha\beta}(t)$
- Large $k, t \Longrightarrow$ Dynamical exponent, scaling functions

2. Multilane exclusion processes

(1) One conservation law:

Asymmetric simple exclusion process (ASEP) [MacDonald, Gibbs, Pipkin (1968); Spitzer (1970)]

- Occupation numbers $n_\ell \in \{0,1\}$ for $\ell \in \{1,\dots,L\}$
- Configuration $\mathrm{n} = \{n_1,\dots,n_L\} \in \{0,1\}^L$
- Markovian jumps with rates g_{10}, g_{01}



- Generator (periodic boundary conditions)

$$\mathcal{L}f(\mathrm{n}) = \sum_{\ell=1}^{L} \left[g_{10} n_\ell (1 - n_{\ell+1}) + g_{01} n_{\ell+1} (1 - n_\ell)\right] \left[f(\mathrm{n}^{\ell,\ell+1}) - f(\mathrm{n})\right]$$

$\Longrightarrow \mathcal{L} n_\ell = j_{\ell-1} - j_\ell$ with instantaneous current

$$j_\ell = g_{10} n_\ell (1 - n_{\ell+1}) - g_{01} n_{\ell+1} (1 - n_\ell)$$

- Fix particle number N: Uniform invariant measure

$\Longrightarrow$ Bernoulli product measures with density $\rho \in [0,1]$

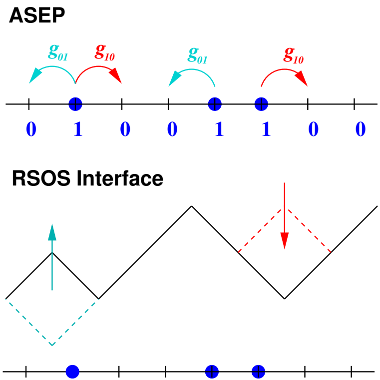
$\Longrightarrow$ Stationary current $j := \langle j_\ell \rangle = f\rho(1-\rho)$

with driving strength $f = g_{10} - g_{01}$

$\Longrightarrow$ Static compressibility $\kappa := \frac{1}{L}\langle (N - \rho L)^2 \rangle = \rho(1-\rho)$

From the ASEP to a RSOS interface

- Map occupation number to slope $s_\ell = 1 - 2n_\ell$ on dual lattice $\Longrightarrow$
- Particle jumps map to interface growth $\Longrightarrow j =$ stationary growth velocity
- Time-integrated particle current across bond $(\ell, \ell+1)$ maps to interface height h_ℓ



Hydrodynamic limit

- Coarse-grain space and rescale $k = x/a$, $t \to t/a$, with lattice spacing $a \to 0$
- Particle conservation, law of large numbers, local stationarity

$\Longrightarrow$ Inviscid Burgers equation

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}j(x,t) = \frac{\partial}{\partial t}\rho(x,t) + j'(\rho(x,t))\frac{\partial}{\partial x}\rho(x,t) = 0$$

with stationary current-density relation $j(\rho)$ and characteristic velocity $j'(\rho) = f(1-2\rho)$

- Kardar-Parisi-Zhang equation for $\rho(x,t) = \partial_x h(x,t)$
- $f = 0$: Diffusive scaling $t \to t/a^2 \Longrightarrow$ Diffusion equation

Fluctuations: Dynamical structure function

$$S(p,t) := \sum_\ell \mathrm{e}^{-2\pi i p\ell/L} \left(\langle\, n_\ell(t) n_0(0)\,\rangle - \rho^2 \right)$$

- Zero bias $g_{10} = g_{01}$ (symmetric simple exclusion process):

$S(p,t) \propto \mathrm{e}^{-Dp^2 t}$

Collective diffusion coefficient $D = (g_{10} + g_{01})/2$ (for SSEP)

Diffusive universality class with dynamical exponent $z = 2$

- Non-zero bias $g_{10} \neq g_{01}$ (ASEP):

$S(p,t) \propto \mathrm{e}^{-ivpt} \hat{f}_{PS}(\lambda p^{3/2} t)$

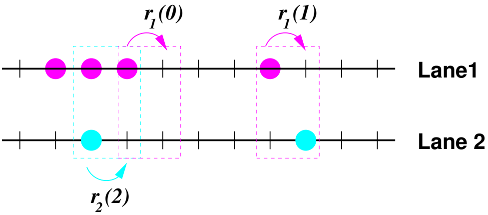
$\hat{f}_{PS}$: FT of universal Prähofer-Spohn scaling function, collective velocity $v = j'(\rho)$, scale factor $\lambda = \sqrt{2}|j''(\rho)|$ [Prähofer, Spohn (2004)]

KPZ universality class with dynamical exponent $z = 3/2$

(2) *M* conservation laws: Multi-lane ASEP

- Interacting multi-lane ASEPs with conserved densities ρ_α
- No jumps between lanes, but rates depend on neighbouring lane

Example: Coupled two-lane TASEP



$$r_1(k) = b_1 + \frac{\gamma}{2}\left(n_k^{(2)} + n_{k+1}^{(2)}\right), \quad r_2(k) = b_2 + \frac{\gamma}{2}\left(n_k^{(1)} + n_{k+1}^{(1)}\right)$$

Stationary state

Theorem (Popkov, Salerno (2004))

Fix particle numbers on torus of L sites. Invariant measure is uniform for any b, γ.

$\Longrightarrow$ Bernoulli product measures with stationary currents:

$$
\begin{aligned}
j_1 &= \rho_1(1-\rho_1)(b_1+\gamma\rho_2) \\
j_2 &= \rho_2(1-\rho_2)(b_2+\gamma\rho_1)
\end{aligned}
$$

• Generalizations: M lanes, other interactions within the lanes, partial or no exclusion, ...

$\Longrightarrow$ Playground for generic current-density relations.

3. Coupled Burgers equations

(1) Nonlinear fluctuating hydrodynamics

- Coarse-grain space and rescale $k = x/a$, $t \to t/a$, with lattice spacing $a \to 0$ (Eulerian scaling)
- Conservation laws, law of large numbers, local stationarity

$\Longrightarrow$ Hyperbolic system of conservation laws

$$\frac{\partial}{\partial t}\vec{\rho}(x,t) + \bar{\mathrm{J}}\frac{\partial}{\partial x}\vec{\rho}(x,t) = 0$$

with current Jacobian $\bar{\mathrm{J}}(x,t) = \mathrm{J}(\vec{\rho}(x,t))$

- Expand around stationary solution $\rho^{\lambda}(x,t) = \rho_{\lambda} + u^{\lambda}(x,t)$

Linearized conservation law (first order expansion in ϕ^α)

$$\frac{\partial}{\partial t}\vec{u}(x,t) + \bar{\mathrm{J}}\frac{\partial}{\partial x}\vec{u}(x,t) = 0$$

with <u>constant</u> $\bar{\mathrm{J}}(\vec{\rho})$.

• Transform to normal modes $\vec{\phi} = \mathrm{R}\vec{u}$ where $\mathrm{RJR}^{-1} = \mathrm{diag}(v_\alpha)$ for $\mathrm{J} \equiv \mathrm{J}(\vec{\rho})$ and R normalized such that $\mathrm{RKR}^T = \mathbb{1}$

$\Longrightarrow$ Normal modes: $\partial_t \phi^\alpha = -\partial_x v_\alpha \phi^\alpha$

• Solution: Travelling waves $\phi^\alpha(x,t) = \phi_0^\alpha(x - v_\alpha t)$ with initial data $\phi^\alpha(x,0) = \phi_0^\alpha(x)$

$\Longrightarrow$ v_α = velocity of fluctuation field α

Nonlinear fluctuating hydrodynamics [Spohn (2014)]

- Second order nonlinearity + phenomenological diffusion + noise:

$\Longrightarrow$ Coupled noisy Burgers equations

$$\partial_t \phi^\alpha = -\partial_x \left(v_\alpha \phi^\alpha + \langle \vec{\phi}, G^\alpha \vec{\phi} \rangle - \partial_x (\mathrm{D}\vec{\phi})^\alpha + \xi^\alpha \right)$$

Mode coupling matrices $\mathrm{G}^\alpha = \frac{1}{2} \sum_\lambda R_{\alpha\lambda} (\mathrm{R}^{-1})^T \mathrm{H}^\lambda \mathrm{R}^{-1}$
determined by the current Hessians H^λ with $\mathrm{H}^\lambda_{\alpha\beta} = \frac{\partial^2}{\partial \rho^\alpha \partial \rho^\beta} j^\lambda$
$\Longrightarrow$ Fluctuations of coarse-grained fluctuation fields

- Equivalent to coupled 1-d KPZ equations with $\phi^\alpha = \partial_x h^\alpha$

Mode-coupling theory [Spohn (2014)]

- Consider strictly hyperbolic case (non-degenerate J)

$\Longrightarrow$ Off-diagonal $S^{\alpha\beta}$ as well as products $S^{\alpha\alpha} S^{\beta\beta}$ decay quickly

$\Longrightarrow$ Mode coupling equation for $S_\alpha \equiv S^{\alpha\alpha}$

$$\partial_t S_\alpha(x,t) = \hat{D}_\alpha S_\alpha(x,t) + \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \, S_\alpha(x-y, t-s) M_\alpha(y,s)$$

- Linear diffusion operator $\hat{D}_\alpha = -v_\alpha \partial_x + D_\alpha \partial_x^2$

- Nonlinear memory kernel $M_\alpha(y,s) = 2\partial_y^2 \sum_\beta \left(G^\alpha_{\beta\beta} S_\beta(y,s) \right)^2$

Fibonacci universality classes [Popkov, Schadschneider, Schmidt, GMS, PNAS (2015)]

- Exact scaling solution of mode-coupling equations

(1) All $G^{\alpha}_{\beta\beta} = 0$: $z_\alpha = 2$, $S_\alpha =$ Gaussian

$\Longrightarrow$ Diffusive scaling

(2) $G^{\alpha}_{\alpha\alpha} \neq 0$: $z_\alpha = 3/2$, $S_\alpha =$ Prähofer-Spohn or modified KPZ

$\Longrightarrow$ KPZ or modified KPZ scaling

(3) $G^{\alpha}_{\alpha\alpha} = 0$ $G^{\alpha}_{\beta\beta} \neq 0$: $z_\alpha = \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots (1+\sqrt{5})/2$, $S_\alpha =$ Lévy

$\Longrightarrow$ Lévy scaling

$\Longrightarrow$ Fibonacci universality classes (Kepler ratios $z_\alpha = F_{\alpha+1}/F_\alpha$)

Applicability of the theory

• Universal tool for translation invariant 1-d systems when

– short-range interactions, local conservation laws and currents

– slow variables relevant for long-time behavior = long-wavelength Fourier components of the conserved densities

$\Longrightarrow$ Applicable to Hamiltonian dynamics, anharmonic chains, stochastic lattice gases, ...

• Quadratic non-linear terms leading, cubic terms only marginally relevant (and only if quadratic terms are absent), quartic and higher order irrelevant in RG sense.

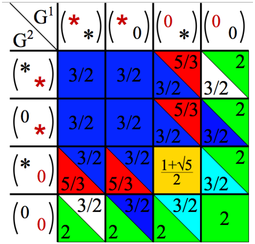
• Coarse-grained evolution equation fully determined by macroscopic diffusion constants, stationary current and static compressibility!

4. Theoretical and simulation results

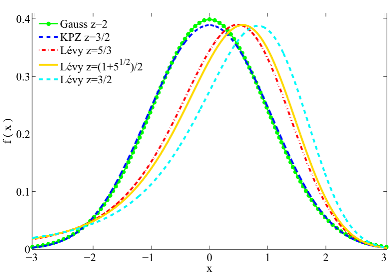
(1) Coupled ASEP's:

Universality classes for two conservation laws

[Popkov, Schmidt, GMS (2015); Spohn, Stoltz (2015)]



Universality classes with dynamical exponents z_α

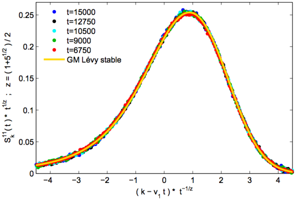


Universal scaling functions with dynamical exponents z_α

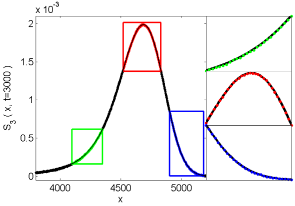
Three lane model

Golden mean mode and one-parameter fit with maximally asymmetric φ–Levy:

Data collapse:



Fixed t=3000



[Popkov, Schadschneider, Schmidt, GMS, PNAS **112** 12645 (2015)]

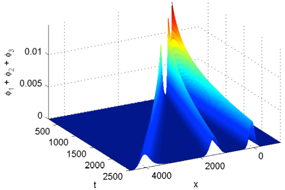
Three lane model (cont')

New Fibonacci universality class: z= 3/2, 5/3, 8/5

Mode 1: 8/5-Fibonacci

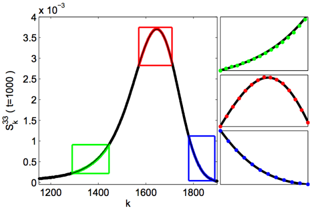
Mode 2: 5/3-Fibonacci,

Mode 3: 3/2-KPZ.



8/5-Fibonacci mode at t=1000:

Fit with max. asym. 8/5−Levy

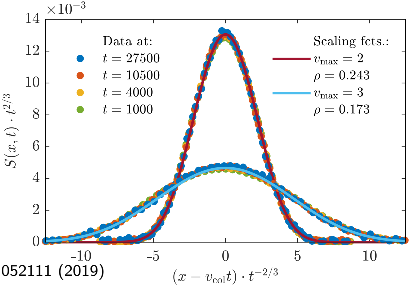


(2) Nagel-Schreckenberg model:

- Cellular automaton for vehicular traffic Nagel and Schreckenberg (1992)

(i) Acceleration: $v'_n = \min(v_n + 1, v_{max})$
(ii) Breaking: $v'_n = \min(d_n, v_n)$
(iii) Randomization: $v'_n = \min(v_n - 1, 0)$ with probability p
(iv) Movement: $x'_n = x_n + v'_n$

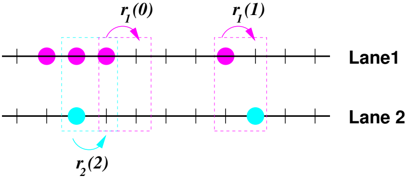
- Basis for more refined traffic flow simulations



$\Longrightarrow$ KPZ universality class

de Gier, Schadschneider, Schmidt, GMS, PRE **100**, 052111 (2019)

(3) Coupled two-lane random walks: [GMS (2018)]



$$r_1(k) = b_1 + \frac{\gamma}{2}\left(n_k^{(2)} + n_{k+1}^{(2)}\right), \qquad r_2(k) = b_2 + \frac{\gamma}{2}\left(n_k^{(1)} + n_{k+1}^{(1)}\right)$$

- Drop exclusion: Invariant product measure

$$\begin{aligned} j_1 &= \rho_1(b_1 + \gamma\rho_2) \\ j_2 &= \rho_2(b_2 + \gamma\rho_1) \end{aligned}$$

- $b_1 = b_2 \implies$ Mode coupling coefficients for modified KPZ class

(4) DAP conditioned on high hopping activity: [Karevski, GMS (2022)]

- Pair deposition and annihilation: Nonconservative dynamics

- Symmetric nearest neighbor jumps: w
- Instantaneous on-site pair annihilation
- Nearest neighbor pair deposition: μ
- Equivalent to exclusion process with

$A0 \leftrightarrow 0A: \; w+\mu, \quad AA \leftrightarrow 00: \; 2w+\mu, \quad 00 \leftrightarrow AA: \; \mu$

- Conditioning on atypical jump activity: Realized by DAP with long-range interaction cf. Popkov, GMS, Simon (2010) for conservative dynamics

- Exact solution: Conformally invariant phase transition line

- Ballistic scaling: $z = 1$, $S = t^{-1} \times$Cauchy distribution

- Observation: $z = 1/1 = F_2/F_1 < 3/2$

5. Main results and open problems

Main results:

• Fibonacci family of dynamical universality classes from nonlinear fluctuating hydrodynamics and mode coupling theory

- Dynamical exponents $z_\alpha = F_{\alpha+1}/F_\alpha$:
$z_1 = 1$, $z_2 = 2$, $z_3 = 3/2$, $z_4 = 5/3$, $z_5 = 8/5$, $\dots$, $z_\infty = \varphi$

- Explicit scaling functions:
$z_1 = 1$: Cauchy, $z_2 = 2$: Gaussian, $z_3 = 3/2$: PS, ?, Lévy,
$\alpha > 3$: Lévy

- M local conservation laws and local interactions: Consecutive dynamical exponents $z_\alpha, z_{\alpha+1}, \dots z_{\alpha+n}$ with $\alpha \in \{2,3\}$, $0 \leq n < M$

• Universality

Open problems:

• Fantastic agreement between theoretical scaling functions and simulation data: Lévy-scaling form obtained from MCT exact?
– Exactly solvable models?
– Rigorous results?

• Modified KPZ universality class?

• $z < 3/2$: Nonlocal interactions?

Also open:

• Other universality classes

• Universal finite-time effects

• Complex mode velocity and onset of phase separation