RANDOM YOUNG TABLEAUX AND THE TANGENT PLANE METHOD

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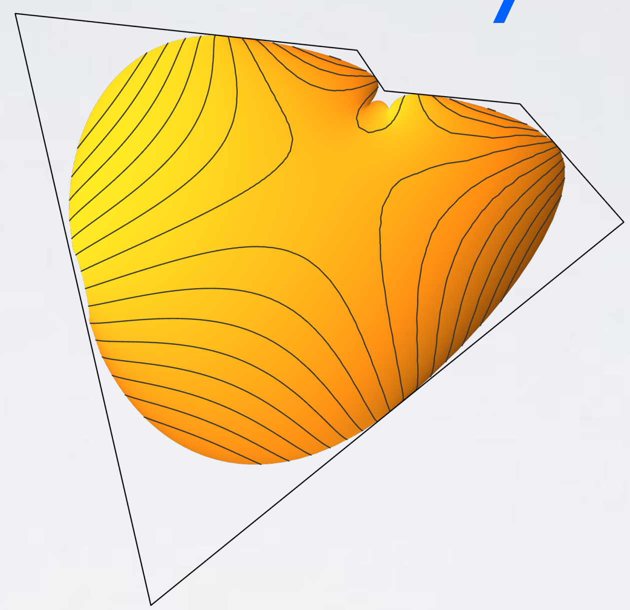
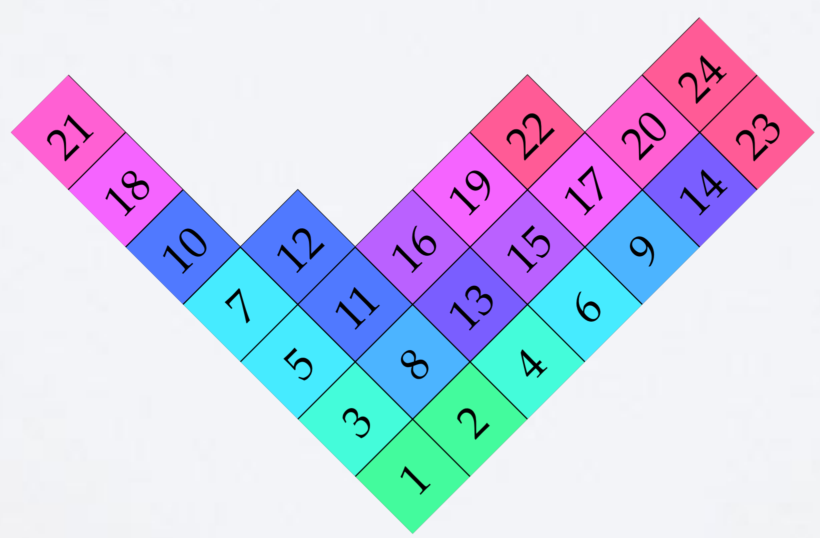
I. THE TANGENT PLANE METHOD

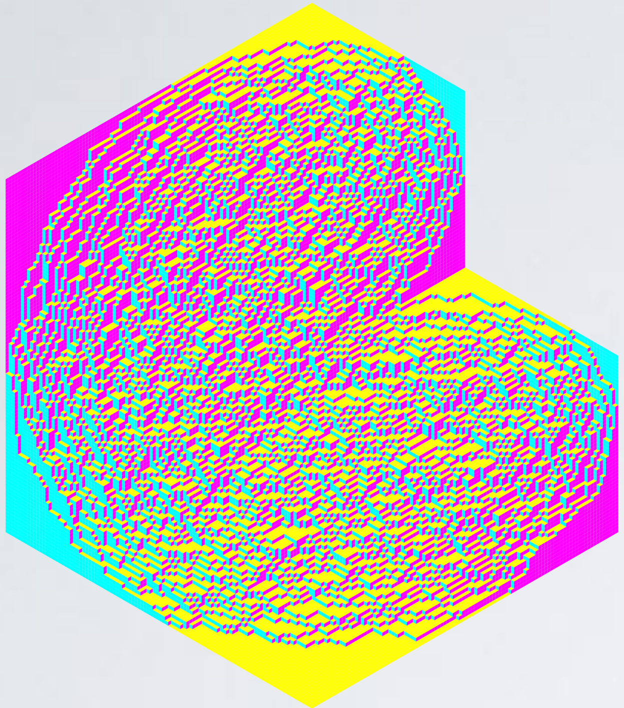
joint work with R. Kenyon (Yale)

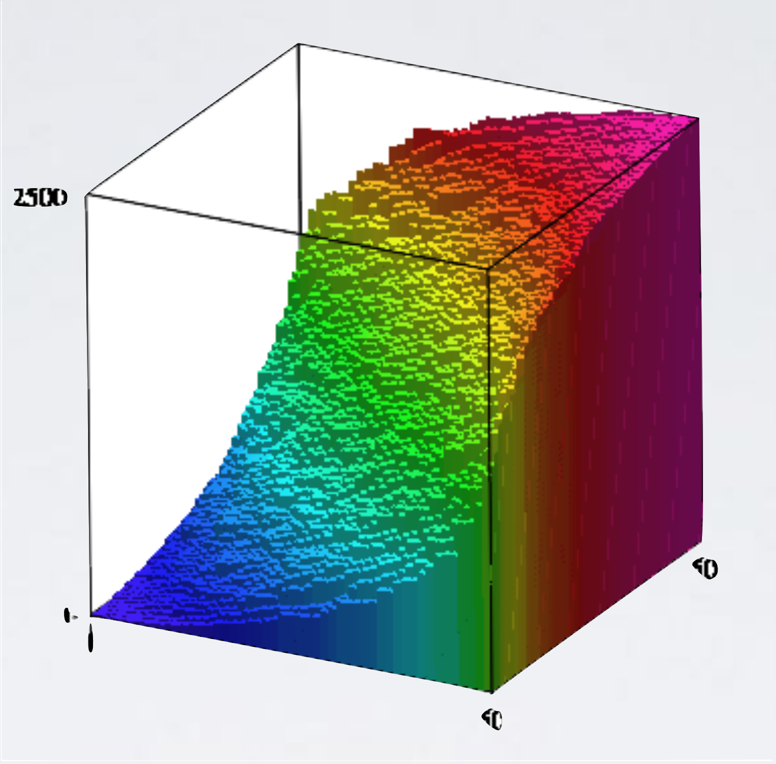


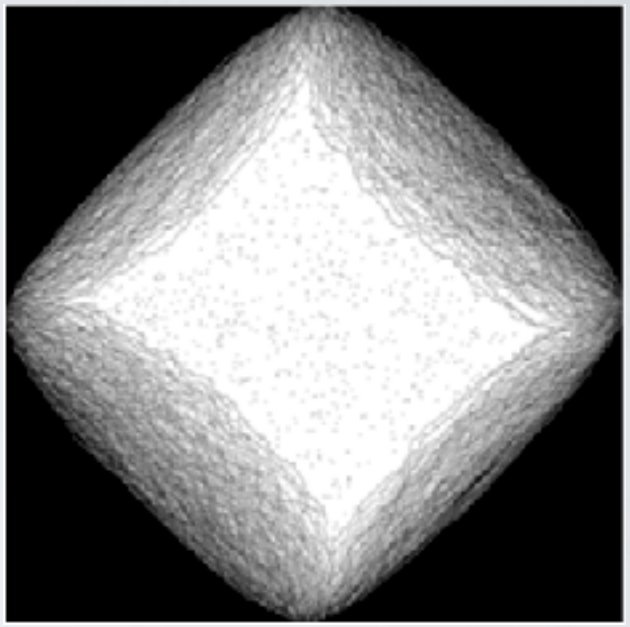
II. RANDOM YOUNG TABLEAUX



ZOO OF LIMIT SHAPES







variational approach

$$\min_{h|_{\partial\Omega}=h_0} \int_\Omega \sigma(\nabla h)$$

surface tension/
local entropy

unifies

general boundary conditions & a variety of models

dimer model
(domino/lozenge tilings
etc)

random
Young tableaux

five-vertex model

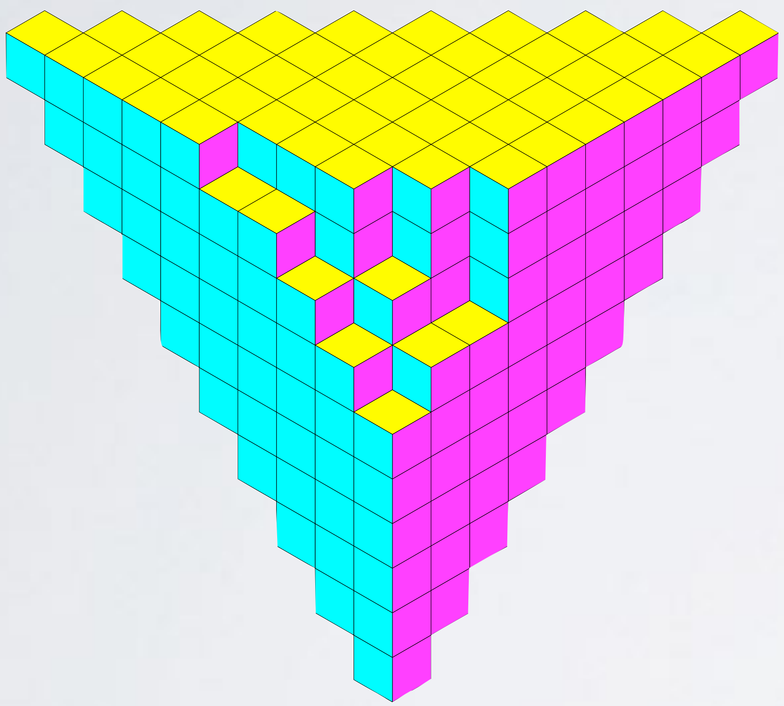
WULFF SHAPE

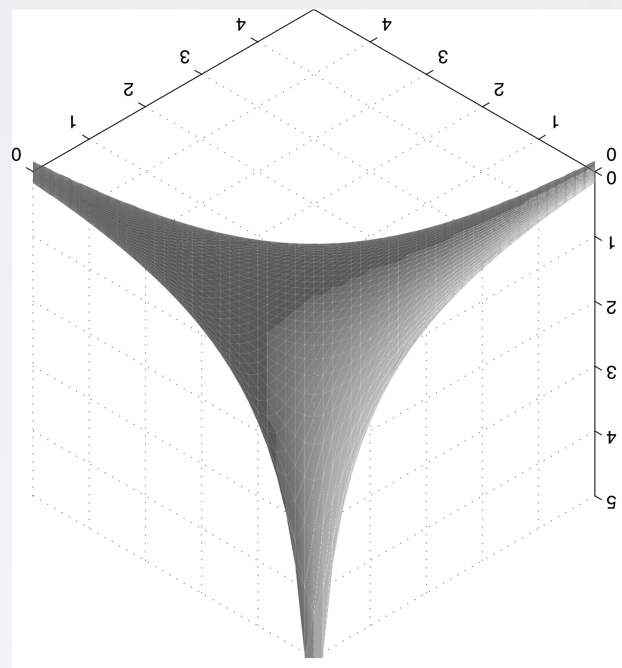
Cerf-Kenyon, Okounkov-Reshetikhin

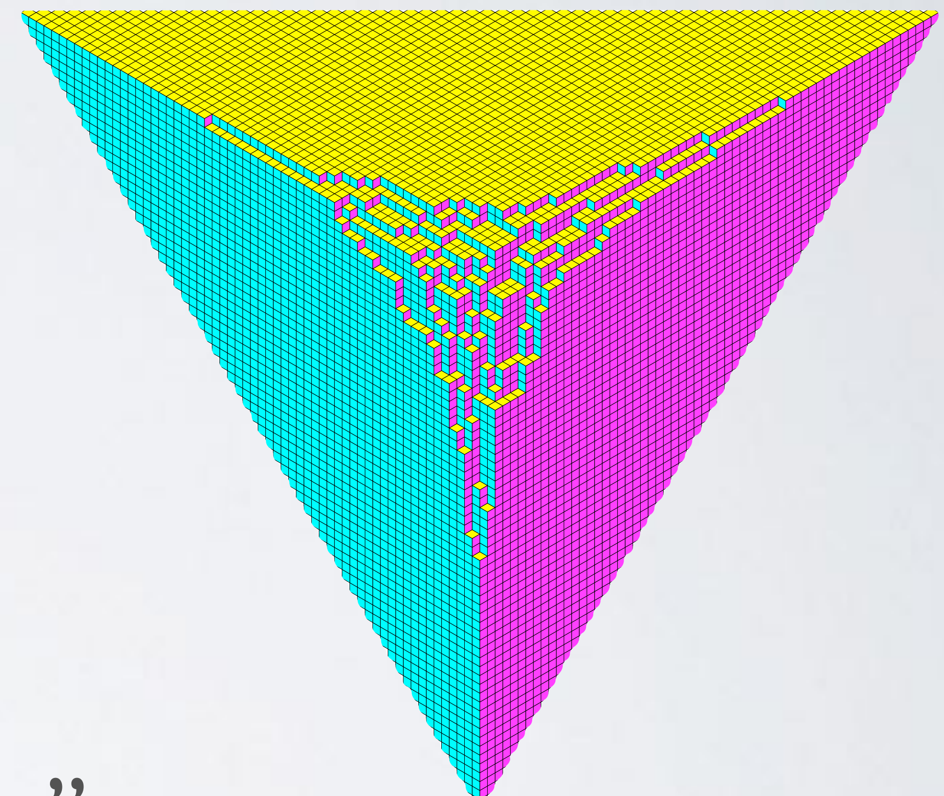
Wulff shape - Legendre dual of surface tension

itself a limit shape

(lozenge tilings $\leftrightarrow$ 3D Young diagram)







“fundamental solution”

(facets, facet-rough transition, phases, algebraic boundary etc)

Kenyon-Okounkov-Sheffield

$\det D^2\sigma \equiv \pi^2$ for the dimer model (“free fermions”)

FREE FERMIONS - $\det D^2\sigma \equiv \pi^2$

"exclusion principle"

"spectral curve is a Harnack curve" (*math*)
"dimers map to free fermions, bosonising the fermions we get $K = 1$ in the fluctuating region" (*physics*)

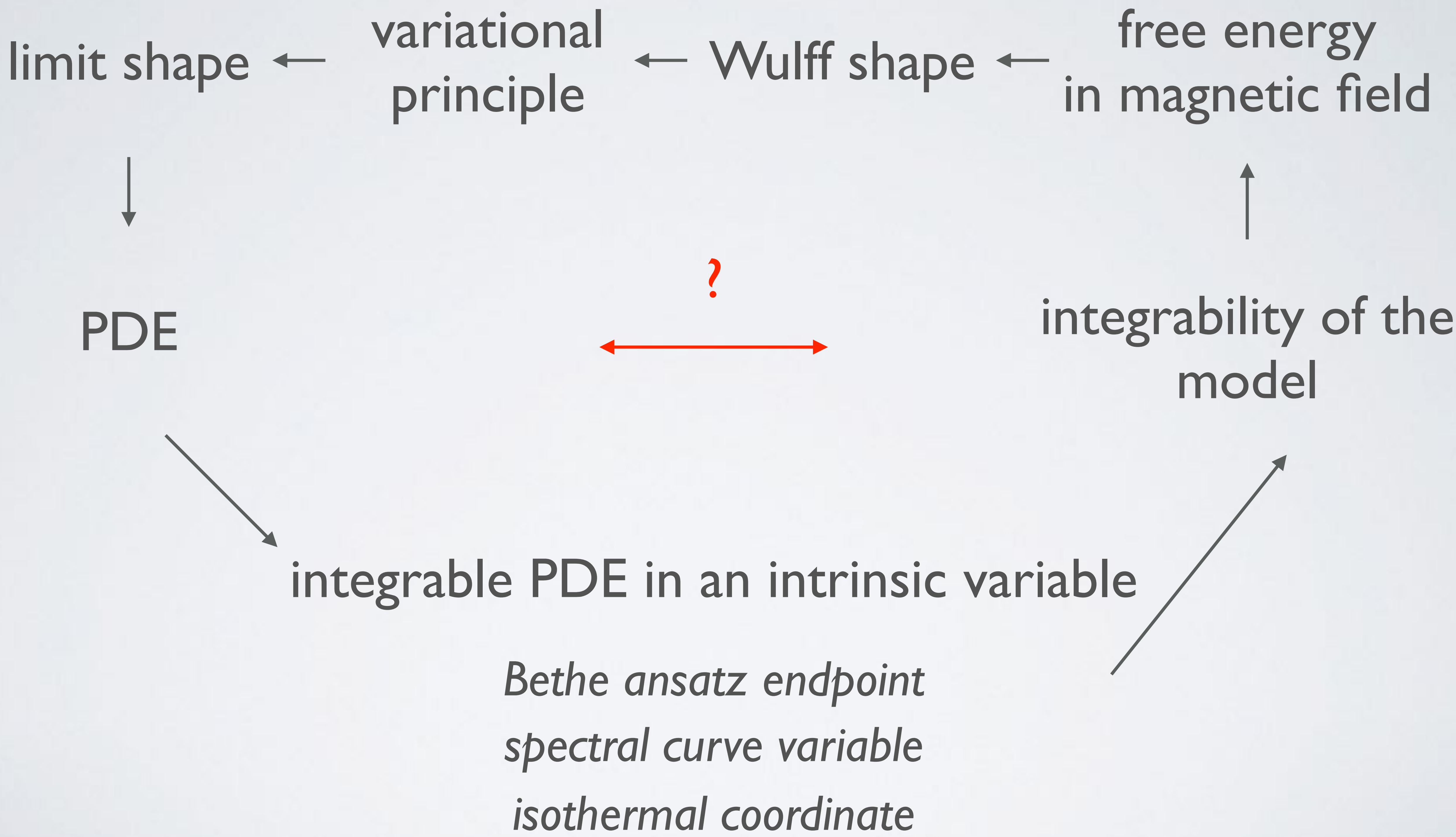
$$K = \frac{\pi}{\sqrt{\det D^2\sigma}}$$

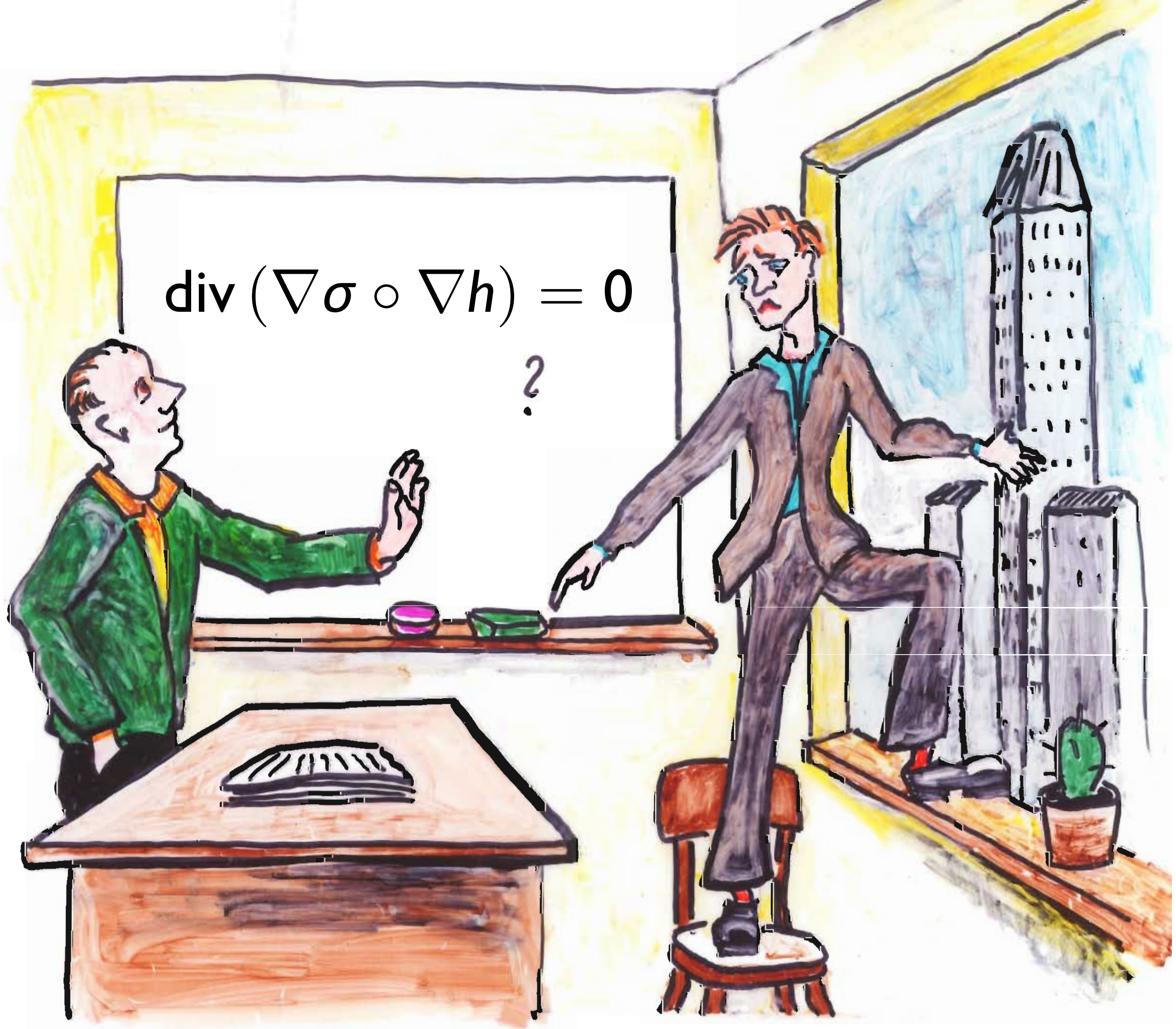
Luttinger parameter
coupling constant
stiffness etc

a strong form of universality

- *universal growth* of height variance
- *homogeneous* (but curved) free field for fluctuations
- *algebraic* boundaries
- *macroscopic universality* of frozen boundaries (Astala-Duse-Prause-Zhong)

INTEGRABLE PDE ?





Don't jump, *"complexify to simplify"* !

a not-so-"hidden" complex variable

(fluctuations, integrability, isothermal)

κ-HARMONIC ENVELOPE

$\sigma(s,t)$
$(s,t) \in \mathcal{N}$

Kenyon-Prause

isothermal on Wulff shape

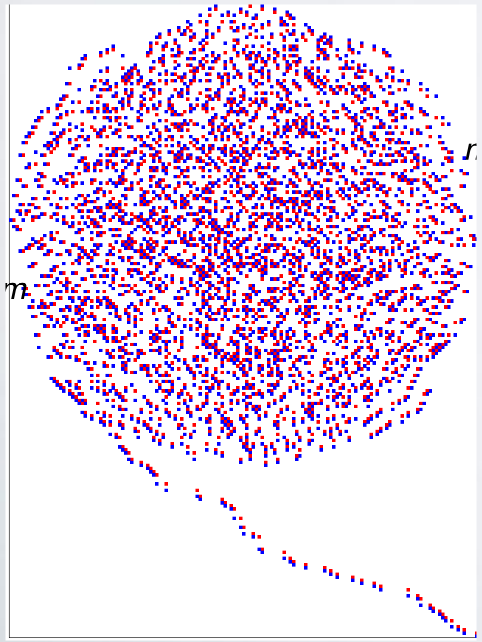
$$\kappa(z) = \sqrt{\det D^2\sigma} \text{ as a function of } z \in \mathbb{H}$$

$$\nabla \cdot \kappa \nabla u = 0$$

Thm: *s*, *t* and *h-(sx+ty)* are all κ-harmonic(z) in the liquid region

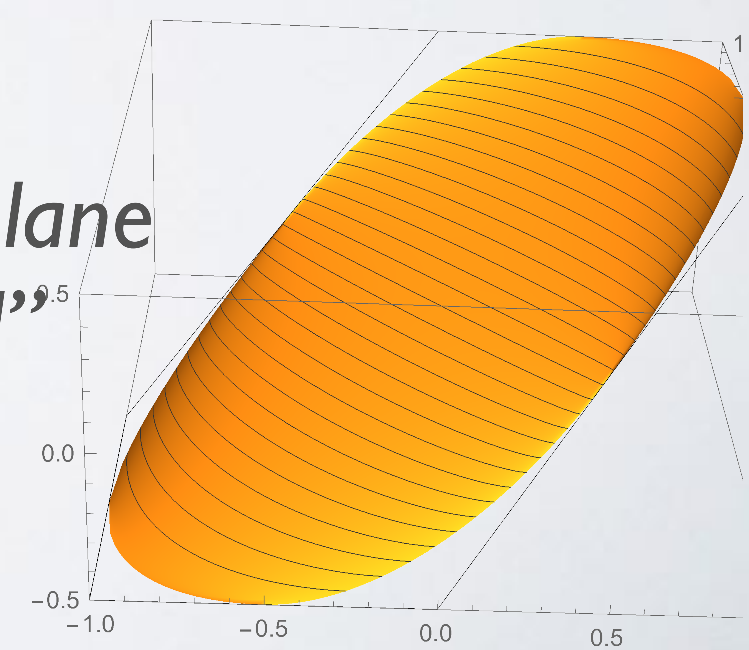
(multi-valued in z)

Corollary: Free-fermionic limit shapes are *envelopes* of harmonically moving planes in R³



tangent method
Colomo-Sportiello

“tangent plane method”



TRIVIAL POTENTIAL

Kenyon-Prause

$$\nabla \cdot \kappa \nabla u = 0$$

reduction to Schrödinger equation

$$(-\Delta + q)(\kappa^{1/2} u) = 0 \qquad q = \frac{\Delta \kappa^{1/2}}{\kappa^{1/2}} \quad \text{potential}$$

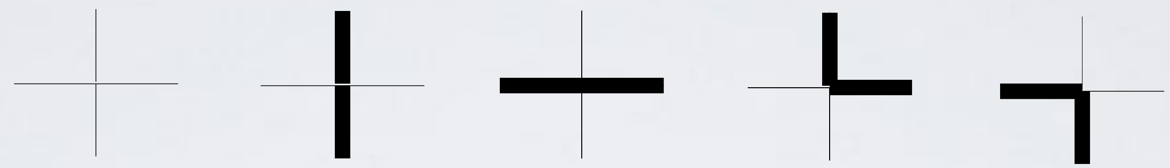
Def: a surface tension has *trivial potential* if $\sqrt[4]{\det D^2 \sigma}$ is a harmonic function of the intrinsic coordinate z

Then κ-harmonic: $$\frac{\text{harmonic}(z)}{\sqrt[4]{\det D^2 \sigma}} \qquad (q=0)$$

5-VERTEX MODEL

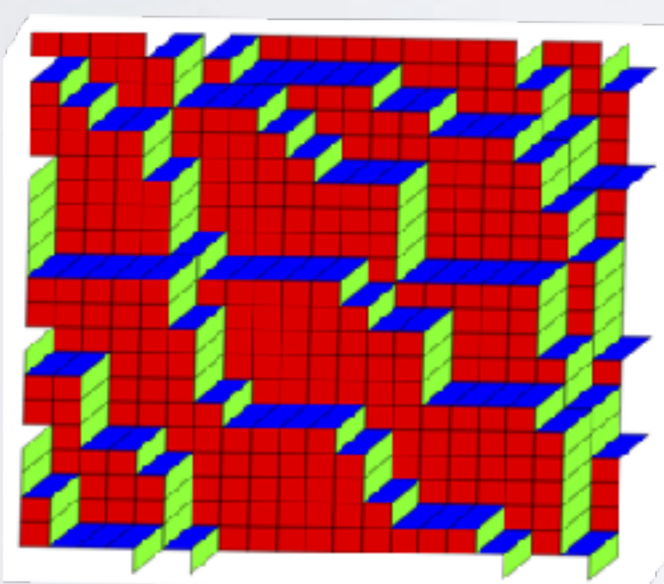
$r \neq 1$ (non-determinantal) *"interacting fermions"*

Wulff shape de Gier-Kenyon-Watson Bethe ansatz solution

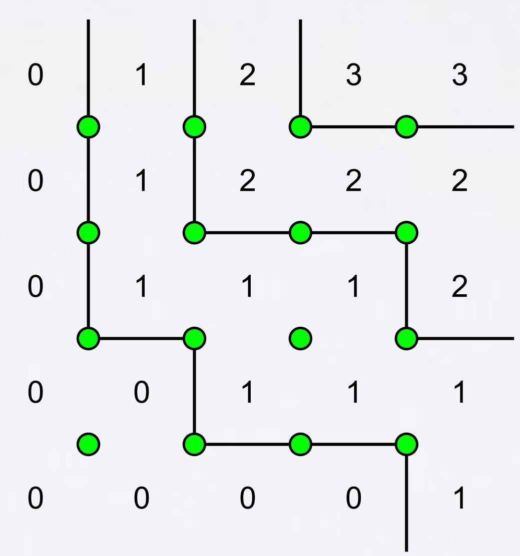


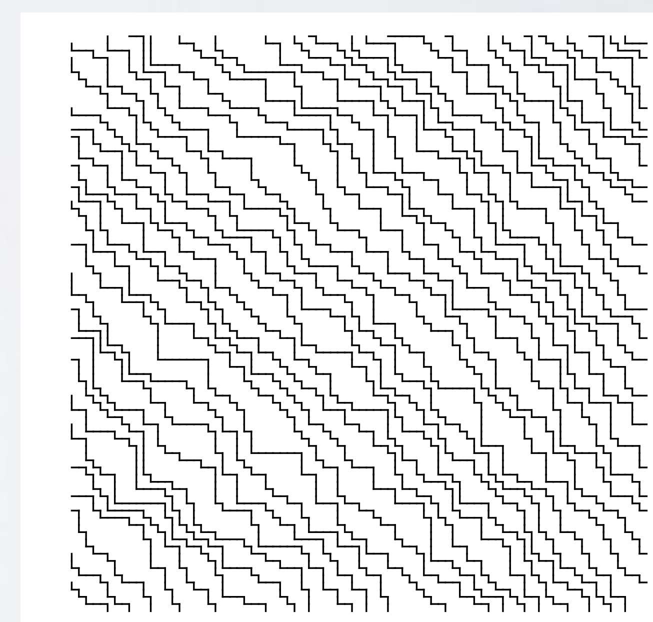
1	1	1	r	r	*(X,Y)*
1	e^X	e^Y	$re^{(X+Y)/2}$	$re^{(X+Y)/2}$	magnetic field

$$P(\textit{configuration}) \propto r^{\#\text{corners}}$$



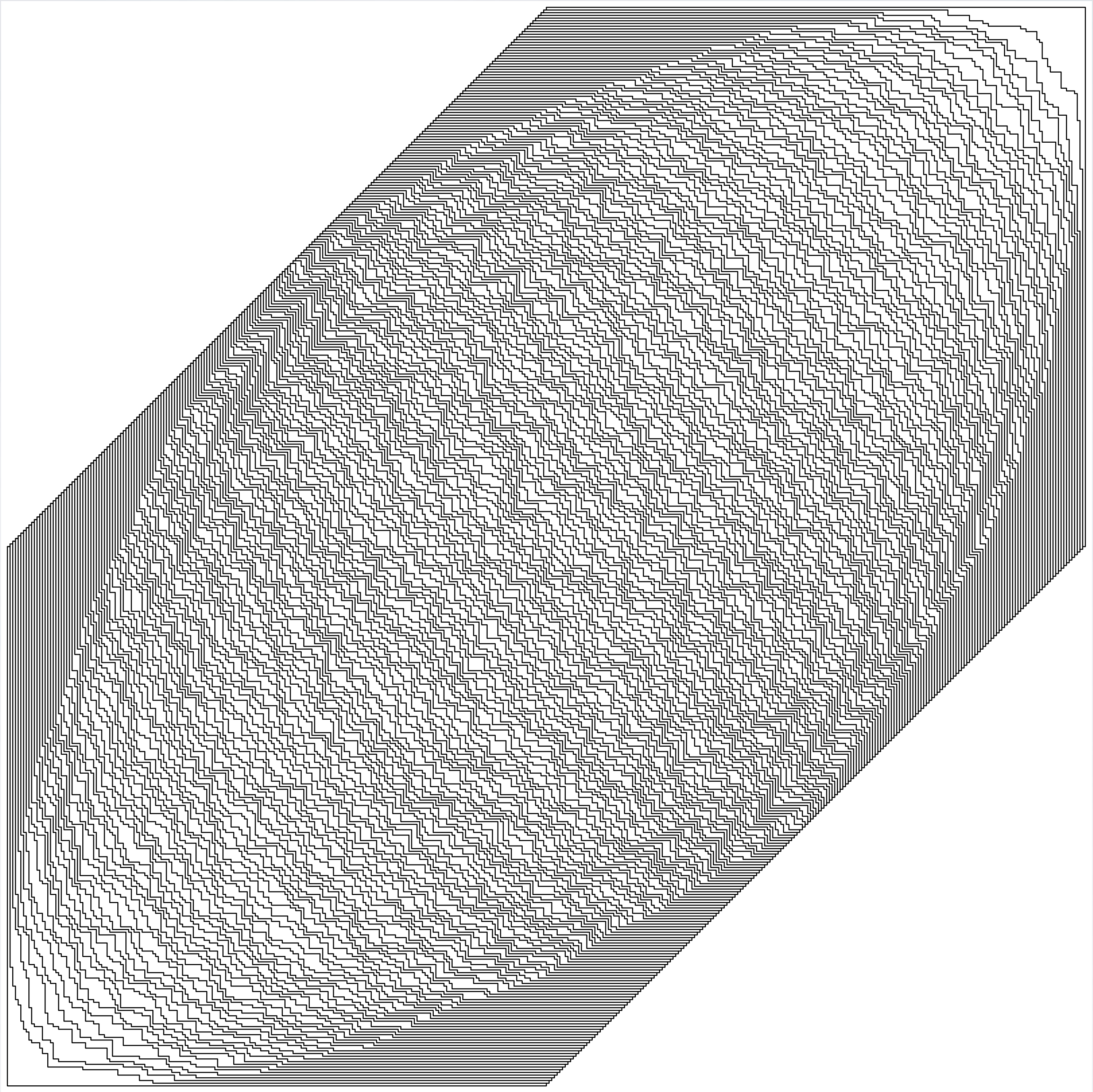
lozenge tilings with
(blue-green) *interaction*





monotone non-intersecting
lattice paths
with corners *penalized*

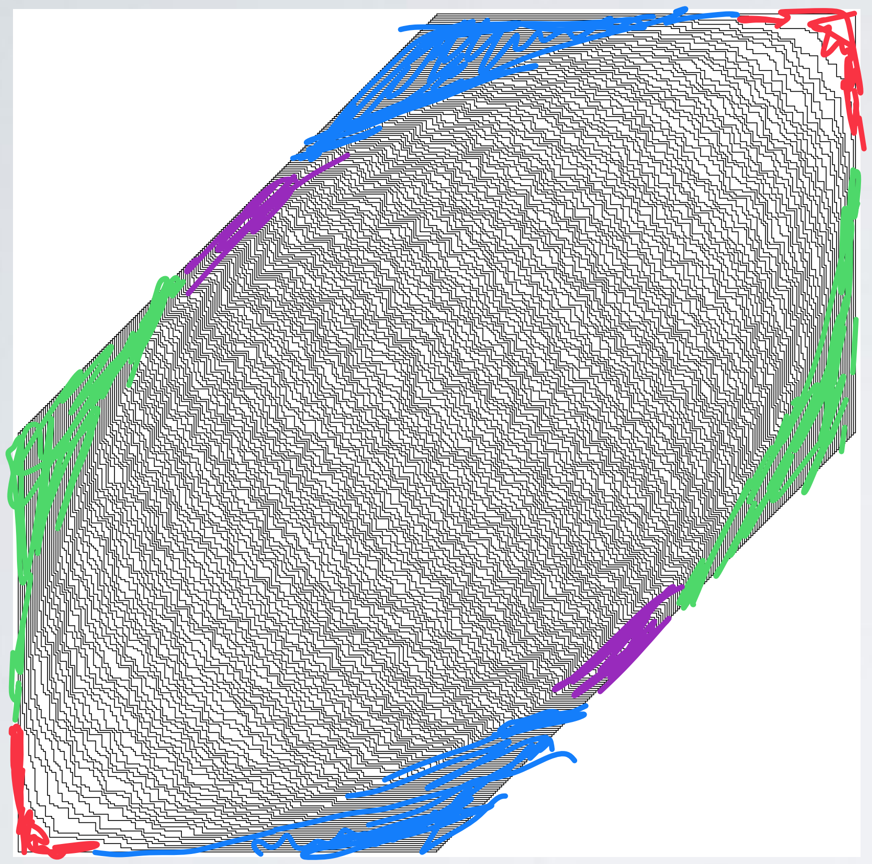
5-VERTEX BOXED PLANE PARTITION



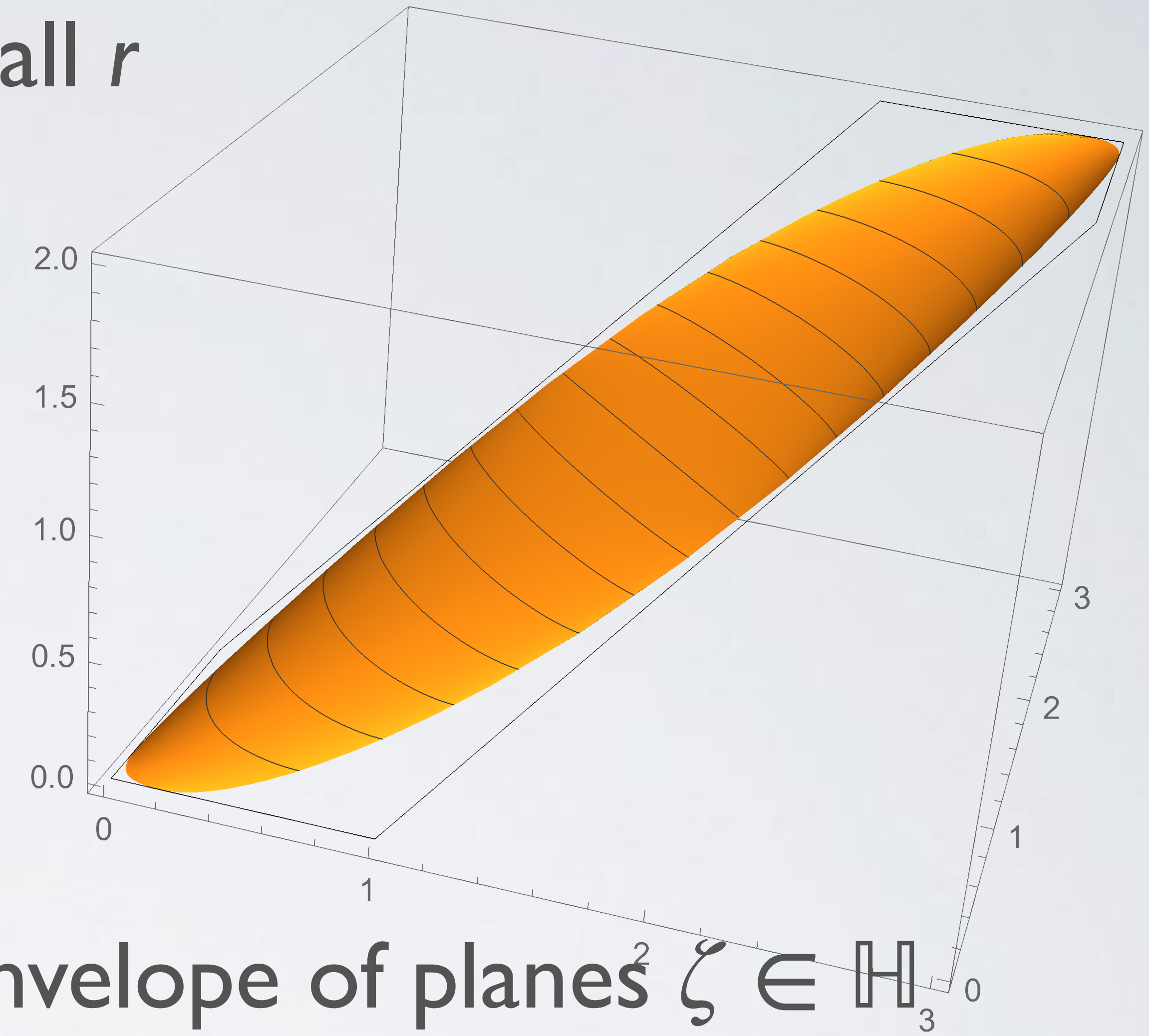
r=0.6

BPP EXAMPLE

small *r*

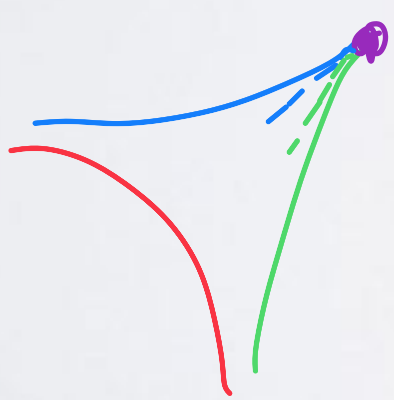


6 facets+ 2 neutral regions



envelope of planes $\zeta \in \mathbb{H}_3$

$(u(\zeta)$ degree 2 cover of $\mathbb{H})$



8 intervals on $\partial\mathbb{H}$

full boundary information

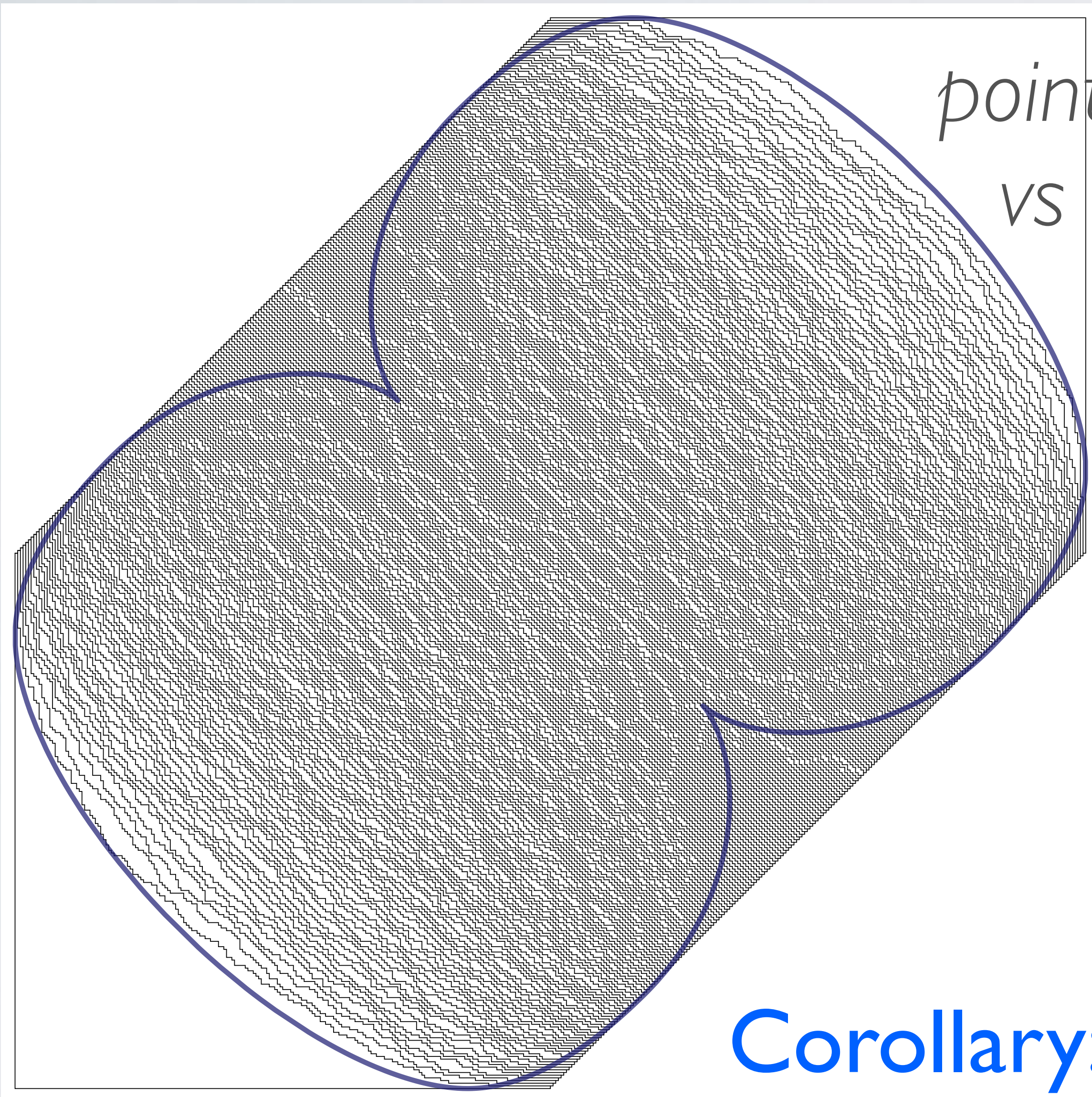
$$x_3 = s(\zeta)x + t(\zeta)y + c(\zeta)$$

trivial potential

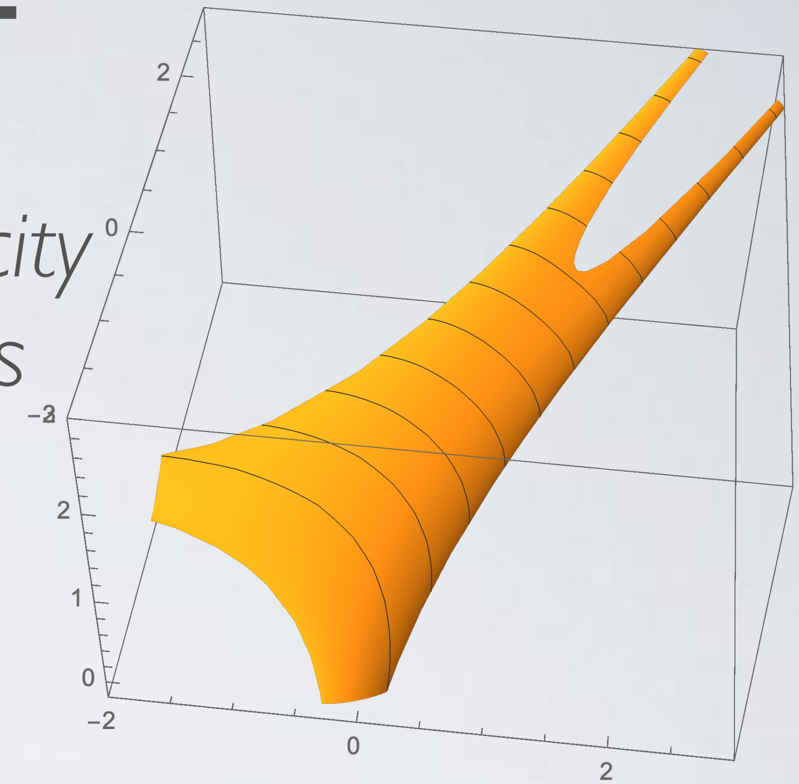
are all *ratios* of linear combinations of *harmonic measures*

BPP EXAMPLE

large *r*



points of non-analycity
vs algebraic curves



free energy/Wulff shape

r=2.5

(*shear phenomenon*)

Corollary: In any component of the liquid region

$$(s\theta)_\zeta x + (t\theta)_\zeta y + G_\zeta - \theta_\zeta h(x,y) = 0$$

$$\theta = \sqrt{\kappa}$$

all holomorphic functions

PART II
RANDOM YOUNG TABLEAUX

work in progress

simple κ

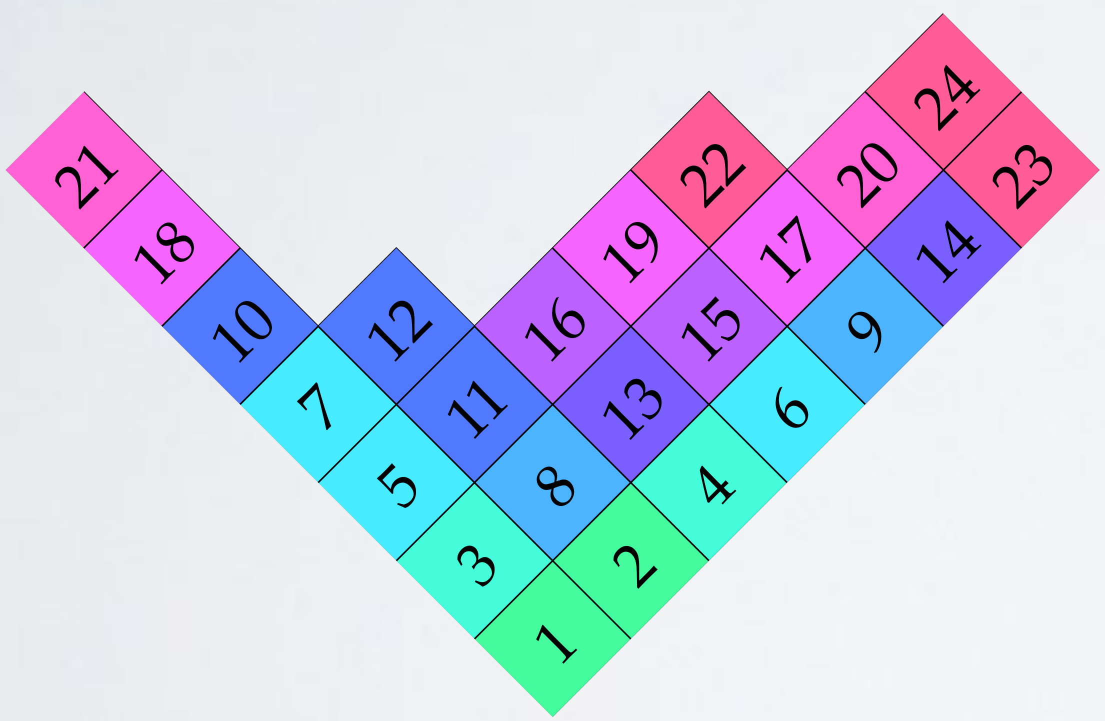
complicated (arbitrary)
boundary conditions

YOUNG DIAGRAMS AND TABLEAUX

10=5+4+1 integer partitions

shape *(5,4,1)* **in** *French notation*

Russian convention: rotated by 45°



A Young tableau of shape *(7,7,5,2,1,1,1)*

monotonic filling in both directions

$|\omega(x_1) - \omega(x_2)| \leqslant |x_1 - x_2|$

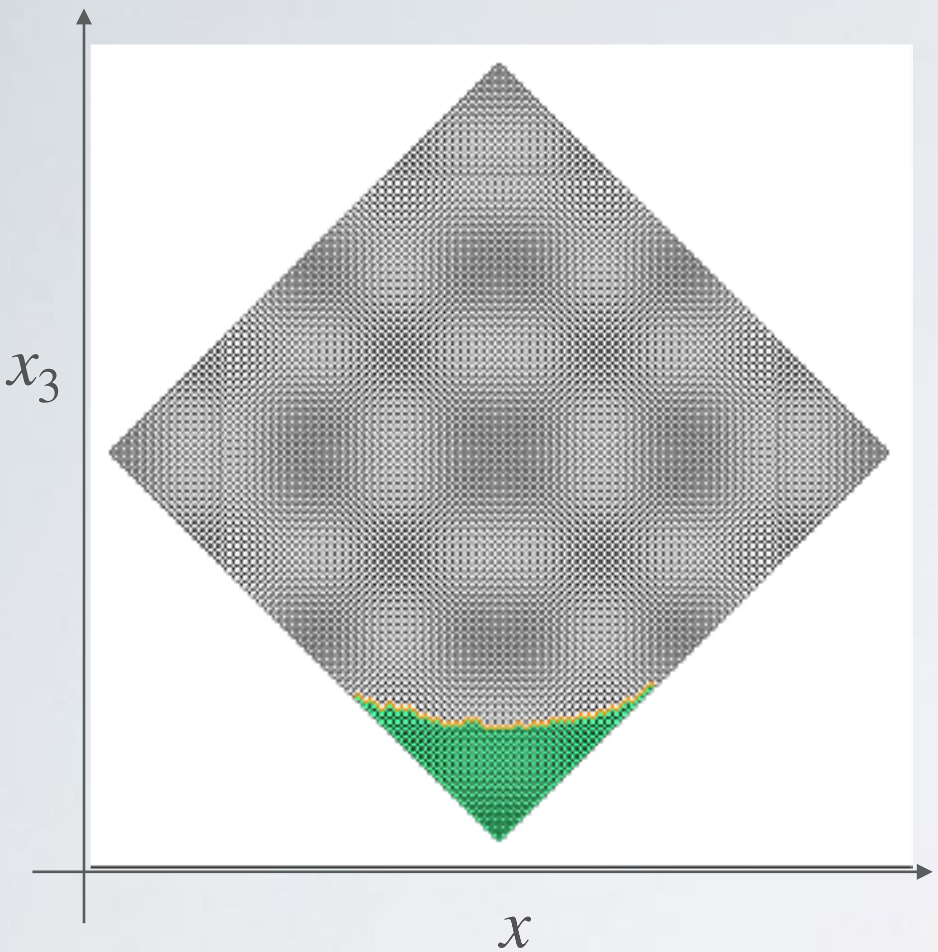
$\omega(x) = |x|$ for large x

profile

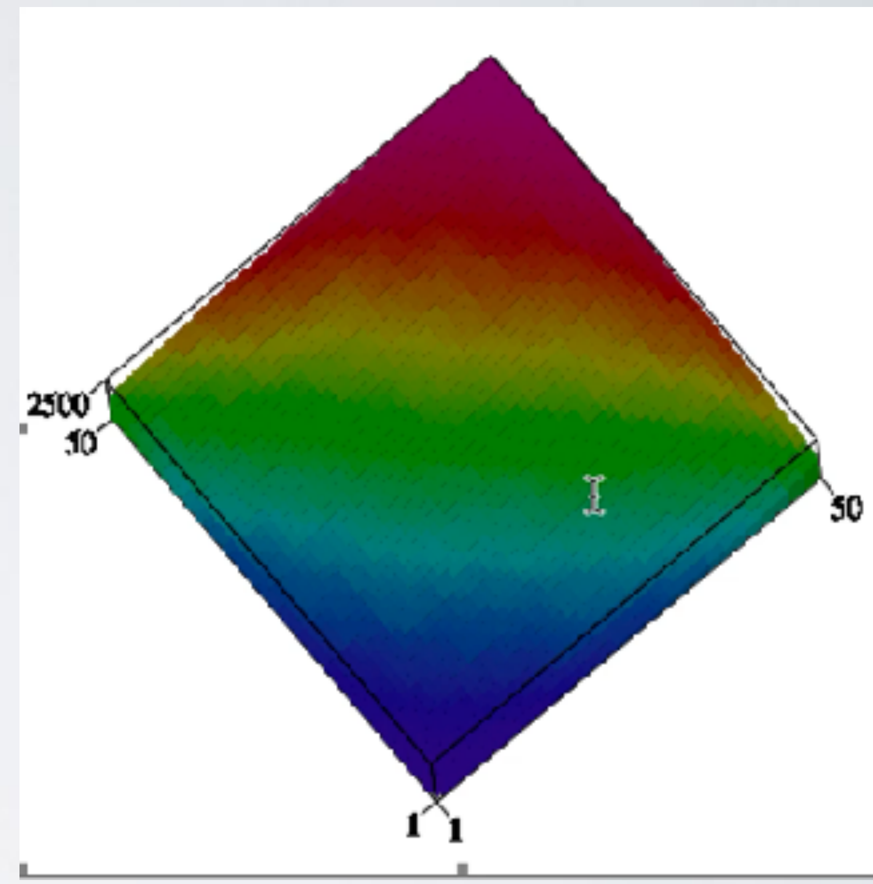
representation theory of symmetric groups

YOUNG TABLEAUX EXAMPLE

square shape

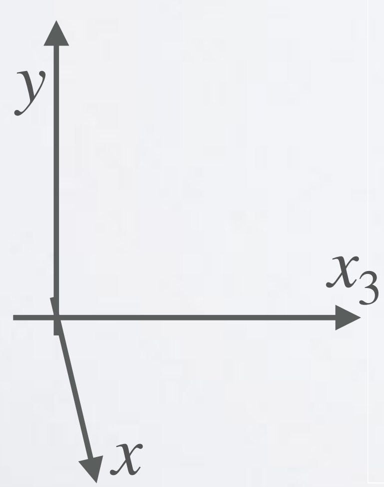


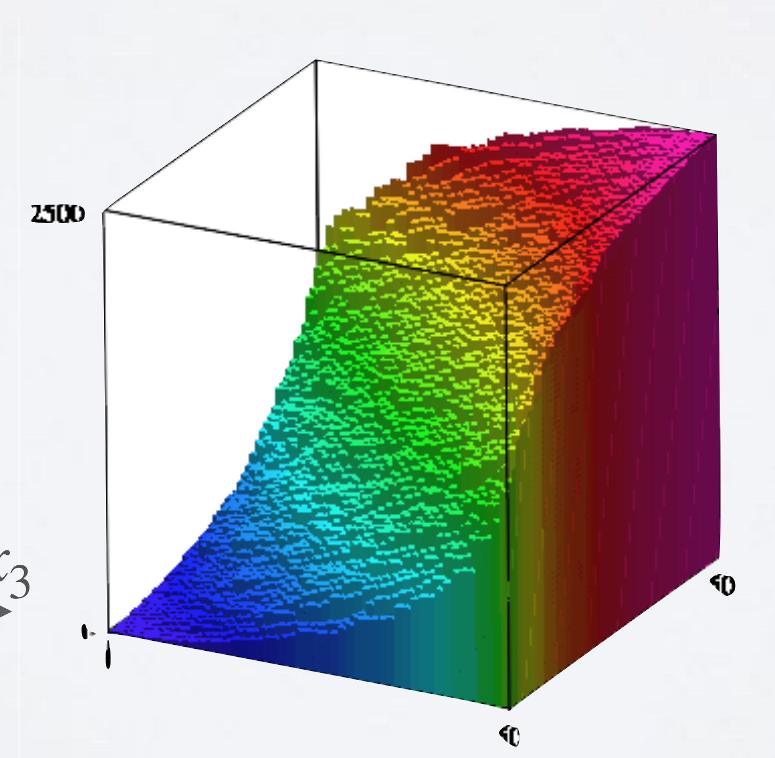
Dan Romik's
MacTableaux



$y \in [0,1]$ "time"

(normalized) asymptotic value of the tableaux



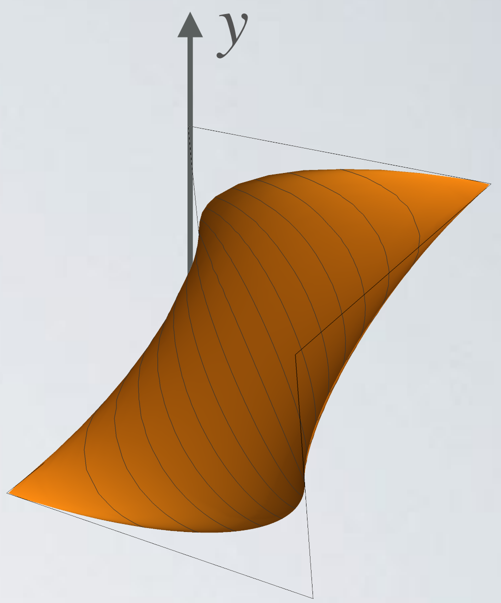


Cyril Banderier et al.'s
YoungPackage

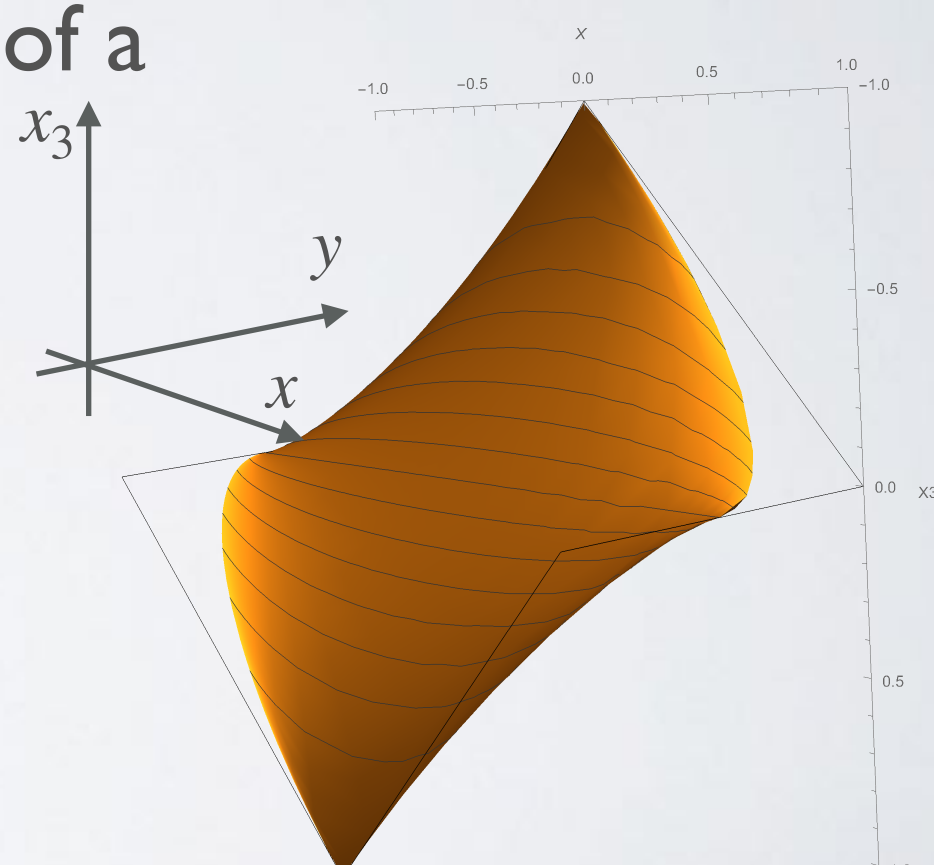
LIMIT SHAPE THEOREM

Biane, W. Sun, Cohn-Kenyon-Propp

The (rescaled) random YT surfaces with a limiting profile ω concentrate around a deterministic surface, called limit shape



The limit shape is a minimizer of a variational problem



'minimal surface' spanning a wire-frame

$x_3 \to h : \Omega \to \mathbb{R}$ height function

$$\min_h \int_\Omega \sigma(\nabla h), \qquad \nabla h \in \mathcal{N}, \quad h(x,0) = |x|, \quad h(x,1) = \omega(x)$$

$(x, y) \to \Omega$

singular and *degenerates* on the boundary

SURFACE TENSION

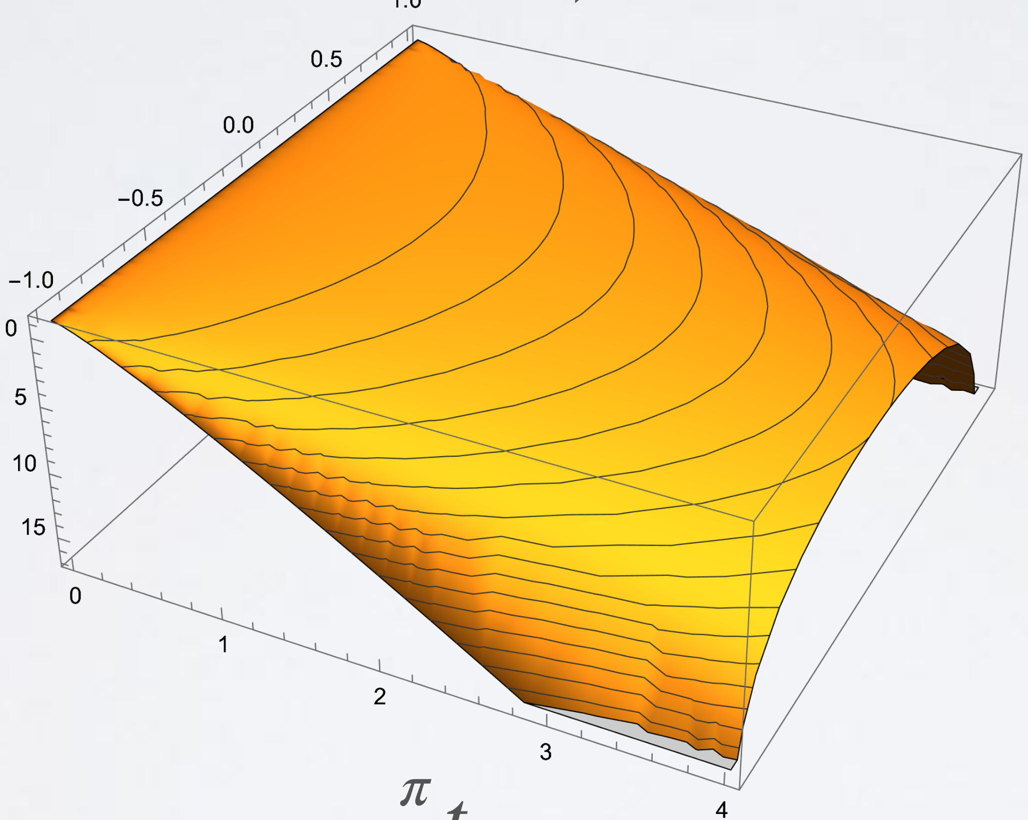
W. Sun

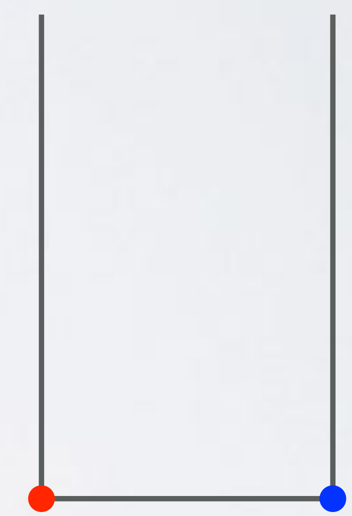
A. Gordenko, Johnston-O'Connel, Shlyakhtenko-Tao

$$\sigma(s,t) = -\left(1 + \log \frac{\cos \frac{\pi}{2} s}{\frac{\pi}{2} t}\right) t$$

$$(s,t) = \nabla h \in \mathcal{N}$$

$$\mathcal{N} = [-1,1] \times [0,\infty)$$



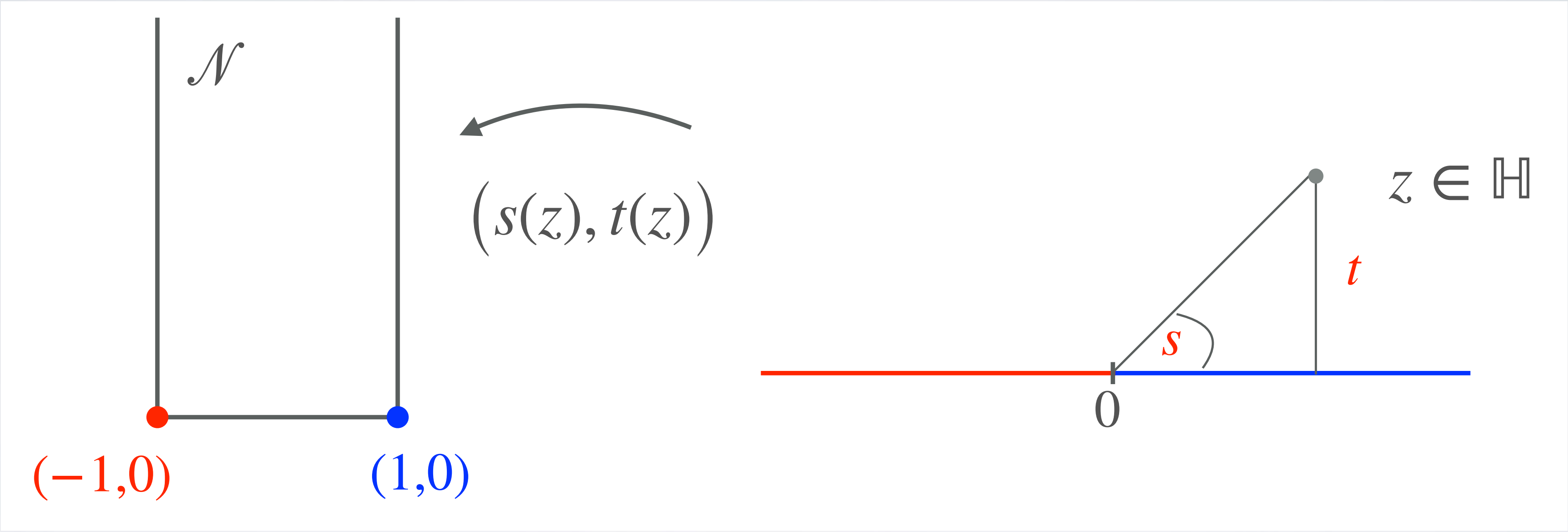


$$\sigma_s = \frac{\pi}{2} t \tan \frac{\pi}{2} s, \quad \sigma_t = \log \frac{\frac{\pi}{2} t}{\cos \frac{\pi}{2} s}$$

$$\det H_\sigma \equiv \pi^2/4$$

HARMONIC COORDINATES

$$z = \frac{\pi}{2} t\left(\tan\left(\frac{\pi}{2} s\right) + i\right) \in \mathbb{H}$$



$$s(z) = -\frac{2}{\pi}\arg z + 1, \quad t(z) = \frac{2}{\pi}\operatorname{Im} z$$

$$\sigma_s = \frac{\pi}{2} t \tan \frac{\pi}{2} s = \operatorname{Re} z, \quad \sigma_t = \log \frac{\frac{\pi}{2} t}{\cos \frac{\pi}{2} s} = \log |z|$$

harmonic conjugates

$$\sigma_s + i\frac{\pi}{2} t = z \quad \text{and} \quad \sigma_t - i\frac{\pi}{2} s = \log z - i\frac{\pi}{2}$$

EULER-LAGRANGE EQUATION

$$\sigma_{ss}h_x^2 + 2\sigma_{st}h_xh_y + \sigma_{tt}h_y^2 = 0$$

$$(x, y) \in \mathscr{L} \mapsto \nabla h = (h_x, h_y) = (s, t) \mapsto z(x, y)$$

$$(\sigma_s(h_x, h_y) + i\frac{\pi}{2}h_y)_x + (\sigma_t(h_x, h_y) - i\frac{\pi}{2}h_x)_y = 0$$

$$z_x + (\log z)_y = z_x + \frac{z_y}{z} = 0$$

$$\frac{z_x}{z_y} = -\frac{1}{z}$$ *(complex Burgers equation)*

$$(h - sx - ty)_{z\bar{z}} = -(s_{z\bar{z}}x + t_{z\bar{z}}y) - (s_zx_{\bar{z}} + t_zy_{\bar{z}}) = -\frac{i}{\pi}\left(-\frac{x_{\bar{z}}}{z} + y_{\bar{z}}\right)$$

$$= -\frac{i}{\pi}\frac{x_{\bar{z}}z_x + y_{\bar{z}}z_y}{z_y} = -\frac{i}{\pi}\frac{z(x, y)_{\bar{z}}}{z_y} = 0$$

TANGENT PLANE METHOD

Kenyon-Prause

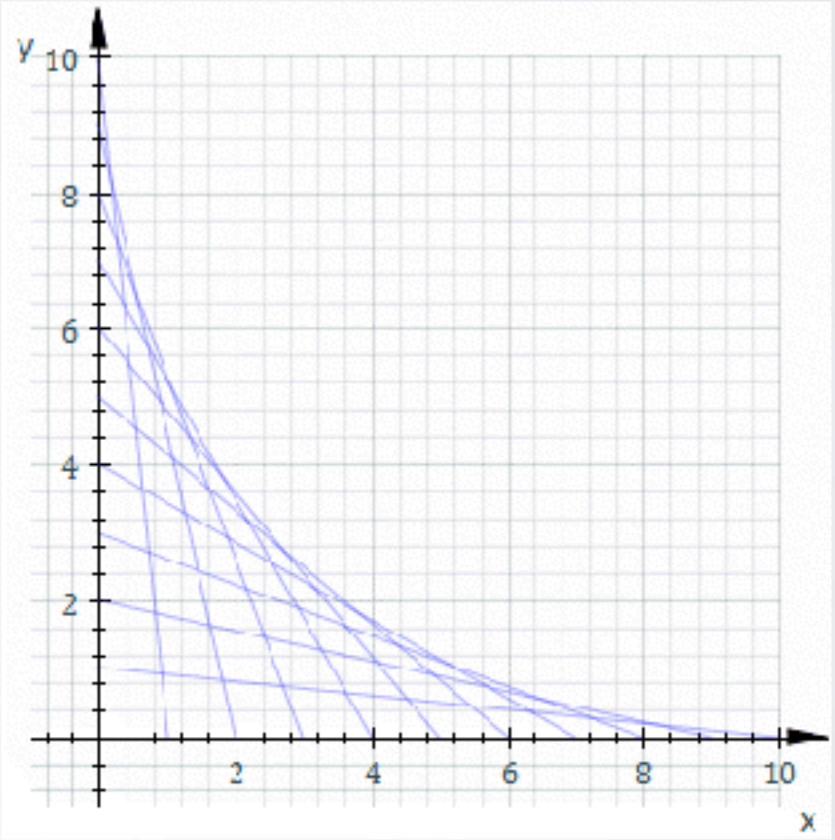
Thm: *s*, *t* and *h-(sx+ty)* are all harmonic(*z*) in the liquid region

(special case of previous thm)

(multi-valued in *z*)

Young tableaux limit shapes are *envelopes* of harmonically moving planes in R^3

$$x_3 = s(z)x + t(z)y + c(z)$$



PREVIOUS RESULTS

on Young tableaux limit shapes

Biane

general profile

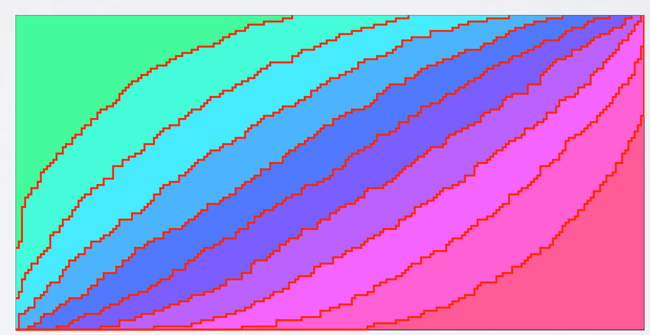
implicit limit shapes
in terms of *free cummulants*

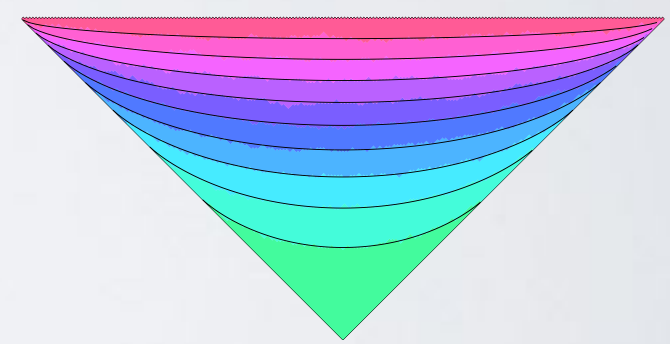
asymptotics of irreducible
representations of symmetric groups

Pittel-Romik
Angel-Holroyd-Romik-Virag

handful of examples
(square, rectangle, staircase)

explicit limit shapes





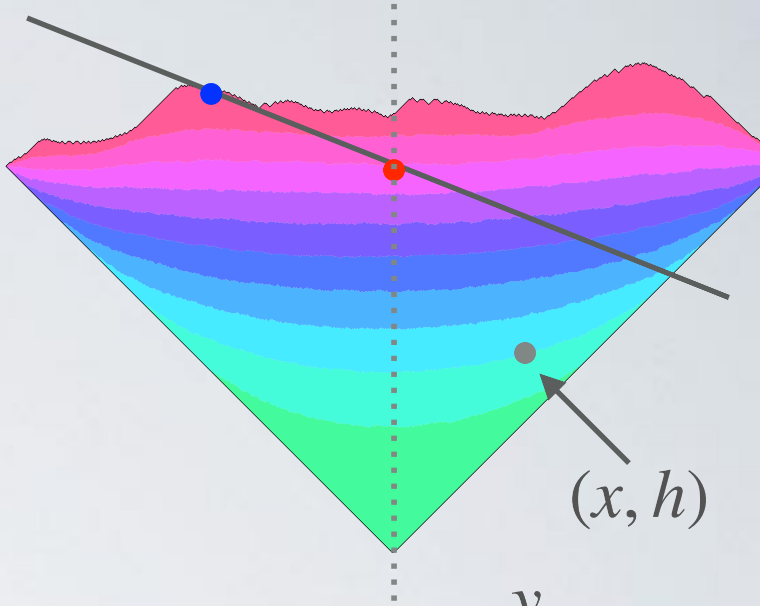
$$x_3'(x) = \frac{2}{\pi} \arctan \frac{x(1-2y)}{\sqrt{4y(1-y)-x^2}}, \quad |x| \le \sqrt{4y(1-y)}$$

family of one-dimensional
variational problems

YOUNG TABLEAUX LIMIT SHAPES

$$\nu(x) = \frac{\omega(x) - |x|}{2}, \quad \mathscr{C}_\nu(u) = \int_{\mathbb{R}} \frac{\nu'(x)dx}{u - x}, \quad G_\omega(u) = \frac{1}{u}\exp(-\mathscr{C}_\nu(u))$$

Kerov, Biane



asymptotic value of the tableau

Thm (P): $z = 1/G_\omega(u), \quad x + \dfrac{1-y}{G_\omega(u)} - u = 0, \quad u \in \mathbb{H}$

Limit surface

$u \in \mathbb{H}$

$$x = \frac{\operatorname{Im} uG(u)}{\operatorname{Im} G(u)}, \quad y - 1 = |G(u)|^2 \frac{\operatorname{Im} u}{\operatorname{Im} G(u)}$$

$$h(x, y) = \left(1 - \frac{2}{\pi}\arg(1/G(u))\right) x + \mathscr{P}\tilde{c}(u)$$

intercept function

$$\tilde{c}(x) = \omega(x) - \omega'(x)x$$

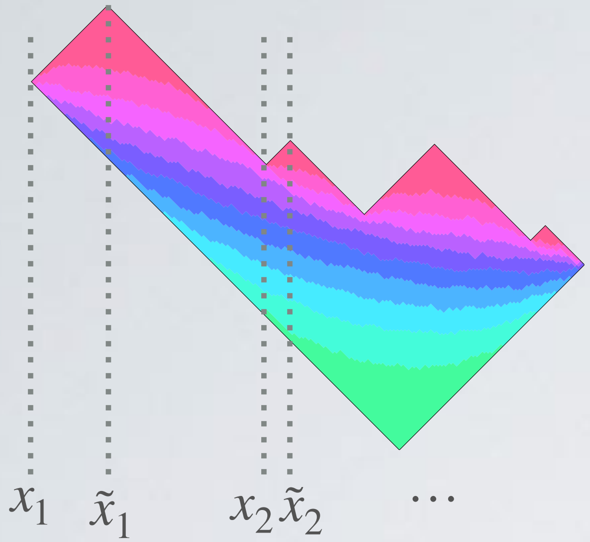
Corollary:

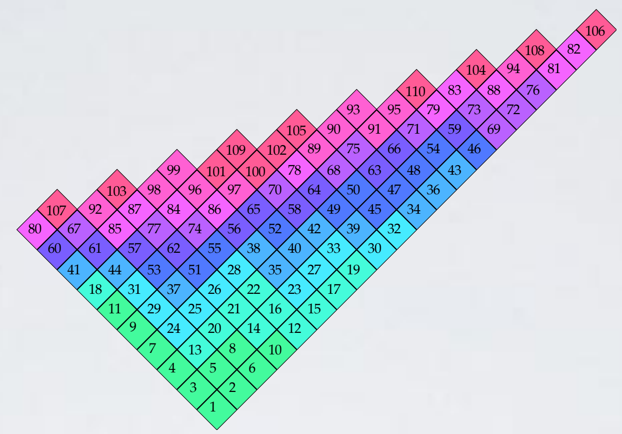
frozen boundary

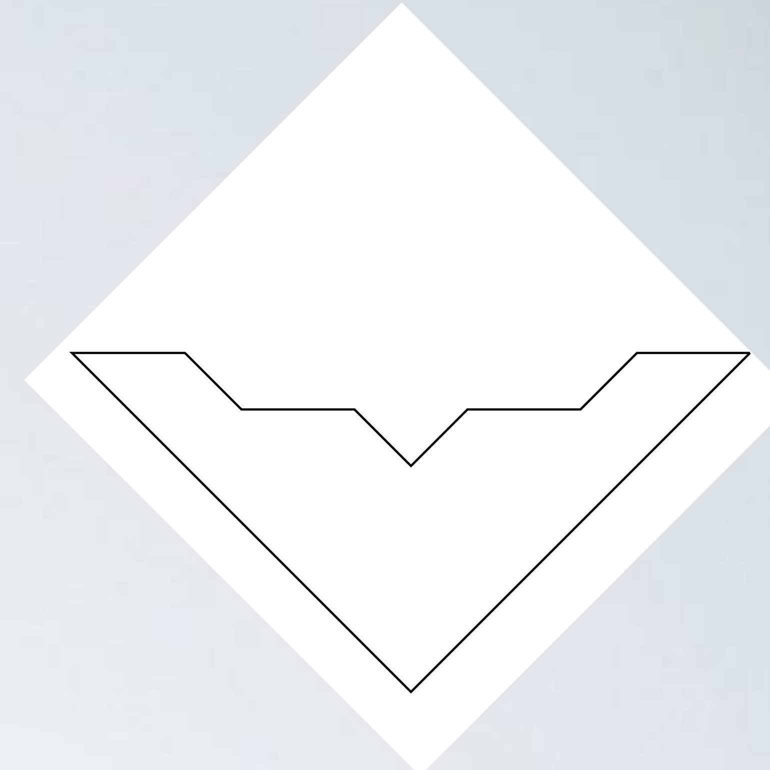
envelope of lines

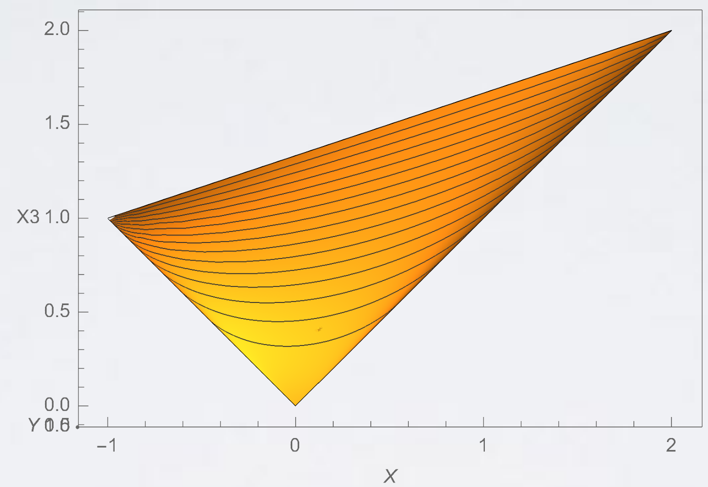
$u \in \mathbb{R} \backslash \operatorname{supp}(\nu'(x)dx)$

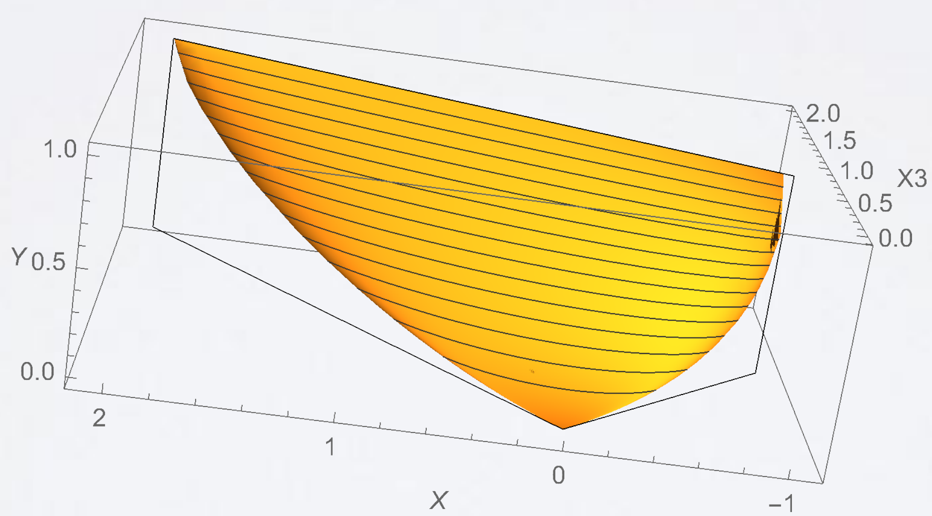
EXAMPLES

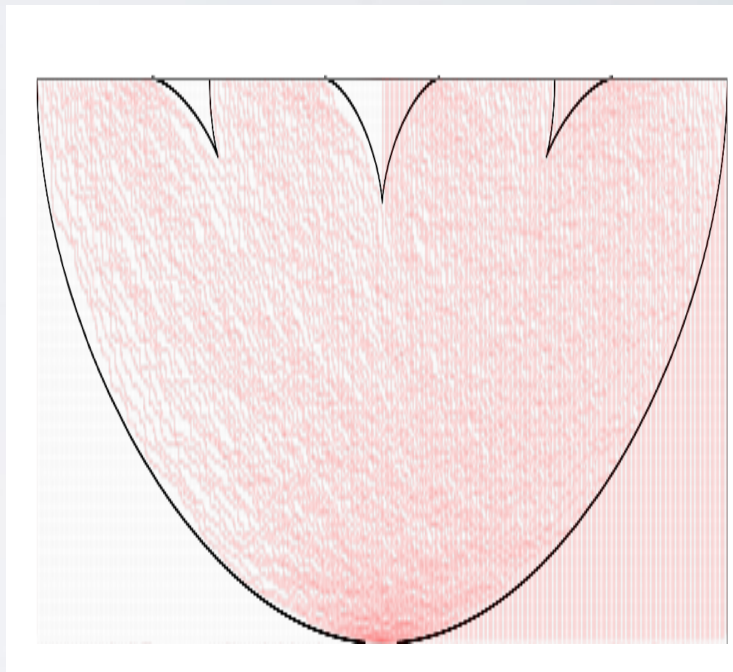




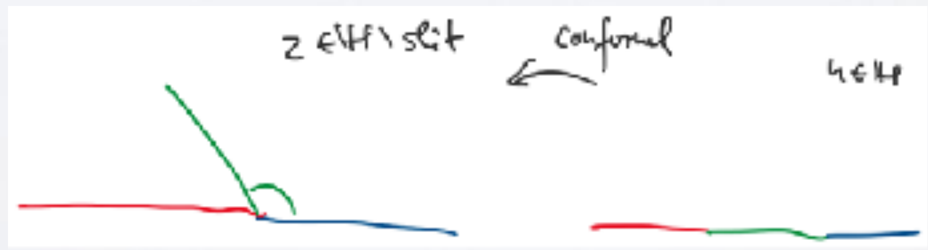








$$G_\omega(u) = \frac{\prod (u - \tilde{x}_i)}{\prod (u - x_i)}$$



PLANCHEREL TABLEAUX

Kerov

$$s_z x + t_z y + c_z = 0$$

$$c(z) = -\frac{2}{\pi}\mathrm{Im}(1/z)$$

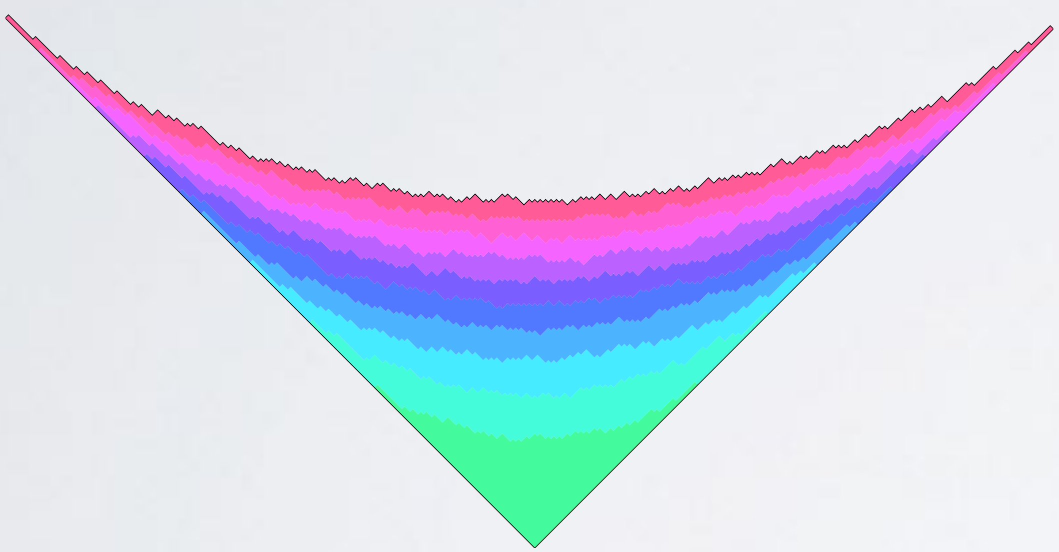
$$\Omega = \{(x,y) \in \mathbb{R}^2 \colon x \in \mathbb{R}, y \geqslant 0\}$$

$$-\frac{x}{z} + y + \frac{1}{z^2} = 0$$

zero boundary values exept at 0

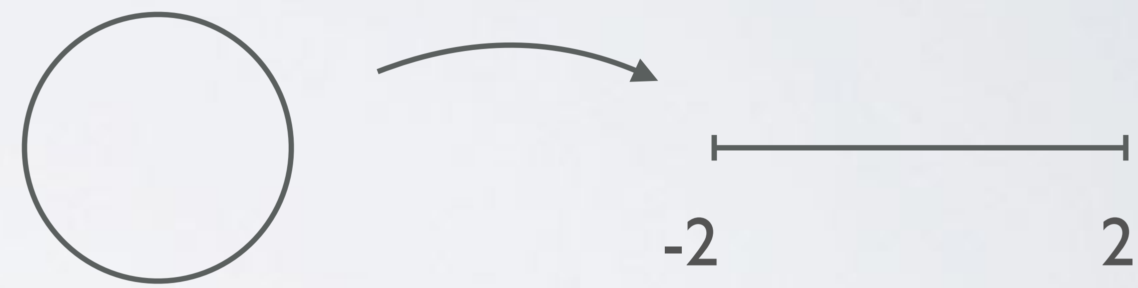
$$h(x,y) = \sqrt{y}\, h\left(\frac{x}{\sqrt{y}}, 1\right)$$

$$G(u) + \frac{1}{G(u)} = u$$

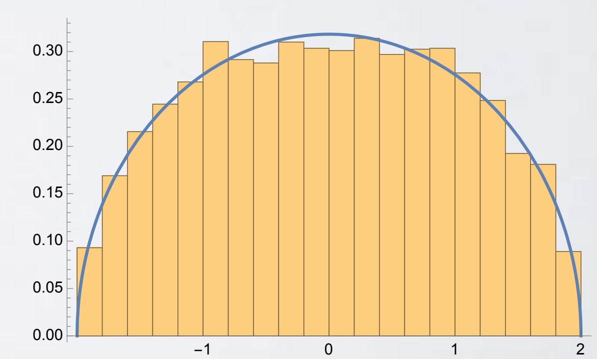


$\mathbb{C}\backslash\mathbb{D}$

$$u = z + 1/z$$



$$G(u) = \mathscr{C}_{\mu_{sc}}(u)$$



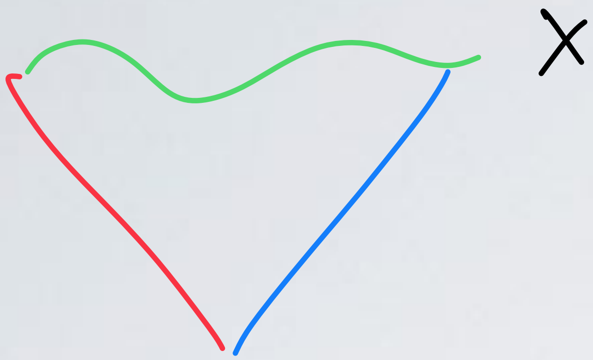
$$h = sx + ty + c = \frac{2}{\pi}\left(x \arcsin(x/2) + \sqrt{4 - x^2}\right)$$

Logan-Shepp-Vershik-Kerov curve

$$x \in [-2,2], \quad y = 1$$

$$d\mu_{sc}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\,dx$$

PROOF

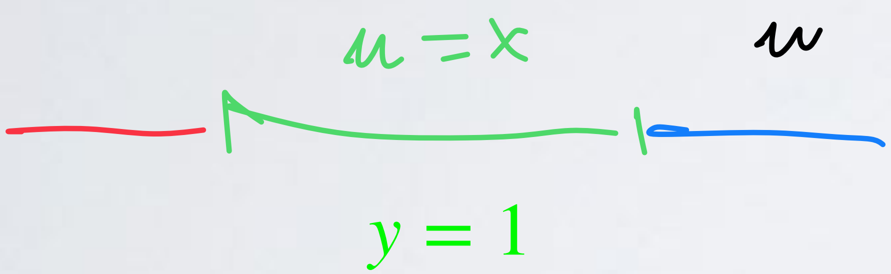


$$s_z x + t_z y + c_z = 0 \quad \textit{(envelope equation)}$$

$$x + \frac{t_z}{s_z}(y - 1) + \frac{(c+t)_z}{s_z} = 0 \qquad z \leftrightarrow (x, y)?$$

$$x - z(u)(y-1) - u = 0$$

$$-\arg z(u) = \frac{\pi}{2}(\omega'(u) - 1)$$



$$u \in \partial \mathbb{H}$$

$$\arg \frac{u}{z(u)} = \pi \nu'(u)$$

$$1/z(u) = \frac{1}{u}\exp(-\mathscr{C}_\nu(u)) =: G_\omega(u) \qquad u \in \mathbb{H}$$

$$h = sx + ty + c = sx + t(y-1) + c + t = sx + \mathscr{P}\tilde{c}(u)$$

$$-\frac{2}{\pi}\operatorname{Im} u \nearrow \qquad \nwarrow \mathscr{P}\tilde{c}(u) + C\operatorname{Im} u$$

OUTLOOK

"calculation" of fractional free convolution power

minor process for random matrices

skew shapes

Related works
similar boundary conditions

Duse-Metcalfe

Bufetov-Gorin

Di Francesco-Reshetikhin

Debin-Ruelle

lecture hall tableaux
Corteel-Keating-Nicoletti

exact same limit shapes?

tangent method		tangent plane method
arctic curve	$\rightsquigarrow$	full limit surface
holomorphic coeff.		harmonic coeff.

(free fermions)