Nonintersecting Brownian bridges in the flat-to-flat geometry

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Randomness, Integrability and Universality
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How to simulate a Brownian motion?

- Let us start with the simple Brownian motion

$$\frac{dx(t)}{dt} = \sqrt{2D}\,\eta(t) \quad , \quad x(0) = a$$

Gaussian white noise with

$$\langle \eta(t) \rangle = 0 \quad , \quad \langle \eta(t)\eta(t') \rangle = \delta(t - t')$$

and D is the diffusion constant

- To simulate it numerically we discretize time with increments $\Delta t \ll 1$

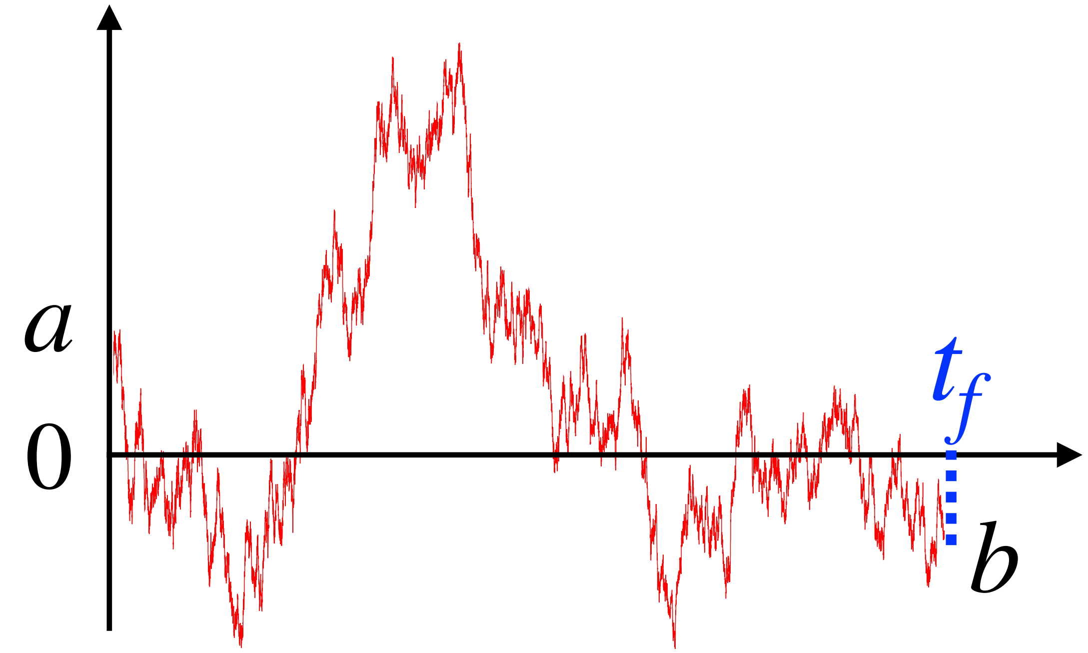
$$x(0) = a$$

$$x(n\,\Delta t) = x((n-1)\Delta t) + \sqrt{2D}\,\eta(t)\,\Delta t \quad , \quad n = 1,2,\cdots$$

Gaussian random variable with zero mean and variance $2D\,\Delta t$

How to simulate a Brownian bridge?

- A Brownian bridge is a Brownian motion, starting from $x_B(0) = a$ and conditioned to end at $x_B(t_f) = b$



- Conditioning stochastic processes is a classical problem in proba. theory

e.g., Doob (1957)

- There exist various efficient ways to simulate a Brownian bridge, e.g.

$$x_B(t) = x(t) + \frac{t}{t_f}(b - x(t_f))$$

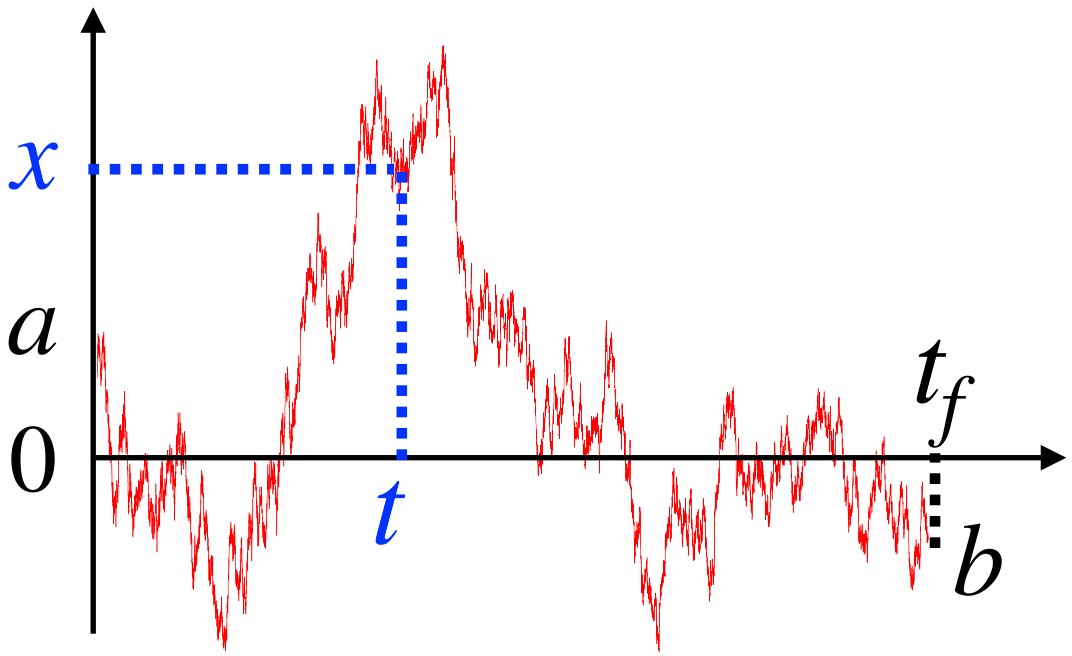
where $x(t)$ is a std Brownian motion starting at $x(0) = a$

- One can also write an effective Langevin equation

Chetrite, Touchette (2015)
Majumdar, Orland (2015)

$$\frac{dx_B(t)}{dt} = \frac{b - x_B(t)}{t_f - t} + \sqrt{2D}\,\eta(t) \quad , \quad x_B(0) = a$$

Effective Langevin Eq. for a Brownian bridge



$$\overbrace{P_{\rm BB}(x,t\,|\,b,a,t_f)}^{\tilde{P}} = \frac{\overbrace{P(x,t\,|\,a,0)}^{P}\,\overbrace{P(b,t_f\,|\,x,t)}^{Q}}{P(b,t_f\,|\,a,0)}$$

where $P(x,t\,|\,a,0) = \dfrac{e^{-\frac{(x-a)^2}{4D\,t}}}{\sqrt{4\pi D\,t}}$ for free Brownian motion

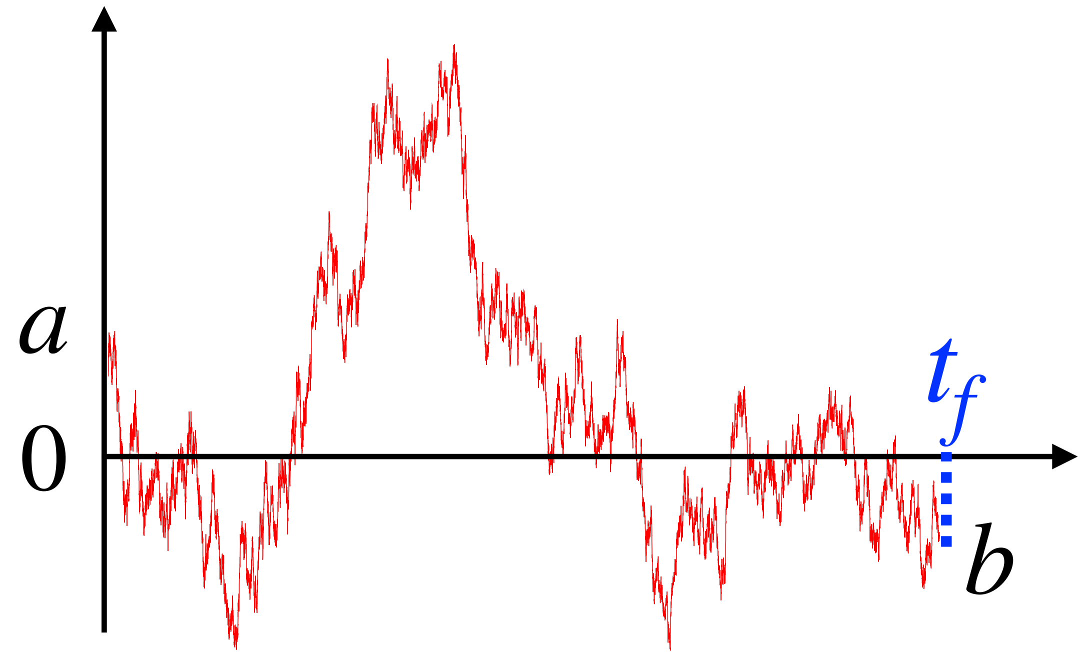
- $P(x,t\,|\,a,0)$ satisfies the forward Fokker-Planck Eq. $\partial_t P = D\partial_x^2 P$
- $Q(b,t_f\,|\,x,t)$ satisfies the backward Fokker-Planck Eq. $\partial_t Q = -\,D\partial_x^2 Q$
- The product $\tilde{P} = P\,Q$ satisfies $\partial_t \tilde{P} = D\partial_x^2 \tilde{P} - 2\,D\,\partial_x\left(\tilde{P}\,\partial_x \ln Q\right)$

⇨ This corresponds to the effective Langevin Eq.

$$\frac{dx_B}{dt} = 2D\partial_x \ln Q + \sqrt{2D}\,\eta(t)$$

with $Q(b,t_f\,|\,x_B,t) = \dfrac{e^{-\frac{(x_B-b)^2}{4D\,(t_f-t)}}}{\sqrt{4\pi D\,(t_f-t)}}$

Effective Langevin Eq. for a Brownian bridge



$$\frac{dx_B(t)}{dt} = \frac{b - x_B(t)}{t_f - t} + \sqrt{2D}\,\eta(t) \quad , \quad x_B(0) = a$$

Discretizing in time $\Longrightarrow$ generates a Brownian bridge trajectory in a rejection free way

Our main motivation

Q: is it possible to generalise the effective Langevin approach from a single Brownian bridge to multiple Brownian bridges with interaction between them?

- A natural setting is the nonintersecting (vicious) Brownian bridges, which has many applications in physics and maths

Karlin & McGregor (1959), de Gennes (1968), Fisher (1984),...

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 48, NUMBER 5 1 MARCH 1968

Soluble Model for Fibrous Structures with Steric Constraints

P.-G. de Gennes

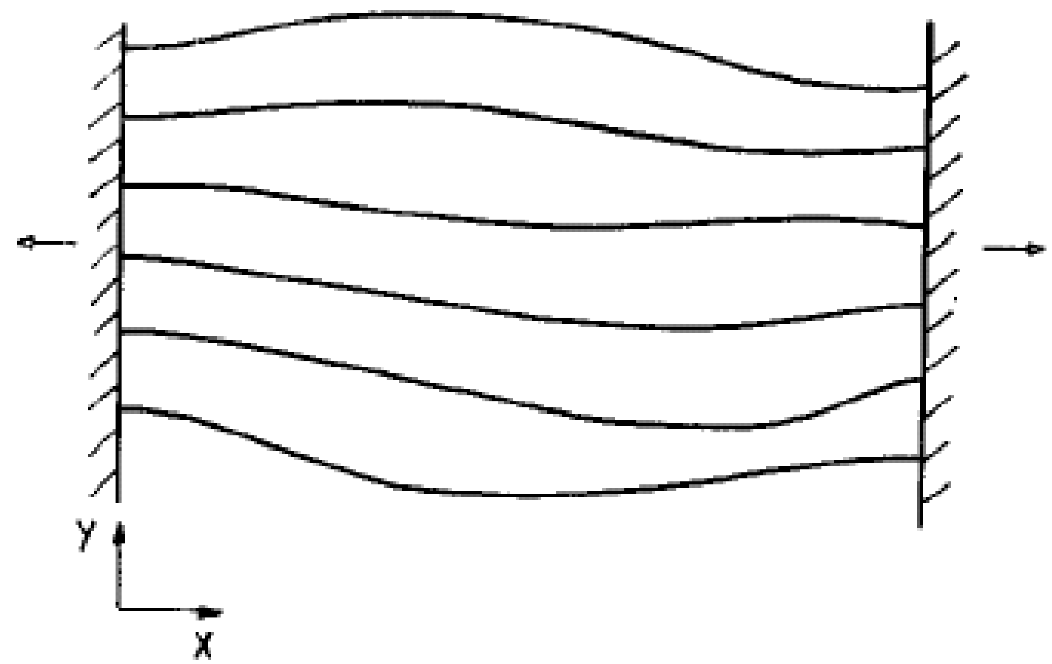
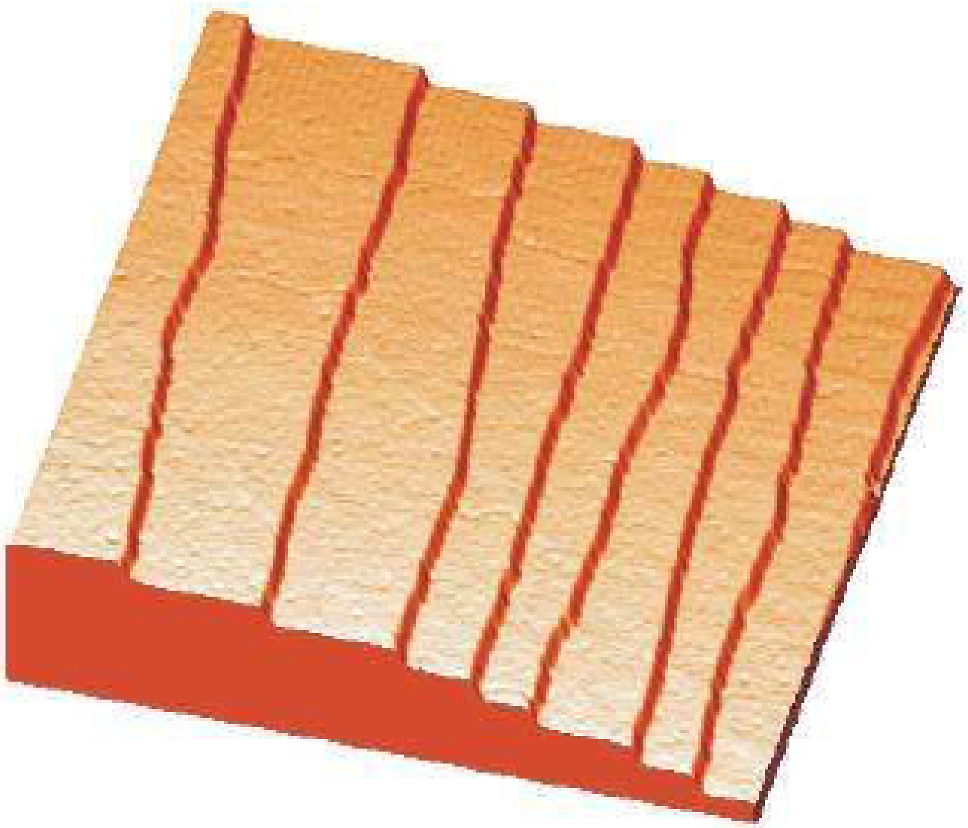
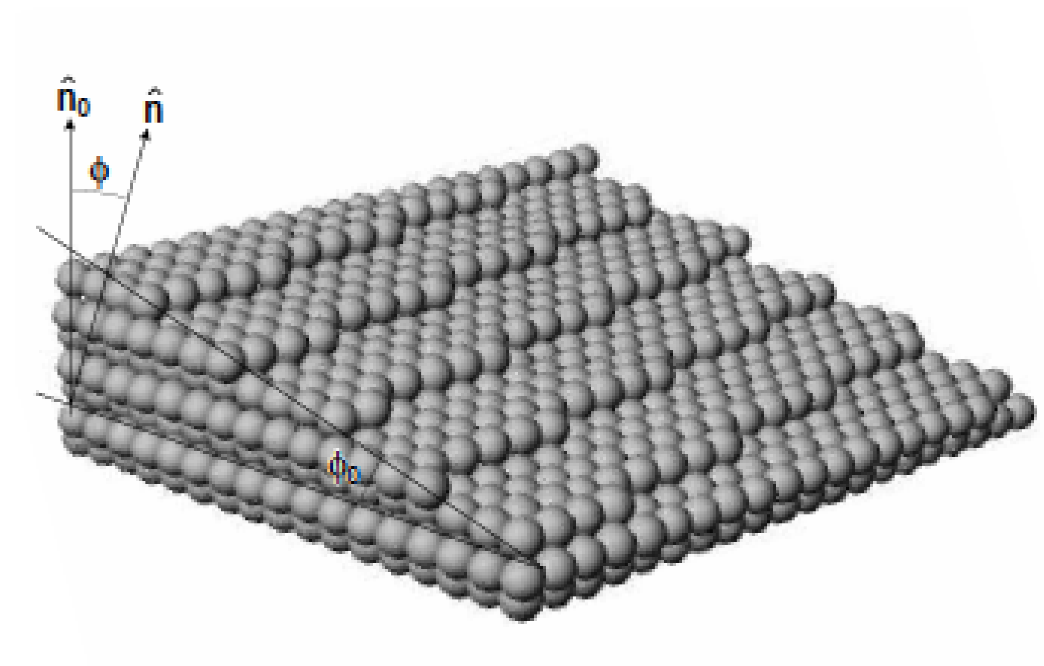
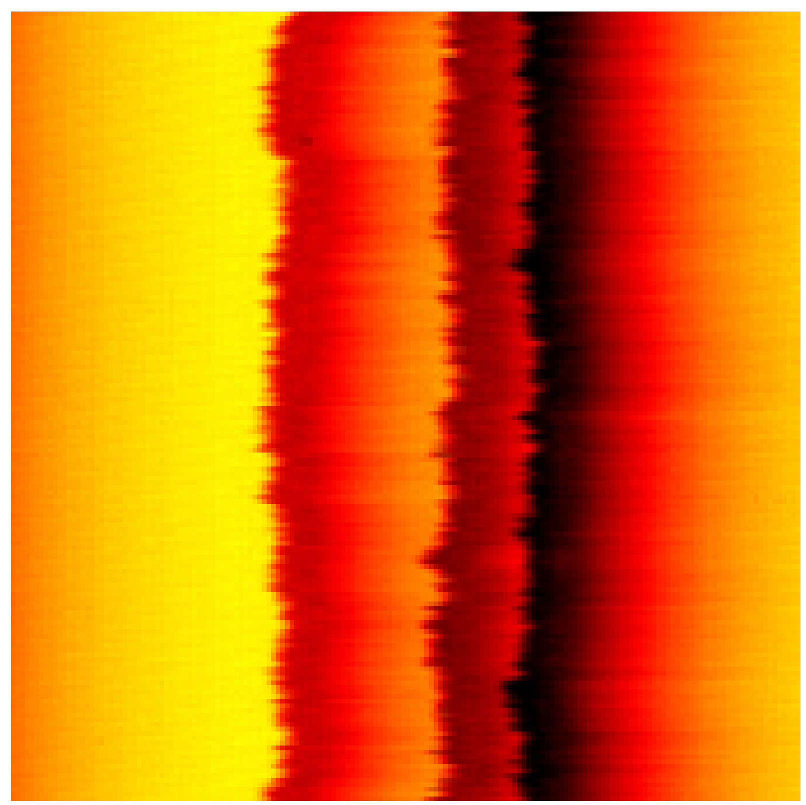


Fig. 1. Model for a two-dimensional fiber structure. The component chains are assumed to be attached to two plates I and F and placed under tension. The chains are bent by thermal fluctuations. Different chains cannot intersect each other.

Step fluctuations on vicinal surfaces







Maryland group (2003)

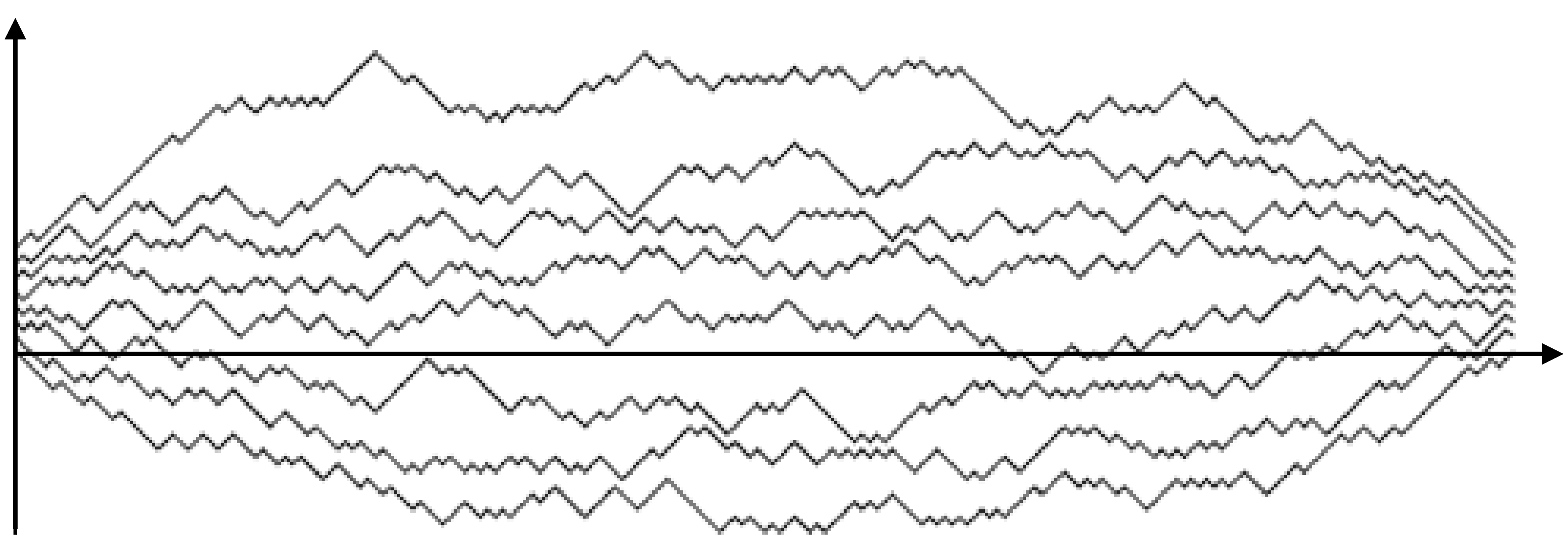
Applications in combinatorics/computer science

Watermelon uniform random generation with applications

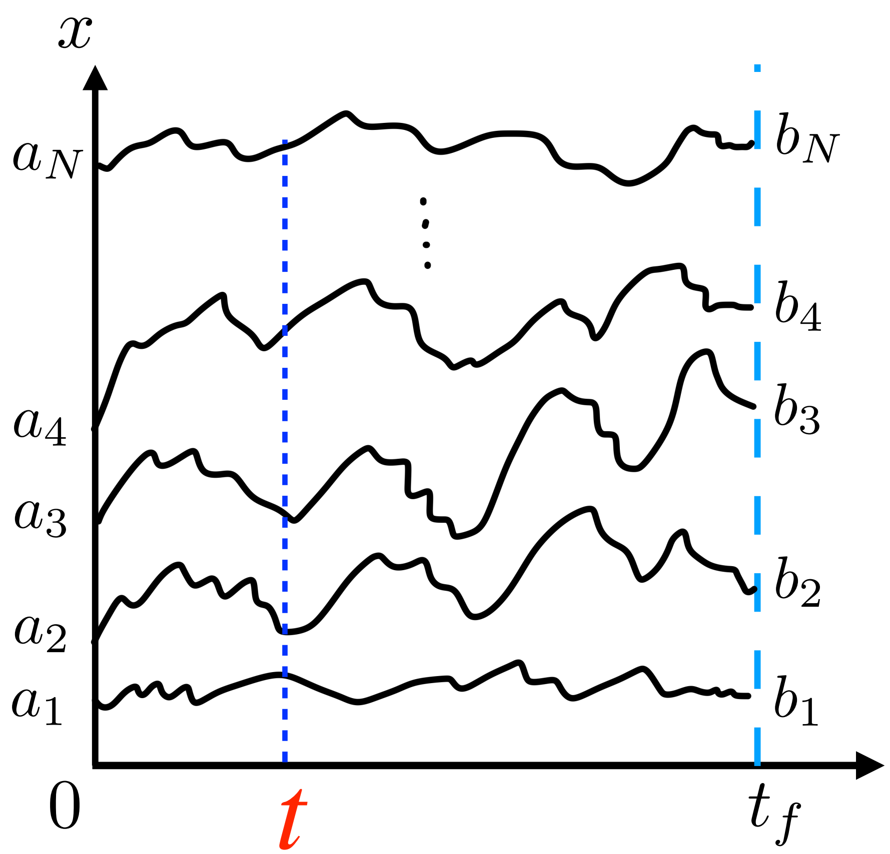
Nicolas Bonichon*, Mohamed Mosbah

LaBRI-Université Bordeaux 1, 351 Cours de la Libération, 33405 Talence, France

- An algorithm to generate discrete time nonintersecting lattice bridges



Nonintersecting (vicious) Brownian bridges



for $a_i = b_i = \frac{i-1}{N}, i = 1, 2, \cdots, N$

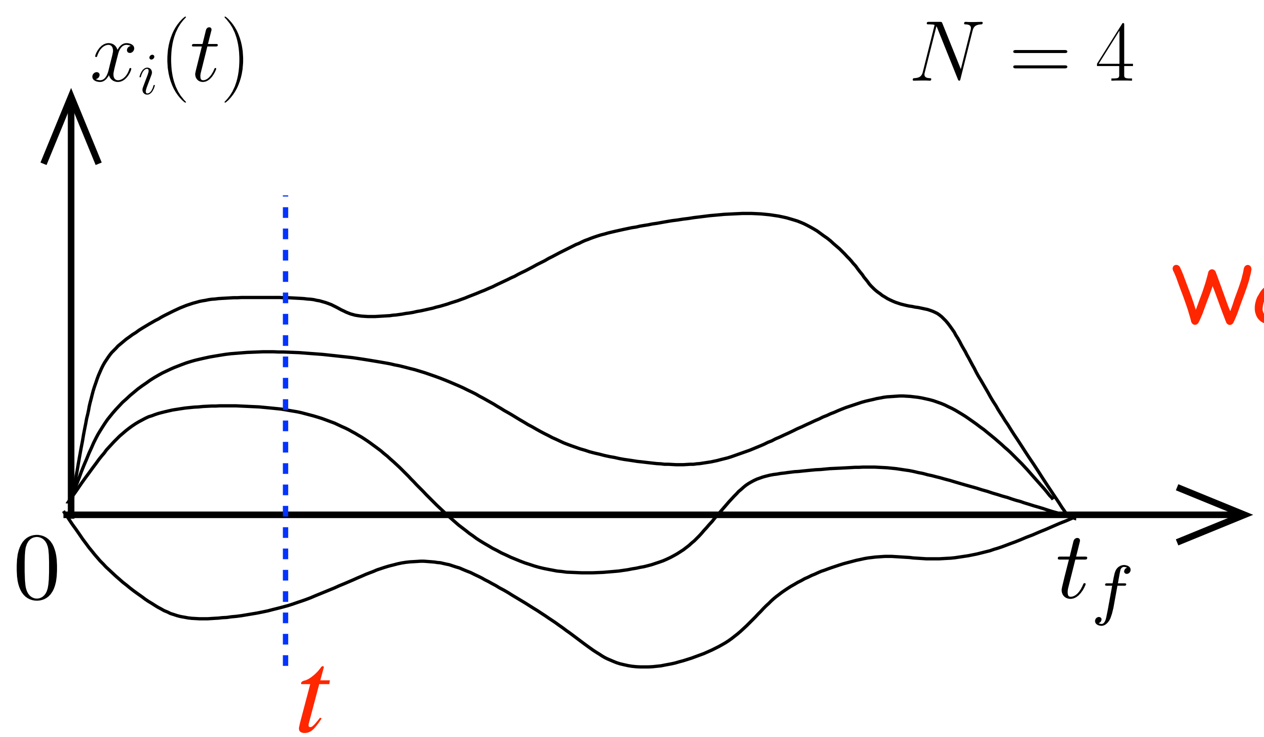
Q1: how to simulate such configurations efficiently (i.e., beyond the costly naive way) ?

Q2: what is the average density at some intermediate time $0 \leq t \leq t_f$?

- A special case $a_i = b_i = 0$ $\Longrightarrow$ ``watermelons''

"Watermelons" and random matrix theory

- Vicious bridges with $a_i = b_i = 0$ for all $i = 1,2,\cdots,N$



Watermelon configuration

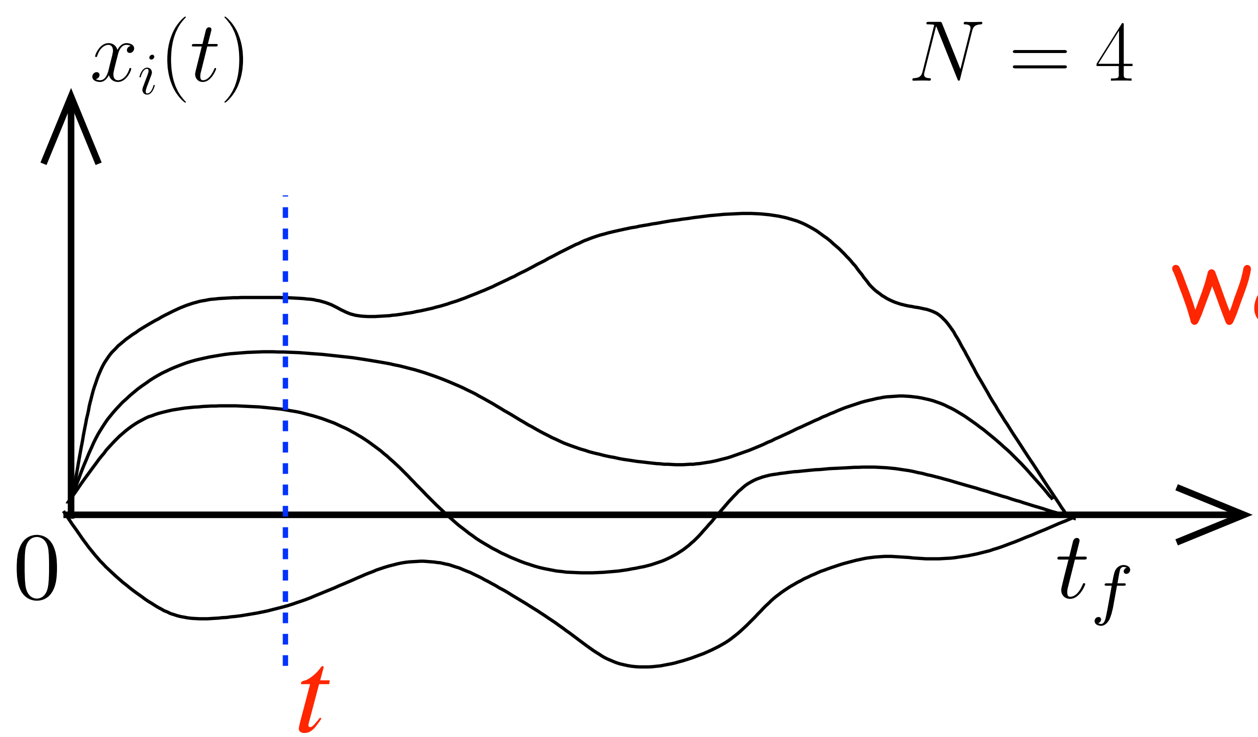
- Joint proba. density function of $x_1(t), x_2(t), \cdots, x_N(t)$ for $0 \leq t \leq t_f$

$$P_{\rm joint}(x_1, x_2, \cdots, x_N; t \,|\, t_f) \propto \prod_{i=1}^{N} (x_i - x_j)^2 e^{-\sum_{i=1}^{N} \frac{x_i^2}{2\sigma^2(t)}} \quad , \quad \sigma^2(t) = \frac{2\,D\,t(t_f - t)}{t_f}$$

⇒ The rescaled positions $\frac{x_i}{\sigma(t)}$ behave like the eigenvalues of the eigenvalues of Gaussian Unitary Ensemble ($\beta = 2$) of random matrices

"Watermelons" and random matrix theory

- Vicious bridges with $a_i = b_i = 0$ for all $i = 1, 2, \cdots, N$



Watermelon configuration

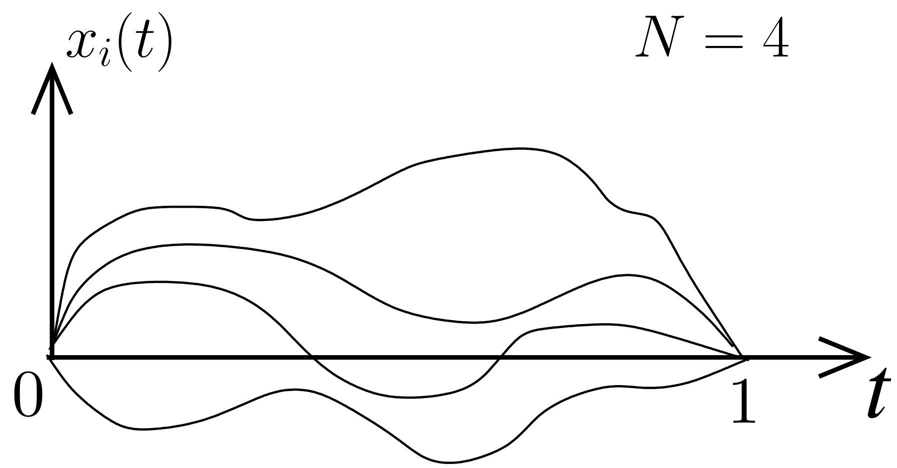
$$\sigma^2(t) = \frac{2\,D\,t(t_f - t)}{t_f}$$

- The average density at any intermediate $0 \leq t \leq t_f$ for large N

$$\rho_N(x;t) \simeq \frac{1}{\pi\sqrt{N}\sigma(t)}\sqrt{2 - \frac{x^2}{N\,\sigma^2(t)}}\,, \quad -\sqrt{2N}\,\sigma(t) \leq x \leq \sqrt{2N}\,\sigma(t)$$

⇒ Wigner semi-circle for all $0 \leq t \leq t_f$

"Watermelons" and Dyson 's Brownian motion



Watermelon configuration

with $t_f = 1$

- Let $H(t)$ be a $N \times N$ random Hermitian matrix with elements

$$H_{mn}(t) = \begin{cases} \frac{1}{\sqrt{2}} \left(x_{mn}^B(t) + i\, \tilde{x}_{mn}^B(t) \right) \ , & m < n \ , \\ x_{mm}^B(t) \ , & m = n \\ \frac{1}{\sqrt{2}} \left(x_{nm}^B(t) - i\, \tilde{x}_{nm}^B(t) \right) \ , & m > n \ , \end{cases}$$

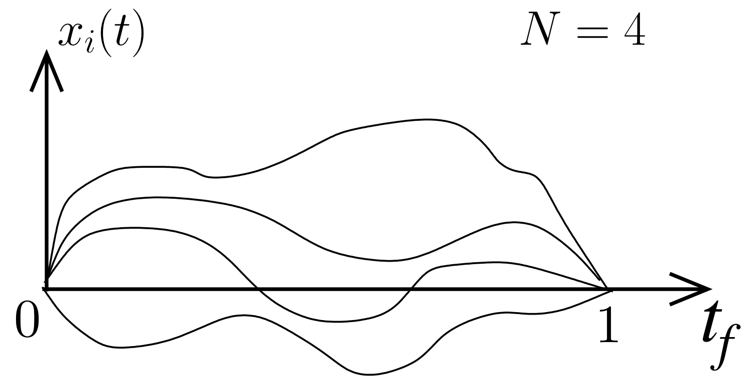
where $x_{mn}^B \, , \tilde{x}_{mn}^B$ are independent Brownian bridges starting and ending at $x = 0$

- Eigenvalues of $H(t)$

$$\underbrace{\{\Lambda_1(t) < \Lambda_2(t) < \cdots < \Lambda_N(t)\}}_{} \overset{d}{=} \{x_1(t) < x_2(t) < \cdots < x_N(t)\}$$

special instance of "Dyson 's Brownian bridge"

``Watermelons" and Dyson 's Brownian motion



Watermelon configuration

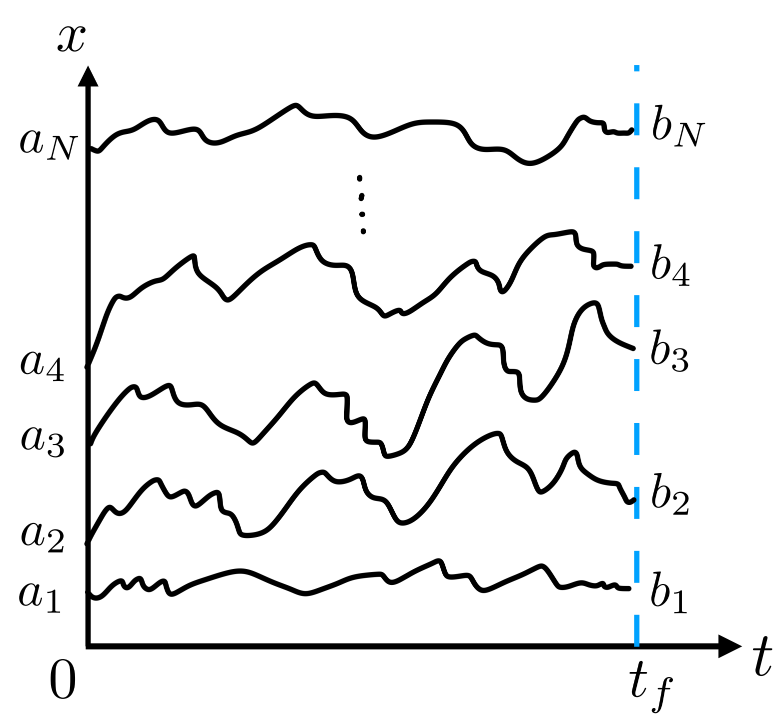
with $t_f = 1$

- Eigenvalues of $H(t)$

$$\underbrace{\{\Lambda_1(t) < \Lambda_2(t) < \cdots < \Lambda_N(t)\} \stackrel{d}{=} \{x_1(t)}_{\text{special instance of ``Dyson 's Brownian bridge"}} < x_2(t) < \cdots < x_N(t)\}$$

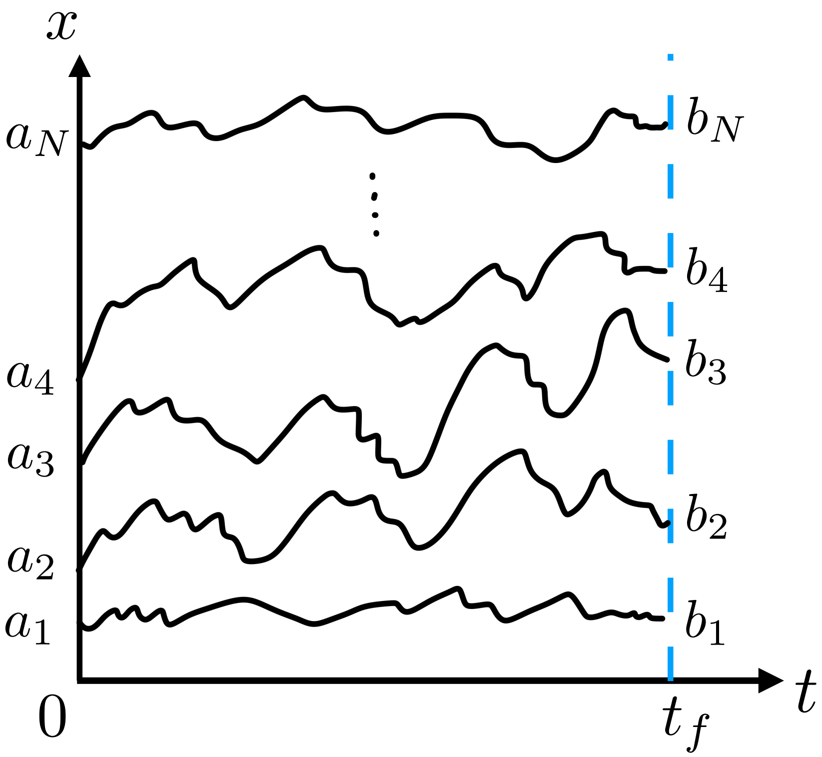
special instance of ``Dyson 's Brownian bridge"

Q: can one generalize such a construction (i.e., connection with Dyson 's Brownian motion) to more general Vicious Brownian Bridges (VBB)?



Main results

Grela, Majumdar, G. S. (2021)



Vicious BB

↓

Dyson BB for $\beta = 2$

↓

Explicit Langevin Eq. for flat-to-flat

↙ ↘

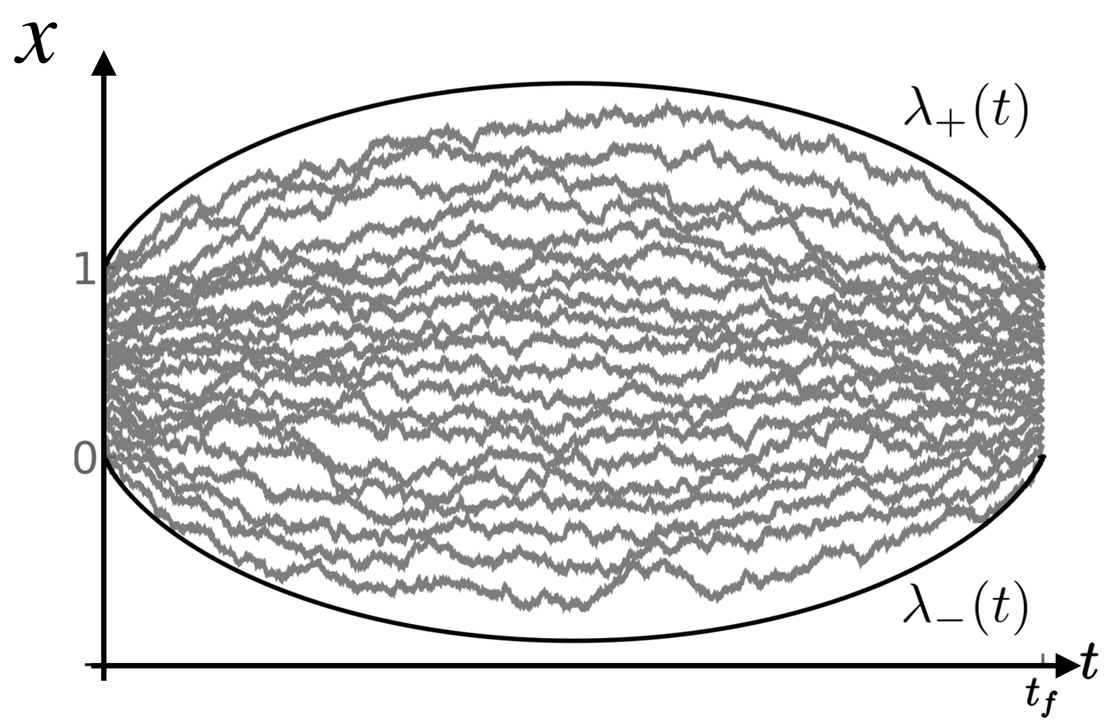
Efficient numerical generation of VBB

Exact results for the density of VBB

Main results

Grela, Majumdar, G. S. (2021)

- Efficient numerical method to simulate N nonintersecting Brownian bridges, e.g. in the "flat-to-flat" geometry, i.e., $a_i = b_i = \dfrac{i-1}{N}$



- In the limit $N \to \infty$, the density $\rho(x;t)$ has a finite support $[\lambda_-(t), \lambda_+(t)]$

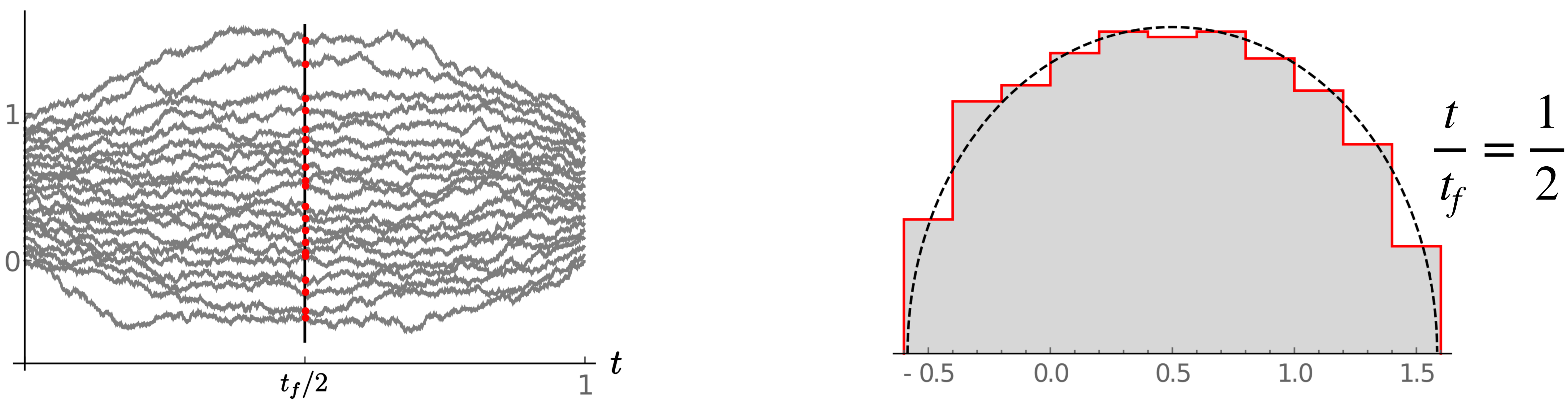
$$\lambda_\pm(t) = \frac{1}{2} \pm \left[t_f \operatorname{arccosh}\left(\frac{1}{\sqrt{T}} \frac{t_f + T(t_f - 2t)}{2(t_f - t)} \right) - t \operatorname{arccosh}\left(\frac{(t_f - t)^2 + t^2 - T(t_f - 2t)^2}{2t(t_f - t)} \right) \right]$$

where $T = e^{-\frac{1}{t_f}}$

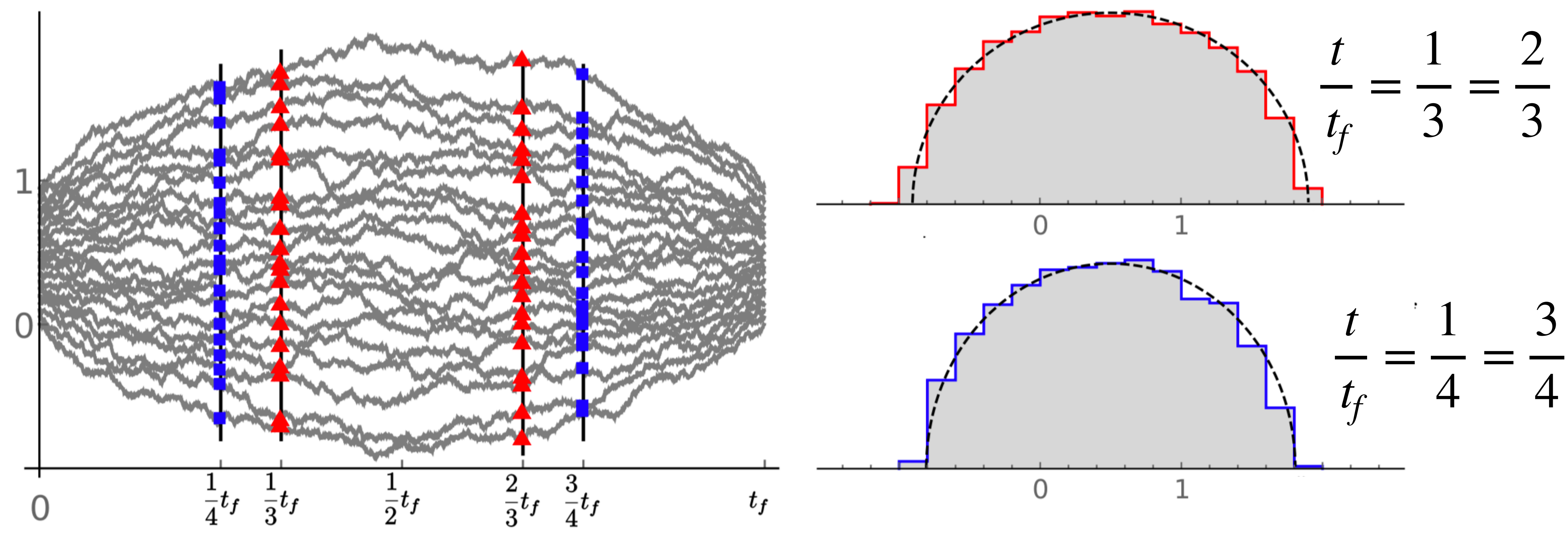
Main results

Grela, Majumdar, G. S. (2021)

- Explicit result for the limiting density $\rho(x;t)$ for some special values of $\frac{t}{t_f}$



see also M. Marino (2005), P. J. Forrester (2021)



Outline

- Vicious Brownian Bridge, Dyson Brownian Bridge and Effective Langevin equation

- Average particle density via Burgers' equation

- Conclusion

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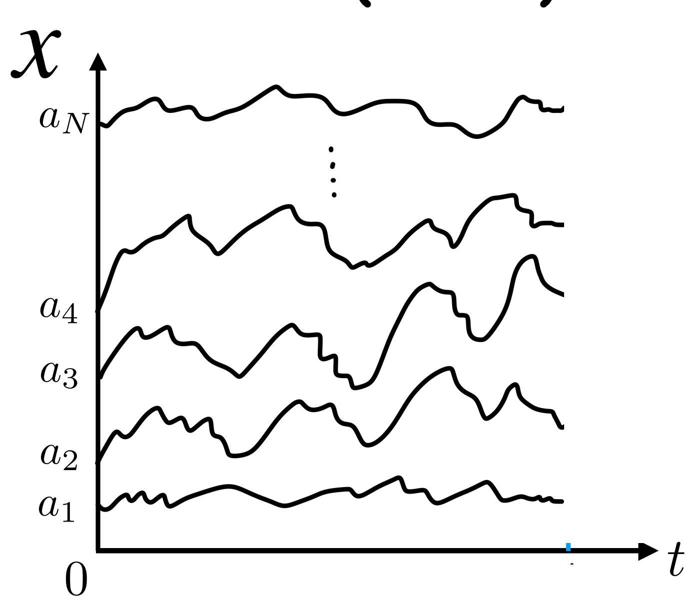
Vicious Brownian motion (VBM)



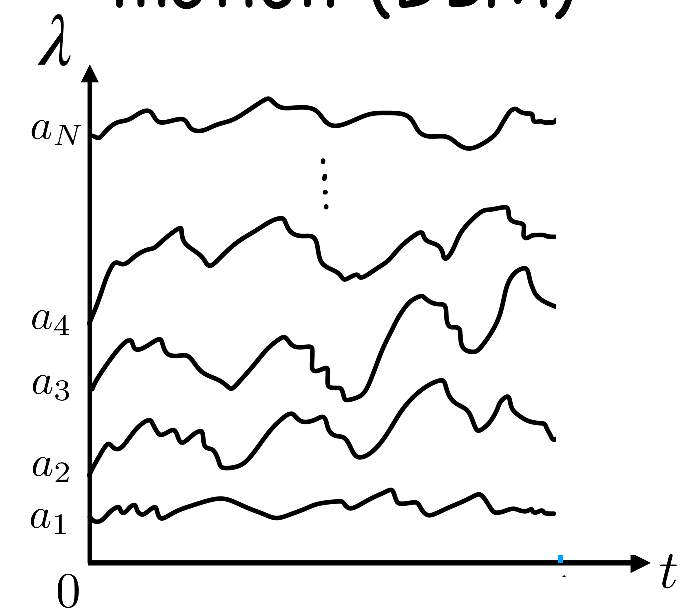
$$\frac{dx_i}{dt} = \frac{1}{\sqrt{N}} \eta_i(t)$$

N indep. Gaussian white noises

conditioned on $x_1(t) \leq x_2(t) \leq \cdots \leq x_N(t)$

init. cond. $x_i(0) = a_i$

Dyson Brownian motion (DBM)



$$\frac{d\lambda_i(t)}{dt} = \frac{1}{N} \sum_{j(\neq i)=1}^{N} \frac{1}{\lambda_i(t) - \lambda_j(t)} + \sqrt{\frac{2}{\beta N}}\, \xi_i(t)$$

N indep. Gaussian white noises

with $\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_N(t)$

init. cond. $\lambda_i(0) = a_i$

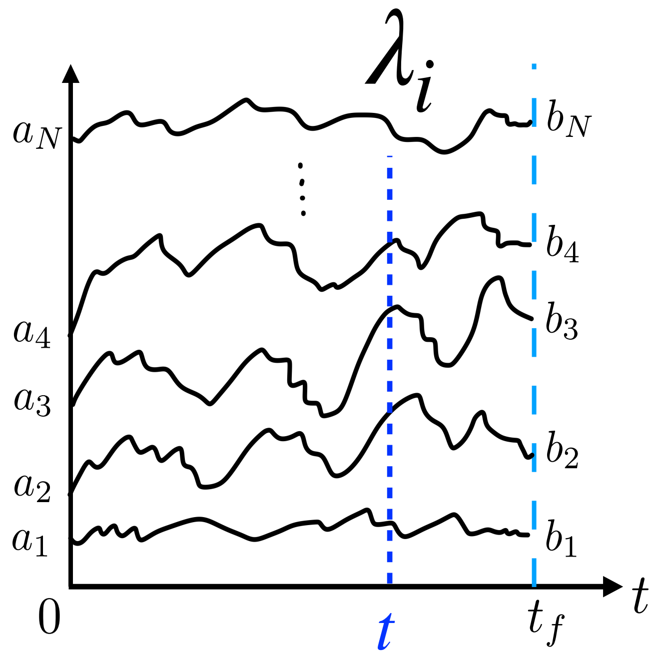
β : Dyson 's index

Johansson (2001)

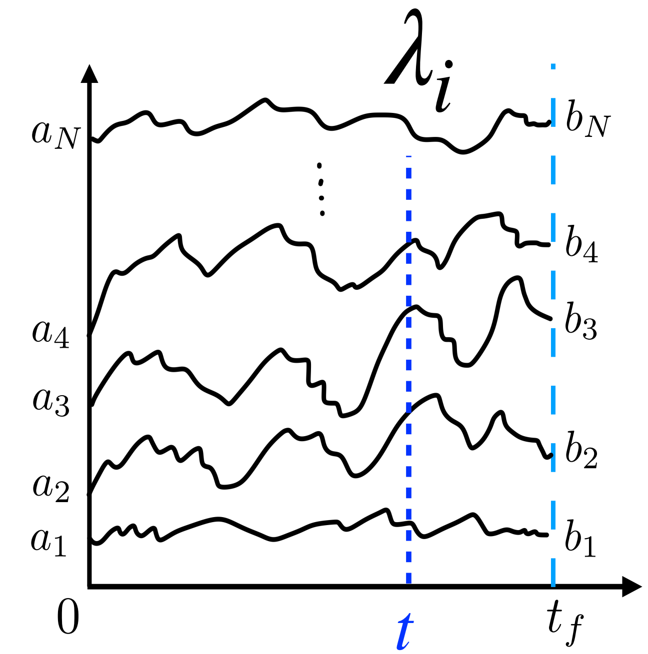
$$P_{\mathrm{VBM},D=\frac{1}{2N}}(\vec{\lambda}, t \,|\, \vec{a}, 0) = \frac{\Delta(\vec{a})}{\Delta(\vec{\lambda})} P_{\mathrm{DBM},\beta=2}(\vec{\lambda}, t \,|\, \vec{a}, 0)$$

where $\Delta(\vec{\lambda}) = \prod_{i<j} (\lambda_j - \lambda_i)$

Vicious Brownian bridge (VBB)



Dyson Brownian bridge (DBB)

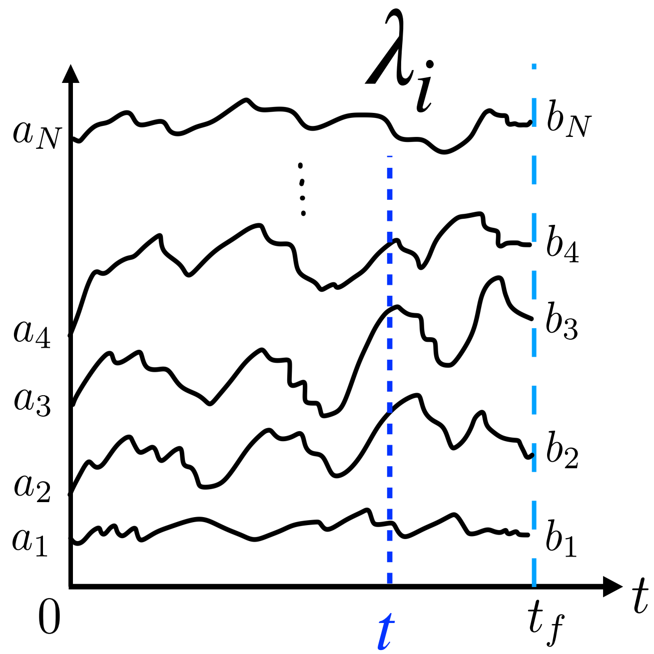


$$P_{\mathrm{VBB},D=\frac{1}{2N}}(\vec{\lambda},t\,|\,\vec{b},\vec{a},t_f) = \frac{P_{\mathrm{VBM},D=\frac{1}{2N}}(\vec{\lambda},t\,|\,\vec{a},0)\,P_{\mathrm{VBM},D=\frac{1}{2N}}(\vec{b},t_f\,|\,\vec{\lambda},t)}{P_{\mathrm{VBM},D=\frac{1}{2N}}(\vec{b},t_f\,|\,\vec{a},0)}$$

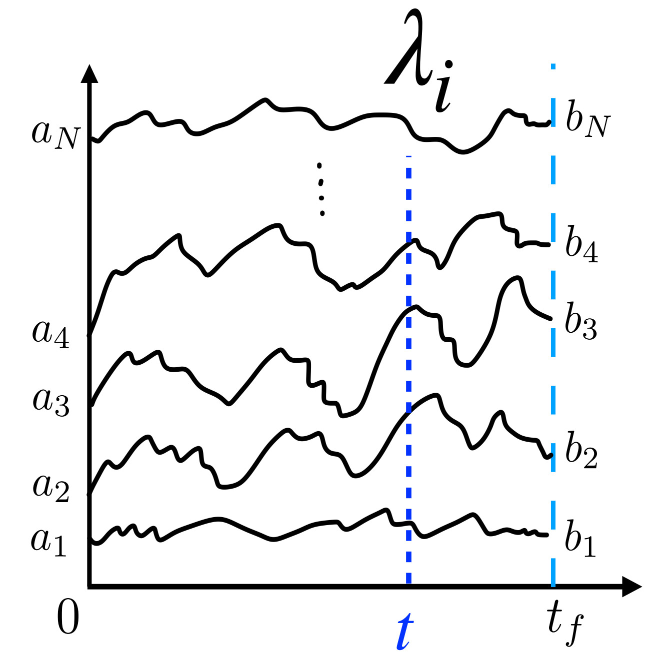
$$= \frac{\frac{\Delta(\vec{a})}{\Delta(\vec{\lambda})}P_{\mathrm{DBM},\beta=2}(\vec{\lambda},t\,|\,\vec{a},0)\,\frac{\Delta(\vec{\lambda})}{\Delta(\vec{b})}P_{\mathrm{DBM},\beta=2}(\vec{b},t_f\,|\,\vec{\lambda},t)}{\frac{\Delta(\vec{a})}{\Delta(\vec{b})}P_{\mathrm{DBM},\beta=2}(\vec{b},t_f\,|\,\vec{a},0)}$$

$$= P_{\mathrm{DBB},\beta=2}(\vec{\lambda},t\,|\,\vec{b},\vec{a},t_f)$$

Vicious Brownian bridge (VBB)



Dyson Brownian bridge (DBB)



$$\Rightarrow \quad P_{\mathrm{VBB},D=\frac{1}{2N}}(\overrightarrow{\lambda}, t \,|\, \overrightarrow{b}, \overrightarrow{a}, t_f) = P_{\mathrm{DBB},\,\beta=2}(\overrightarrow{\lambda}, t \,|\, \overrightarrow{b}, \overrightarrow{a}, t_f)$$

$$\Rightarrow \quad \{\lambda_1, \lambda_2, \cdots, \lambda_N\}_{\mathrm{VBB},D=\frac{1}{2N}} \overset{d}{=} \{\lambda_1, \lambda_2, \cdots, \lambda_N\}_{\mathrm{DBB},\,\beta=2}$$

Grela, Majumdar, G. S. (2021)

Effective Langevin Eq. for DBB $(\beta = 2)$

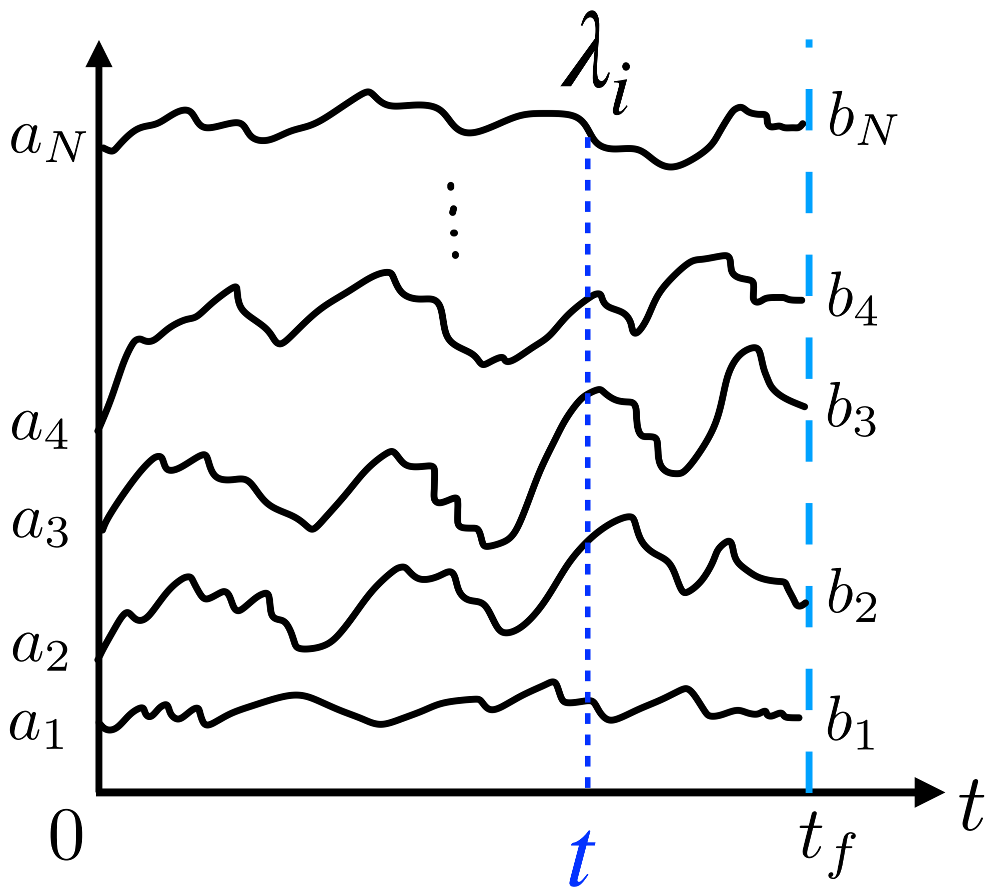
$$\overbrace{P_{\mathrm{DBB},\beta=2}(\vec{\lambda},t\,|\,\vec{b},\vec{a},t_f)}^{\tilde{P}} = \frac{\overbrace{P_{\mathrm{DBM},\beta=2}(\vec{\lambda},t\,|\,\vec{a},0)}^{P}\;\overbrace{P_{\mathrm{DBM},\beta=2}(\vec{b},t_f\,|\,\vec{\lambda},t)}^{Q}}{P_{\mathrm{DBM},\beta=2}(\vec{b},t_f\,|\,\vec{a},0)}$$

- Write explicit Fokker-Planck Eq. for P and Q separately
- Write the Fokker-Planck Eq. for the product $\tilde{P} = P\,Q$
- For the flat final config. $b_i = \dfrac{i-1}{N}$, $i = 1,2,\cdots,N$

  This FP Eq. for $\tilde{P}$ simplifies

$\Rightarrow$ One can read off the effective Langevin Eq. associated to $\tilde{P}$

Effective Langevin Eq. for DBB $(\beta = 2)$



Flat final config. $b_i = \frac{i-1}{N}$

Grela, Majumdar, G. S. (2021)

$$\frac{d\lambda_i}{dt} = -\frac{\lambda_i}{t_f - t} + \frac{1}{N(t_f - t)} \sum_{j(\neq i)} \frac{e^{\frac{\lambda_i}{t_f - t}}}{e^{\frac{\lambda_i}{t_f - t}} - e^{\frac{\lambda_j}{t_f - t}}} + \frac{1}{\sqrt{N}} \xi_i(t) \qquad i = 1, \cdots, N$$

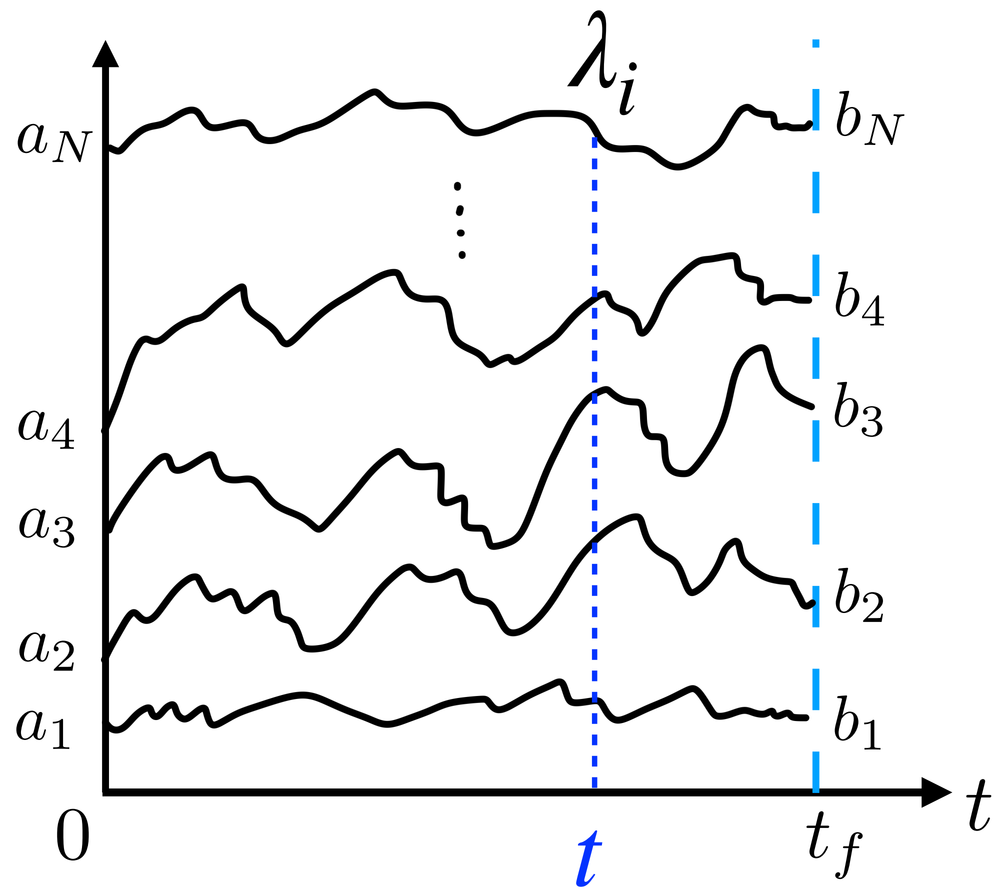
N indep. Gaussian white noises

valid for any initial condition $\vec{\lambda}(t=0) = \vec{a}$

it automatically ensures (i) final flat config.

(ii) non-crossing during $[0, t_f]$

Effective Langevin Eq. for DBB $(\beta = 2)$



Flat final config. $b_i = \frac{i-1}{N}$

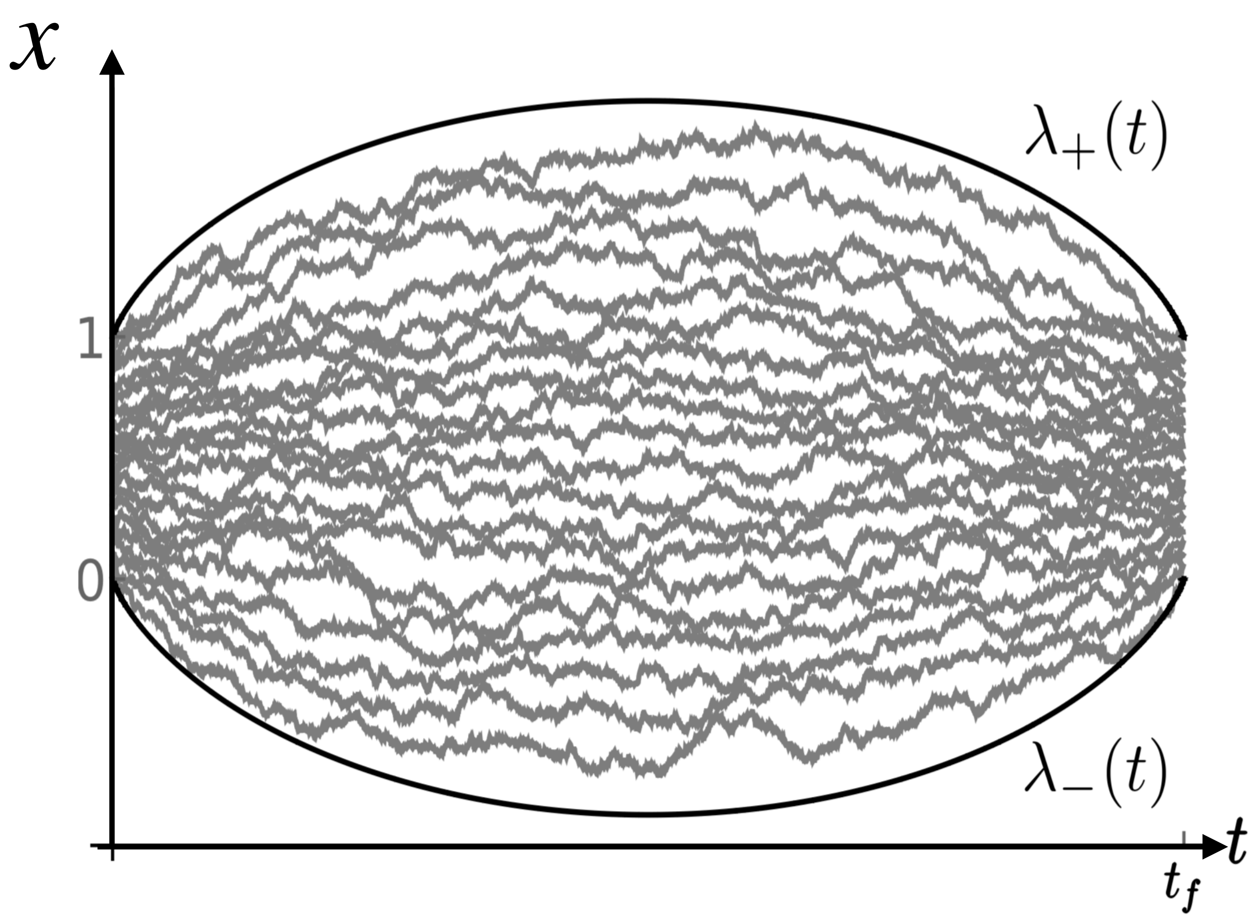
Grela, S. N. M., Schehr (2021)

$$\frac{d\lambda_i}{dt} = -\frac{\lambda_i}{t_f - t} + \frac{1}{N(t_f - t)} \sum_{j(\neq i)} \frac{e^{\frac{\lambda_i}{t_f - t}}}{e^{\frac{\lambda_i}{t_f - t}} - e^{\frac{\lambda_j}{t_f - t}}} + \frac{1}{\sqrt{N}} \xi_i(t) \qquad i = 1, \cdots, N$$

N indep. Gaussian white noises

Discretizing in time $\Longrightarrow$ generates VBB trajectories in a flat to flat geometry in a rejection free way

Effective Langevin Eq. for DBB $(\beta = 2)$



Flat final config. $b_i = \frac{i-1}{N}$

Grela, S. N. M., Schehr (2021)

$$\frac{d\lambda_i}{dt} = -\frac{\lambda_i}{t_f - t} + \frac{1}{N(t_f - t)} \sum_{j(\neq i)} \frac{e^{\frac{\lambda_i}{t_f - t}}}{e^{\frac{\lambda_i}{t_f - t}} - e^{\frac{\lambda_j}{t_f - t}}} + \frac{1}{\sqrt{N}} \xi_i(t) \qquad i = 1, \cdots, N$$

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Outline

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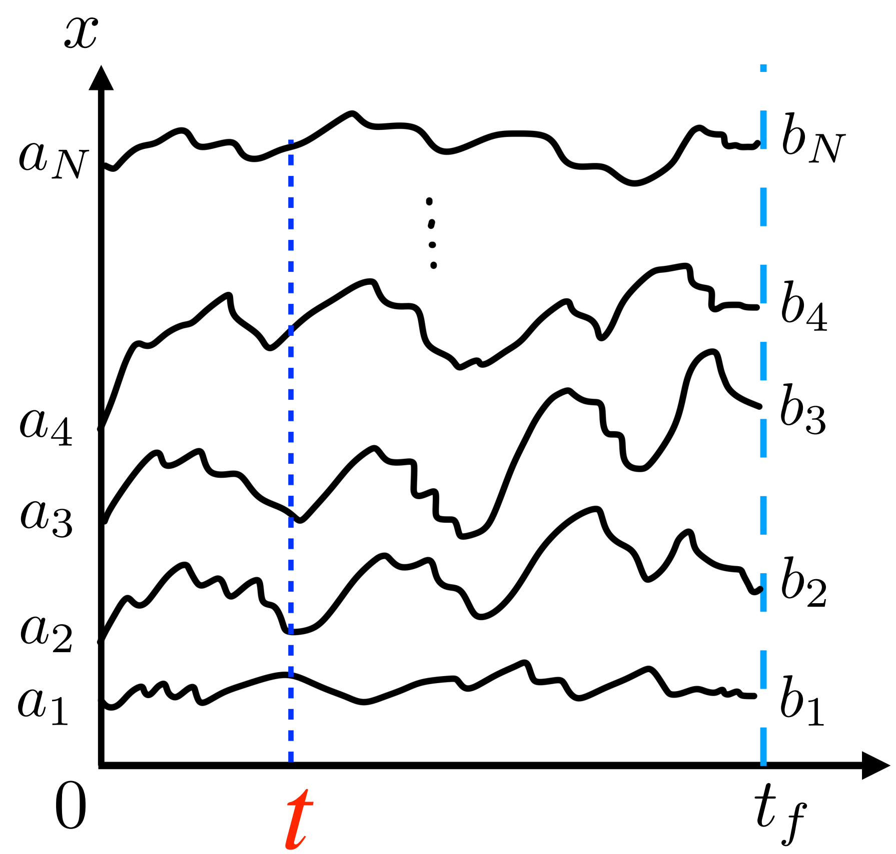
- Average particle density via Burgers' equation
- Conclusion

Nonintersecting (vicious) Brownian bridges



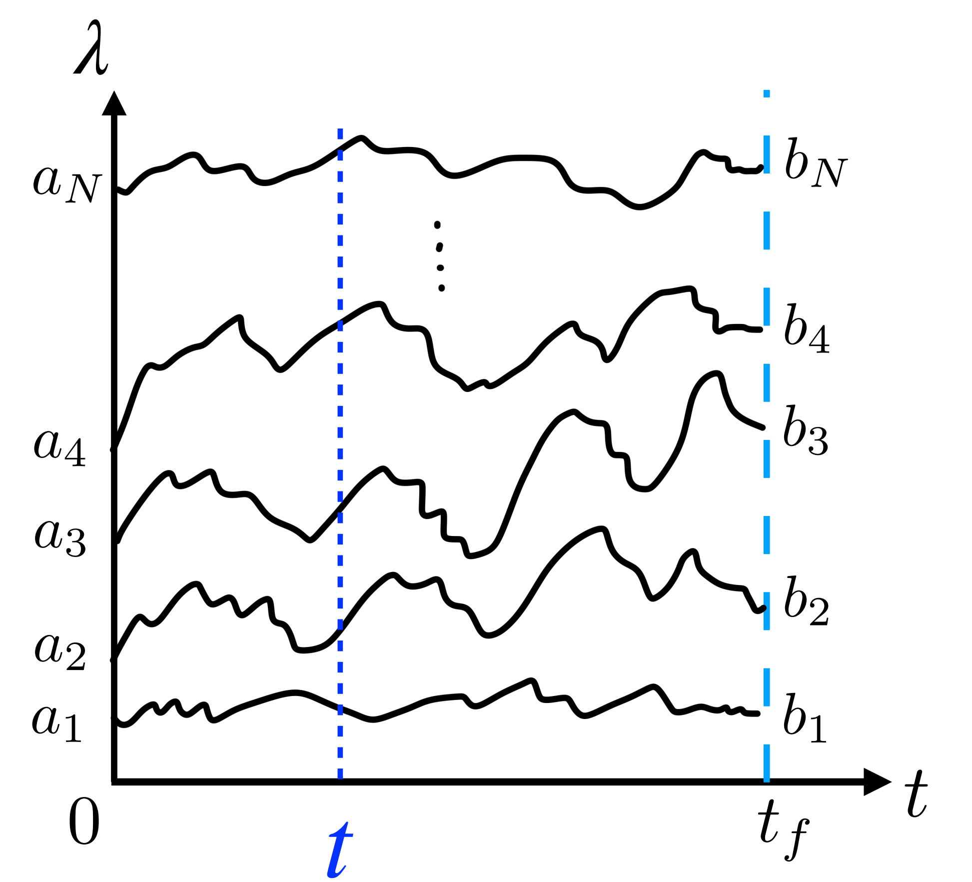
for $a_i = b_i = \frac{i-1}{N}, i = 1,2,\cdots,N$

Q2: what is the average density at some intermediate time $0 \leq t \leq t_f$?

Joint distribution of the positions of the VBB

- Vicious Brownian Bridges in flat-to-flat geometry

$$P_{\rm VBB}(\vec{\lambda}, t \,|\, \vec{b}, \vec{a}, t_f) = \frac{P_{\rm VBM}(\vec{\lambda}, t \,|\, \vec{a}, 0)\, P_{\rm VBM}(\vec{b}, t_f \,|\, \vec{\lambda}, t)}{P_{\rm VBM}(\vec{b}, t_f \,|\, \vec{a}, 0)}$$



using Karlin-Mc Gregor formula

for $a_i = b_i = \dfrac{i-1}{N}\,, i = 1,2,\cdots, N$

$$P_{\rm VBB}\left(\vec{\lambda}, t \,|\, \vec{a}, 0; \vec{b}, t_f\right) \propto \det_{1\le i,j \le N}\left(e^{-\frac{N}{2t}\left(\lambda_i - a_j\right)^2}\right) \det_{1\le i,j \le N}\left(e^{-\frac{N}{2(t_f-t)}\left(b_i - \lambda_j\right)^2}\right)$$

$$P_{\rm VBB}\left(\vec{\lambda}, t \,|\, \vec{a}, 0; \vec{b}, t_f\right) \propto \prod_{i=1}^{N} e^{-\frac{N t_f}{2t(t_f-t)}\lambda_i^2 + \frac{(N-1)t_f}{2t(t_f-t)}\lambda_i} \prod_{i<j} \sinh\left(\frac{\lambda_i - \lambda_j}{2t}\right) \prod_{i<j} \sinh\left(\frac{\lambda_i - \lambda_j}{2(t_f - t)}\right)$$

⇨ Chern-Simons model (Mariño 2005), bi-orthogonal Stieltjes-Wigert polynomials (Dolivet & Tierz 2007, Katori & Takahashi 2012)

Joint distribution of the positions of the VBB

$$P_{\rm VBB}\left(\vec{\lambda}, t\,|\,\vec{a}, 0; \vec{b}, t_f\right) \propto \prod_{i=1}^{N} e^{-\frac{N t_f}{2t(t_f-t)}\lambda_i^2 + \frac{(N-1)t_f}{2t(t_f-t)}\lambda_i} \prod_{i<j} \sinh\left(\frac{\lambda_i - \lambda_j}{2t}\right) \prod_{i<j} \sinh\left(\frac{\lambda_i-\lambda_j}{2(t_f-t)}\right)$$

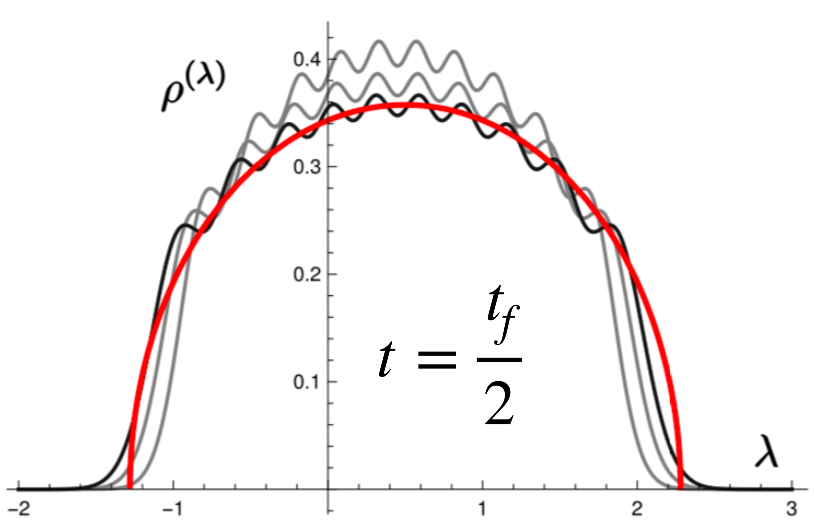
- Muttalib-Borodin (MB) ensembles in the (X, θ) variables, $X_i = e^{\theta \frac{\lambda_i}{t}}$, $\theta = \dfrac{t}{t_f - t}$

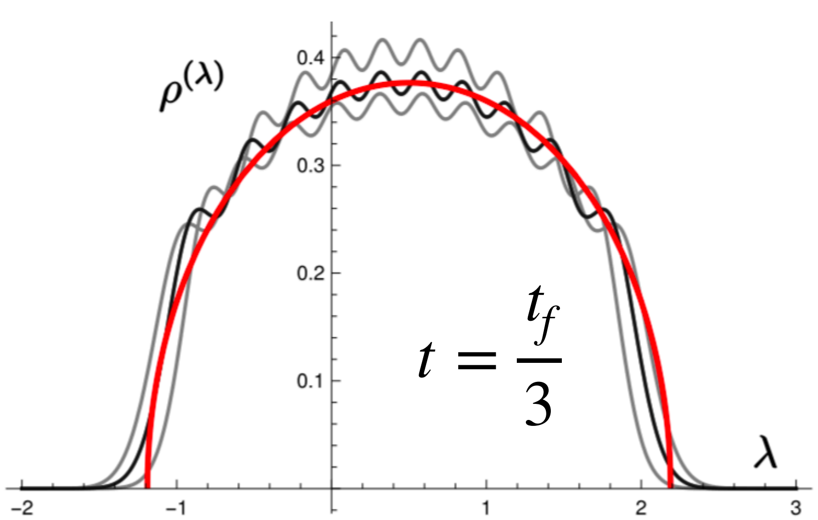
$$\mathscr{P}_{\rm VBB}\left(\vec{X}, \theta\,|\,\vec{a}, 0; \vec{b}, t_f\right) \propto \prod_{i=1}^{N} e^{-\frac{N t_f}{2\theta}(\log X_i)^2 - \log X_i} \prod_{i<j}(X_i - X_j) \prod_{i<j}\left(X_i^{1/\theta} - X_j^{1/\theta}\right)$$

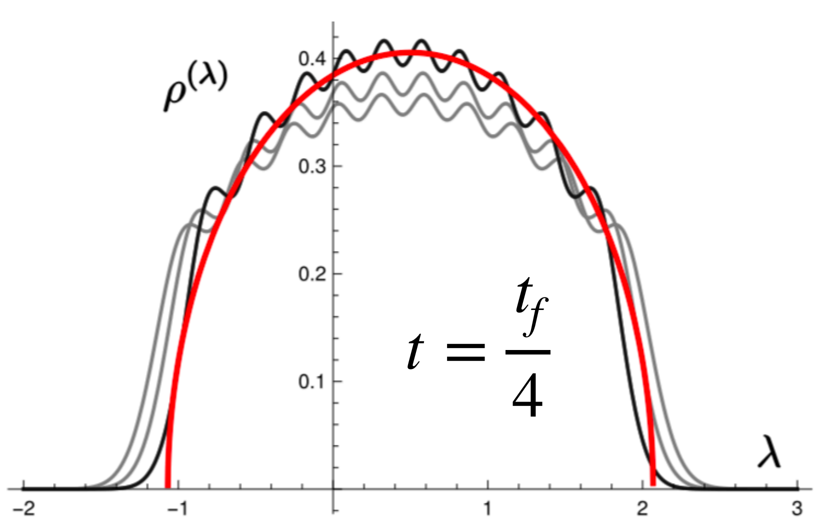
see also Claeys, Romano '14, Claeys, Wang '22

with parameter $1/\theta$

- Formal exact results for finite N were obtained by Takahashi & Katori (2012), e.g. for the density... but extracting the large N limit seems very difficult







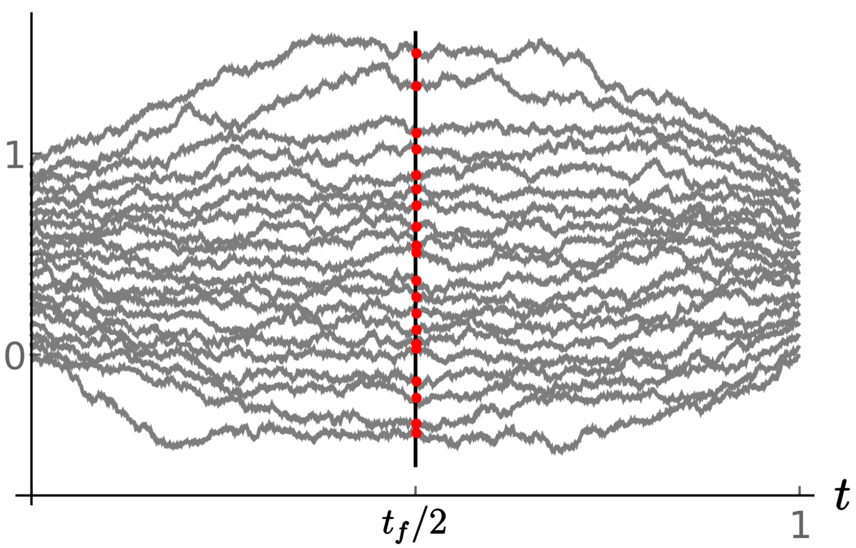
Average density in the large N limit

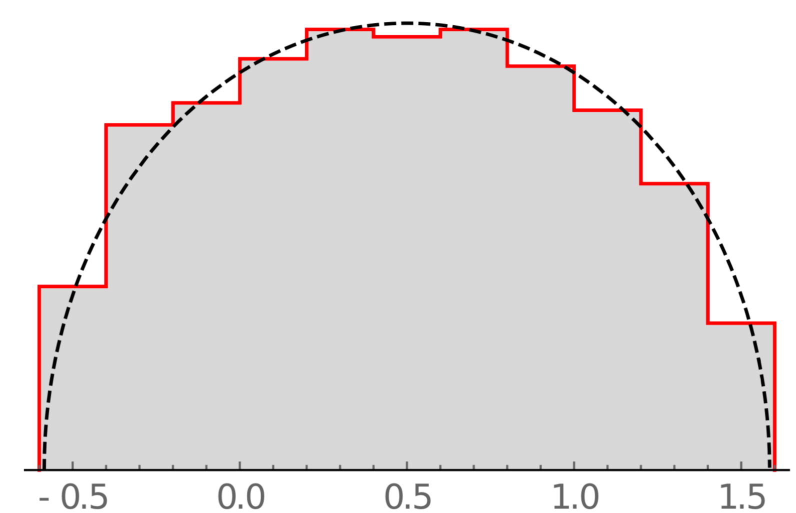
$$\mathscr{P}_{\rm VBB}\left(\vec{X},\theta\,|\,\vec{a},0;\vec{b},t_f\right) \propto \prod_{i=1}^{N} e^{-\frac{N t_f}{2\theta}(\log X_i)^2 - \log X_i} \prod_{i<j}(X_i - X_j) \prod_{i<j}\left(X_i^{1/\theta} - X_j^{1/\theta}\right)$$

$$X_i = e^{\theta\,\frac{\lambda_i}{t}}\,,\ \theta = \frac{t}{t_f - t}$$

with parameter $1/\theta$

- For the special case $t = t_f/2$, i.e., $\theta = 1$, this becomes an orthogonal polynomial ensemble and the density for large N can be computed: NOT a semi-circle !





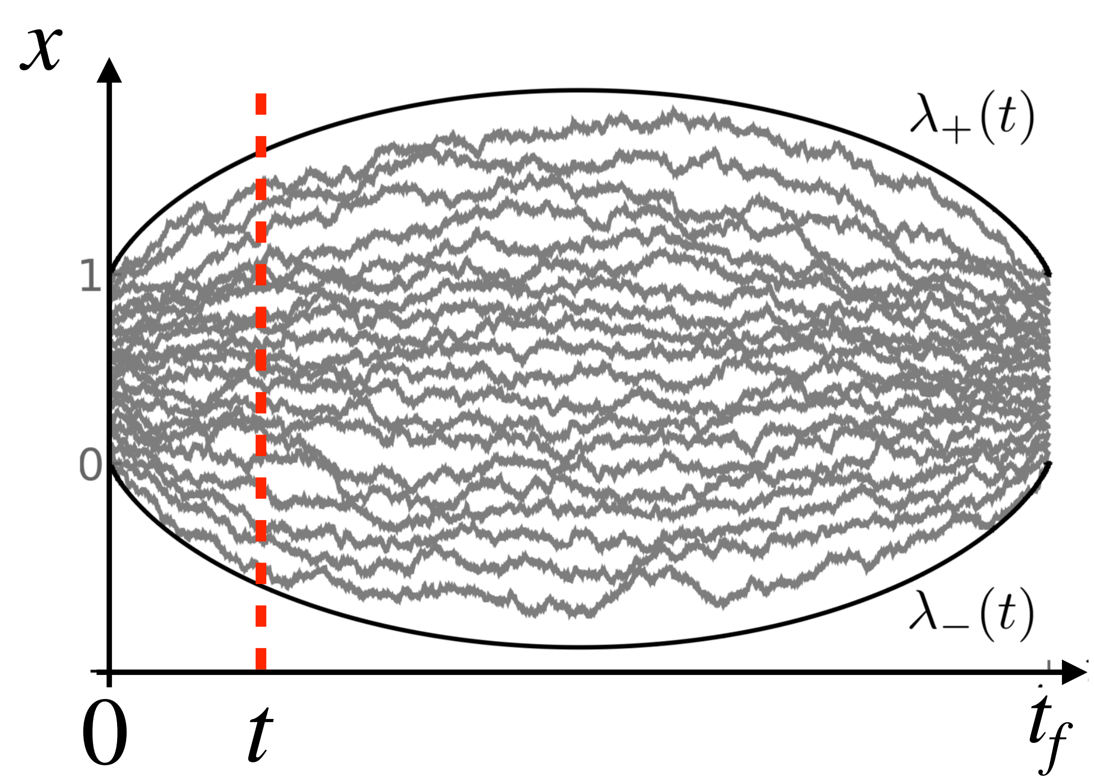
$$\frac{t}{t_f} = \frac{1}{2}$$

M. Marino (2005), P. J. Forrester (2021)

Average density in the large N limit

$$\mathscr{P}_{\rm VBB}\left(\vec{X},\theta\,|\,\vec{a},0;\vec{b},t_f\right) \propto \prod_{i=1}^{N} e^{-\frac{N t_f}{2\theta}(\log X_i)^2 - \log X_i} \prod_{i<j}(X_i - X_j) \prod_{i<j}\left(X_i^{1/\theta} - X_j^{1/\theta}\right)$$

$$X_i = e^{\theta \frac{\lambda_i}{t}}\ ,\ \theta = \frac{t}{t_f - t}$$



- For $t \neq t_f/2$, i.e., $\theta \neq 1$, hard to compute the average large N density

- However, our effective Langevin equation gives access to this average density, in principle for any $t \in [0,t_f]$

Towards computing the average density from the effective Langevin equation

- For the equi-spaced/flat final condition: $b_i = \dfrac{i-1}{N}, i = 1,2,\cdots,N$

$$\frac{d\lambda_i}{dt} = -\frac{\lambda_i}{t_f - t} + \frac{1}{N(t_f - t)} \sum_{j(\neq i)} \frac{e^{\frac{\lambda_i}{t_f - t}}}{e^{\frac{\lambda_i}{t_f - t}} - e^{\frac{\lambda_j}{t_f - t}}} + \frac{1}{\sqrt{N}} \xi_i(t) \qquad i = 1,\cdots,N$$

- To compute the density, a convenient change of variables

$$\begin{cases} \lambda_i = \dfrac{t_f}{1+\theta} \log X_i \quad , \quad X_i > 0 \\ t = t_f \dfrac{\theta}{1+\theta} \end{cases} ,$$

$$\boxed{\frac{dX_i}{d\theta} = \frac{1}{Nt_f} \sum_{j(\neq i)} \frac{X_i^2}{X_i - X_j} + \frac{1+\theta}{t_f} X_i \, \tilde{\xi}_i(\theta)} , \quad X_i(\theta = 0) = e^{\frac{a_i}{t_f}}$$

N indep. Gaussian white noises

⇨ multiplicative noise (Ito prescription)

From Langevin to Burgers' equation

- The goal is to compute the density in the original variables (λ, t)

$$\frac{d\lambda_i}{dt} = -\frac{\lambda_i}{t_f - t} + \frac{1}{N(t_f - t)} \sum_{j(\neq i)} \frac{e^{\frac{\lambda_i}{t_f - t}}}{e^{\frac{\lambda_i}{t_f - t}} - e^{\frac{\lambda_j}{t_f - t}}} + \frac{1}{\sqrt{N}} \xi_i(t) \quad , \quad \lambda_i(t = 0) = a_i$$

$$\rho_N^{(\lambda)}(\lambda; t) = \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta(\lambda - \lambda_i(t)) \right\rangle \quad = \frac{e^{\frac{\lambda}{t_f - t}}}{t_f - t} \rho_N \left(X = e^{\frac{\lambda}{t_f - t}}; \theta = \frac{t}{t_f - t} \right)$$

- But it is easier to compute the density in the variables (X, θ)

$$\frac{dX_i}{d\theta} = \frac{1}{Nt_f} \sum_{j(\neq i)} \frac{X_i^2}{X_i - X_j} + \frac{1 + \theta}{t_f} X_i \tilde{\xi}_i(\theta) \quad , \quad X_i(\theta = 0) = e^{\frac{a_i}{t_f}}$$

$$\rho_N(X; \theta) = \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta\left(X - X_i(\theta)\right) \right\rangle$$

From Langevin to Burgers' equation

$$\rho_N(X;\theta) = \frac{1}{N}\left\langle \sum_{i=1}^{N} \delta\left(X - X_i(\theta)\right) \right\rangle$$

Q: how to compute it ?

- Introduce the resolvent (or Green 's function) with $X = e^y$

$$G_N(y;\theta) = \frac{e^y}{Nt_f}\left\langle \sum_{i=1}^{N} \frac{1}{e^y - X_i(\theta)} \right\rangle$$

defined on the complex $y-$plane

- The density is obtained from the Sochocki-Plemelj formula

$$\rho_N(X;\theta) = \frac{t_f}{\pi} \lim_{\epsilon\to 0_+} \mathrm{Im}\left[\frac{1}{y} G_N(\ln y;\theta)\right]_{y=X-i\epsilon}$$

⇨

Can one derive an evolution equation (i.e., a PDE) for G_N ?

Average particle density via Burgers' equation

- One can show that as $N \to \infty$

$$G(y;\theta) = \lim_{N\to\infty} G_N(y;\theta)$$

satisfies

$$\partial_\theta G + G\,\partial_y G = 0$$

Inviscid complex Burgers' equation

with initial condition

$$G(y,0) = G_0(y) = \frac{e^y}{t_f} \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{e^y - e^{\frac{a_i}{t_f}}}$$

- This is different from the « standard » Burgers' equation for DBM

Blaizot, Nowak (2010), Allez, Bouchaud, Guionnet (2012),
Blaizot, Grela, Nowak, Warchol (2015), Krajenbrinck, Le Doussal, O'Connel (2020)

Solving the Burgers' equation in "flat-to-flat" geometry

$$\partial_\theta G + G\,\partial_y G = 0 \quad , \quad G(y,0) = G_0(y)$$

- The Burgers' equation can be solved via the method of characteristics
- Solution for the « flat-to-flat » geometry $a_i = b_i = \frac{i-1}{N}\, , i = 1,2,\cdots,N$

$$H(y;\theta) = e^{-G(y;\theta)} \quad , \quad \text{where} \quad X = e^{y} \quad \text{and} \quad T = e^{-1/t_f}$$

$$H^{\theta} = \frac{1}{X}\frac{1-H}{T-H} \quad , \quad \theta = \frac{t}{t_f - t}$$

- Limiting density: $\rho(X;\theta) = \lim_{N\to\infty} \rho_N(X;\theta) = \frac{t_f}{\pi} \lim_{\epsilon\to 0_+} \mathrm{Im}\left[\frac{1}{y} G(\ln y;\theta)\right]_{y=X-i\epsilon}$
- From $\rho(X;\theta)$ we can compute the average density in the original (λ, t) coordinates

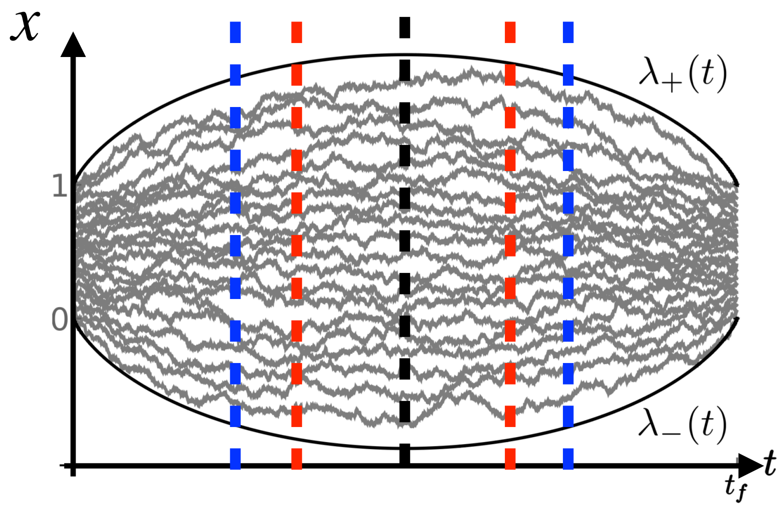
Solving the Burgers' equation in "flat-to-flat" geometry

Grela, Majumdar, G. S. (2021)

$$\partial_\theta G + G\,\partial_y G = 0 \quad , \quad G(y,0) = G_0(y)$$

- The Burgers' equation can be solved via the method of characteristics

⇨ allows to compute the edges of the support $[\lambda_-(t), \lambda_+(t)]$



$$\lambda_\pm(t) = \frac{1}{2} \pm \left[t_f \operatorname{arccosh}\left(\frac{1}{\sqrt{T}} \frac{t_f + T(t_f - 2t)}{2(t_f - t)} \right) - t \operatorname{arccosh}\left(\frac{(t_f - t)^2 + t^2 - T(t_f - 2t)^2}{2t(t_f - t)} \right) \right]$$

- By solving $H^\theta = \dfrac{1}{X}\dfrac{1-H}{T-H}$, $\theta = \dfrac{t}{t_f - t}$, for $\theta = 2,3,4$ we obtain the density explicitly for $t = \dfrac{t_f}{4}, \dfrac{t_f}{3}, \dfrac{t_f}{2}, \dfrac{2t_f}{3}, \dfrac{3t_f}{4}$

Outline

- Vicious Brownian Bridge, Dyson Brownian Bridge and Effective Langevin equation
- Average particle density via Burgers' equation
- Conclusion

Conclusions

- Mapping between N nonintersecting Brownian Bridges and N Dyson Brownian Bridges of index $\beta = 2$

- Efficient numerical method to generate Dyson Brownian Bridges via an effective Langevin equation

⇨ can be extended to generate discrete time random bridges

De Bruyne, Majumdar, G. S. (2021)

- Exploit this effective Langevin equation to derive a Burgers' equation (in the inviscid limit) for the Green 's function (resolvent)

⇨ exact results for the density in the limit $N \to \infty$

- Extensions to other models (e.g. deformed GOE...)

Mergny, Majumdar (2022)