Modern Non-Perturbative Techniques in QFT

GGI lectures 2022, Denis Karateev

1. Two-point correlation functions (Euclidean, Wightman and time-ordered), spectral densities, Little group vs. Lorentz group, c-theorems in $2d$: [1]

2. Scattering amplitudes, form factors and unitarity conditions: [1, 2]

3. Concise review of the numerical S-matrix bootstrap approach: sections 1 and 4.1 in [3]

4. Computing observables in $2d$ ϕ^4 model using Hamiltonian Truncation and S-matrix bootstrap: [4, 5]

5. Relating scattering amplitudes and the UV CFT data: [1, 6]

6. Bounds from dispersion relations vs. bounds from the numerical S-matrix bootstrap: [7]

References

[1] D. Karateev, *Two-point functions and bootstrap applications in quantum field theories*, *JHEP* **02** (2022) 186, [2012.08538].

[2] D. Karateev, S. Kuhn and J. a. Penedones, *Bootstrapping Massive Quantum Field Theories*, *JHEP* **07** (2020) 035, [1912.08940].

[3] A. Hebbar, D. Karateev and J. Penedones, *Spinning S-matrix bootstrap in 4d*, *JHEP* **01** (2022) 060, [2011.11708].

[4] H. Chen, A. L. Fitzpatrick and D. Karateev, *Form factors and spectral densities from Lightcone Conformal Truncation*, *JHEP* **04** (2022) 109, [2107.10285].

[5] H. Chen, A. L. Fitzpatrick and D. Karateev, *Bootstrapping 2d ϕ^4 theory with Hamiltonian truncation data*, *JHEP* **02** (2022) 146, [2107.10286].

[6] D. Karateev, J. Marucha, J. a. Penedones and B. Sahoo, *Bootstrapping the a-anomaly in 4d QFTs*, 2204.01786.

[7] H. Chen, A. L. Fitzpatrick and D. Karateev, *Bounds on Scattering Amplitudes in Fractional Dimensions*, to appear.

day 1

* non-perturbative def. of QFTs
* examples
* observables
* techniques
* some results

day 2

* define CFTs
* examples
* method
* interesting direction

① QFT

energy

g

UV

CFT + deformation

$\downarrow$ compute all observables (correlation functions)

IR

CFT

(UV-complete)

$\Delta < d$

$$A_{QFT} = A_{UVCFT} + A_{def}$$

$$\underset{[0]}{A_{def}} = g M^{d-\Delta} \int d^d x \, \underbrace{O_\Delta(x)}_{\Delta - d}$$

② examples:

a) $\mathcal{L} = (\partial\varphi)^2 - V(\varphi)$

free bosonic CFT

deformation

$$V(\varphi) = m_0^2 \varphi^2 + m_0^{4-d} \frac{\bar\lambda}{4!} \varphi^4 + \text{counter terms}$$

mass-like parameter

coupling

b) Ising Field Theory in 2d or 3d

$2 \leq d < 4$

(in $d=4$ EFT)
Landau pole

c) 4d QCD

non-UV complete ϕ^4 ""
* $d=4$ "ϕ^4"
* $d=4$ QED

~~#~~

observable(r)

$$\boxed{\langle 0|O(x)\,O(0)|0\rangle_T} = f(x)$$

$x \to 0$ (energy $\to \infty$)

$$\lim_{x\to 0} f(x) = \text{const} \neq \infty$$

Scattering $\longrightarrow$ semi-classical potent. $\xrightarrow{r=0}$

$O($

$$A(x)\,B(y) \underset{|x-y|}{=} \sum_{O} g(x,y,d)\, O(x)$$

- c-theorem in 2d

UV CFT $\quad c_T^{UV}$

$\uparrow x=0$

$$\langle 0|T^{\mu\nu}(x)\,T^{\rho\delta}(0)|0\rangle_E = f(x) \longrightarrow \rho_T(s)$$

$\downarrow x=\infty$

IR CFT $\quad c_T^{IR}$

* Observables in QFTs

$\rightarrow$ correlation functions

Eucl.: $\langle 0 | \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \dots | 0 \rangle_E$

Lorentz: * $\langle 0 | \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \dots | 0 \rangle_W$ (Wightman)

$$x_1^\mu = \{ x_1^0 - i\epsilon, \vec{x} \}$$

$$\epsilon > 0$$

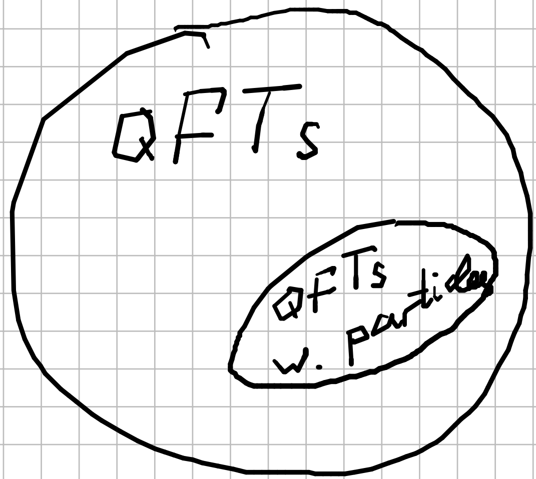
$X_1 = X_2$

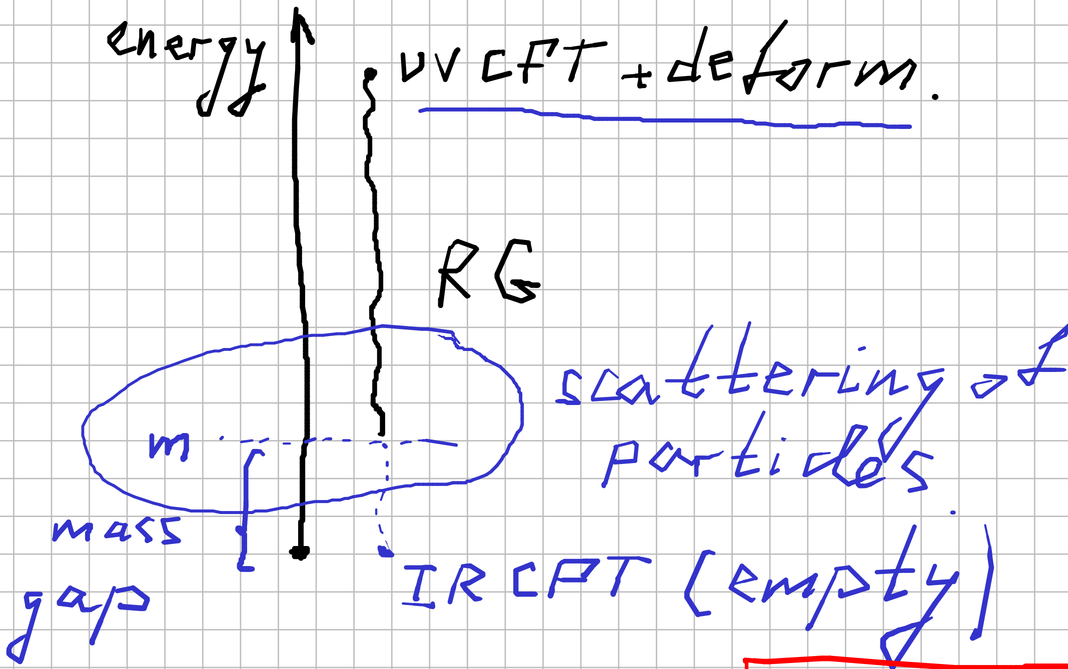
* $\langle 0 | \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) | 0 \rangle_T =$

$$= \Theta(x_1^0 - x_2^0) \langle 0 | \underline{\mathcal{O}_1(x_1)}\, \mathcal{O}_2(x_2) | 0 \rangle_W$$

$$\underline{\pm}\ \Theta(x_2^0 - x_1^0) \langle 0 | \underline{\mathcal{O}_2(x_2)}\, \mathcal{O}_1(x_1) | 0 \rangle_W$$

* R or A

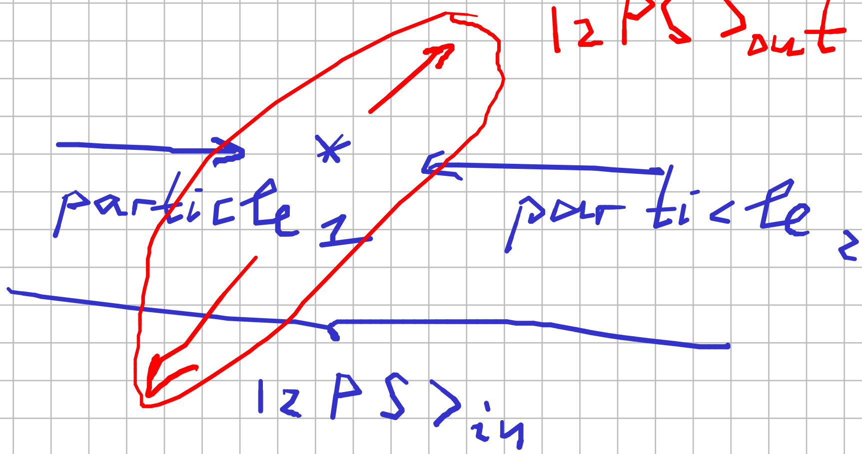




$\boxed{\langle n\text{pf} \rangle_T}$

asymptotic in and out states

$|nPS\rangle_{in}$ or $|nPS\rangle_{out}$



LSZ reduct. formula

$$\boxed{S(s,t,u) \times \delta(\dots) \equiv {}_{out}\langle 2PS | 2PS \rangle_{in}}$$

Observables

① scattering ampl. $\quad S(s,t,u)$

$\vec{p}_1 \vec{p}_2 \to \vec{p}_3 \vec{p}_4$

② form factor

$$\langle 2PS|\, O(x=0)\,|0\rangle = F_O(s,\cancel{t})$$

out, in

$\vec{p}_1, \vec{p}_2$

$s \to 0, \infty$

$$\boxed{\begin{aligned} s &\equiv -(p_1+p_2)^2 \\ t &\equiv -(p_1-p_2)^2 \end{aligned}}$$

$$s = t - 4m^2$$

particles: $p_1^{\mu} = \{p_1^0, \vec{p}_1\}$, $\quad \mu = 0,\dots,d-1$

$$\boxed{p_1^2 = -m^2}$$

(on-shell constr)

$$p_1^0 = \sqrt{m^2 + \vec{p}_1^{\,2}}$$

spectral dens.

③ $FT\left(\langle 0|\, T^{\mu\nu}(x)\, T^{\rho\delta}(0)\,|0\rangle_W\right)$ $\to \rho_T^0(s)$, $\rho_T^2(s)$

$s \to 0, \infty$

Little group spins (reps of $SO(d-1)$)

$SO(1,d-1)$ reps. $\xrightarrow{\text{Hilbert space}}$ $SO(d-1)$ Little group

Methods

① Lattice (expensive)

② Hamiltonian Truncation (inexpensive)

solve a particular model

combine

③ Bootstrap methods (symmetries and fundamental principles)

S-matrix bootstrap

numerical conf. bootstrap

(c-theorem, a-theorem, 't Hooft anomaly matching)

derive generic constraints on the space of QFTs (model independent)

Results:

2d $\quad \mathcal{L}^{QFT} = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_0^2\varphi^2 - m_0^2\frac{\bar{\lambda}}{4!}\varphi^4$ + counter terms

$\underline{m_0, \bar{\lambda}}$ $\qquad m_0 = 1$

$\bar{\lambda}$ free parameter

Hamiltonian Trunc. + S-matrix bootstrap

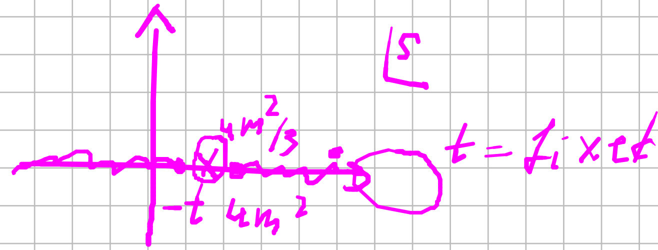
compute

$$\boxed{\begin{array}{l} \mathcal{S}(s,\cancel{t},\cancel{u}) \\ F_T(s) \\ \rho_T(s), \end{array}} \quad \cancel{\rho_T^2(s)}$$

in perturbation theory $\bar{\lambda} \ll 1$

how to compute them $\bar{\lambda} \sim 1$? (non-pert.)

S-matrix bootstrap



d

$$\begin{cases} \mathcal{S}(s,t,u) \leftarrow \text{interacting part} \\ \mathcal{T}(s,t,u) \end{cases}$$

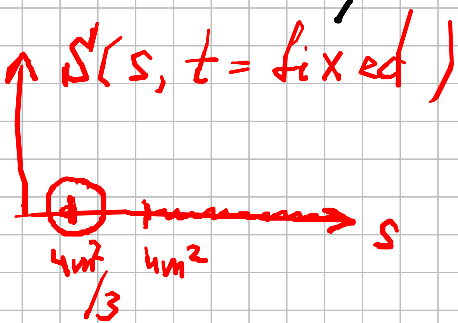
$$s+t+u=4m^2$$

(generalized to particles with general masses and spins)

describe my scattering

$$\Lambda_{k,\ell} \equiv \frac{m^{d-4+2(k+\ell)}}{k!\,\ell!}\,\partial_s^k\,\partial_t^\ell\,\mathcal{S}_{2\to 2}\left(s=t=u=4m^2/3\right)$$

dimensionless

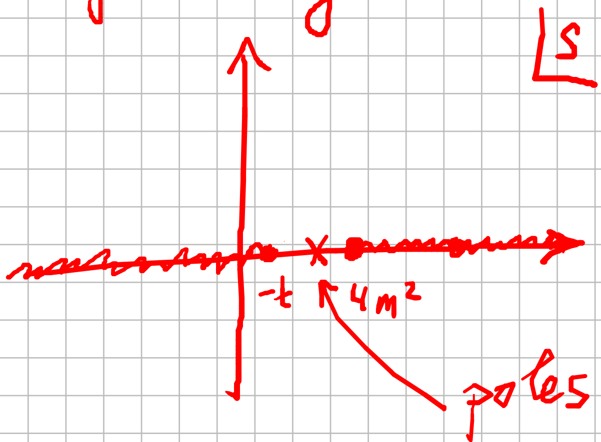


Assumptions:

① $f(s,t,u) = f(t,s,u) = f(u,t,s)$ crossing symmetry

$s \in [4m^2, \infty)$ & $t \leq 0$ physical domain

② analyticity



t = fixed

(maximal analyticity)

③ unitarity: $|\psi\rangle$

$\langle \psi | \psi \rangle \geq 0$ matrix$(\langle \alpha | \beta \rangle) \succeq 0$

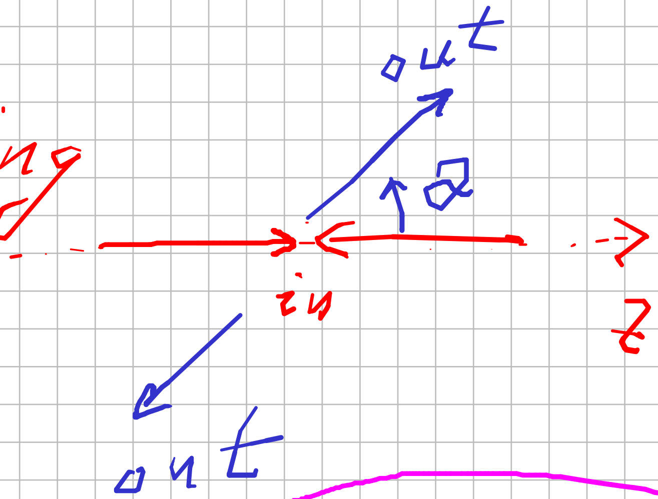
$$\begin{array}{c|cc} & |2PS\rangle^{\ell}_{in} & |2PS\rangle^{\ell}_{out} \\ \hline {}^{\ell}_{in}\langle 2PS| & 1 & S_\ell(s) \\ {}^{\ell}_{out}\langle 2PS| & S_\ell^*(s) & 1 \end{array} \succeq 0 \qquad \ell = 0, \cancel{1}, 2, \cancel{3}, \dots$$

$$\boxed{\begin{pmatrix} 1 & S_\ell(s) \\ S^*_\ell(s) & 1 \end{pmatrix} \geq 0} \qquad s \geq 4m^2$$

partial amplitudes

$$\underline{S_\ell(s)} = 1 + \frac{i}{\sqrt{s}\,(s-4m^2)^{\frac{3-d}{2}}} \int_{-1}^{+1} \underline{dx}\, P_\ell(x)\, \underline{T(s,t,u)}$$

& $x = \cos\theta \leftarrow$ scattering angle



$$\begin{cases} t = t(s,x) \\ u = u(s,x) \end{cases}$$

$$1 - |S_\ell(s)|^2 \geq 0 \Rightarrow \boxed{|S_\ell(s)|^2 \leq 1}$$

$|mm\rangle$ $\quad |m_1 m_2\rangle \quad |m_1 m_1\rangle \quad |m_2 m_2\rangle \quad T(x)T(y)|0\rangle$

$|3PS\rangle$

$|4PS\rangle$

$$||2PS\rangle^{\ell}_{in} \qquad |2PS\rangle^{\ell}_{out} \qquad \int d^4x\, e^{ip\cdot x}\, T^{\mu\nu}(x)|0\rangle \equiv |\Psi\rangle$$

	$\vert 2PS\rangle^{\ell}_{in}$	$\vert 2PS\rangle^{\ell}_{out}$	$\vert\Psi\rangle$
${}^{\ell}_{in}\langle 2PS\vert$	1	$\boxed{S_\ell(s)}$	$\underline{F_T(s)}$
${}^{\ell}_{out}\langle 2PS\vert$	$S^*_\ell(s)$	1	$\underline{F^*_T(s)}$
$\langle\Psi\vert$	$F^*_T(s)$	$F_T(s)$	$\rho_T(s)$

$\succeq 0$

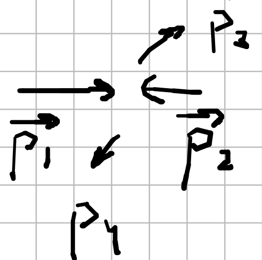
2d

$\rho_T(s)$ $\quad 3\times 3 \quad$ $\underline{s \geq 4m^2}$

C^{UV} & $C^{IR} = 0$

m

$m\ m \to m\ m$



(mostly plus sign.)

$$\eta^{\mu\nu} = \{\underbrace{-, +, +, \ldots, +}_{d}\}$$

$$\boxed{\begin{aligned} p_i^2 &= -m^2 \\ &\parallel \\ -(p_i^0)^2 &+ (\vec{p}_i)^2 \end{aligned}}$$

$$S(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\nu) = {}_{out}\langle 2PS | 2PS \rangle_{in}$$

$$\boxed{p_1 + p_2 = p_3 + p_4}$$

$$S(s, t, \cancel{u})$$

$$\begin{cases} s = -(p_1 + p_2)^2 \\ t = -(p_1 - p_3)^2 \\ u = -(p_1 - p_4)^2 \end{cases}$$

$$\underline{s + t + u = 4m^2}$$

4d

$$A_\mu(x) = \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\, \delta(p^2) \left(a_\lambda\, \underline{\epsilon^\lambda_\mu(p)}\, e^{ip\cdot x} + a^\dagger_\lambda\, \underline{\epsilon^{*\lambda}_\mu(p)}\, e^{-ip\cdot x} \right)$$

$\mu = 0, 1, 2, 3$

$$a^\dagger_\lambda |0\rangle = |\vec{p}, \lambda\rangle \quad \text{Hilbert space}$$

λ ↑ helicity $\quad SO(2)$

S-matrix bootstrap

$$p_i^2 = -m^2$$

Axioms

① crossing

② analyticity (maximal)

③ unitarity

$$s + t + u = 4m^2$$

m

$$\vec{p}_1 \vec{p}_2 \longrightarrow \vec{p}_3 \vec{p}_4$$

$$\begin{cases} s = -(p_1+p_2)^2 \\ t = -(p_1-p_3)^2 \\ u = -(p_1-p_4)^2 \end{cases}$$

$$\underbrace{S(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)}_{\text{full scat. amp}} = \delta\delta'' + \underbrace{T(s, t, u)}_{\text{interact. part}}$$

$\downarrow$ project to def. spin

$$T(s,t,\cancel{x}) = \# \sum_{j=0,\cancel{1},2,\cancel{3},4,\ldots} P_j(x)\, T_j(s)$$

$$S_j(s) = 1 + i\tilde{T}_j(s)$$

4d

$$\underline{4d} \qquad \tilde{T}_j(s) = \# \int_{-1}^{+1} dx\, \boxed{P_j(x)}\, \tilde{T}(s, t(s,x), u(s,x))$$

$P_j(x) \xrightarrow{4d} d^j_{\lambda\lambda'}(x)$

$P_j(x) \xrightarrow{\text{general } d} C_j(x)$

$$x = \cos\theta$$

$$\begin{cases} t = -\dfrac{s-4m^2}{2}(1-x) \\ u = -\dfrac{s-4m^2}{2}(1+x) \end{cases}$$

unitarity $\underbrace{\begin{pmatrix} 1 & S_j(s) \\ S_j^*(s) & 1 \end{pmatrix}}_{} \succeq 0 \Rightarrow \underline{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i\tau_j(s) \\ -i\tau_j^*(s) & 0 \end{pmatrix} \succeq 0}$ ✓ ✓

SDPB

* a) Dispersion relations („positivity")

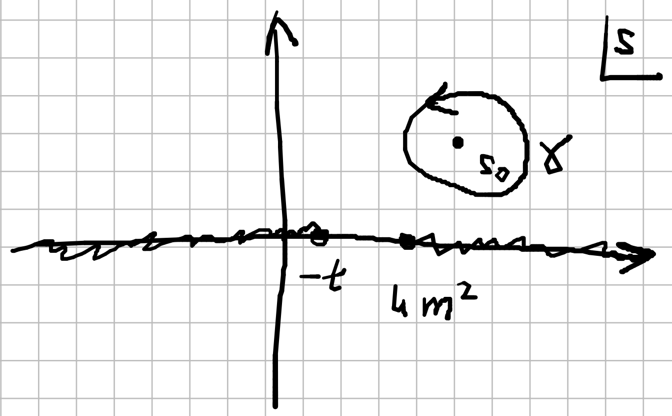
b) numerical methods which SDPB

$$|S_j(s)|^2 \leq 1 \qquad S_j = 1 + i\tau_j$$

$$0 \leq \underline{|\tau_j|^2} \leq 2\,\mathrm{Im}\,\tau_j \leq 1 + \underline{|\tau_j|^2} \Rightarrow \boxed{\mathrm{Im}\,\tau_j(s) \geq 0}$$

positivity

„φ^4": $\tau = -\lambda + O(\lambda^2)$

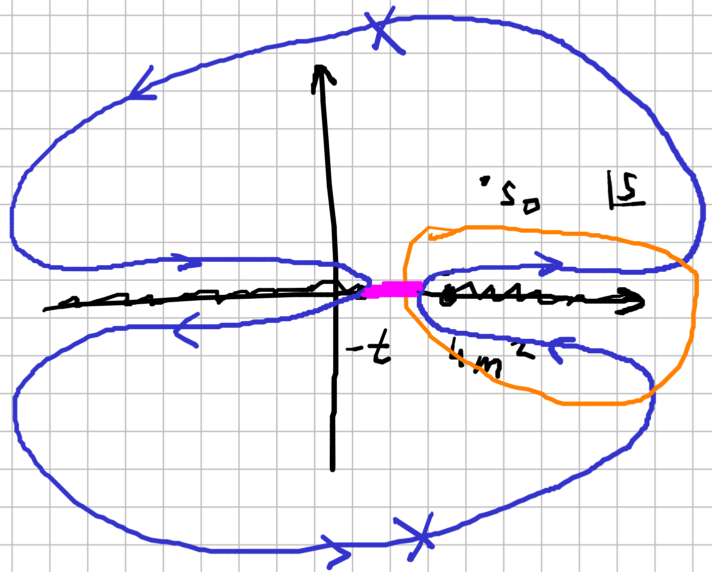


$$\mathcal{T}(\underline{s}, t=\text{fixed})$$

$$\oint_\gamma \frac{\mathcal{T}(s,t=\text{fixed})}{s-s_0}\, ds = 2\pi i\, \mathcal{T}(s_0, t=\text{fixed}) \sim \Lambda_{0,0}$$

$$\begin{cases} s_0 = 4m^2/3 \\ t = 4m^2/3 \end{cases}$$

$$\boxed{\Lambda_{k,\ell} \equiv \#\, \partial_s^k \partial_t^\ell\, \mathcal{T}(s=t=u=4m^2/3)}$$



$$\overbrace{\mathcal{T}(s+i\epsilon,t) - \mathcal{T}(s-i\epsilon,t)}^{\text{disc}_s \mathcal{T}} =$$

$$= \mathcal{T}(s+i\epsilon) - \mathcal{T}^*(s+i\epsilon) = 2i\, \mathrm{Im}\, \mathcal{T}(s)$$

$$t \qquad t \qquad t$$

$$\sum_j \underbrace{\mathcal{T}_j(s)}_{\geq 0} P_j(x)$$

$$\underbrace{\Lambda_{0,0}}_{\geq 0} = \int_{4m^2}^{\infty} \frac{\mathrm{Im}\,\mathcal{T}(s,t)}{s-\underbrace{s_0}_{4m^2/3}} + (\text{second } (s \to u))$$

$$\tau(s,t) \sim s \log^2 s$$

$$s \gg 4m^2$$

$$\lim_{s \to \infty} \frac{\tau(s,t)}{s^2} = 0$$

$$\boxed{-\frac{5}{16}\Lambda_{2,0} \leq \Lambda_{2,1} \leq \underline{1.15215}\,\Lambda_{2,0}}$$

b)

$$\rho_s = \frac{\sqrt{4m^2 - s_0} - \sqrt{4m^2 - s}}{\sqrt{4m^2 - s_0} + \sqrt{4m^2 - s}} \qquad s \geqslant 4m^2$$

$$s_0 = 4m^2/3$$

$$T(s,t,u) = \sum_{a,b,c=0}^{\cancel{\infty}\, N_{max}} \alpha_{abc}\, \underline{\rho_s^a \rho_t^b \rho_u^c} + \text{poles}$$

coefficients

analyticity ✓

project

crossing ✓ $\alpha_{(abc)}$

$s+t+u = 4m^2$

$\Downarrow$ kill some α_{abc}

$$T_j(s) = \sum_{a,b,c=0}^{\infty} \alpha_{(abc)}\, \underline{\mathrm{Int}_j^{abc}(s)}$$

$$= \vec{\alpha} \cdot \vec{\mathrm{Int}}_j(s)$$

$$\boxed{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \vec{\alpha} \cdot \begin{pmatrix} 0 & i\,\vec{\mathrm{Int}}(s) \\ -i\,\vec{\mathrm{Int}}(s) & 0 \end{pmatrix} \succcurlyeq 0} \qquad \underline{s \geqslant 4m^2}$$

$$\Lambda_{0,0} = T(s=t=u=4m^2/3) = \underline{\sum \#_{abc}\,\alpha_{abc}}$$

$\to \max \Lambda_{0,0}$

$\to \min \Lambda_{0,0}$

$$\boxed{-600 \lesssim \Lambda_{0,0} \lesssim +200}$$

* 2d S-matrix bounds
  (reproduce known integrable models)

  2d S-matrix + $H\overline{T} \rightarrow$ „?"

* 4d pions (massless & <u>massive</u>)
  photons

* 10 sugra

S-matrix + something else $\rightarrow$ FF & SP (generalized)

$\updownarrow$

unitarity conditions

$\rightarrow$ 4d probe (a-anomaly) (dilatone) energy

∞ — • UV CFT + deform.

scattering

m_1, m_2

0

$$\begin{pmatrix} 1 & S_j^* & F \\ S_j^* & 1 & F^* \\ F^* & F & \rho \end{pmatrix} \succeq 0$$

sensitive to c

$$c_{UV} = \# \int_{4m^2}^{\infty} ds \, \frac{\rho(s)}{s^{3}}$$

2d sine-Gordon $\qquad [\varphi] = 0$

$$\mathcal{L} = \underbrace{-\frac{1}{2}(\partial\varphi)^2}_{\text{free CFT}} + \underbrace{\frac{m_0^2}{2\beta}\cos(\beta\varphi) + \text{cont. term}}_{}$$

free CFT

$c = 1$

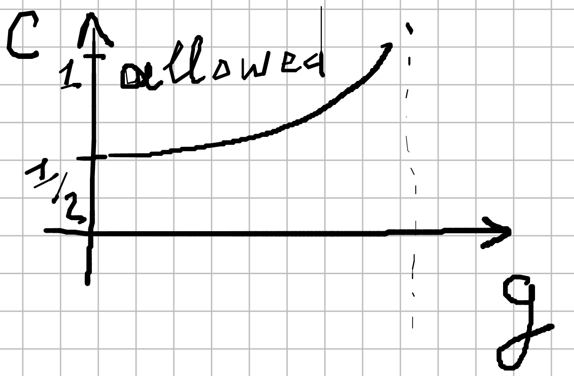
β — deformation parameter

$m_0 = 1$

$m_1, m_2, m_3, \ldots$

$$T(s,t,u) = -\frac{g^2}{s - m_2^2} + \text{"rho"}$$

$m_1 m_1 \to m_1 m_2$



* particle with mass m $\quad \mathbb{Z}_2$ odd

$$A$$

* Probe particle B $\quad (m_B = 0)$

(dilatone)

$$f \to 0 \quad \text{(decouples)}$$

$$4d \quad \begin{cases} AA \to AA \\ AB \to AB \\ BB \to BB \Rightarrow T_{BB \to BB} = a^{UV} \times (s^2 + t^2 + u^2) + O(s^3) \end{cases}$$

KS

Global symmetry $U(1) \longrightarrow J_\mu$

UV CFT

2d $\langle J_\mu J_\nu \rangle = \underline{C_J} T_1 + \underline{C_J'} T_2$

tensor structure

3d $\times$

t'Hooft

4d $\langle J_\mu J_\nu J_\rho \rangle = \lambda T$

Summary

* Non-pert. def. of QFTs (UV-complete)
  - UV CFT
  - IR CFT (empty)

mass gap (particles)

scat. amp. methods

* Lattice

  HT + bootstrap $\longrightarrow$ „φ^4"-theories

* Unitarity (semi-posit. def. matrices)

  SDPB

* S-matrix bootstrap $\xrightarrow{\text{probe UV CFT}}$ data

conformal bootstrap

classify all the CFTs