Over the past 25 years, there has been an explosion of interest in the area of random tilings. The first book devoted to the topic, this timely text describes the mathematical theory of tilings. It starts from the most basic questions (which planar domains are tileable?), before discussing advanced topics about the local structure of very large random tessellations. The author explains each feature of random tilings of large domains, discussing several different points of view and leading on to open problems in the field. The book is based on upper-division courses taught to a variety of students but it also serves as a self-contained introduction to the subject. Test your understanding with the exercises provided and discover connections to a wide variety of research areas in mathematics, theoretical physics, and computer science, such as conformal invariance, determinantal point processes, Gibbs measures, high-dimensional random sampling, symmetric functions, and variational problems.
Lozenge tilings via the dynamic loop equation.

Vadim Gorin

May 20, 2022
Part 1: A general interacting Markov chain

State space: \[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \in \mathbb{Z}, \quad x = \{x_i\} = \{\lambda_i - i\theta\} \]

Transition probabilities: \[ P(x + e|x) \sim \prod_{1 \leq i < j \leq N} \frac{b(x_i + \theta e_i) - b(x_j + \theta e_j)}{b(x_i) - b(x_j)} \prod_{i=1}^{N} \phi^+(x_i)^{e_i} \phi^-(x_i)^{1-e_i} \]
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\]

Example: non-intersecting independent random walks

\[
\theta = 1, \\
b(x) = x, \\
\phi^{+}(x) = p, \\
\phi^{-}(x) = 1 - p.
\]
The dynamical loop equation

**Transition probabilities:** for \( e \in \{0, 1\}^N \)

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P(x+e|x) \sim \prod_{1 \leq i < j \leq N} \frac{b(x_i + \theta e_i) - b(x_j + \theta e_j)}{b(x_i) - b(x_j)} \prod_{i=1}^{N} \phi^+(x_i)^{e_i} \phi^-(x_i)^{1-e_i}
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**Theorem.** (G.-Huang-22) Assume holomorphic \( b, \phi^\pm \). Then so is

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\mathbb{E} \left[ \phi^+(z) \prod_{j=1}^{N} \frac{b(z + \theta) - b(x_j + \theta e_j)}{b(z) - b(x_j)} + \phi^-(z) \prod_{j=1}^{N} \frac{b(z) - b(x_j + \theta e_j)}{b(z) - b(x_j)} \right].
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- **Equation** is a statement about the cancellation of the poles.
The dynamical loop equation

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- **Equation** is a statement about the cancellation of the poles.
- A new relative of Dyson-Schwinger / Nekrasov / loop equations for $\beta$–ensembles of random matrices and log-gases.
The dynamical loop equation

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- A basic block for asymptotics (cf. Yang–Baxter relation).
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- **Equation** is a statement about the cancellation of the poles.
- A new relative of Dyson-Schwinger / Nekrasov / loop equations for $\beta$–ensembles of random matrices and log-gases.
- A basic block for asymptotics (cf. Yang–Baxter relation).
- In fact, there are far ancestors in the Baxter’s book.
The dynamical loop equation: examples

**Transition probabilities:** for $e \in \{0, 1\}^N$

$$\mathbb{P}(x+e|x) \sim \prod_{1 \leq i < j \leq N} \frac{b(x_i + \theta e_i) - b(x_j + \theta e_j)}{b(x_i) - b(x_j)} \prod_{i=1}^{N} \phi^+(x_i)^{e_i} \phi^-(x_i)^{1-e_i}$$

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1. Non-intersecting Bernoulli and Poisson random walks;
2. Dyson Brownian Motion (at general $\beta$);
3. Random lozenge and domino tilings;
4. Corners process of self-adjoint random matrices (at general $\beta$);
5. Macdonald / Koornwinder processes (principal specialization).
Part II: Application to \((q, \kappa)\)-distributions on tilings

Lozenge tilings of planar domains ("trapezoids")
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- **Uniformly random tilings** are well-understood by now.
- Today: **a non-uniform and inhomogeneous case.**
Part II: Application to \((q, \kappa)\)-distributions on tilings

Lozenge tilings of planar domains ("trapezoids")

- Uniformly random tilings are well-understood by now.
- Today: a non-uniform and inhomogeneous case. Why?
Part II: Application to \((q, \kappa)\)-distributions on tilings

Motivations for studying inhomogeneous random tilings:

**A priori:** Probing universality  
Does the homogeneous case phenomenology extend?

**A posteriori:** Discovering integrability  
Koornwinder polynomials; conformal invariance; algebraic answers.
$(q, \kappa)$–distributions on lozenge tilings

$$\mathbb{P}(\mathcal{T}) = \frac{1}{Z} \prod_{\diamondsuit \in \mathcal{T}} w(\diamondsuit),$$

$$w(\diamondsuit) = \kappa q^x - \kappa^{-1} q^{-x},$$

Width: $T$
Right/left boundaries:
$$N = \sum_{i=1}^{r} (b_i - a_i)$$
\((q, \kappa)\)–distributions on lozenge tilings

\[
P(\mathcal{T}) = \frac{1}{\mathcal{Z}} \prod_{\diamondsuit \in \mathcal{T}} w(\diamondsuit),
\]

\[
w(\diamondsuit) = \kappa q^x - \kappa^{-1} q^{-x},
\]

**Degenerations:**
- \(q = 1\): Uniform measure
- \(\kappa \to \infty\): Measure \(q^{\text{volume}}\)
- \(\kappa \to 0\): Measure \(q^{-\text{volume}}\)
- \(q, \kappa \to 1\): Linear \(w(\diamondsuit)\)

**Positivity:**
- Real \(\kappa\) and \(q\).
- Imaginary \(\kappa\), real \(q\).
- Complex \(\kappa\) and \(q\) with \(|\kappa| = |q| = 1\).
Random samples for $100 \times 100 \times 100$ hexagon

- Sampler of [Borodin-Gorin-Rains-10] for any $(q, \kappa)$.
Random samples for $100 \times 100 \times 100$ hexagon

- Sampler of [Borodin-Gorin-Rains-10] for any $(q, \kappa)$.
- Law of Large Numbers + local bulk limit theorem for the hexagon in [Borodin-Gorin-Rains-10].
- CLT for global fluctuation along a single slice of the hexagon in [Dimitrov-Knizel-19]; several slices in [Duits-Liu-22+].
Random samples for $100 \times 100 \times 100$ hexagon

- Sampler of [Borodin-Gorin-Rains-10] for any $(q, \kappa)$.

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- More general polygons were not accessible before today.
Arctic boundary: algebraic parameterization

**Task.** Prove that tilings are asymptotically frozen outside a curve. Find this **Arctic curve.**
Arctic boundary: algebraic parameterization

\[ w(\diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2}, \]

Small parameter \( \varepsilon \to 0 \) and

\[ \varepsilon N \to N, \quad \varepsilon T \to T, \quad \varepsilon \ln(q) \to \ln(q), \]
\[ \varepsilon a_i \to a_i, \quad \varepsilon b_i \to b_i, \quad 1 \leq i \leq r. \]

**Theorem.** (G.-Huang-22) The Arctic curve \((t(u), x(u))\) is:

\[ q^t = \frac{V'(u)}{U'(u)}, \quad q^x = \frac{w \pm \sqrt{w^2 - 4\kappa^2 q^{-t}}}{2\kappa^2 q^{-t}}, \quad w = V(u) - \frac{U(u)}{q^t}, \]

where \( q^{-u} + \kappa^2 q^u \) runs over the real line \( \mathbb{R} \) and

\[ f_0(u) = \frac{(q^N - q^u)(\kappa^2 q^{-T} - q^{-u})}{(\kappa^2 q^N - q^{-u})(q^{-T} - q^u)} \prod_{i=1}^{r} \frac{(q^{a_i} - q^u)(\kappa^2 q^{b_i} - q^{-u})}{(\kappa^2 q^{a_i} - q^{-u})(q^{b_i} - q^u)}, \]
\[ U(u) = \frac{f_0(u)q^{-u} - \kappa^2 q^u}{1 - f_0(u)}, \quad V(u) = \frac{q^{-u} - f_0(u)\kappa^2 q^u}{1 - f_0(u)}. \]
Examples of Arctic curves
Examples of Arctic curves
**Task.** Identify the **limit shape** in terms of the asymptotic proportions of lozenges.

Find \((t, x) \mapsto (p_{\Box}, p_{\Diamond}, p_{\bigtriangleup})\) or \((t, x) \mapsto f\).
Algebraic Limit Shape

\[ w(\triangle) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2}, \]

Small parameter \( \varepsilon \to 0 \) and

\[ \varepsilon N \to N, \quad \varepsilon T \to T, \quad \varepsilon \ln(q) \to \ln(q), \]

\[ \varepsilon a_i \to a_i, \quad \varepsilon b_i \to b_i, \quad 1 \leq i \leq r. \]

**Theorem.** (G.-Huang-22) Complex slope in the liquid region:

\[ f(t, x) = \frac{(q^{-u} + \kappa^2 q^u) - (q^{-x} + \kappa^2 q^x)}{(q^{-u} + \kappa^2 q^u) - (q^{-x+t} + \kappa^2 q^{x-t})}, \]

where \( u \) solves \( q^{-x} + \kappa^2 q^{x-t} = V(u) - \frac{U(u)}{q^t} \) with

\[ f_0(u) = \frac{(q^N - q^u)(\kappa^2 q^{-T} - q^{-u})}{(\kappa^2 q^N - q^{-u})(q^{-T} - q^u)} \prod_{i=1}^{r} \frac{(q^a_i - q^u)(\kappa^2 q^b_i - q^{-u})}{(\kappa^2 q^{a_i} - q^{-u})(q^{b_i} - q^u)}, \]

\[ U(u) = \frac{f_0(u)q^{-u} - \kappa^2 q^u}{1 - f_0(u)}, \quad V(u) = \frac{q^{-u} - f_0(u)\kappa^2 q^u}{1 - f_0(u)}. \]
Algebraic Limit Shape

\[ w(\diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2}, \]

Small parameter \( \varepsilon \to 0 \) and

\[ \varepsilon N \to N, \quad \varepsilon T \to T, \quad \varepsilon \ln(q) \to \ln(q), \]

\[ \varepsilon a_i \to a_i, \quad \varepsilon b_i \to b_i, \quad 1 \leq i \leq r. \]

**Corollary.** (G.-Huang-22) For a polynomial \( Q \), in the liquid region

\[ Q \left( \frac{f q^{t-x} - \kappa^2 q^x}{1 - f}, \frac{q^{-x} - f \kappa^2 q^{x-t}}{1 - f} \right) = 0. \]
Algebraic Limit Shape

\[ w(\Diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2}, \]

Small parameter $\varepsilon \to 0$ and

\[ \varepsilon N \to N, \quad \varepsilon T \to T, \quad \varepsilon \ln(q) \to \ln(q), \]

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**Conjecture.** Same statement for tilings of arbitrary polygons.

[Kenyon-Okounkov-05] discovered this for $\kappa = \infty$ case ($q^{\text{volume}}$).
**Task.** Compute asymptotic fluctuations of the centered heights. Find the field \( \lim_{\varepsilon \to 0} \left[ h(\varepsilon^{-1} t, \varepsilon^{-1} x) - \mathbb{E} h(\varepsilon^{-1} t, \varepsilon^{-1} x) \right] \).
Gaussian Free Field fluctuations

\[ w(\diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2}, \]

Small parameter \( \varepsilon \to 0 \) and

\[ \varepsilon N \to N, \quad \varepsilon T \to T, \quad \varepsilon \ln(q) \to \ln(q), \]

\[ \varepsilon a_i \to a_i, \quad \varepsilon b_i \to b_i, \quad 1 \leq i \leq r. \]

**Theorem.** (G.-Huang-22) Inside the liquid region, we have

\[ \lim_{\varepsilon \to 0} \sqrt{\pi} \left( h(\varepsilon^{-1} t, \varepsilon^{-1} x) - \mathbb{E}[h(\varepsilon^{-1} t, \varepsilon^{-1} x)] \right) = \mathfrak{G}(t, x); \]

\( \mathfrak{G}(t, x) \) is a (generalized) centered Gaussian field of covariance

\[ \mathbb{E} \mathfrak{G}(t, x) \mathfrak{G}(t', x') = -\frac{1}{2\pi} \ln \left| \frac{\Omega(t, x) - \Omega(t', x')}{\Omega(t, x) - \Omega(t', x')} \right|; \]

\[ \Omega(t, x) = q^{-u} + \kappa^2 q^u, \text{ and } u \text{ solves } q^{-x} + \kappa^2 q^{x-t} = V(u) - \frac{U(u)}{q^t}. \]
Gaussian Free Field fluctuations

\[ w(\diamond) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2}, \]

Small parameter \( \varepsilon \to 0 \) and

\[ \varepsilon N \to N, \quad \varepsilon T \to T, \quad \varepsilon \ln(q) \to \ln(q), \]

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Corollary. (G.-Huang-22) Inside the liquid region, we have

\[ \lim_{\varepsilon \to 0} \sqrt{\pi} \left( h(\varepsilon^{-1} t, \varepsilon^{-1} x) - \mathbb{E}[h(\varepsilon^{-1} t, \varepsilon^{-1} x)] \right) = \mathcal{G}(t, x); \]

\( \mathcal{G}(t, x) \) is the **Gaussian Free Field** in the complex structure of

either \[ \frac{f q^{t-x} - \kappa^2 q^x}{1 - f} \quad \text{or} \quad \frac{q^{-x} - f \kappa^2 q^{x-t}}{1 - f}, \]

where \( f(t, x) \) is the **complex slope** at \((t, x)\).
Gaussian Free Field fluctuations

\[ w(\bigcirc) = \kappa q^{x-t/2} - \kappa^{-1} q^{-x+t/2}, \]

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\]

where \( f(t, x) \) is the **complex slope** at \((t, x)\).

**Conjecture.** Same is true for arbitrary domains.
For the particular case of the uniform measure:

- Rational parameterization of the Arctic curve for trapezoids. 
  \[ [\text{Petrov-14}] \]

- Limit shape via algebraic equations for trapezoids. 
  \[ [\text{Kenyon–Okounkov-07}], \ [\text{Petrov-14}], \ [\text{Duse-Metcalfe-15}] \]

- Gaussian Free Field fluctuations for trapezoids. 
  \[ [\text{Petrov-15}], \ [\text{Bufetov-Gorin-18}], \ [\text{Huang-20}] \]
A glimpse into the proofs

**Step 1: Partition Function**

- [Borodin–Gorin–Rains-10] by advanced determinantal evaluations;
- Or, by quasi-branching rules and principal specialization formulas of [Rains-05] for **Koornwinder symmetric polynomials**.

**Proposition.** For any \( x_1 < \cdots < x_N \),

\[
\sum_{\mathcal{T}} \prod_{\langle \rangle \in \mathcal{T}} \left( \kappa q^{x_{\langle \rangle}} - \kappa^{-1} q^{-x_{\langle \rangle}} \right) = Z_N \cdot \prod_{i<j} \left[ \kappa q^{x_i} - \kappa^{-1} q^{-x_i} - \kappa q^{x_j} + \kappa^{-1} q^{-x_j} \right]
\]
A glimpse into the proofs

**Step 2:** Particle-hole involution and **Markov chain structure**

**Proposition.** \( \{ x(t) \} \) for the \((q, \kappa)\)-distributions on lozenge tilings is a Markov chain with transition probabilities:

\[
\sim \prod_{1 \leq i < j \leq N} \frac{b_t(x_i + \theta e_i) - b_t(x_j + \theta e_j)}{b_t(x_i) - b_t(x_j)} \prod_{i=1}^{N} \phi^+_t(x_i)^{e_i} \phi^-_t(x_i)^{1-e_i},
\]

where \( e \in \{0, 1\}^N \), \( \theta = 1 \), \( b_t(x) = q^{-x} + \kappa^2 q^{x-t} \), and

\[
\phi^+_t(x) = q^{T+N-1-t}(1-q^{x-N+1})(1-\kappa^2 q^{x-T+1}), \quad \phi^-_t(x) = -(1-q^{x+T-t})(1-\kappa^2 q^{x+N-t}).
\]
A glimpse into the proofs

**Step 3:** Apply **dynamical loop equations** to

\[
\prod_{i<j} b_t(x_i + \theta e_i) - b_t(x_j + \theta e_j) \frac{b_t(x_i) - b_t(x_j)}{b_t(x_i) - b_t(x_j)} \prod_{i=1}^{N} \phi_t^+(x_i) e_i \phi_t^-(x_i)^{1-e}
\]

Holomorphic

\[
\mathbb{E} \left[ \phi^+(z) \prod_{j=1}^{N} \frac{b(z + \theta) - b(x_j + \theta e_j)}{b(z) - b(x_j)} + \phi^-(z) \prod_{j=1}^{N} \frac{b(z) - b(x_j + \theta e_j)}{b(z) - b(x_j)} \right].
\]

Leads to the decomposition of the time increment as

“**deterministic drift**” + “**Gaussian stochastic part**” + “**small error**”
A glimpse into the proofs

**Step 4: Solve** the stochastic evolution

"deterministic drift" + "Gaussian part"

Key roles played by:

- Analytic continuation of the **complex slope** $f(t, x)$ from real $x$ to complex $z$: "doubly complex slope"
- Characteristic flow of the first-order PDE in $(\mathbb{R} \times \mathbb{C} \mapsto \mathbb{C})$

\[
\partial_t \ln f(t, z) + \partial_z \ln(1 - f(t, z)) = \ln(q) \frac{\kappa^2 q^{z-t} + q^{-z}}{\kappa^2 q^{z-t} - q^{-z}}.
\]
Summary

• The dynamical loop equation

\[ \prod_{i<j} \frac{b_t(x_i + \theta e_i) - b_t(x_j + \theta e_j)}{b_t(x_i) - b_t(x_j)} \prod_{i=1}^{N} \phi^+(x_i)^{e_i} \phi^-(x_i)^{1-e_i} \]

\[ \mathbb{E} \left[ \phi^+(z) \prod_{j=1}^{N} \frac{b(z + \theta) - b(x_j + \theta e_j)}{b(z) - b(x_j)} + \phi^-(z) \prod_{j=1}^{N} \frac{b(z) - b(x_j + \theta e_j)}{b(z) - b(x_j)} \right]. \]

• Application to \((q, \kappa)\)-lozenge tilings: parameterized Arctic curve, algebraic limit shape, GFF fluctuations.