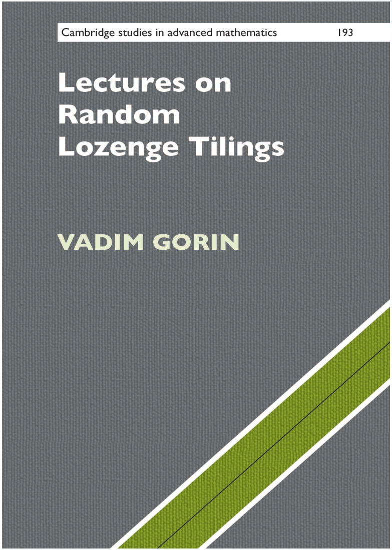

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Lozenge tilings via the dynamic loop equation.

Vadim Gorin

May 20, 2022

Part 1: A general interacting Markov chain

State space: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \in \mathbb{Z}, \quad \boldsymbol{x} = \{x_i\} = \{\lambda_i - i\theta\}$

Transition probabilities: for $\boldsymbol{e} \in \{0,1\}^N$

$$\mathbb{P}(\boldsymbol{x}+\boldsymbol{e}|\boldsymbol{x}) \sim \prod_{1\leq i<j\leq N} \frac{b(x_i+\theta e_i) - b(x_j+\theta e_j)}{b(x_i) - b(x_j)} \prod_{i=1}^{N} \phi^+(x_i)^{e_i} \phi^-(x_i)^{1-e_i}$$

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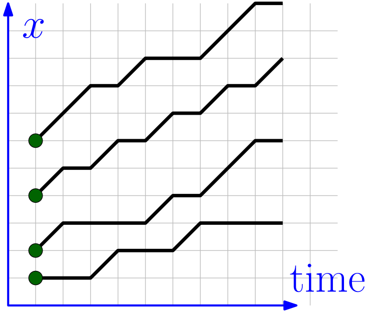
Example: *non-intersecting* independent random walks



$\theta = 1$,
$b(x) = x$,
$\phi^+(x) = p$,
$\phi^-(x) = 1 - p$.

The dynamical loop equation

Transition probabilities: for $\boldsymbol{e} \in \{0,1\}^N$

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Theorem. (G.-Huang-22) Assume holomorphic b, $\phi^{\pm}$. Then so is

$$\mathbb{E}\left[\phi^+(z)\prod_{j=1}^{N}\frac{b(z+\theta)-b(x_j+\theta e_j)}{b(z)-b(x_j)} + \phi^-(z)\prod_{j=1}^{N}\frac{b(z)-b(x_j+\theta e_j)}{b(z)-b(x_j)}\right].$$

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- A new relative of Dyson-Schwinger / Nekrasov / loop equations for β–ensembles of random matrices and log-gases.

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- A new relative of Dyson-Schwinger / Nekrasov / loop equations for β–ensembles of random matrices and log-gases.
- A basic block for asymptotics (cf. Yang–Baxter relation).
- In fact, there are far ancestors in the Baxter's book.

The dynamical loop equation: examples

Transition probabilities: for $\boldsymbol{e} \in \{0,1\}^N$

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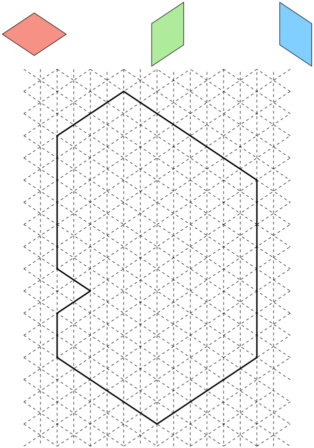
1. Non-intersecting Bernoulli and Poisson random walks;
2. Dyson Brownian Motion (at general β);
3. Random lozenge and domino tilings;
4. Corners process of self-adjoint random matrices (at general β);
5. Macdonald / Koornwinder processes (principal specialization).

Part II: Application to (q, κ)–distributions on tilings

Lozenge tilings of planar domains ("trapezoids")



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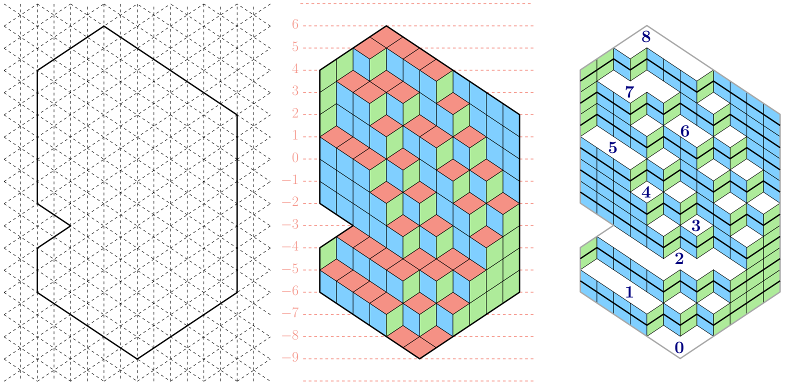

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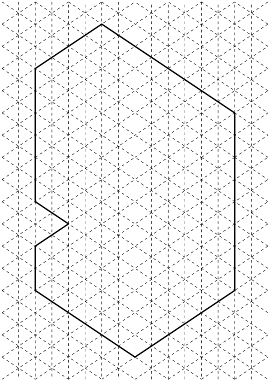

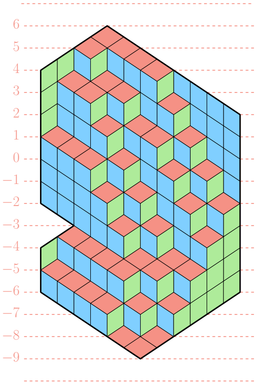

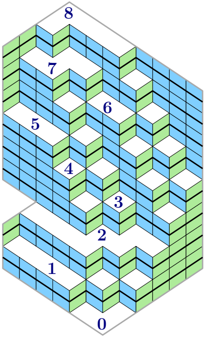

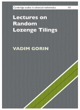

- *Uniformly random tilings* are well-understood by now.
- Today: **a non-uniform and inhomogeneous case**.

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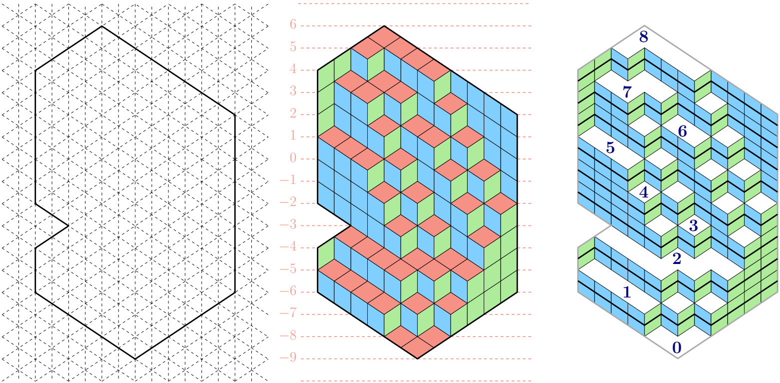
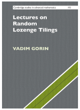
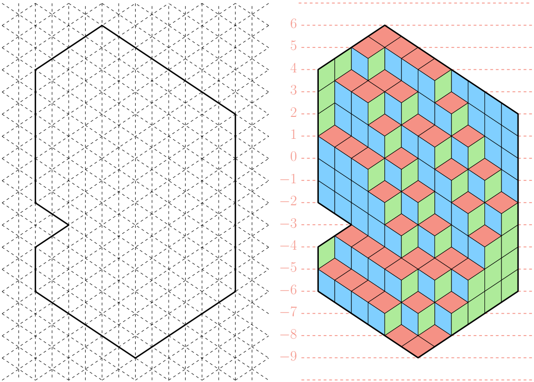
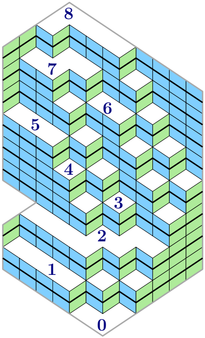




- *Uniformly random tilings* are well-understood by now.
- Today: **a non-uniform and inhomogeneous case**. **Why?**

Part II: Application to (q, κ)–distributions on tilings





Motivations for studying inhomogeneous random tilings:

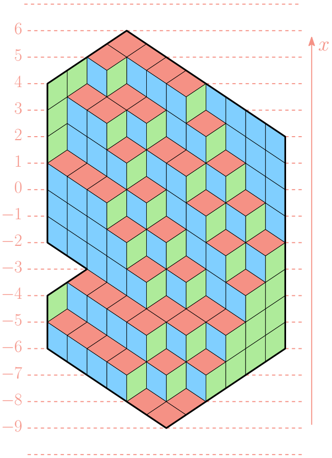
A priory: *Probing universality*

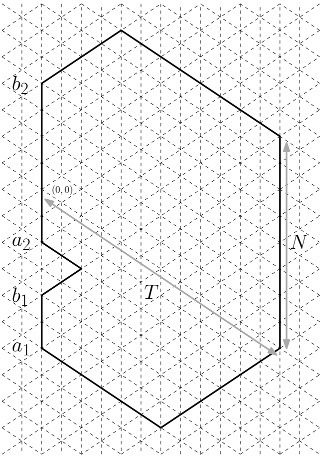
Does the homogeneous case phenomenology extend?

A posteriori: *Discovering integrability*

Koornwinder polynomials; conformal invariance; algebraic answers.

(q,κ)–distributions on lozenge tilings





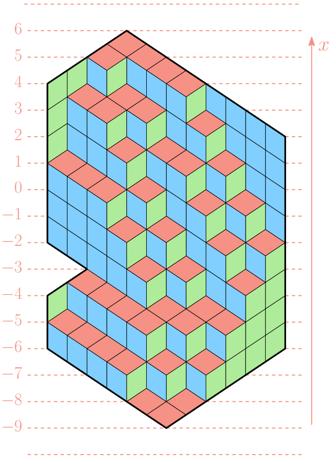
$$\mathbb{P}(\mathcal{T}) = \frac{1}{\mathcal{Z}} \prod_{\diamond \in \mathcal{T}} w(\diamond),$$

$$w(\diamond) = \kappa q^x - \kappa^{-1} q^{-x},$$

Width: T

Right/left boundaries:
$N = \sum_{i=1}^{r}(b_i - a_i)$

(q, κ)–distributions on lozenge tilings



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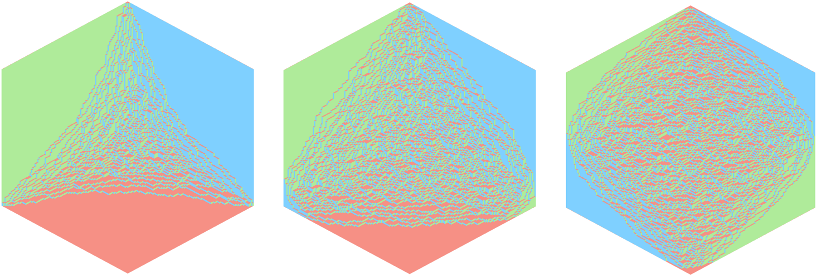
Degenerations:

- $q = 1$: Uniform measure
- $\kappa \to \infty$: Measure q^{volume}
- $\kappa \to 0$: Measure $q^{-\text{volume}}$
- $q, \kappa \to 1$: Linear $w(\diamond)$

Positivity:

- Real κ and q.
- Imaginary κ, real q.
- Complex κ and q with $|\kappa| = |q| = 1$.

Random samples for $100 \times 100 \times 100$ hexagon



- Sampler of [Borodin-Gorin-Rains-10] for any (q, κ).

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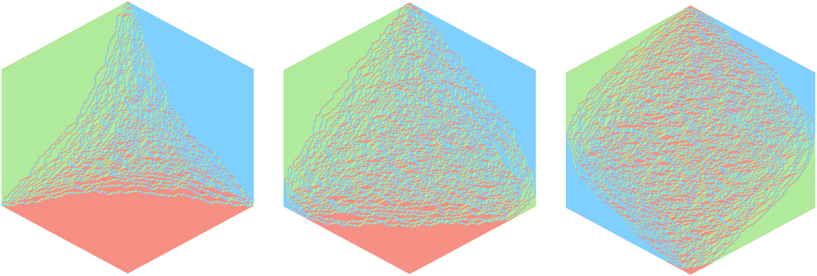

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- Law of Large Numbers + local bulk limit theorem for the hexagon in [Borodin-Gorin-Rains-10].
- CLT for global fluctuation along a single slice of the hexagon in [Dimitrov-Knizel-19]; several slices in [Duits-Liu-22+].

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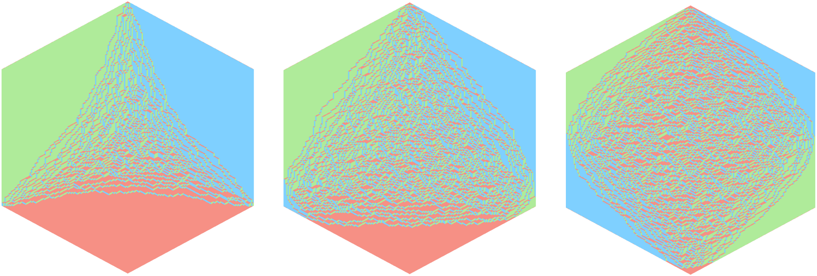

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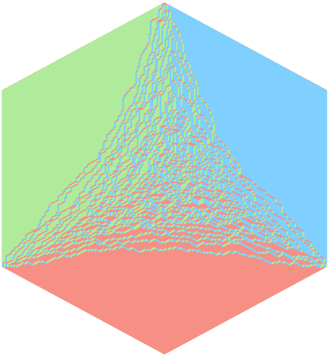
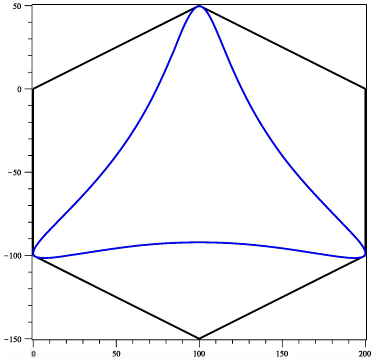
- More general polygons were not accessible before today.

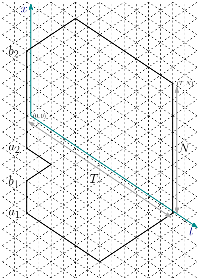
Arctic boundary: algebraic parameterization





Task. Prove that tilings are asymptotically frozen outside a curve.
Find this **Arctic curve.**

Arctic boundary: algebraic parameterization



$$w(\diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2},$$

Small parameter $\varepsilon \to 0$ and

$$\varepsilon N \to \boldsymbol{N}, \quad \varepsilon T \to \boldsymbol{T}, \quad \varepsilon \ln(q) \to \ln(\boldsymbol{q}),$$

$$\varepsilon a_i \to \boldsymbol{a}_i, \quad \varepsilon b_i \to \boldsymbol{b}_i, \quad 1 \le i \le r.$$

Theorem. (G.-Huang-22) The Arctic curve $(\boldsymbol{t}(u), \boldsymbol{x}(u))$ is:

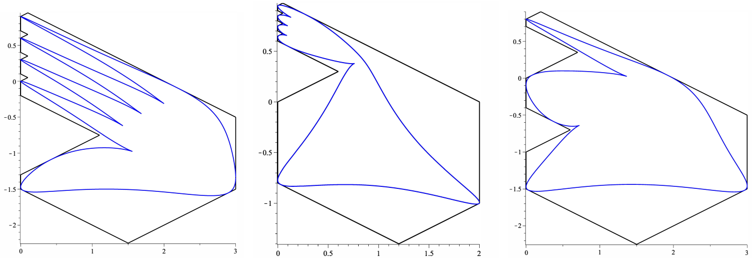
$$\boldsymbol{q}^{\boldsymbol{t}} = \frac{V'(u)}{U'(u)}, \qquad \boldsymbol{q}^{\boldsymbol{x}} = \frac{w \pm \sqrt{w^2 - 4\kappa^2 \boldsymbol{q}^{-\boldsymbol{t}}}}{2\kappa^2 \boldsymbol{q}^{-\boldsymbol{t}}}, \quad w = V(u) - \frac{U(u)}{\boldsymbol{q}^{\boldsymbol{t}}},$$

where $\boldsymbol{q}^{-u} + \kappa^2 \boldsymbol{q}^u$ runs over the real line $\mathbb{R}$ and

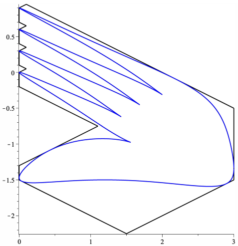
$$f_0(u) = \frac{(\boldsymbol{q}^{\boldsymbol{N}} - \boldsymbol{q}^u)(\kappa^2 \boldsymbol{q}^{-\boldsymbol{T}} - \boldsymbol{q}^{-u})}{(\kappa^2 \boldsymbol{q}^{\boldsymbol{N}} - \boldsymbol{q}^{-u})(\boldsymbol{q}^{-\boldsymbol{T}} - \boldsymbol{q}^u)} \prod_{i=1}^{r} \frac{(\boldsymbol{q}^{\boldsymbol{a}_i} - \boldsymbol{q}^u)(\kappa^2 \boldsymbol{q}^{\boldsymbol{b}_i} - \boldsymbol{q}^{-u})}{(\kappa^2 \boldsymbol{q}^{\boldsymbol{a}_i} - \boldsymbol{q}^{-u})(\boldsymbol{q}^{\boldsymbol{b}_i} - \boldsymbol{q}^u)},$$

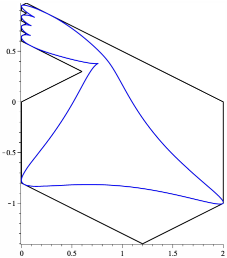
$$U(u) = \frac{f_0(u)\boldsymbol{q}^{-u} - \kappa^2 \boldsymbol{q}^u}{1 - f_0(u)}, \qquad V(u) = \frac{\boldsymbol{q}^{-u} - f_0(u)\kappa^2 \boldsymbol{q}^u}{1 - f_0(u)}.$$

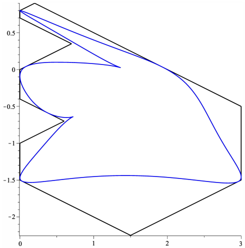
Examples of Arctic curves

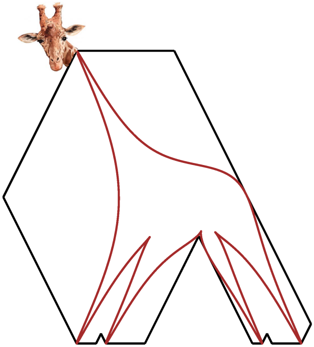


Examples of Arctic curves

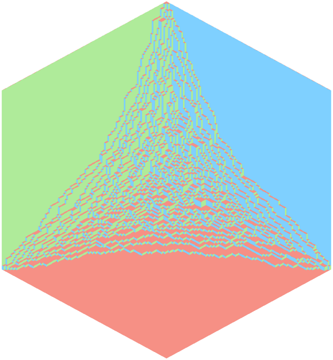


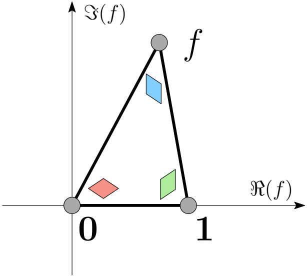






Algebraic Limit Shape



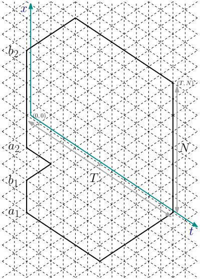


Encoding via **complex slope**
motivated by [Kenyon–Okounkov-05]

Task. Identify the **limit shape** in terms of the asymptotic proportions of lozenges.

Find $(\boldsymbol{t}, \boldsymbol{x}) \mapsto (p_{\Diamond}, p_{\square}, p_{\square})$ or $(\boldsymbol{t}, \boldsymbol{x}) \mapsto f$.

Algebraic Limit Shape



$$w(\diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2},$$

Small parameter $\varepsilon \to 0$ and

$$\varepsilon N \to \boldsymbol{N}, \quad \varepsilon T \to \boldsymbol{T}, \quad \varepsilon \ln(q) \to \ln(\boldsymbol{q}),$$

$$\varepsilon a_i \to \boldsymbol{a}_i, \quad \varepsilon b_i \to \boldsymbol{b}_i, \quad 1 \le i \le r.$$

Theorem. (G.-Huang-22) Complex slope in the **liquid region**:

$$f(\boldsymbol{t},\boldsymbol{x}) = \frac{(\boldsymbol{q}^{-u} + \kappa^2 \boldsymbol{q}^{u}) - (\boldsymbol{q}^{-\boldsymbol{x}} + \kappa^2 \boldsymbol{q}^{\boldsymbol{x}})}{(\boldsymbol{q}^{-u} + \kappa^2 \boldsymbol{q}^{u}) - (\boldsymbol{q}^{-\boldsymbol{x}+\boldsymbol{t}} + \kappa^2 \boldsymbol{q}^{\boldsymbol{x}-\boldsymbol{t}})},$$

where u solves $\quad \boldsymbol{q}^{-\boldsymbol{x}} + \kappa^2 \boldsymbol{q}^{\boldsymbol{x}-\boldsymbol{t}} = V(u) - \frac{U(u)}{\boldsymbol{q}^{\boldsymbol{t}}} \quad$ with

$$f_0(u) = \frac{(\boldsymbol{q}^{\boldsymbol{N}} - \boldsymbol{q}^{u})(\kappa^2 \boldsymbol{q}^{-\boldsymbol{T}} - \boldsymbol{q}^{-u})}{(\kappa^2 \boldsymbol{q}^{\boldsymbol{N}} - \boldsymbol{q}^{-u})(\boldsymbol{q}^{-\boldsymbol{T}} - \boldsymbol{q}^{u})} \prod_{i=1}^{r} \frac{(\boldsymbol{q}^{\boldsymbol{a}_i} - \boldsymbol{q}^{u})(\kappa^2 \boldsymbol{q}^{\boldsymbol{b}_i} - \boldsymbol{q}^{-u})}{(\kappa^2 \boldsymbol{q}^{\boldsymbol{a}_i} - \boldsymbol{q}^{-u})(\boldsymbol{q}^{\boldsymbol{b}_i} - \boldsymbol{q}^{u})},$$

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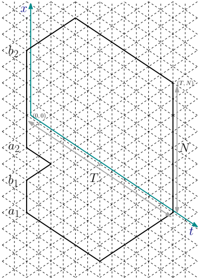
Algebraic Limit Shape



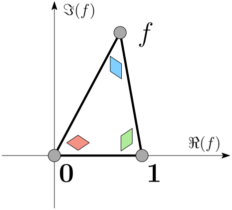
$$w(\diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2},$$

Small parameter $\varepsilon \to 0$ and

$$\varepsilon N \to \boldsymbol{N}, \quad \varepsilon T \to \boldsymbol{T}, \quad \varepsilon \ln(q) \to \ln(\boldsymbol{q}),$$

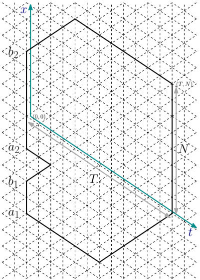
$$\varepsilon a_i \to \boldsymbol{a}_i, \quad \varepsilon b_i \to \boldsymbol{b}_i, \quad 1 \le i \le r.$$

Corollary. (G.-Huang-22) For a **polynomial** Q, in the liquid region



$$Q\left(\frac{f\boldsymbol{q}^{\boldsymbol{t}-\boldsymbol{x}} - \kappa^2 \boldsymbol{q}^{\boldsymbol{x}}}{1-f}, \frac{\boldsymbol{q}^{-\boldsymbol{x}} - f\kappa^2 \boldsymbol{q}^{\boldsymbol{x}-\boldsymbol{t}}}{1-f}\right) = 0.$$

Algebraic Limit Shape



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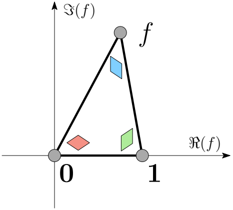
$$\varepsilon N \to \boldsymbol{N}, \quad \varepsilon T \to \boldsymbol{T}, \quad \varepsilon \ln(q) \to \ln(\boldsymbol{q}),$$

$$\varepsilon a_i \to \boldsymbol{a}_i, \quad \varepsilon b_i \to \boldsymbol{b}_i, \quad 1 \le i \le r.$$

Corollary. (G.-Huang-22) For a **polynomial** Q, in the liquid region

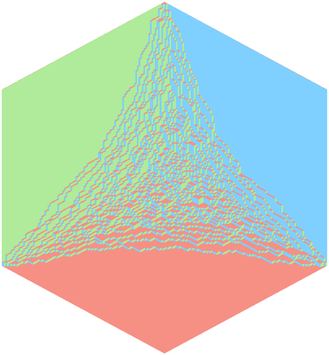


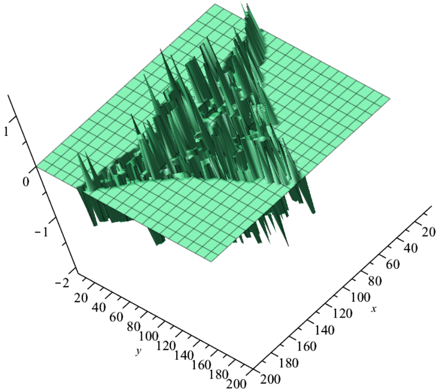
$$Q\left(\frac{f\boldsymbol{q}^{\boldsymbol{t}-\boldsymbol{x}} - \kappa^2 \boldsymbol{q}^{\boldsymbol{x}}}{1-f}, \frac{\boldsymbol{q}^{-\boldsymbol{x}} - f\kappa^2 \boldsymbol{q}^{\boldsymbol{x}-\boldsymbol{t}}}{1-f}\right) = 0.$$

Conjecture. Same statement for tilings of **arbitrary polygons**.

[Kenyon-Okounkov-05] discovered this for $\kappa = \infty$ case (q^{volume}).

Height fluctuations

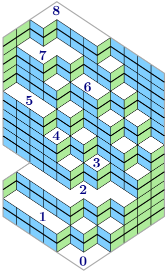




Task. Compute **asymptotic fluctuations** of the centered heights.

Find the field $\lim_{\varepsilon \to 0} \left[h(\varepsilon^{-1}\boldsymbol{t}, \varepsilon^{-1}\boldsymbol{x}) - \mathbb{E}h(\varepsilon^{-1}\boldsymbol{t}, \varepsilon^{-1}\boldsymbol{x}) \right]$.

Gaussian Free Field fluctuations



$$w(\diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2},$$

Small parameter $\varepsilon \to 0$ and

$$\varepsilon N \to \boldsymbol{N}, \quad \varepsilon T \to \boldsymbol{T}, \quad \varepsilon \ln(q) \to \ln(\boldsymbol{q}),$$

$$\varepsilon a_i \to \boldsymbol{a}_i, \quad \varepsilon b_i \to \boldsymbol{b}_i, \quad 1 \le i \le r.$$

Theorem. (G.-Huang-22) Inside the liquid region, we have

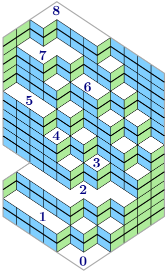
$$\lim_{\varepsilon \to 0} \sqrt{\pi}\Big(h(\varepsilon^{-1}\boldsymbol{t}, \varepsilon^{-1}\boldsymbol{x}) - \mathbb{E}[h(\varepsilon^{-1}\boldsymbol{t}, \varepsilon^{-1}\boldsymbol{x})]\Big) = \mathfrak{G}(\boldsymbol{t}, \boldsymbol{x});$$

$\mathfrak{G}(\boldsymbol{t}, \boldsymbol{x})$ is a (generalized) centered Gaussian field of covariance

$$\mathbb{E}\mathfrak{G}(\boldsymbol{t}, \boldsymbol{x})\mathfrak{G}(\boldsymbol{t}', \boldsymbol{x}') = -\frac{1}{2\pi} \ln \left| \frac{\Omega(\boldsymbol{t}, \boldsymbol{x}) - \Omega(\boldsymbol{t}', \boldsymbol{x}')}{\overline{\Omega}(\boldsymbol{t}, \boldsymbol{x}) - \Omega(\boldsymbol{t}', \boldsymbol{x}')} \right|;$$

$\Omega(\boldsymbol{t}, \boldsymbol{x}) = \boldsymbol{q}^{-u} + \kappa^2 \boldsymbol{q}^{u}$, and u solves $\boldsymbol{q}^{-\boldsymbol{x}} + \kappa^2 \boldsymbol{q}^{\boldsymbol{x}-\boldsymbol{t}} = V(u) - \frac{U(u)}{\boldsymbol{q}^{\boldsymbol{t}}}$

Gaussian Free Field fluctuations



$$w(\diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2},$$

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Corollary. (G.-Huang-22) Inside the liquid region, we have

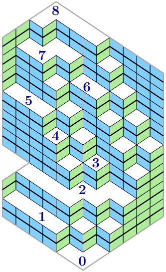
$$\lim_{\varepsilon \to 0} \sqrt{\pi} \big(h(\varepsilon^{-1}\boldsymbol{t}, \varepsilon^{-1}\boldsymbol{x}) - \mathbb{E}[h(\varepsilon^{-1}\boldsymbol{t}, \varepsilon^{-1}\boldsymbol{x})] \big) = \mathfrak{G}(\boldsymbol{t}, \boldsymbol{x});$$

$\mathfrak{G}(\boldsymbol{t}, \boldsymbol{x})$ is the **Gaussian Free Field** in the complex structure of

either $\quad \dfrac{f \boldsymbol{q}^{\boldsymbol{t}-\boldsymbol{x}} - \kappa^2 \boldsymbol{q}^{\boldsymbol{x}}}{1-f} \quad$ or $\quad \dfrac{\boldsymbol{q}^{-\boldsymbol{x}} - f \kappa^2 \boldsymbol{q}^{\boldsymbol{x}-\boldsymbol{t}}}{1-f},$

where $f(\boldsymbol{t}, \boldsymbol{x})$ is the **complex slope** at $(\boldsymbol{t}, \boldsymbol{x})$.

Gaussian Free Field fluctuations



$$w(\diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2},$$

Small parameter $\varepsilon \to 0$ and

$$\varepsilon N \to \boldsymbol{N}, \quad \varepsilon T \to \boldsymbol{T}, \quad \varepsilon \ln(q) \to \ln(\boldsymbol{q}),$$

$$\varepsilon a_i \to \boldsymbol{a}_i, \quad \varepsilon b_i \to \boldsymbol{b}_i, \quad 1 \le i \le r.$$

Corollary. (G.-Huang-22) Inside the liquid region, we have

$$\lim_{\varepsilon \to 0} \sqrt{\pi}\big(h(\varepsilon^{-1}\boldsymbol{t}, \varepsilon^{-1}\boldsymbol{x}) - \mathbb{E}[h(\varepsilon^{-1}\boldsymbol{t}, \varepsilon^{-1}\boldsymbol{x})]\big) = \mathfrak{G}(\boldsymbol{t}, \boldsymbol{x});$$

$\mathfrak{G}(\boldsymbol{t}, \boldsymbol{x})$ is the **Gaussian Free Field** in the complex structure of

either $\quad \dfrac{f\boldsymbol{q}^{\boldsymbol{t}-\boldsymbol{x}} - \kappa^2 \boldsymbol{q}^{\boldsymbol{x}}}{1-f} \quad$ or $\quad \dfrac{\boldsymbol{q}^{-\boldsymbol{x}} - f\kappa^2 \boldsymbol{q}^{\boldsymbol{x}-\boldsymbol{t}}}{1-f},$

where $f(\boldsymbol{t}, \boldsymbol{x})$ is the **complex slope** at $(\boldsymbol{t}, \boldsymbol{x})$.

Conjecture. Same is true for **arbitrary** domains.

Literature for the uniform measure

$$\mathbb{P}(\mathcal{T}) = \frac{1}{\mathcal{Z}} \prod_{\diamond \in \mathcal{T}} w(\diamond), \qquad w(\diamond) = \kappa q^x - \kappa^{-1} q^{-x}$$

For (q, κ)–distributions (even q^{volume}), all three types of results are new.

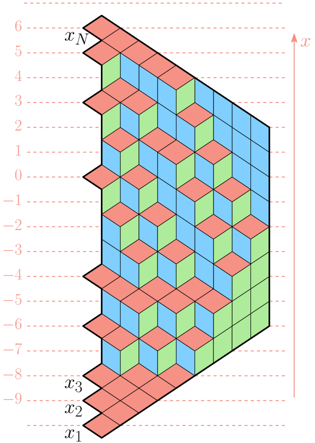
For the particular case of the **uniform measure**:

- Rational parameterization of the Arctic curve for trapezoids. *[Petrov-14]*
- Limit shape via algebraic equations for trapezoids. *[Kenyon–Okounkov-07], [Petrov-14], [Duse-Metcalfe-15]*
- Gaussian Free Field fluctuations for trapezoids. *[Petrov-15], [Bufetov-Gorin-18], [Huang-20]*

A glimpse into the proofs

Step 1: Partition Function

- [Borodin–Gorin–Rains-10] by advanced determinantal evaluations;
- Or, by quasi-branching rules and principal specialization formulas of [Rains-05] for **Koornwinder symmetric polynomials**.



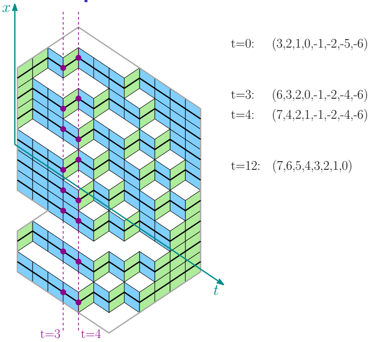
Proposition. For any $x_1 < \cdots < x_N$,

$$\sum_{\mathcal{T}} \prod_{\diamond \in \mathcal{T}} \left(\kappa q^{x(\diamond)} - \kappa^{-1} q^{-x(\diamond)} \right)$$

$$= Z_N \cdot \prod_{i<j} \left[\kappa q^{x_i} - \kappa^{-1} q^{-x_i} - \kappa q^{x_j} + \kappa^{-1} q^{-x_j} \right]$$

A glimpse into the proofs

Step 2: Particle-hole involution and **Markov chain structure**



Proposition. $\{x(t)\}$ for the (q,κ)-distributions on lozenge tilings is a Markov chain with transition probabilities:

$$\sim \prod_{1\le i<j\le N} \frac{b_t(x_i+\theta e_i)-b_t(x_j+\theta e_j)}{b_t(x_i)-b_t(x_j)} \prod_{i=1}^{N} \phi_t^{+}(x_i)^{e_i}\phi_t^{-}(x_i)^{1-e_i},$$

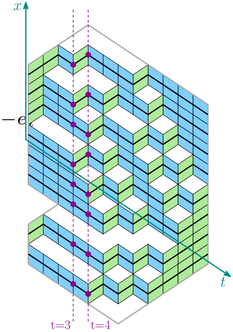
where $\boldsymbol{e}\in\{0,1\}^N$, $\theta=1$, $b_t(x)=q^{-x}+\kappa^2 q^{x-t}$, and

$$\phi_t^{+}(x)=q^{T+N-1-t}(1-q^{x-N+1})(1-\kappa^2 q^{x-T+1}),\quad \phi_t^{-}(x)=-(1-q^{x+T-t})(1-\kappa^2 q^{x+N-t}).$$

A glimpse into the proofs

Step 3: Apply **dynamical loop equations** to

$$\prod_{i<j} \frac{b_t(x_i+\theta e_i) - b_t(x_j+\theta e_j)}{b_t(x_i) - b_t(x_j)} \prod_{i=1}^{N} \phi_t^+(x_i)^{e_i} \phi_t^-(x_i)^{1-e_i}$$



Holomorphic

$$\mathbb{E}\left[\phi^+(z) \prod_{j=1}^{N} \frac{b(z+\theta) - b(x_j+\theta e_j)}{b(z) - b(x_j)} + \phi^-(z) \prod_{j=1}^{N} \frac{b(z) - b(x_j+\theta e_j)}{b(z) - b(x_j)}\right].$$

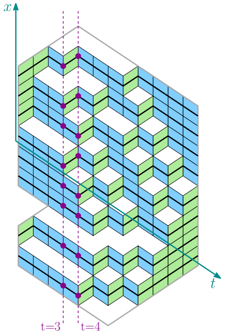
Leads to the decomposition of the time increment as

"deterministic drift" + "Gaussian stochastic part" + "small error"

A glimpse into the proofs

Step 4: **Solve** the stochastic evolution

"deterministic drift" + "Gaussian part"



Key roles played by:

- Analytic continuation of the **complex slope** $f(\boldsymbol{t}, \boldsymbol{x})$ from real $\boldsymbol{x}$ to complex $\boldsymbol{z}$: "doubly complex slope"
- Characteristic flow of the first-order PDE in $(\mathbb{R} \times \mathbb{C} \mapsto \mathbb{C})$

$$\partial_{\boldsymbol{t}} \ln f(\boldsymbol{t}, \boldsymbol{z}) + \partial_{\boldsymbol{z}} \ln\big(1 - f(\boldsymbol{t}, \boldsymbol{z})\big) = \ln(\boldsymbol{q}) \frac{\kappa^2 \boldsymbol{q}^{\boldsymbol{z}-\boldsymbol{t}} + \boldsymbol{q}^{-\boldsymbol{z}}}{\kappa^2 \boldsymbol{q}^{\boldsymbol{z}-\boldsymbol{t}} - \boldsymbol{q}^{-\boldsymbol{z}}}.$$

Summary

- The dynamical loop equation

$$\prod_{i<j} \frac{b_t(x_i+\theta e_i) - b_t(x_j+\theta e_j)}{b_t(x_i) - b_t(x_j)} \prod_{i=1}^{N} \phi_t^+(x_i)^{e_i} \phi_t^-(x_i)^{1-e_i}$$

$$\mathbb{E}\left[\phi^+(z)\prod_{j=1}^{N}\frac{b(z+\theta)-b(x_j+\theta e_j)}{b(z)-b(x_j)} + \phi^-(z)\prod_{j=1}^{N}\frac{b(z)-b(x_j+\theta e_j)}{b(z)-b(x_j)}\right].$$

- Application to (q,κ)–lozenge tilings: parameterized Arctic curve, algebraic limit shape, GFF fluctuations.