Arctic Boundaries in Ice Models

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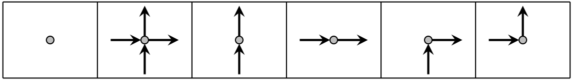
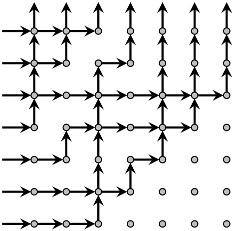
Six-Vertex Ensembles and Ice Models

Let $\Lambda \subset \mathbb{Z}^2$ be finite, and assign each vertex in Λ one of the following six edge configurations



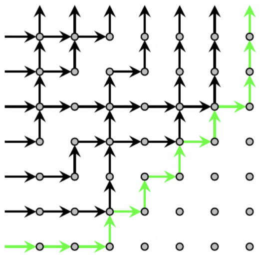
- **Domain-wall boundary conditions** arise when $\Lambda = [1, N] \times [1, N]$, and arrows enter from the left boundary and exit through the top.
- **Ice model**: Assignment is chosen uniformly at random



Six-vertex ensembles are collections of non-crossing directed (up-right) paths.

Arctic Boundary of Six-Vertex Ensembles

- Six-vertex ensemble $\mathcal{E}$ on Λ
- A vertex $v \in \Lambda$ is in the **frozen region** of $\mathcal{E}$ if one of the following holds
  - Every vertex northwest of v is packed in $\mathcal{E}$
  - Every vertex northeast of v is vertical in $\mathcal{E}$
  - Every vertex southwest of v is horizontal in $\mathcal{E}$
  - Every vertex southeast of v is empty in $\mathcal{E}$
- The boundary of the frozen region is called the **arctic boundary**



The **bottommost path** of a domain-wall six-vertex ensemble traces the southeast boundary of the frozen region

Limiting Boundary Parameterization

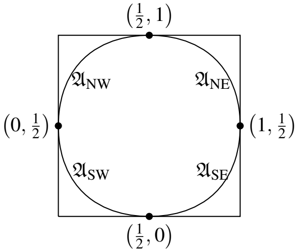
Define the portion of an ellipse

$$\mathfrak{A}_{\mathrm{SE}} = \left\{(x, y) \in \mathbb{R}^2 : (2x-1)^2 + (2y-1)^2 - 4(1-x)y = 1\right\} \cap \left(\left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]\right),$$

and its reflections

$$\mathfrak{A}_{\mathrm{SW}} = \left\{(x, y) \in \mathbb{R}^2 : (1-x, y) \in \mathfrak{A}_{\mathrm{SE}}\right\}; \qquad \mathfrak{A}_{\mathrm{NE}} = \left\{(x, y) \in \mathbb{R}^2 : (x, 1-y) \in \mathfrak{A}_{\mathrm{SE}}\right\};$$

$$\mathfrak{A}_{\mathrm{NW}} = \left\{(x, y) \in \mathbb{R}^2 : (1-x, 1-y) \in \mathfrak{A}_{\mathrm{SE}}\right\}.$$



- Let $\mathfrak{A} = \mathfrak{A}_{\mathrm{SE}} \cup \mathfrak{A}_{\mathrm{SW}} \cup \mathfrak{A}_{\mathrm{NE}} \cup \mathfrak{A}_{\mathrm{NW}}$.
- Then $\mathfrak{A}$ is **not smooth** at its four tangency points with $[0, 1] \times [0, 1]$.
  - Different from what one observers in dimers

Arctic Boundaries for Ice Model

- Let $N \in \mathbb{Z}_{>0}$ be a large integer.
- Let $\mathcal{E}$ denote a sample of the ice model on $\Lambda = [1, N] \times [1, N]$
- Let $(i, j) \in [1, N] \times [1, N]$ be an integer pair, and set $z = \left(\frac{i}{N}, \frac{j}{N}\right) \in [0, 1] \times [0, 1]$.
- Fix a real number $\varepsilon > 0$, and assume that $\operatorname{dist}(z, \mathfrak{A}) > \varepsilon$.

Theorem (A., 2018)

There exists $\delta = \delta(\varepsilon) > 0$ such that, with probability at least $1 - e^{-\delta N}$, (i, j) is in the frozen region of $\boldsymbol{M}$ if and only if z is outside of $\mathfrak{A}$.

- Eloranta (1999), Zinn-Justin (2000), Allison–Reshetikhin (2005), Sylijuåsen–Zvonarev (2004): Predicted existence of arctic boundary following its realization for domino tilings by Jockush–Propp–Shor (1995)
- Colomo–Pronko (2010): Predicted above explicit form of arctic boundary
- Colomo–Sportiello (2016): Reproduced prediction through **tangent method**
  - Di Francesco–Guitter (2018), Debin–Ruelle (2018), Corteel–Keating–Nicoletti (2019), . . .: Predicts arctic boundaries of other statistical mechanical models

Trajectory of the Bottom Path of the Ice Model

- Let $\mathcal{E}$ denote a sample of the ice model on $\Lambda = [1, N] \times [1, N]$.
- Denote the non-crossing paths in $\mathcal{E}$, from bottom to top, by $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$.
- Define $I_1 = \left[0, \frac{1}{2}\right] \times \{0\}$ and $I_2 = \{1\} \times \left[\frac{1}{2}, 1\right]$, and let $\mathfrak{P} = I_1 \cup \mathfrak{A}_{\mathrm{SE}} \cup I_2$.

By symmetry, we must show the following theorem.

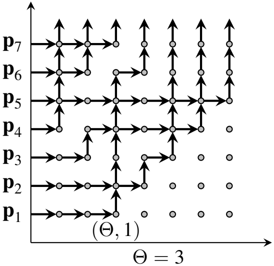
Theorem

For any $\varepsilon > 0$*, there exists* $\delta = \delta(\varepsilon) > 0$ *such that* $\operatorname{dist}\left(N^{-1}\boldsymbol{p}_1, \mathfrak{P}\right) < \varepsilon$ *holds with probability at least* $1 - e^{-\delta N}$.

- Proof based on a justification of the **(geometric) tangent method**, a general heuristic introduced by Colomo–Sportiello (2016) for deriving arctic boundaries of statistical mechanical models
- Proof is not very model-dependent and also should apply to other families of statistical mechanical systems

Refined Partition Function

- Domain-wall six-vertex ensemble $\mathcal{E}$ with paths $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$
- Let $\Theta = \Theta(\mathcal{E}) \in [1, N]$ be such that $\mathbf{p}_1$ exits the bottom row at $(\Theta, 1)$



- The **partition function** Z_N counts domain-wall six-vertex ensembles $\mathcal{E}$.
- The **refined partition function** $Z_N(K)$ counts those with $\Theta(\mathcal{E}) = K$.
- Define the K**-refined correlation function** $H_N(K)$ by

$$H_N(K) = \mathbb{P}\big[\Theta(\mathcal{E}) = K\big] = \frac{Z_N(K)}{Z_N}.$$

Refined Enumeration

Required integrable input: Asymptotics for refined partition function

- Zeilberger (1996): $H_N(K) = \binom{N+K-2}{N-1}\binom{2N-K-1}{N-1}\binom{3N-2}{N-1}^{-1}$
- Thus, for fixed $\kappa > 0$, we have for large N that

$$H_N(\kappa N) = \exp\Big(-\big(\mathfrak{h}(\kappa) + o(1)\big)N\Big),$$

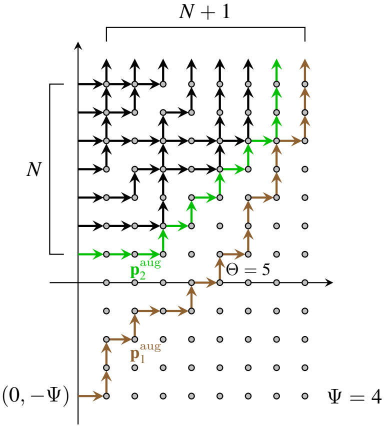
for an **explicit** $\mathfrak{h}(\kappa)$ given by

$$\mathfrak{h}(\kappa) = (1+\kappa)\log(1+\kappa) + (2-\kappa)\log(2-\kappa) - \kappa\log\kappa \\ - (1-\kappa)\log(1-\kappa) - 3\log 3 + 2\log 2$$

- **Tangency point**: $\mathfrak{h}(\kappa)$ minimized at $\kappa = \frac{1}{2}$, so we likely have $\Theta \approx \frac{N}{2}$
  - If the arctic boundary exists, it should meet the bottom boundary of $[0,1]\times[0,1]$ at $\left(\frac{N}{2}, 0\right)$
- Colomo–Sportiello (2016): Use the function $\mathfrak{h}$ to predict a parameterization for the limiting trajectory of $\mathbf{p}_1$ (entire arctic boundary)

Augmented Domains and Ensembles

For $\Psi \in \mathbb{Z}_{\geq 0}$, a Ψ**-augmented ensemble** is a domain-wall six-vertex ensemble on $[1, N] \times [1, N]$, with an additional path entering at $(0, -\Psi)$ and exiting at $(N + 1, N)$.

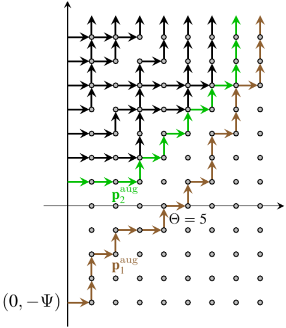


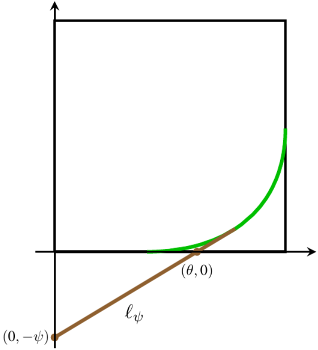
- Denote the paths in this ensemble, from bottom to top, by $\mathbf{p}_1^{\text{aug}}, \mathbf{p}_2^{\text{aug}}, \ldots, \mathbf{p}_{N+1}^{\text{aug}}$.
- Let Θ denote be such that $\mathbf{p}_1^{\text{aug}}$ exits the x-axis at $(\Theta, 0)$

Tangency Assumption

- Fix $\psi > 0$, and let $\Psi \approx \psi N$
- Select a Ψ-augmented ensemble $\mathcal{E}_\Psi$ uniformly at random
- With high probability, we will have $\Theta = \Theta(\mathcal{E}_\Psi) \approx \theta N$, for some $\theta = \theta(\psi) > 0$

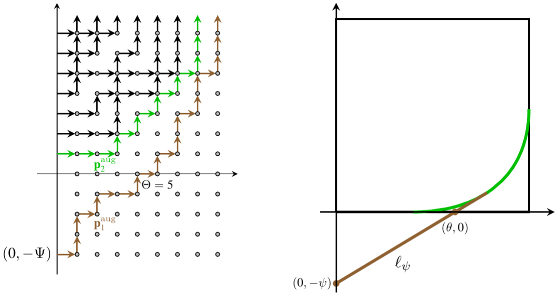
Belief: As N tends to ∞, $\mathbf{p}_1^{\text{aug}}$ first approximates a line ℓ_ψ **tangent to the arctic boundary** of the domain-wall ice model and then merges with it.





Determining the Arctic Boundary

- If we could determine $\theta = \theta(\psi)$ for each $\psi > 0$, then we would determine ℓ_ψ.
- Convex envelope obtained by varying over ψ gives $\mathfrak{A}_{\mathrm{SE}}$.



- The number of augmented ensembles $\mathcal{E}_\Psi$ with $\Theta(\mathcal{E}_\Psi) = \Phi \approx \varphi N$ is proportional to

$$\binom{\Phi + \Psi - 1}{\Psi} H_{N+1}(\Phi) = \exp\Big(\big(g_\psi(\varphi) + o(1)\big)N\Big),$$

where $g_\psi(\varphi) = (\varphi + \psi)\log(\varphi + \psi) - \varphi \log \varphi - \psi \log \psi - \mathfrak{h}(\varphi)$.

- The maximizer $\varphi = \theta$ of $g_\psi(\varphi)$ determines $\theta = \theta(\psi) > 0$.

Tangent Method Heuristic

The tangent method (Colomo–Sportiello, 2016)

- Using exact asymptotics for $H_N(K)$, find explicit $\mathfrak{h} : \mathbb{R}_{>0} \to \mathbb{R}$ such that

$$H_N(\kappa N) = \exp\Big(- \big(\mathfrak{h}(\kappa) + o(1)\big)N\Big)$$

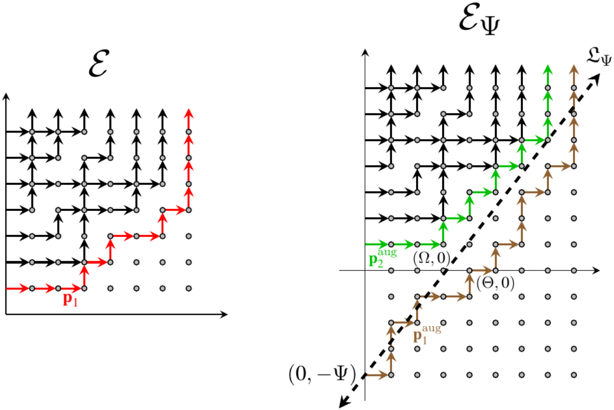
- Define $g_\psi(\varphi) = (\varphi + \psi)\log(\varphi + \psi) - \varphi \log \varphi - \psi \log \psi - \mathfrak{h}(\varphi)$
- Let $\theta = \theta(\psi)$ denote the maximizer of g_ψ
- For each ψ, let ℓ_ψ denote the line through $(0, -\psi)$ and $(0, \theta)$
- Then the arctic boundary is the convex envelope formed by the ℓ_ψ after varying over ψ, which is $\mathfrak{A}_{\mathrm{SE}}$

Issues

- Must justify the tangency assumption
- It is not transparent that arctic boundary exists (namely, that $\mathbf{p}_1$ in the original model or $\mathbf{p}_2^{\mathrm{aug}}$ in the augmented model have limiting trajectories)
- The introduction of the new path $\mathbf{p}_1^{\mathrm{aug}}$ in the augmented model might change the trajectory of $\mathbf{p}_1$ in the original model

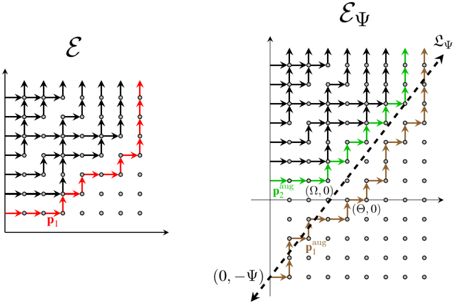
Notation

- Let $\mathcal{E}$ and $\mathcal{E}_\Psi$ be a domain-wall six-vertex ensemble and a Ψ-augmented ensemble, respectively, both chosen uniformly at random.
- Let $\mathfrak{L} = \mathfrak{L}_\Psi$ be the tangent line to $\mathbf{p}_2^{\text{aug}}$ through $(0, -\Psi)$.
- Let $(\Omega, 0) = \mathfrak{L}_\Psi \cap \{y = 0\}$, and let $\mathbf{p}_1^{\text{aug}}$ exit the x-axis at $(\Theta, 0)$.



Proof Outline

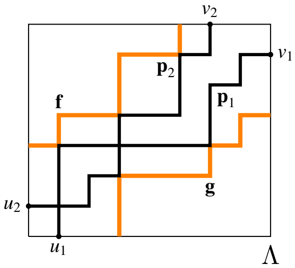
1. *Tangency*: $\mathbb{P}\Big[|\Omega - \Theta| < \varepsilon N\Big] > 1 - C\exp(-c\varepsilon^2 N)$
2. *Concentration Estimate*: $\mathbb{P}\Big[|\Theta - \theta N| < \varepsilon N\Big] > 1 - C\exp(-c\varepsilon^2 N)$
3. *Comparing* $\mathbf{p}_1$ *and* $\mathbf{p}_2^{\mathrm{aug}}$: Stochastically bound $\mathbf{p}_1$ approximately above and approximately below by $\mathbf{p}_2^{\mathrm{aug}}$
    1. Couple $\mathcal{E}$ and $\mathcal{E}_\Psi$ in two ways, such that $\mathbf{p}_1$ is (weakly) below $\mathbf{p}_2^{\mathrm{aug}}$ under the first and $\mathbf{p}_2$ is (weakly) above $\mathbf{p}_2^{\mathrm{aug}}$ under the second
    2. $\mathbb{P}\Big[\operatorname{dist}(\mathbf{p}_1, \mathbf{p}_2) < \varepsilon N\Big] > 1 - C\exp(-c\varepsilon^2 N)$



Concentration estimate follows from exact enumeration: $\mathbb{P}[\Theta = \Phi] = Z_\Psi^{-1}\binom{\Phi+\Psi-1}{\Psi}H_{N+1}(\Phi)$

Boundary Data

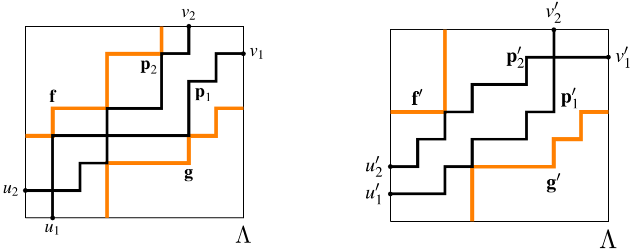
If X and Y are vertices / noncrossing paths, with X northwest of Y, we say $X \leq Y$.



- Rectangle Λ
- "Barrier paths" $\mathbf{f}$ and $\mathbf{g}$ with $\mathbf{f} \leq \mathbf{g}$
- "Entrance vertices" $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ with $u_1 \geq u_2 \geq \cdots \geq u_m$
- "Exit vertices" $\mathbf{v} = (v_1, v_2, \ldots, v_m)$, with $v_1 \geq v_2 \geq \cdots \geq v_m$
- Let $\mathfrak{E}_{\mathbf{f};\mathbf{g}}^{\mathbf{u};\mathbf{w}}$ denote set of six-vertex ensembles on Λ whose paths $\mathbf{p}_1 \geq \mathbf{p}_2 \geq \cdots \mathbf{p}_m$ satisfy $\mathbf{f} \leq \mathbf{p}_i \leq \mathbf{g}$, such that $\mathbf{p}_i$ enters Λ through u_i and exits Λ through v_i

Monotone Couplings

- Assume boundary data $(\mathbf{f}, \mathbf{g}; \mathbf{u}, \mathbf{v})$ and $(\mathbf{f}', \mathbf{g}'; \mathbf{u}', \mathbf{v}')$ satisfy $\mathbf{f} \geq \mathbf{f}'$, $\mathbf{g} \geq \mathbf{g}'$, $\mathbf{u} \geq \mathbf{u}'$, $\mathbf{v} \geq \mathbf{v}'$
- Uniformly random ensembles $\mathcal{E}$ and $\mathcal{E}'$ in $\mathfrak{E} = \mathfrak{E}_{\mathbf{f};\mathbf{g}}^{\mathbf{u};\mathbf{w}}$ and $\mathfrak{E}' = \mathfrak{E}_{\mathbf{f}';\mathbf{g}'}^{\mathbf{u}';\mathbf{w}'}$, respectively
- Paths of $\mathcal{E}$ and $\mathcal{E}'$ are $\mathbf{p}_1 \geq \mathbf{p}_2 \geq \cdots \mathbf{p}_m$ and $\mathbf{p}_1' \geq \mathbf{p}_2' \geq \cdots \mathbf{p}_m'$, respectively



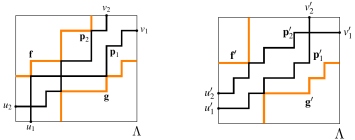
Lemma

The laws of $\mathcal{E}$ and $\mathcal{E}'$ can be coupled so that each $\boldsymbol{p}_i \geq \boldsymbol{p}_i'$, almost surely.

- Allows $\mathbf{f}, \mathbf{f}' = -\infty$ and / or $\mathbf{g}, \mathbf{g}' = \infty$
- Proof uses monotonicity of Glauber dynamics (used by Corwin–Hammond, 2014)
- Essentially only place where ice weights are used (outside of integrable input)
  - Sometimes known as Fortuin–Kasteleyn–Ginibre (FKG) type condition
    - Holds for a broad class of statistical mechanical models (such as six-vertex at $\Delta \leq \frac{1}{2}$)

Proof Outline for Monotonicity

There exist $\mathcal{E}(0) \in \mathfrak{E}$ and $\mathcal{E}'(0) \in \mathfrak{E}'$ with paths $\mathbf{p}_i(0)$ and $\mathbf{p}'_i(0)$, respectively, so that $\mathbf{p}_i(0) \geq \mathbf{p}'_i(0)$.



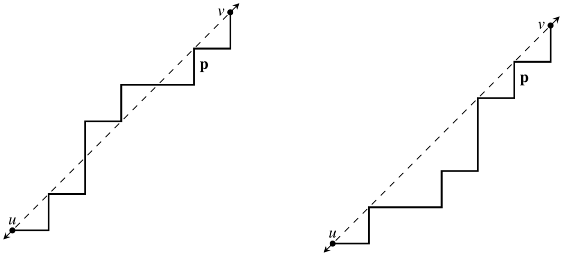
Run the **Glauber dynamics** on $\big(\mathcal{E}(0), \mathcal{E}'(0)\big)$

- Select a face F of Λ uniformly at random
- With probability $\frac{1}{2}$, perform "up-flip" (if possible) in $\mathcal{E}(0)$ and $\mathcal{E}'(0)$ at F
- Otherwise perform "down-flip" in $\mathcal{E}(0)$ and $\mathcal{E}'(0)$ at F
- This produces new (random, coupled) six-vertex ensembles $\mathcal{E}(1) \in \mathfrak{E}$ and $\mathcal{E}'(1) \in \mathfrak{E}'$
- Repeating this, we obtain random, coupled $\mathcal{E}(1), \mathcal{E}(2), \ldots \in \mathfrak{E}$ and $\mathcal{E}'(1), \mathcal{E}'(2), \ldots \in \mathfrak{E}'$

- **Monotone preserving property**: If each $\mathbf{p}_i(t) \geq \mathbf{p}'_i(t)$, then each $\mathbf{p}_i(t+1) \geq \mathbf{p}'_i(t+1)$
- Then $\mathcal{E}(\infty) = \lim_{t \to \infty} \mathcal{E}(t)$ and $\mathcal{E}'(\infty) = \lim_{t \to \infty} \mathcal{E}'(t)$ are uniform on $\mathfrak{E}$ and $\mathfrak{E}'$, respectively, since the Glauber dynamics are stationary with respect to these uniform measures, and each $\mathbf{p}_i(\infty) \geq \mathbf{p}'_i(\infty)$ almost surely

Linearity Estimates

- Let $u, v \in \mathbb{Z}^2$, with v northeast of u, and set $\text{dist}(u, v) = M$.
- Let $\ell = \ell(u, v)$ denote the line through u and v.



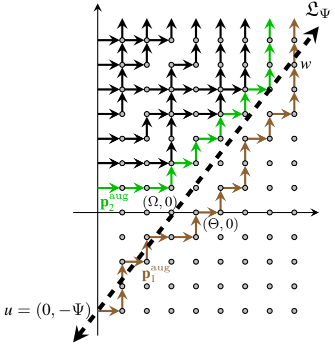
Standard estimates for linearity of (possibly conditioned) random walks

1. For a uniformly random path $\mathbf{p}$ from u to v,
$\mathbb{P}\left[\text{dist}(\mathbf{p}, \ell) < \varepsilon M\right] > 1 - C\exp(-c\varepsilon^2 M)$.
2. For a uniformly random path $\mathbf{p}$ from u to v conditioned to lie weakly below (or above) ℓ, $\mathbb{P}\left[\text{dist}(\mathbf{p}, \ell) < \varepsilon M\right] > 1 - C\exp(-c\varepsilon^2 M)$.

Second statement can formally be deduced from first and monotonicity

Proof of $\Theta \approx \Omega$

Set $u = (0, -\Psi)$, and let w be the first vertex in $\mathbf{p}_1^{\mathrm{aug}}$ above the x-axis such that w is (weakly) below $\mathfrak{L}_\Psi$ but the next vertex in $\mathbf{p}_1^{\mathrm{aug}}$ is not.



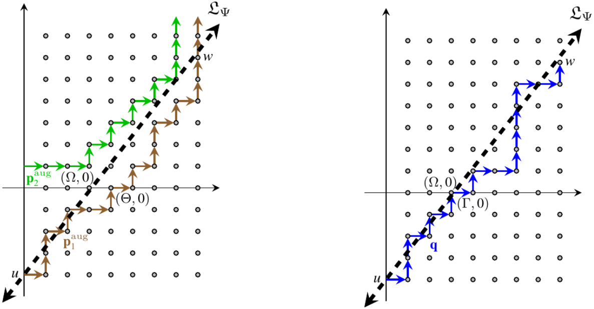
We condition on the following.

- The paths $\mathbf{p}_2^{\mathrm{aug}}, \mathbf{p}_3^{\mathrm{aug}}, \ldots, \mathbf{p}_{N+1}^{\mathrm{aug}}$
- The event that $\mathbf{p}_1^{\mathrm{aug}}$ passes through w, and the part of $\mathbf{p}_1^{\mathrm{aug}}$ northeast of w

Gibbs property: The law of $\mathbf{p}_1^{\mathrm{aug}}$ southwest of w is given by a uniformly random path from u to w, conditioned to remain weakly below $\mathbf{p}_2^{\mathrm{aug}}$.

Proof of $\Theta \approx \Omega$

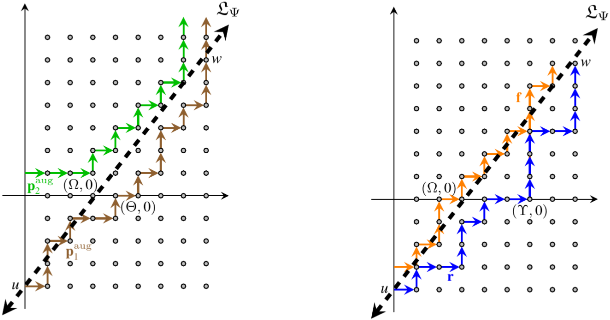
Gibbs property: The law of $\mathbf{p}_1^{\mathrm{aug}}$ is given by a uniformly random path in $\mathfrak{E}^{u;w}_{\mathbf{p}_2^{\mathrm{aug}};\infty}$.



- Let $\mathbf{q}$ be a uniformly random path in $\mathfrak{E}^{u;w}_{-\infty,\infty}$ (from u to w without barriers)
- By the linearity estimate, $\mathbf{q}$ is εN-linear with probability $1 - C\exp(-c\varepsilon^2 N)$
- So, if $\mathbf{q}$ exits the x-axis at $(\Gamma, 0)$, then $\mathbb{P}\big[|\Gamma - \Omega| < \varepsilon N\big] \geq 1 - C\exp(-c\varepsilon^2 N)$
- By monotonicity, we may couple $\mathbf{p}_1^{\mathrm{aug}}$ and $\mathbf{q}$ so that $\mathbf{p}_1^{\mathrm{aug}} \geq \mathbf{q}$ almost surely
- Thus, $\mathbb{P}\big[\Theta \geq \Omega - \varepsilon N\big] \geq \mathbb{P}\big[\Gamma \geq \Omega - \varepsilon N\big] \geq 1 - C\exp(-c\varepsilon^2 N)$

Proof of $\Theta \approx \Omega$

Gibbs property: The law of $\mathbf{p}_1^{\mathrm{aug}}$ is given by a uniformly random path in $\mathfrak{E}^{u;w}_{\mathbf{p}_2^{\mathrm{aug}};\infty}$.



- Let $\mathbf{r}$ be a uniformly random path from u to v, conditioned to lie weakly below $\mathfrak{L}_\Psi$ (so it is uniform on $\mathfrak{E}^{u;w}_{\mathbf{f},\infty}$, for some $\mathbf{f} \geq \mathbf{p}_2^{\mathrm{aug}}$)
- By the linearity estimate, $\mathbf{r}$ is εN-linear with probability $1 - C\exp(-c\varepsilon^2 N)$
- So, if $\mathbf{r}$ exits the x-axis at $(\Upsilon, 0)$, then $\mathbb{P}\big[|\Upsilon - \Omega| < \varepsilon N\big] \geq 1 - C\exp(-c\varepsilon^2 N)$
- By monotonicity, we may couple $\mathbf{p}_1^{\mathrm{aug}}$ and $\mathbf{r}$ so that $\mathbf{p}_1^{\mathrm{aug}} \leq \mathbf{r}$ almost surely
- Thus, $\mathbb{P}\big[\Theta \leq \Omega + \varepsilon N\big] \geq \mathbb{P}\big[\Upsilon \leq \Omega + \varepsilon N\big] \geq 1 - C\exp(-c\varepsilon^2 N)$

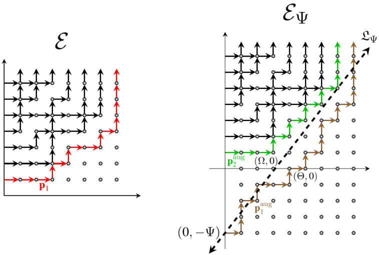
Comparing $\mathbf{p}_1$ and $\mathbf{p}_2^{\text{aug}}$

Seek to stochastically bound $\mathbf{p}_1$ approximately above / below by $\mathbf{p}_2^{\text{aug}}$

1. Couple $\mathcal{E}$ and $\mathcal{E}_\Psi$ in two ways, such that $\mathbf{p}_1$ is (weakly) below $\mathbf{p}_2^{\text{aug}}$ under the first and $\mathbf{p}_2$ is (weakly) above $\mathbf{p}_2^{\text{aug}}$ under the second
2. $\mathbb{P}\big[\operatorname{dist}(\mathbf{p}_1, \mathbf{p}_2) < \varepsilon N\big] > 1 - C\exp(-c\varepsilon^2 N)$

First part follows from monotonicity

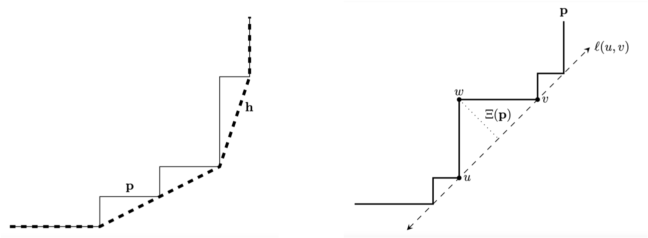
- View top path in $\mathcal{E}_\Psi$ as barrier: Remaining paths below correpsonding $\mathcal{E}$ paths
  - Monotonicity implies coupling so that $\mathbf{p}_2 \leq \mathbf{p}_2^{\text{aug}}$
- View bottom path in $\mathcal{E}_\Psi$ as barrier: Remaining paths above $\mathcal{E}$ paths
  - Montonocity implies coupling so that $\mathbf{p}_2 \geq \mathbf{p}_2^{\text{aug}}$



Proximity of $\mathbf{p}_1$ and $\mathbf{p}_2$

Seek to show $\mathbb{P}\big[\,\mathrm{dist}(\mathbf{p}_1, \mathbf{p}_2) < \varepsilon N\big] > 1 - C\exp(-c\varepsilon^2 N)$

1. Show $\mathbf{p}_1$ and $\mathbf{p}_2$ are likely "approximately convex"
2. Show approximate convexity of $\mathbf{p}_2$ likely implies $\mathrm{dist}(\mathbf{p}_1, \mathbf{p}_2) < \varepsilon N$



- Let $\mathbf{h} = \mathbf{h}(\mathbf{p})$ denote the convex envelope of any path $\mathbf{p}$
- Let $\Xi = \Xi(\mathbf{p}) = \max_{v \in \mathbf{p}} \mathrm{dist}\big(v, \mathbf{h}(\mathbf{p})\big)$

Define event $\mathcal{E} = \mathcal{E}(\varepsilon) = \big\{\Xi(\mathbf{p}_1) < \varepsilon N\big\} \cap \big\{\Xi(\mathbf{p}_2) < \varepsilon N\big\}$

- On $\mathcal{E}$, the paths $\mathbf{p}_1$ and $\mathbf{p}_2$ are "approximately convex"

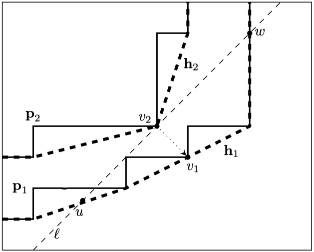
1. Show $\mathbb{P}[\mathcal{E}] > 1 - C\exp(-c\varepsilon^2 N)$
2. Show $\mathbb{P}\big[\mathbf{1}_{\mathcal{E}}\,\mathrm{dist}(\mathbf{p}_1, \mathbf{p}_2) < 5\varepsilon N\big] > 1 - \exp(C\varepsilon^2 N)$

Convexity Implies Proximity

Set $\mathbf{h}_1 = \mathbf{h}(\mathbf{p}_1)$ and $\mathbf{h}_2 = \mathbf{h}(\mathbf{p}_2)$

- On convexity event $\mathcal{E}$, we have $\operatorname{dist}(\mathbf{p}_1, \mathbf{p}_2) \leq \operatorname{dist}(\mathbf{h}_1, \mathbf{h}_2) + 2\varepsilon N$

Suffices to show $\mathbb{P}\left[\mathbf{1}_{\mathcal{E}} \operatorname{dist}(\mathbf{h}_1, \mathbf{h}_2) < 3\varepsilon N\right] > 1 - C\exp(c\varepsilon^2 N)$



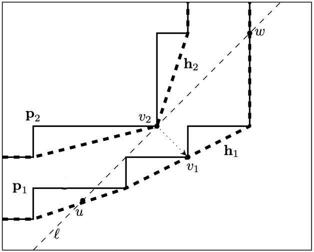
- Fix $v_1 \in \mathbf{h}_1$, and let $v_2 \in \mathbf{h}_2$ be such that $\operatorname{dist}(v_1, v_2) = \operatorname{dist}(v_1, \mathbf{h}_2)$

Must show $\mathbb{P}\left[\mathbf{1}_{\mathcal{E}} \operatorname{dist}(v_1, v_2) < \varepsilon N\right] > 1 - C\exp(-c\varepsilon^2 N)$

- Let ℓ be line through v_2 orthogonal to line through (v_1, v_2)
- Convexity of $\mathbf{h}_2$ implies $\mathbf{h}_2 \subset \mathrm{NW}(\ell)$ (is northwest of ℓ)
    - Assume for simplicity that $\mathbf{p}_2 \subset \mathrm{NW}(\ell)$
        - Holds after shifting ℓ down by εN, since $\mathbf{1}_{\mathcal{E}} \operatorname{dist}(\mathbf{p}_2, \mathbf{h}_2) < \varepsilon N$ and $\mathbf{h}_2 \subset \mathrm{NW}(\ell)$
- Let ℓ meet $\mathbf{h}_1$ at (u, w), and assume for simplicity that $u, w \in \mathbf{p}_1$

Convexity Implies Proximity

Must show that $\mathbb{P}\left[\mathbf{1}_{\mathcal{E}}\,\mathrm{dist}(v_1, v_2) < \varepsilon N\right] > 1 - C\exp(-c\varepsilon^2 N)$



- Condition on $\mathbf{p}_2$ and on $\mathbf{p}_1$ outside of interval (u, w)
  - Gibbs property: Then $\mathbf{p}_1$ is a uniformly random path starting at u and ending at w, and conditioned to lie above $\mathbf{p}_2$
- Montonicity: Replacing $\mathbf{p}_2$ with ℓ only "pushes v down"
- Linearity: With probability $1 - C\exp(-c\varepsilon^2 N)$, a uniformly random path from u to w conditioned to stay below ℓ does not go below ℓ by more than εN
- Shows $\mathbb{P}\left[\mathbf{1}_{\mathcal{E}}\,\mathrm{dist}(v_1, v_2) < \varepsilon N\right] > 1 - C\exp(-c\varepsilon^2 N)$

Summary

- Established arctic boundaries for domain-wall ice model
- Proceeds by **justification of tangent method** of Colomo–Sportiello
    - Involves inserting an augmented path in the domain
    - Path should be tangent to arctic boundary
    - Refined partition function asymptotics identify trajectory of the path
        - Integrability only involved through understanding these asymptotics
        - Full solvability / determinantality of the model not required
- Proof involves analysis of non-intersecting path ensembles (reminiscent of ideas used by Corwin–Hammond in very different context)
    - Prove approximate tangency of additional path to arctic boundary of augmented ensemble
        - Gibbs property
        - Monotonicity
    - Prove additional path does not substantially affect arctic boundary
        - Convexity (and Gibbs property / monotonicity)