Let $\Lambda \subset \mathbb{Z}^2$ be finite, and assign each vertex in $\Lambda$ one of the following six edge configurations:

- **Domain-wall boundary conditions** arise when $\Lambda = [1, N] \times [1, N]$, and arrows enter from the left boundary and exit through the top.

- **Ice model**: Assignment is chosen uniformly at random.

Six-vertex ensembles are collections of non-crossing directed (up-right) paths.
Arctic Boundary of Six-Vertex Ensembles

- Six-vertex ensemble $\mathcal{E}$ on $\Lambda$
- A vertex $v \in \Lambda$ is in the **frozen region** of $\mathcal{E}$ if one of the following holds
  - Every vertex northwest of $v$ is packed in $\mathcal{E}$
  - Every vertex northeast of $v$ is vertical in $\mathcal{E}$
  - Every vertex southwest of $v$ is horizontal in $\mathcal{E}$
  - Every vertex southeast of $v$ is empty in $\mathcal{E}$
- The boundary of the frozen region is called the **arctic boundary**

The **bottommost path** of a domain-wall six-vertex ensemble traces the southeast boundary of the frozen region
Define the portion of an ellipse

\[ \mathcal{A}_{SE} = \{ (x, y) \in \mathbb{R}^2 : (2x - 1)^2 + (2y - 1)^2 - 4(1 - x)y = 1 \} \cap \left( \left[ \frac{1}{2}, 1 \right] \times \left[ 0, \frac{1}{2} \right] \right), \]

and its reflections

\[ \mathcal{A}_{SW} = \{ (x, y) \in \mathbb{R}^2 : (1 - x, y) \in \mathcal{A}_{SE} \}; \quad \mathcal{A}_{NE} = \{ (x, y) \in \mathbb{R}^2 : (x, 1 - y) \in \mathcal{A}_{SE} \}; \quad \mathcal{A}_{NW} = \{ (x, y) \in \mathbb{R}^2 : (1 - x, 1 - y) \in \mathcal{A}_{SE} \}. \]

Let \( \mathcal{A} = \mathcal{A}_{SE} \cup \mathcal{A}_{SW} \cup \mathcal{A}_{NE} \cup \mathcal{A}_{NW} \).

Then \( \mathcal{A} \) is not smooth at its four tangency points with \([0, 1] \times [0, 1]\).

- Different from what one observes in dimers...
Let $N \in \mathbb{Z}_{>0}$ be a large integer.

Let $E$ denote a sample of the ice model on $\Lambda = [1, N] \times [1, N]$.

Let $(i, j) \in [1, N] \times [1, N]$ be an integer pair, and set $z = \left( \frac{i}{N}, \frac{j}{N} \right) \in [0, 1] \times [0, 1]$.

Fix a real number $\varepsilon > 0$, and assume that $\operatorname{dist}(z, \mathcal{A}) > \varepsilon$.

**Theorem (A., 2018)**

There exists $\delta = \delta(\varepsilon) > 0$ such that, with probability at least $1 - e^{-\delta N}$, $(i, j)$ is in the frozen region of $\mathcal{M}$ if and only if $z$ is outside of $\mathcal{A}$.


Colomo–Pronko (2010): Predicted above explicit form of arctic boundary

Colomo–Sportiello (2016): Reproduced prediction through tangent method

Let $\mathcal{E}$ denote a sample of the ice model on $\Lambda = [1, N] \times [1, N]$.

Denote the non-crossing paths in $\mathcal{E}$, from bottom to top, by $p_1, p_2, \ldots, p_N$.

Define $I_1 = [0, \frac{1}{2}] \times \{0\}$ and $I_2 = \{1\} \times [\frac{1}{2}, 1]$, and let $\mathcal{P} = I_1 \cup \mathcal{A}_{SE} \cup I_2$.

By symmetry, we must show the following theorem.

**Theorem**

*For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\text{dist} \left( N^{-1}p_1, \mathcal{P} \right) < \varepsilon$ holds with probability at least $1 - e^{-\delta N}$.***

- Proof based on a justification of the *(geometric) tangent method*, a general heuristic introduced by Colomo–Sportiello (2016) for deriving arctic boundaries of statistical mechanical models
- Proof is not very model-dependent and also should apply to other families of statistical mechanical systems
Refined Partition Function

- Domain-wall six-vertex ensemble $\mathcal{E}$ with paths $p_1, p_2, \ldots, p_N$
- Let $\Theta = \Theta(\mathcal{E}) \in [1, N]$ be such that $p_1$ exits the bottom row at $(\Theta, 1)$

The partition function $Z_N$ counts domain-wall six-vertex ensembles $\mathcal{E}$.

The refined partition function $Z_N(K)$ counts those with $\Theta(\mathcal{E}) = K$.

Define the $K$-refined correlation function $H_N(K)$ by

$$H_N(K) = \mathbb{P}[\Theta(\mathcal{E}) = K] = \frac{Z_N(K)}{Z_N}.$$
Required integrable input: Asymptotics for refined partition function

- **Zeilberger (1996):** \( H_N(K) = \binom{N+K-2}{N-1} \binom{2N-K-1}{N-1} \binom{3N-2}{N-1}^{-1} \)

- Thus, for fixed \( \kappa > 0 \), we have for large \( N \) that

\[
H_N(\kappa N) = \exp \left( - (\mathcal{h}(\kappa) + o(1))N \right),
\]

for an explicit \( \mathcal{h}(\kappa) \) given by

\[
\mathcal{h}(\kappa) = (1 + \kappa) \log(1 + \kappa) + (2 - \kappa) \log(2 - \kappa) - \kappa \log \kappa
- (1 - \kappa) \log(1 - \kappa) - 3 \log 3 + 2 \log 2
\]

- **Tangency point:** \( \mathcal{h}(\kappa) \) minimized at \( \kappa = \frac{1}{2} \), so we likely have \( \Theta \approx \frac{N}{2} \)
  - If the arctic boundary exists, it should meet the bottom boundary of \([0, 1] \times [0, 1] \) at \( \left( \frac{N}{2}, 0 \right) \)

- **Colomo–Sportiello (2016):** Use the function \( \mathcal{h} \) to predict a parameterization for the limiting trajectory of \( \mathbf{p}_1 \) (entire arctic boundary)
For $\Psi \in \mathbb{Z}_{\geq 0}$, a $\Psi$-augmented ensemble is a domain-wall six-vertex ensemble on $[1, N] \times [1, N]$, with an additional path entering at $(0, -\Psi)$ and exiting at $(N + 1, N)$.

- Denote the paths in this ensemble, from bottom to top, by $p_{1}^{\text{aug}}, p_{2}^{\text{aug}}, \ldots, p_{N+1}^{\text{aug}}$.
- Let $\Theta$ denote be such that $p_{1}^{\text{aug}}$ exits the $x$-axis at $(\Theta, 0)$. 
Tangency Assumption

- Fix $\psi > 0$, and let $\Psi \approx \psi N$
- Select a $\Psi$-augmented ensemble $\mathcal{E}_\Psi$ uniformly at random
- With high probability, we will have $\Theta = \Theta(\mathcal{E}_\Psi) \approx \theta N$, for some $\theta = \theta(\psi) > 0$

**Belief:** As $N$ tends to $\infty$, $p_{1}^{\text{aug}}$ first approximates a line $\ell_\psi$ **tangent to the arctic boundary** of the domain-wall ice model and then merges with it.
Determining the Arctic Boundary

- If we could determine $\theta = \theta(\psi)$ for each $\psi > 0$, then we would determine $\mathcal{E}_\psi$.
- Convex envelope obtained by varying over $\psi$ gives $\mathcal{A}_{SE}$.

The number of augmented ensembles $\mathcal{E}_\Psi$ with $\Theta(\mathcal{E}_\Psi) = \Phi \approx \varphi N$ is proportional to

$$
\left(\frac{\Phi + \Psi - 1}{\Psi}\right) H_{N+1}(\Phi) = \exp \left( (g_\psi(\varphi) + o(1)) N \right),
$$

where $g_\psi(\varphi) = (\varphi + \psi) \log(\varphi + \psi) - \varphi \log \varphi - \psi \log \psi - h(\varphi)$.

- The maximizer $\varphi = \theta$ of $g_\psi(\varphi)$ determines $\theta = \theta(\psi) > 0$. 

Tangent Method Heuristic

The tangent method (Colomo–Sportiello, 2016)

- Using exact asymptotics for $H_N(K)$, find explicit $\eta : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ such that
  $$H_N(\kappa N) = \exp \left( - (\eta(\kappa) + o(1))N \right)$$

- Define $g_\psi(\varphi) = (\varphi + \psi) \log(\varphi + \psi) - \varphi \log \varphi - \psi \log \psi - \eta(\varphi)$
- Let $\theta = \theta(\psi)$ denote the maximizer of $g_\psi$
- For each $\psi$, let $\ell_\psi$ denote the line through $(0, -\psi)$ and $(0, \theta)$
- Then the arctic boundary is the convex envelope formed by the $\ell_\psi$ after varying over $\psi$, which is $\mathcal{A}_{SE}$

Issues

- Must justify the tangency assumption
- It is not transparent that arctic boundary exists (namely, that $p_1$ in the original model or $p_{2}^{\text{aug}}$ in the augmented model have limiting trajectories)
- The introduction of the new path $p_{1}^{\text{aug}}$ in the augmented model might change the trajectory of $p_1$ in the original model
Notation

- Let $\mathcal{E}$ and $\mathcal{E}_\Psi$ be a domain-wall six-vertex ensemble and a $\Psi$-augmented ensemble, respectively, both chosen uniformly at random.
- Let $\mathcal{L} = \mathcal{L}_\Psi$ be the tangent line to $p_{2}^{\text{aug}}$ through $(0, -\Psi)$.
- Let $(\Omega, 0) = \mathcal{L}_\Psi \cap \{y = 0\}$, and let $p_{1}^{\text{aug}}$ exit the $x$-axis at $(\Theta, 0)$. 

\[ (0, -\Psi) \]
Proof Outline

1. **Tangency:** $\mathbb{P}[|\Omega - \Theta| < \varepsilon N] > 1 - C \exp(-c\varepsilon^2 N)$

2. **Concentration Estimate:** $\mathbb{P}[|\Theta - \theta N| < \varepsilon N] > 1 - C \exp(-c\varepsilon^2 N)$

3. **Comparing $p_1$ and $p_2^{\text{aug}}$:** Stochastically bound $p_1$ approximately above and approximately below by $p_2^{\text{aug}}$

   1. Couple $\mathcal{E}$ and $\mathcal{E}_\Psi$ in two ways, such that $p_1$ is (weakly) below $p_2^{\text{aug}}$ under the first and $p_2$ is (weakly) above $p_2^{\text{aug}}$ under the second

2. $\mathbb{P} \left[ \text{dist}(p_1, p_2) < \varepsilon N \right] > 1 - C \exp(-c\varepsilon^2 N)$

Concentration estimate follows from exact enumeration: $\mathbb{P}[\Theta = \Phi] = Z^{-1}_\Psi \left( \Phi + \Psi^{-1} \right) H_{N+1}(\Phi)$
Boundary Data

If $X$ and $Y$ are vertices / noncrossing paths, with $X$ northwest of $Y$, we say $X \leq Y$.

- Rectangle $\Lambda$
- “Barrier paths” $f$ and $g$ with $f \leq g$
- “Entrance vertices” $u = (u_1, u_2, \ldots, u_m)$ with $u_1 \geq u_2 \geq \cdots \geq u_m$
- “Exit vertices” $v = (v_1, v_2, \ldots, v_m)$, with $v_1 \geq v_2 \geq \cdots \geq v_m$
- Let $\mathcal{E}_{f,g}^{u,v}$ denote set of six-vertex ensembles on $\Lambda$ whose paths $p_1 \geq p_2 \geq \cdots p_m$ satisfy $f \leq p_i \leq g$, such that $p_i$ enters $\Lambda$ through $u_i$ and exits $\Lambda$ through $v_i$
Monotone Couplings

- Assume boundary data \((f, g; u, v)\) and \((f', g'; u', v')\) satisfy \(f \geq f',\ g \geq g',\ u \geq u',\ v \geq v'\)
- Uniformly random ensembles \(\mathcal{E}\) and \(\mathcal{E}'\) in \(\mathcal{E} = \mathcal{E}_{f;g}^{u;w}\) and \(\mathcal{E}' = \mathcal{E}_{f';g'}^{u';w'}\), respectively
- Paths of \(\mathcal{E}\) and \(\mathcal{E}'\) are \(p_1 \geq p_2 \geq \cdots p_m\) and \(p'_1 \geq p'_2 \geq \cdots p'_m\), respectively

\[
\begin{array}{c}
\text{Lemma}
\end{array}
\]

The laws of \(\mathcal{E}\) and \(\mathcal{E}'\) can be coupled so that each \(p_i \geq p'_i\), almost surely.

- Allows \(f, f' = -\infty\) and / or \(g, g' = \infty\)
- Proof uses monotonicity of Glauber dynamics (used by Corwin–Hammond, 2014)
- Essentially only place where ice weights are used (outside of integrable input)
  - Sometimes known as Fortuin–Kasteleyn–Ginibre (FKG) type condition
    - Holds for a broad class of statistical mechanical models (such as six-vertex at \(\Delta \leq \frac{1}{2}\))
Proof Outline for Monotonicity

There exist $\mathcal{E}(0) \in \mathcal{E}$ and $\mathcal{E}'(0) \in \mathcal{E}'$ with paths $p_i(0)$ and $p'_i(0)$, respectively, so that $p_i(0) \geq p'_i(0)$.

- Run the Glauber dynamics on $(\mathcal{E}(0), \mathcal{E}'(0))$
  - Select a face $F$ of $\Lambda$ uniformly at random
  - With probability $\frac{1}{2}$, perform “up-flip” (if possible) in $\mathcal{E}(0)$ and $\mathcal{E}'(0)$ at $F$
  - Otherwise perform “down-flip” in $\mathcal{E}(0)$ and $\mathcal{E}'(0)$ at $F$
  - This produces new (random, coupled) six-vertex ensembles $\mathcal{E}(1) \in \mathcal{E}$ and $\mathcal{E}'(1) \in \mathcal{E}'$
  - Repeating this, we obtain random, coupled $\mathcal{E}(1), \mathcal{E}(2), \ldots \in \mathcal{E}$ and $\mathcal{E}'(1), \mathcal{E}'(2), \ldots \in \mathcal{E}'$

- **Monotone preserving property**: If each $p_i(t) \geq p'_i(t)$, then each $p_i(t + 1) \geq p'_i(t + 1)$
  - Then $\mathcal{E}(\infty) = \lim_{t \to \infty} \mathcal{E}(t)$ and $\mathcal{E}'(\infty) = \lim_{t \to \infty} \mathcal{E}'(t)$ are uniform on $\mathcal{E}$ and $\mathcal{E}'$, respectively, since the Glauber dynamics are stationary with respect to these uniform measures, and each $p_i(\infty) \geq p'_i(\infty)$ almost surely
Let $u, v \in \mathbb{Z}^2$, with $v$ northeast of $u$, and set $\text{dist}(u, v) = M$.

Let $\ell = \ell(u, v)$ denote the line through $u$ and $v$.

Standard estimates for linearity of (possibly conditioned) random walks

1. For a uniformly random path $p$ from $u$ to $v$, 
   \[ \mathbb{P}\left[ \text{dist}(p, \ell) < \varepsilon M \right] > 1 - C \exp(-c\varepsilon^2 M). \]

2. For a uniformly random path $p$ from $u$ to $v$ conditioned to lie weakly below (or above) $\ell$, 
   \[ \mathbb{P}\left[ \text{dist}(p, \ell) < \varepsilon M \right] > 1 - C \exp(-c\varepsilon^2 M). \]

Second statement can formally be deduced from first and monotonicity.
Proof of $\Theta \approx \Omega$

Set $u = (0, -\Psi)$, and let $w$ be the first vertex in $p_{1}^{\text{aug}}$ above the $x$-axis such that $w$ is (weakly) below $\mathcal{L}_\Psi$ but the next vertex in $p_{1}^{\text{aug}}$ is not.

We condition on the following.

- The paths $p_{2}^{\text{aug}}, p_{3}^{\text{aug}}, \ldots, p_{N+1}^{\text{aug}}$
- The event that $p_{1}^{\text{aug}}$ passes through $w$, and the part of $p_{1}^{\text{aug}}$ northeast of $w$

**Gibbs property**: The law of $p_{1}^{\text{aug}}$ southwest of $w$ is given by a uniformly random path from $u$ to $w$, conditioned to remain weakly below $p_{2}^{\text{aug}}$. 
Proof of $\Theta \approx \Omega$

**Gibbs property**: The law of $p_{1}\text{aug}$ is given by a uniformly random path in $\mathcal{E}_{-\infty,\infty}^{u:w, p_{2}\text{aug}}$.

- Let $q$ be a uniformly random path in $\mathcal{E}_{-\infty,\infty}^{u:w}$ (from $u$ to $w$ without barriers).
- By the linearity estimate, $q$ is $\varepsilon N$-linear with probability $1 - C \exp(-c\varepsilon^2 N)$.
- So, if $q$ exits the $x$-axis at $(\Gamma, 0)$, then $\mathbb{P}[|\Gamma - \Omega| < \varepsilon N] \geq 1 - C \exp(-c\varepsilon^2 N)$.
- By monotonicity, we may couple $p_{1}\text{aug}$ and $q$ so that $p_{1}\text{aug} \geq q$ almost surely.
- Thus, $\mathbb{P}[\Theta \geq \Omega - \varepsilon N] \geq \mathbb{P}[\Gamma \geq \Omega - \varepsilon N] \geq 1 - C \exp(-c\varepsilon^2 N)$.
Proof of $\Theta \approx \Omega$

**Gibbs property:** The law of $p_{1}^{\text{aug}}$ is given by a uniformly random path in $\mathcal{C}_{p_{2}^{\text{aug}};\infty}^{u,w}$.  

Let $r$ be a uniformly random path from $u$ to $v$, conditioned to lie weakly below $\mathcal{L}_{\Psi}$ (so it is uniform on $\mathcal{C}_{f,\infty}^{u,w}$, for some $f \geq p_{2}^{\text{aug}}$).

By the linearity estimate, $r$ is $\varepsilon N$-linear with probability $1 - C \exp(-c\varepsilon^2 N)$.

So, if $r$ exits the $x$-axis at $(\Upsilon, 0)$, then $\mathbb{P}[|\Upsilon - \Omega| < \varepsilon N] \geq 1 - C \exp(-c\varepsilon^2 N)$.

By monotonicity, we may couple $p_{1}^{\text{aug}}$ and $r$ so that $p_{1}^{\text{aug}} \leq r$ almost surely.

Thus, $\mathbb{P}[\Theta \leq \Omega + \varepsilon N] \geq \mathbb{P}[\Upsilon \leq \Omega + \varepsilon N] \geq 1 - C \exp(-c\varepsilon^2 N)$.  

- Let $r$ be a uniformly random path from $u$ to $v$, conditioned to lie weakly below $\mathcal{L}_{\Psi}$ (so it is uniform on $\mathcal{C}_{f,\infty}^{u,w}$, for some $f \geq p_{2}^{\text{aug}}$).
- By the linearity estimate, $r$ is $\varepsilon N$-linear with probability $1 - C \exp(-c\varepsilon^2 N)$.
- So, if $r$ exits the $x$-axis at $(\Upsilon, 0)$, then $\mathbb{P}[|\Upsilon - \Omega| < \varepsilon N] \geq 1 - C \exp(-c\varepsilon^2 N)$.
- By monotonicity, we may couple $p_{1}^{\text{aug}}$ and $r$ so that $p_{1}^{\text{aug}} \leq r$ almost surely.
- Thus, $\mathbb{P}[\Theta \leq \Omega + \varepsilon N] \geq \mathbb{P}[\Upsilon \leq \Omega + \varepsilon N] \geq 1 - C \exp(-c\varepsilon^2 N)$.
Comparing $p_1$ and $p_2^{\text{aug}}$

Seek to stochastically bound $p_1$ approximately above / below by $p_2^{\text{aug}}$

1. Couple $E$ and $E_\Psi$ in two ways, such that $p_1$ is (weakly) below $p_2^{\text{aug}}$ under the first and $p_2$ is (weakly) above $p_2^{\text{aug}}$ under the second

2. $\mathbb{P}[\text{dist}(p_1, p_2) < \varepsilon N] > 1 - C \exp(-c\varepsilon^2 N)$

First part follows from monotonicity

- View top path in $E_\Psi$ as barrier: Remaining paths below corresponding $E$ paths
  - Monotonicity implies coupling so that $p_2 \leq p_2^{\text{aug}}$
- View bottom path in $E_\Psi$ as barrier: Remaining paths above $E$ paths
  - Monotonicity implies coupling so that $p_2 \geq p_2^{\text{aug}}$
Proximity of $p_1$ and $p_2$

Seek to show $\mathbb{P}\left[ \text{dist}(p_1, p_2) < \varepsilon N \right] > 1 - C \exp(-c\varepsilon^2 N)$

1. Show $p_1$ and $p_2$ are likely “approximately convex”
2. Show approximate convexity of $p_2$ likely implies $\text{dist}(p_1, p_2) < \varepsilon N$

Let $h = h(p)$ denote the convex envelope of any path $p$

Let $\Xi = \Xi(p) = \max_{v \in p} \text{dist}(v, h(p))$

Define event $\mathcal{E} = \mathcal{E}(\varepsilon) = \{\Xi(p_1) < \varepsilon N\} \cap \{\Xi(p_2) < \varepsilon N\}$

On $\mathcal{E}$, the paths $p_1$ and $p_2$ are “approximately convex”

1. Show $\mathbb{P}[\mathcal{E}] > 1 - C \exp(-c\varepsilon^2 N)$
2. Show $\mathbb{P}\left[ 1_{\mathcal{E}} \text{dist}(p_1, p_2) < 5\varepsilon N \right] > 1 - \exp(C\varepsilon^2 N)$
Convexity Implies Proximity

Set $h_1 = h(p_1)$ and $h_2 = h(p_2)$

- On convexity event $\mathcal{E}$, we have $\text{dist}(p_1, p_2) \leq \text{dist}(h_1, h_2) + 2\epsilon N$

Suffices to show $\mathbb{P}[\mathbf{1}_\mathcal{E} \text{dist}(h_1, h_2) < 3\epsilon N] > 1 - C \exp(c\epsilon^2 N)$

- Fix $v_1 \in h_1$, and let $v_2 \in h_2$ be such that $\text{dist}(v_1, v_2) = \text{dist}(v_1, h_2)$

Must show $\mathbb{P}[\mathbf{1}_\mathcal{E} \text{dist}(v_1, v_2) < \epsilon N] > 1 - C \exp(-c\epsilon^2 N)$

- Let $\ell$ be line through $v_2$ orthogonal to line through $(v_1, v_2)$

- Convexity of $h_2$ implies $h_2 \subset NW(\ell)$ (is northwest of $\ell$)
  - Assume for simplicity that $p_2 \subset NW(\ell)$
    - Holds after shifting $\ell$ down by $\epsilon N$, since $\mathbf{1}_\mathcal{E} \text{dist}(p_2, h_2) < \epsilon N$ and $h_2 \subset NW(\ell)$

- Let $\ell$ meet $h_1$ at $(u, w)$, and assume for simplicity that $u, w \in p_1$
Convexity Implies Proximity

Must show that $\mathbb{P}[\mathbf{1}_\varepsilon \text{ dist}(v_1, v_2) < \varepsilon N] > 1 - C \exp(-c \varepsilon^2 N)$

- Condition on $p_2$ and on $p_1$ outside of interval $(u, w)$
  - Gibbs property: Then $p_1$ is a uniformly random path starting at $u$ and ending at $w$, and conditioned to lie above $p_2$
  - Montonicity: Replacing $p_2$ with $\ell$ only “pushes $v$ down”
  - Linearity: With probability $1 - C \exp(-c \varepsilon^2 N)$, a uniformly random path from $u$ to $w$ conditioned to stay below $\ell$ does not go below $\ell$ by more than $\varepsilon N$
  - Shows $\mathbb{P}[\mathbf{1}_\varepsilon \text{ dist}(v_1, v_2) < \varepsilon N] > 1 - C \exp(-c \varepsilon^2 N)$
Established arctic boundaries for domain-wall ice model

Proceeds by **justification of tangent method** of Colomo–Sportiello
- Involves inserting an augmented path in the domain
- Path should be tangent to arctic boundary
- Refined partition function asymptotics identify trajectory of the path
  - Integrability only involved through understanding these asymptotics
  - Full solvability / determinantality of the model not required

Proof involves analysis of non-intersecting path ensembles (reminiscent of ideas used by Corwin–Hammond in very different context)
- Prove approximate tangency of additional path to arctic boundary of augmented ensemble
  - Gibbs property
  - Monotonicity
- Prove additional path does not substantially affect arctic boundary
  - Convexity (and Gibbs property / monotonicity)