Global asymptotics of particle systems at high temperature

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Based on work w/ **Florent Benaych-Georges** & **Vadim E. Gorin**

Plan of the talk

Eigenvalues of Gaussian beta ensemble (GβE)

General theorems and proof ideas

Further questions

- γ-Semifree Probability
- Discrete Ensembles

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Gaussian Unitary Ensemble (GUE)

GUE_N is the probability measure on $\mathcal{H}_N := \{A \in \mathbb{C}^{N\times N} \mid A^* = A\}$ on Hermitian matrices with density

$$\frac{1}{\mathcal{Z}_N} \cdot \exp\left\{-\frac{\mathrm{Trace}(A^2)}{2}\right\} \prod_{i=1}^{N} da_{ii} \prod_{1\le i<j\le N} d\Re a_{ij}\, d\Im a_{ij}.$$

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If $A \in \mathcal{H}_N$ is GUE_N-distributed, its real eigenvalues

$$x_1 \le x_2 \le \cdots \le x_{N-1} \le x_N$$

are random and their distribution is:

$$\boxed{\text{Eigen}_N^{(2)}(x_1, \cdots, x_N) = \frac{1}{Z_N^{(2)}} \prod_{1\le i<j\le N} (x_i - x_j)^2 \prod_{k=1}^{N} e^{-\frac{1}{2}x_k^2}}$$

We call this the GUE (eigenvalue) density.

Global asymptotics of Hermite N–particle ensemble

Consider the empirical measures

$$\mu_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{\frac{x_i}{\sqrt{N}}}, \quad \text{where } x_1 \le \cdots \le x_N \text{ is Eigen}_N^{(2)}\text{–distributed.}$$

Theorem (Wigner '55)

The (random) probability measures μ_N converge weakly, in probability, to the semicircle distribution — with density

$$s(t) := \mathbf{1}_{\{-2\le t\le 2\}} \cdot \frac{\sqrt{4-t^2}}{2\pi},$$

i.e. for any $f \in C_b(\mathbb{R})$:

$$\lim_{N\to\infty} \mathbb{E}_{x_1\le\cdots\le x_N}\left[\int_{\mathbb{R}} f(t)\mu_N(dt)\right] = \int_{-2}^{2} f(t)s(t)\,dt.$$

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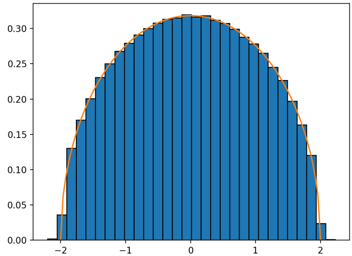

Eigenvalues of Gaussian Beta Ensemble (GβE)

For general $\beta \geq 0$, we study the random N-tuple

$$x_1 \leq x_2 \leq \cdots \leq x_{N-1} \leq x_N$$

determined by the probability measure

$$\boxed{\mathrm{Eigen}_N^{(\beta)}(x_1, \cdots, x_N) = \frac{1}{Z_N^{(\beta)}} \prod_{1 \leq i < j \leq N} (x_i - x_j)^\beta \prod_{k=1}^{N} e^{-\frac{1}{2} x_k^2}}$$

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Why?

1. For $\beta = 1 \,\&\, 4$, it's the eigenvalue density of Gaussian Orthogonal Ensemble (GOE) & Gaussian Symplectic Ensemble (GSE).

2. Relation with particle systems in physics (log-gas); β is called the inverse temperature.

Global asymptotics of GβE eigenvalues

Nothing changes if $\beta > 0$ is fixed: as $N \to \infty$, then

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The outlier $\beta = 0$ case: In this case, the density is

$$\mathrm{Eigen}_N^{(\beta=0)}(x_1, \cdots, x_N) = \frac{1}{(2\pi)^{N/2}} \prod_{k=1}^{N} e^{-\frac{1}{2}x_k^2}.$$

Then $x_1, \cdots, x_N$ are i.i.d. standard Gaussian r.v.'s. Hence if $\beta = 0$, $N \to \infty \implies \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \longrightarrow$ Gaussian distribution.

Global asymptotics of GβE eigenvalues at high temp

Theorem (Duy, Shirai '15 & Benaych-Georges, C, Gorin '22)

Consider the empirical measures

$$\mu_{N,\beta} := \frac{1}{N}\sum_{i=1}^{N} \delta_{\frac{x_i}{N}}, \quad \textit{where } x_1 \le \cdots \le x_N \textit{ is Eigen}_N^{(\beta)}\textit{–distributed.}$$

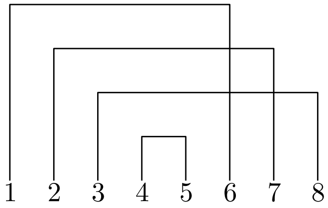
In the limit

$$N \to \infty, \quad \beta \to 0^+, \quad \frac{N\beta}{2} \to \gamma \in (0,\infty),$$

the measures $\mu_{N,\beta}$ *converge weakly, in probability, to certain probability measure* μ_γ *which can be completely described.*

Global asymptotics of Hermite N–particle β–ensemble at high temperature

For a *perfect matching* $\pi = \{B_1, \cdots, B_n\}$ of $\{1, \cdots, 2n\}$, draw the *arc diagram*. Define $\mathrm{roof}(\pi) :=$ # roofs with no intersections.



$$\pi = \{1, 6\} \sqcup \{2, 7\} \sqcup \{3, 8\} \sqcup \{4, 5\}.$$

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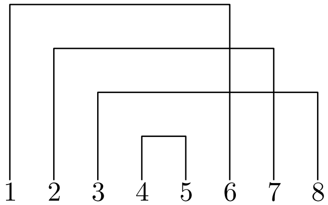

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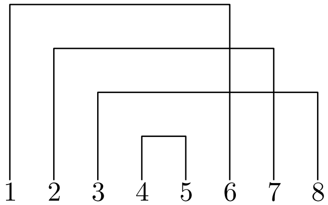

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Theorem (Benaych-Georges, Cuenca, Gorin '22)

The limiting measure μ_γ is uniquely determined by its moments:

$$\int_{-\infty}^{\infty} x^k \, \mu_\gamma(dx) \quad = \sum_{\text{perfect matchings } \pi \text{ of } \{1,\cdots,k\}} (\gamma+1)^{roof(\pi)}.$$

Limits as $\gamma \to 0^+$ and $\gamma \to \infty$

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Comments:

1. If k is odd, the k-th moment of μ_γ is zero.
2. If $k = 2n$ and $\gamma \to 0^+$, then
$$\begin{aligned}\text{RHS} &= \text{number of perfect matchings of } \{1, 2, \cdots, 2n\} \\ &= (2n-1)(2n-3)\cdots 3 \cdot 1.\end{aligned}$$
3. If $k = 2n$ and $\gamma \to \infty$ (need to divide by γ^n first), then
RHS = number of <u>*noncrossing*</u> perfect matchings of $\{1, 2, \cdots, 2n\}$
$$= \text{Catalan number } C_n = \frac{(2n)!}{(n+1)!\,n!}.$$

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Recall: Levy's continuity theorem

Let $\{\mu_N\}_{N\geq 1}$, μ be probability measures on $\mathbb{R}^d$.
The *Fourier transform* of μ_N is

$$\phi_N(\vec{\mathbf{x}}) := \int_{\mathbb{R}^d} K(\vec{\mathbf{a}}, \vec{\mathbf{x}})\, \mu_N(\vec{\mathbf{a}}).$$

where $\vec{\mathbf{x}} := (x_1, \cdots, x_d)$, $\vec{\mathbf{a}} := (a_1, \cdots, a_d)$,

$$K(\vec{\mathbf{a}}, \vec{\mathbf{x}}) := e^{\mathbf{i}(a_1 x_1 + \cdots + a_d x_d)}.$$

Similarly, let $\phi(\vec{\mathbf{x}})$ be the Fourier transform of μ.

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Theorem

$$\mu_N \to \mu \textit{ weakly} \Longleftrightarrow \phi_N(\vec{\mathbf{x}}) \to \phi(\vec{\mathbf{x}}) \textit{ pointwise.}$$

Intuition: At least when all measures are compactly supported, use

$$\mathbb{E}_\mu \left[a_1^{k_1} \cdots a_d^{k_d} \right] = \frac{\partial^{k_1 + \cdots + k_d}}{\partial x_1^{k_1} \cdots x_d^{k_d}} \phi(\vec{\mathbf{x}}) \Big|_{x_1 = \cdots = x_d = 0}.$$

Multivariate Bessel functions

Theorem (Benaych-Georges, Cuenca, Gorin '22)

(Abbreviated) LLN for empirical measures of $x_1 \leq \cdots \leq x_N \iff$ *Taylor coeffs of the logarithm of* β*-Fourier transforms converge.*

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Our β-Fourier transform = Dunkl transform relies on a kernel $K(\vec{\mathbf{a}}, \vec{\mathbf{x}})$ that depends on β, the multivariate Bessel function:

$$\boxed{K(\vec{\mathbf{a}}, \vec{\mathbf{x}}) = B_N^{(\beta)}(\vec{\mathbf{a}}, \vec{\mathbf{x}}), \quad \beta \geq 0}$$

defined from the (differential, symmetrized) Dunkl operators $P_k^{(\beta)}$:

$B_N^{(\beta)}(\vec{\mathbf{a}}, \vec{\mathbf{x}})$ is symmetric in the variables $x_1, \cdots, x_N$,

$B_N^{(\beta)}(\vec{\mathbf{a}}, \vec{\mathbf{0}}) = 1$,

$$P_k^{(\beta)} B_N^{(\beta)}(\vec{\mathbf{a}}, \vec{\mathbf{x}}) = \left(\sum_{i=1}^N a_i^k\right) \cdot B_N^{(\beta)}(\vec{\mathbf{a}}, \vec{\mathbf{x}}), \ \forall k = 1, 2, \cdots .$$

How to think of the Bessel generating function?

- When $\beta = 0$:

$$P_k^{(\beta=0)} = \left(\frac{\partial}{\partial x_1}\right)^k + \cdots + \left(\frac{\partial}{\partial x_N}\right)^k,$$

$$B_N^{(\beta=0)}(\vec{\mathbf{a}}, \vec{\mathbf{x}}) = \frac{1}{N!} \sum_{\sigma \in S(N)} e^{a_1 x_{\sigma(1)} + \cdots + a_N x_{\sigma(N)}}.$$

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- When $\beta = 2$: they are the HCIZ, spherical integral
$$B_N^{(\beta=2)}(\vec{\mathbf{a}}, \vec{\mathbf{x}}) := \int_{U(N)} e^{\mathrm{Trace}(U D(\vec{\mathbf{a}}) U^* D(\vec{\mathbf{x}}))} \mathrm{Haar}(dU),$$
where $D(\vec{\mathbf{a}}) := \mathrm{diag}(\vec{\mathbf{a}})$, $D(\vec{\mathbf{x}}) := \mathrm{diag}(\vec{\mathbf{x}})$; the integral is over the *Haar probability measure* on $U(N)$.

- When $\beta = 1, 4$: $B_N^{(\beta=1)}(\vec{\mathbf{a}}, \vec{\mathbf{x}})$, $B_N^{(\beta=4)}(\vec{\mathbf{a}}, \vec{\mathbf{x}})$ are spherical integrals over orthogonal and symplectic compact groups.

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- They are limits of *Macdonald polynomials*.

Our general approach

The idea is to apply moment / operator method to the β-Fourier transform

$$G_N^{(\beta)}(\vec{\mathbf{x}}) := \int_{\mathbb{R}^N} B_N^{(\beta)}(\vec{\mathbf{a}}, \vec{\mathbf{x}})\, \mathrm{d}\mu_N(\vec{\mathbf{a}}),$$

by analogy with classical theory

$$e^{\mathbf{i}(a_1 x_1 + \cdots + a_N x_N)} \longrightarrow B_N^{(\beta)}(\vec{\mathbf{a}}, \vec{\mathbf{x}})$$

$$\frac{\partial^k}{\partial x_1^k} + \cdots + \frac{\partial^k}{\partial x_N^k} \longrightarrow P_k^{(\beta)}$$

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$$\frac{\partial^k}{\partial x_1^k} + \cdots + \frac{\partial^k}{\partial x_N^k} \longrightarrow P_k^{(\beta)}$$

$$\boxed{\prod_{i=1}^{s} P_{k_i}^{(\beta)} \left(G_N^{(\beta)} \right) \Bigg|_{x_1 = \cdots = x_N = 0} = \mathbb{E}_{\mu_N} \left[\prod_{i=1}^{s} (a_1^{k_i} + \cdots + a_N^{k_i}) \right]}$$

Our general approach

$$\boxed{\prod_{i=1}^{s} P_{k_i}^{(\beta)}\left(e^{\ln(G_N^{\beta})}\right)\Big|_{x_1=\cdots=x_N=0} = \mathbb{E}_{\mu_N}\left[\prod_{i=1}^{s}(a_1^{k_i}+\cdots+a_N^{k_i})\right]}$$

These equations link:
analytic info of $G_N^{(\beta)} \leftrightarrow$ **probabilistic info of** μ_N.
In the high temperature limit, they link:
limits of Taylor coeffs of $\ln(G_N^{(\beta)}) \leftrightarrow$ **limits of moments of** μ_N.

The first main theorem

$$\prod_{i=1}^{s} P_{k_i}^{(\beta)}\left(e^{\ln(G_N^{\beta})}\right)\Big|_{x_1=\cdots=x_N=0} = \mathbb{E}_{\mu_N}\left[\prod_{i=1}^{s}(a_1^{k_i}+\cdots+a_N^{k_i})\right]$$

Theorem (Benaych-Georges, Cuenca, Gorin '22)

LLN $\Longleftrightarrow$ *limits of Taylor coeffs of* $\ln(G_N^{(\beta)})$, *i.e. TFAE:*

(1) There exist $m_1, m_2, \cdots$ *such that*

$\lim_{N\to\infty} \mathbb{E}_{\mu_N}\left[N^{-s}\prod_{i=1}^{s}(a_1^{k_i}+\cdots+a_N^{k_i})\right] = \prod_{i=1}^{s} m_{k_i}$.

(2) There exist $\kappa_1, \kappa_2, \cdots$ *such that*

$$\lim_{N\to\infty,\,\beta\to 0^+} \frac{1}{\ell!}\cdot\frac{\partial^\ell}{\partial x_1^\ell}\ln\left(G_N^{(\beta)}\right)\Big|_{x_1=\cdots=x_N=0} = \kappa_\ell/\ell, \quad \forall\, \ell\in\mathbb{Z}_{\geq 1},$$

$$\lim_{N\to\infty,\,\beta\to 0^+} \frac{\partial^r}{\partial x_{i_1}\cdots\partial x_{i_r}}\ln\left(G_N^{(\beta)}\right)\Big|_{x_1=\cdots=x_N=0} = 0,\ \text{if } |\{i_1,\cdots,i_r\}|\geq 2.$$

The moments of the limiting measure in the LLN are $m_1, m_2, \cdots$.

Moments of the limiting measure

In example of eigenvalues of $G\beta E_N$, the β-Fourier Transform is:

$$G_N^{(\beta)}(x_1, \cdots, x_N) = \exp\left(\frac{x_1^2 + \cdots + x_N^2}{2}\right),$$

so $\kappa_2 = 1$; $\kappa_1 = \kappa_3 = \kappa_4 = \cdots = 0$.

What are the corresponding moments m_k's?

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The first few relations m_k's $\leftrightarrow$ κ_ℓ's are:

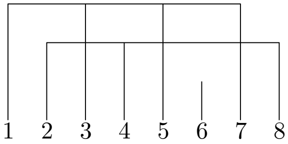
$$\begin{aligned} m_1 &= \kappa_1 \\ m_2 &= (\gamma+1)\kappa_2 + \kappa_1^2 \\ m_3 &= (\gamma+1)(\gamma+2)\kappa_3 + 3(\gamma+1)\kappa_2\kappa_1 + \kappa_1^3 \\ m_4 &= (\gamma+1)(\gamma+2)(\gamma+3)\kappa_4 + (\gamma+1)(2\gamma+3)\kappa_2^2 + \cdots \\ &\cdots \end{aligned}$$

The second main theorem of [BG – C – G] is an explicit formula.

The second main theorem: moments of limiting measure

For a *set partition* $\pi = \{B_1, \cdots, B_m\}$ of $\{1, 2, \cdots, k\}$, draw the arc diagram of π and define the weight

$$W_\gamma(\pi) := \prod_{i=1}^{m} \frac{p(i)!\,(\gamma + |B_i| - 1)!}{(\gamma + p(i))!}.$$



$$\{1, \cdots, 8\} = \{1, 3, 5, 7\} \sqcup \{2, 4, 8\} \sqcup \{6\}.$$

$p(i) := \#$ roofs of B_i with some intersection.

$|B_i| :=$ size of the block B_i.

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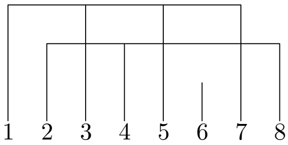

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$$p(1) = p(3) = 0,\ p(2) = 2 \Rightarrow W_\gamma(\pi) = 2(\gamma + 1)(\gamma + 2)(\gamma + 3).$$

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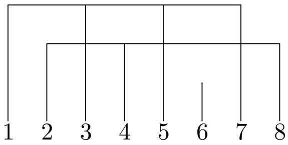

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Theorem (Benaych-Georges, Cuenca, Gorin '22)

$$m_k = \sum_{\textit{set partitions } \pi \textit{ of } \{1,\cdots,k\}} W_\gamma(\pi) \prod_{B\in\pi} \kappa_{|B|}, \quad \forall\, k \geq 1.$$

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γ-cumulants and γ-semifree probability

The relation m_k's $\leftrightarrow$ κ_ℓ's generalizes the relation between **moments** $\leftrightarrow$ **cumulants** of a probability measure (at $\gamma = 0$), and between **moments** $\leftrightarrow$ **free cumulants** (at $\gamma = \infty$).

We call κ_ℓ's the γ-**semifree cumulants**.

Problem

Study the γ-Semifree Probability.

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For example,

1. **Conjecture:** Given probability measures μ, ν of compact support, and γ-semifree cumulants $\{\kappa_\ell^\mu\}_{\ell \geq 1}$, $\{\kappa_\ell^\nu\}_{\ell \geq 1}$, there exists a unique probability measure $\mu \boxplus_\gamma \nu$ of compact support such that

$$\kappa_\ell^{\mu \boxplus_\gamma \nu} = \kappa_\ell^\mu + \kappa_\ell^\mu, \quad \ell \geq 1.$$

This would be the γ-Semifree Convolution of μ and ν.

γ-cumulants and γ-semifree probability

2. Assuming the conjecture, classify the *infinitely divisible laws* with respect to the operation of γ-semifree convolution.

3. **Theorem (BG – C – G '22).** (LLN and connection to RMT) If the empirical measures of $(a_1 \leq \cdots \leq a_N)$, $(b_1 \leq \cdots \leq b_N)$ converge weakly to μ, ν, then the eigenvalues $(c_1 \leq \cdots \leq c_N)$ of the β-sum $A_N +_\beta B_N$ converge in the high temperature regime to the γ-semifree convolution $\mu \boxplus_\gamma \nu$.

etc, etc.

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Further questions

- γ-Semifree Probability
- Discrete Ensembles

Discrete ensembles: The Jack-Plancherel measure

Begin with the Burnside identity for symmetric group S_N:

$$\sum_{\lambda \in \mathbb{Y}_N} \dim(\lambda)^2 = N!,$$

where λ ranges over partitions of size N:

$$\mathbb{Y}_N := \{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0) \text{ s.t. } |\lambda| = \sum_{i \geq 1} \lambda_i = N\},$$

$$\dim(\lambda) = \frac{N!}{\prod_{s \in \lambda}(a(s) + l(s) + 1)} = \text{dim. of an irred. } S_N\text{-module.}$$

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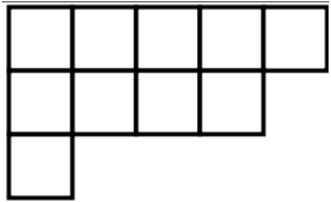
Definition

The Plancherel measure is the measure on $\mathbb{Y}_N$ is:

$$Planch^{(1)}(\lambda) := \frac{\dim(\lambda)^2}{N!} = \frac{N!}{\prod_{s \in \lambda}(a(s) + l(s) + 1)^2}.$$

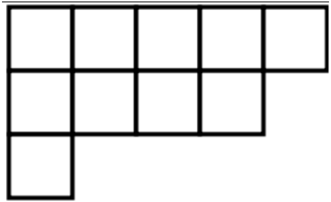
Limits of the Plancherel measure



Young diagram of partition $(5, 4, 1)$

Large $\frac{1}{\sqrt{N}}$-normalized partition

Limits of the Plancherel measure



Young diagram of partition $(5, 4, 1)$

Large $\frac{1}{\sqrt{N}}$-normalized partition

Theorem (Vershik-Kerov '77 & Logan-Shepp '77)

The $\frac{1}{\sqrt{N}}$*–normalized profiles of Planch*$^{(1)}$*–distributed partitions* $\lambda \in \mathbb{Y}_N$ *converge as* $N \to \infty$ *to*

$$\omega(u) := \begin{cases} \frac{2}{\pi}(u \arcsin(u/2) + \sqrt{4 - u^2}), & \text{if } |u| \leq 2, \\ |u|, & \text{if } |u| \geq 2. \end{cases}$$

The α-Jack-Plancherel measure

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$$Planch^{(\alpha)}(\lambda) := \frac{\alpha^N \cdot N!}{\prod_{s\in\lambda}(\alpha a(s) + l(s) + 1)(\alpha a(s) + l(s) + \alpha)}.$$

This is a "β-ensemble-type measure" with $\beta = 2/\alpha$.
Use α-anisotropic partitions with boxes $\sqrt{\alpha} \times (1/\sqrt{\alpha})$.

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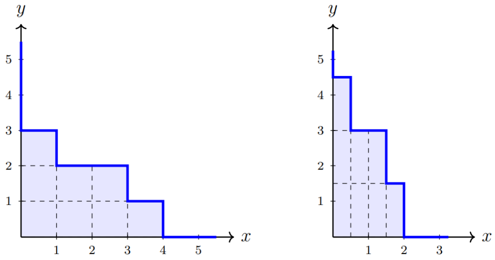

α-anisotropic partition $(4,3,1)$ with $\alpha = 1/4$

Image taken from [Dolega-Sniady, Gaussian fluctuations of Jack-deformed random Young diagrams, PTRF '19].

High temp limits of α-Jack-Plancherel measure

Theorem (Dolega-Sniady '19)

The $\frac{1}{\sqrt{N}}$-normalized profiles of α-anisotropic Planch$^{(\alpha)}$-distributed partitions $\lambda \in \mathbb{Y}_N$ converge in the high temperature regime

$$N \to \infty, \quad \alpha \to \infty, \quad N/\alpha \to g \in (0, \infty),$$

to certain limit shape $\omega_g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $\omega_g(u) = |u|$, whenever $|u| \gg 0$.

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But what is the limit shape ω_g?

Where are the beautiful formulas for the moments?

[Cuenca, Dolega, Moll '22+] to the rescue!

High temp limits of α-Jack-Plancherel measure

The process is a bit indirect: consider the Markov-Krein correspondence $\omega \to \nu_\omega$, which makes an association:

$$\text{shapes } \omega : \mathbb{R} \to \mathbb{R}_{\geq 0} \longrightarrow \text{probability measures } K[\omega] \text{ on } \mathbb{R}.$$

Our heroes are the Kerov's transition measures $K[\omega_g]$

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Theorem (Cuenca-Dolega-Moll '22+)

The limit shapes ω_g are uniquely determined by the moments of ν_{ω_g}, which have nice combinatorial formulas:

$$\int_{\mathbb{R}} x^m \, K[\omega_g](dx) = \sum_{\text{Motzkin paths } P \text{ of length } m} g^{-|H^{\to}(P)|/2} \cdot \prod_{j \geq 0} j^{|H^{\to}(P;j)|}.$$

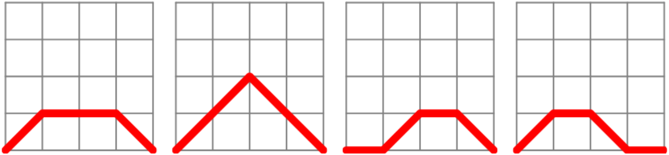
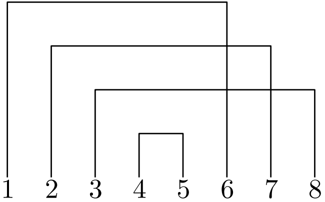


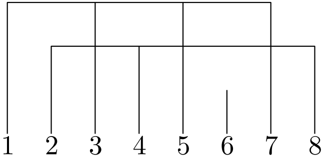
Image taken from Wolfram MathWorld - Motzkin number

Thank you for your attention)



$\mathrm{roof}(\pi) = 2$

$W_\gamma(\pi) = (\gamma+1)^{\mathrm{roof}(\pi)} = (\gamma+1)^2$



$p(1) = 0,\ p(2) = 2,\ p(3) = 0$

$W_\gamma(\pi) = \prod_{i=1}^{m} \frac{p(i)!\,(\gamma+|B_i|-1)!}{(\gamma+p(i))!} = 2(\gamma+1)(\gamma+2)(\gamma+3)$