Crosscap States in Integrable Theories

João Caetano

Based on 2111.09901 with Shota Komatsu

Florence
23 May 2022

Crosscap states in 2D

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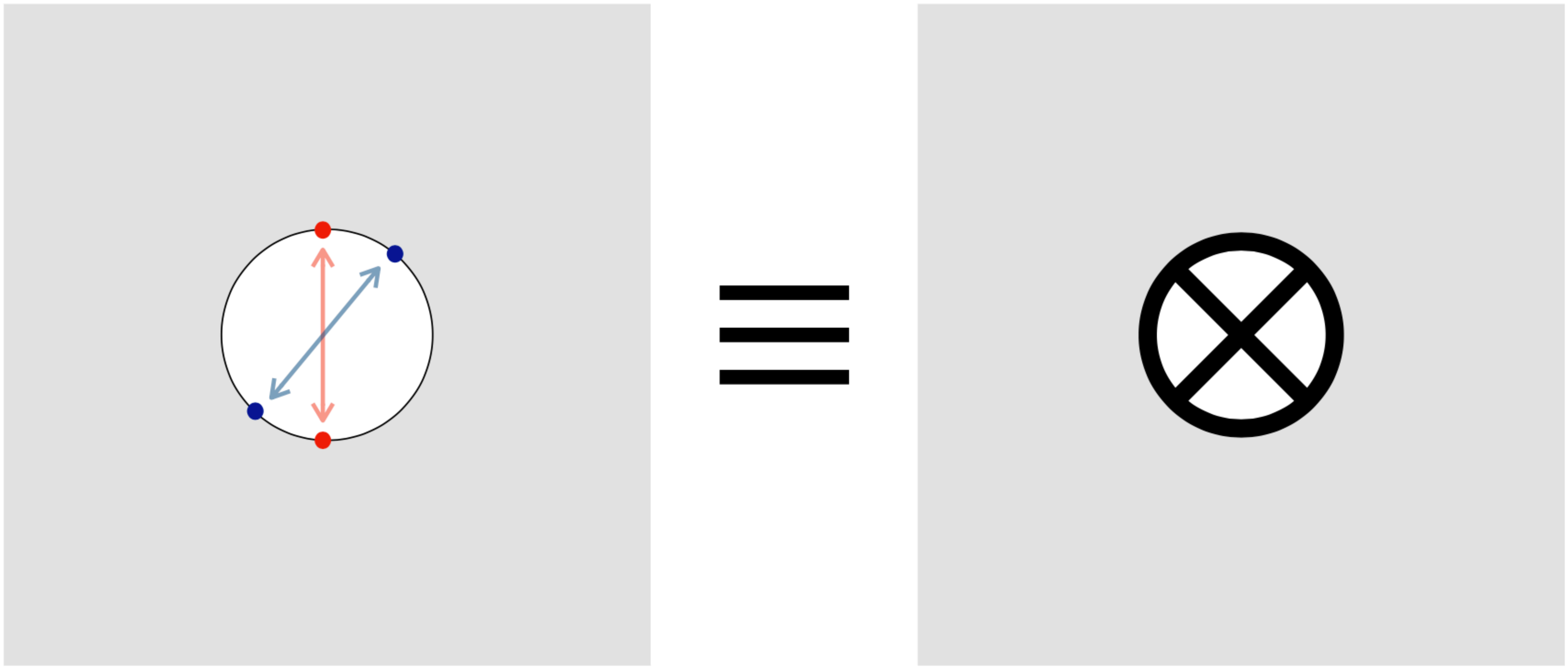
$$z \sim -1/\bar{z}$$



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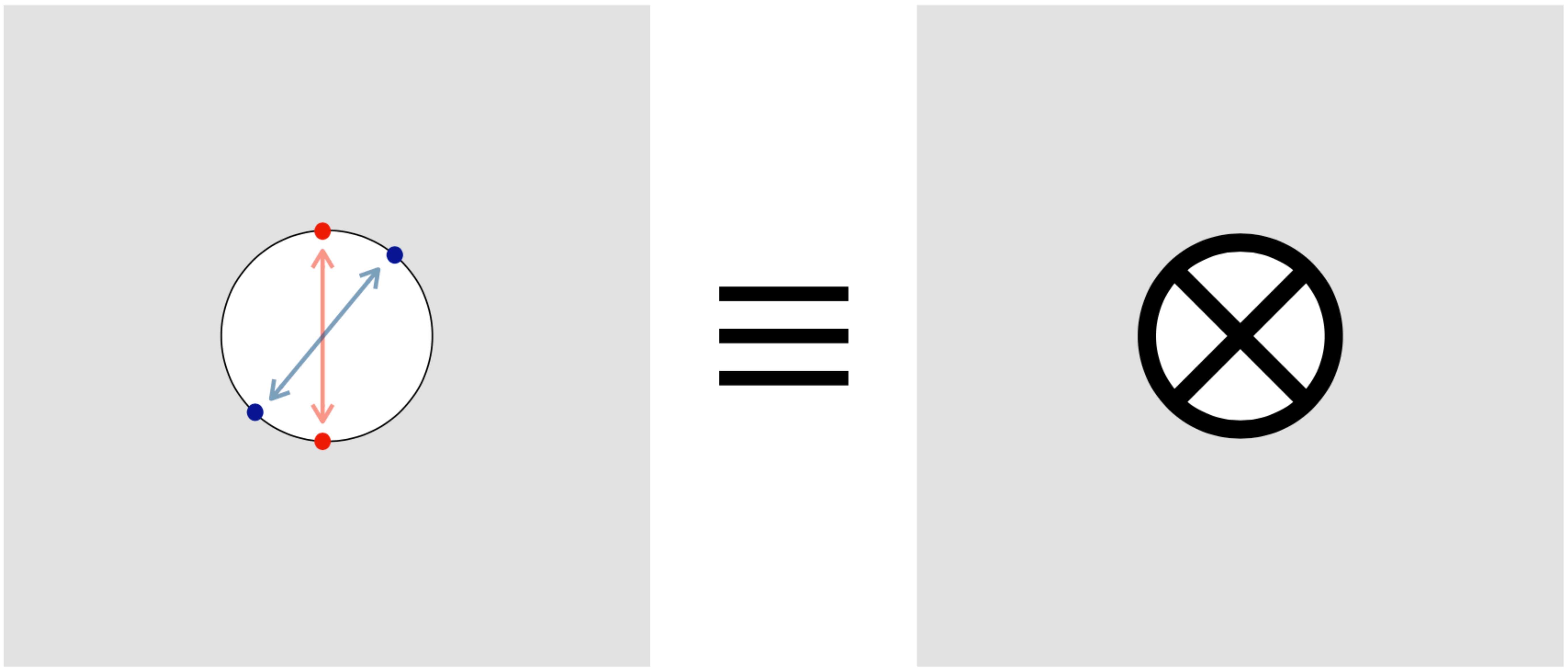

- Cut out a disk from a 2d surface + identify points at the boundary of the disk
- The state created by this procedure is the **crosscap state**

Crosscap states in 2D

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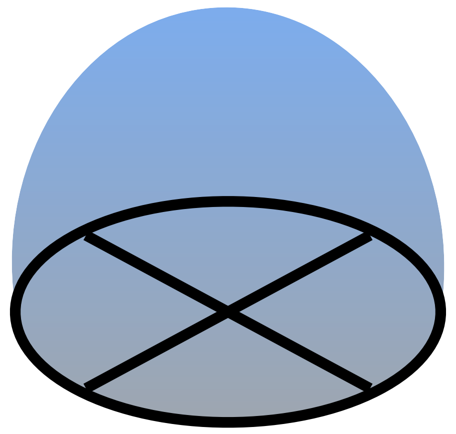
- Insert one crosscap state on S^2: $\mathbb{RP}^2$



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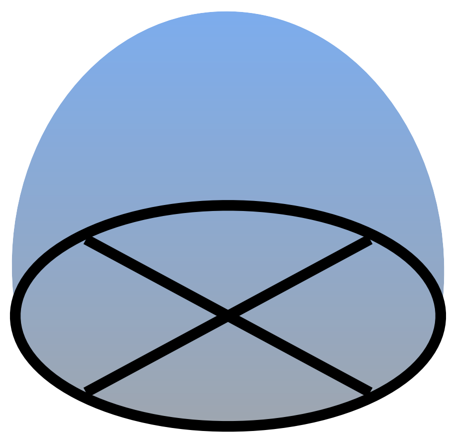


Non-orientable manifolds

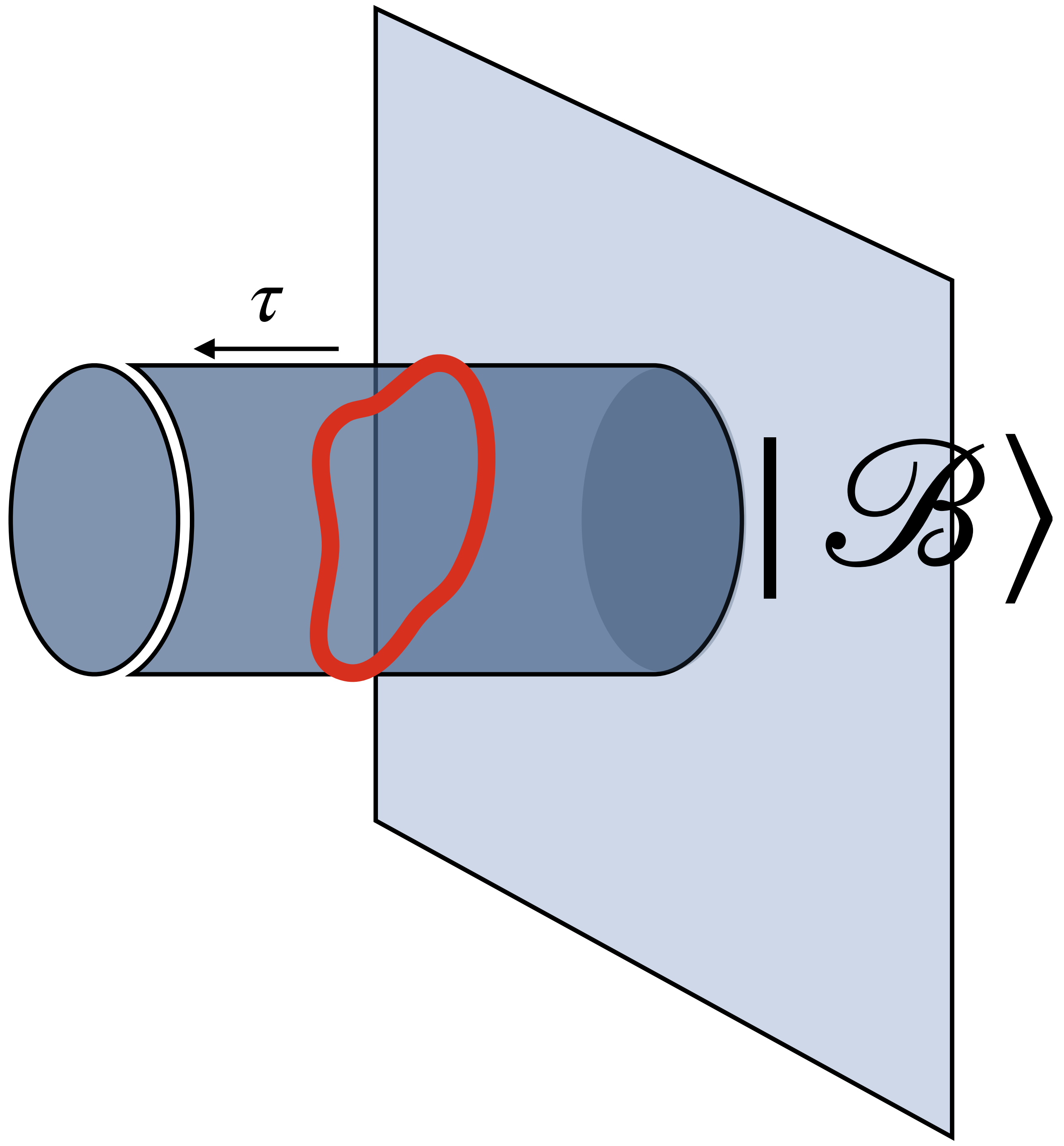
- Insert two crosscap states on S^2: Klein bottle

Boundaries & defects are great

- Wilson/'t Hooft loops in gauge theories: order parameter for **confinement**

- In 2D, boundaries and interfaces appear naturally as low energy description of **lattice systems** with impurities (e.g. Kondo effect)

- Strings and holography



D-brane = boundary state on the worldsheet

Boundary states

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In 1+1 D QFTs:

- Fixed points of RG and use 2D CFT techniques

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- Fixed points of RG and use 2D CFT techniques

Systematic construction of conformal boundary conditions:

THE BOUNDARY AND CROSSCAP STATES IN CONFORMAL FIELD THEORIES

NOBUYUKI ISHIBASHI

Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan

Received 20 June 1988

A method to obtain the boundary states and the crosscap states explicitly in various conformal field theories, is presented. This makes it possible to construct and analyse open string theories in several closed string backgrounds. We discuss the construction of such theories in the case of the backgrounds corresponding to the conformal field theories with SU(2) current algebra symmetry.

BOUNDARY CONDITIONS, FUSION RULES AND THE VERLINDE FORMULA

John L. CARDY

Department of Physics, University of California, Santa Barbara, CA 93106, USA

Received 27 February 1989

Boundary operators in conformal field theory are considered as arising from the juxtaposition of different types of boundary conditions. From this point of view, the operator content of the theory in an annulus may be related to the fusion rules. By considering the partition function in such a geometry, we give a simple derivation of the Verlinde formula.

Boundary states

In 1+1 D QFTs:

- Use integrable models (∞ conserved charges)

Boundary states

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- Use integrable models (∞ conserved charges)

For special boundaries, called **integrable boundaries**, one can follow their RG flow

BOUNDARY S MATRIX AND BOUNDARY STATE IN TWO-DIMENSIONAL INTEGRABLE QUANTUM FIELD THEORY

SUBIR GHOSHAL* and ALEXANDER ZAMOLODCHIKOV†‡

Department of Physics and Astronomy, Rutgers University, PO Box 849, Piscataway, NJ 08855-0849, USA

Received 29 November 1993

We study integrals of motion and factorizable S matrices in two-dimensional integrable field theory with boundary. We propose the "boundary cross-unitarity equation," which is the boundary analog of the crossing-symmetry condition of the "bulk" S matrix. We derive the boundary S matrices for the Ising field theory with boundary magnetic field and for the boundary sine–Gordon model.

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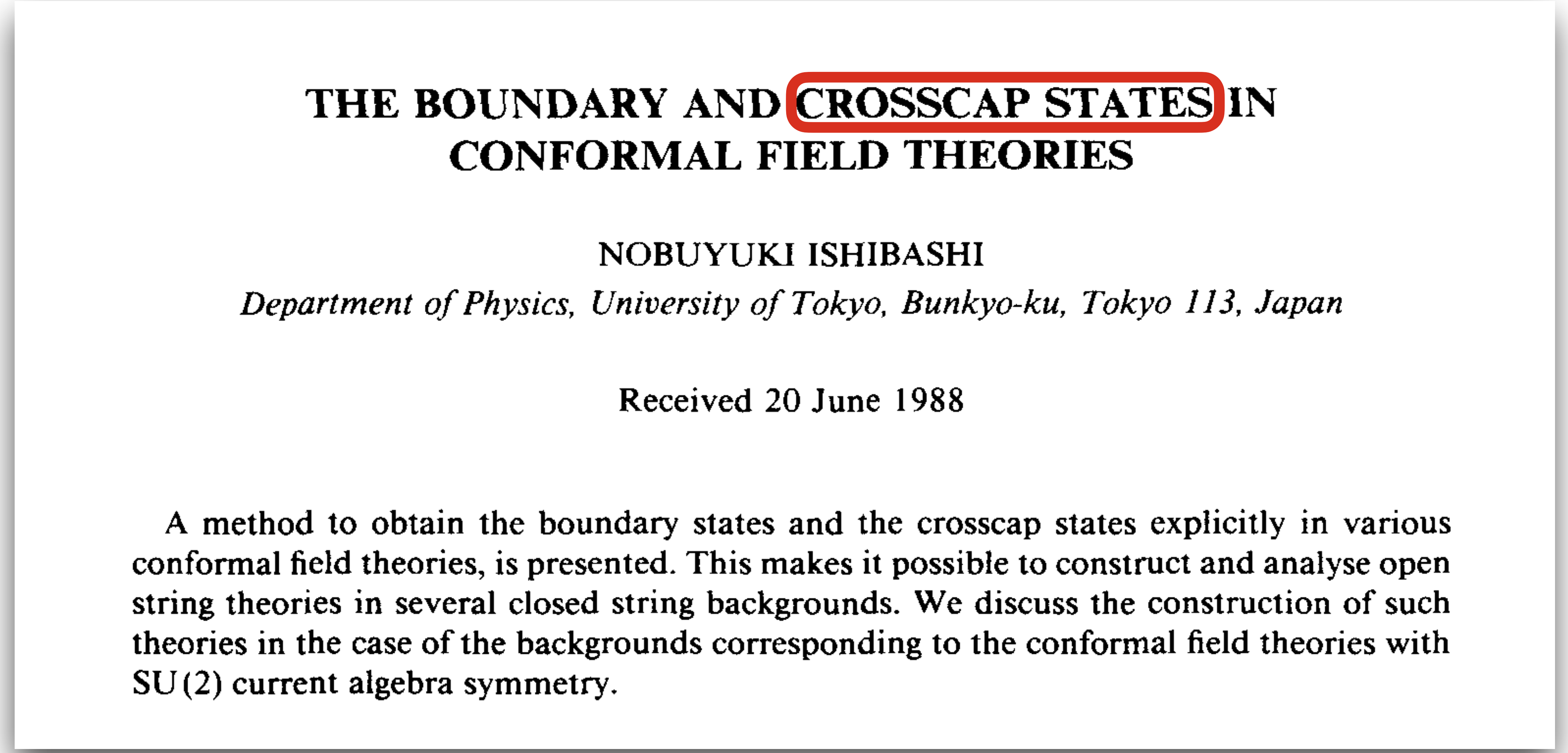
- Common in string compactifications, e.g. de Sitter vacua construction

Crosscap states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- **Never studied in integrable models!**

Outline

- Exact crosscap overlaps & p-function in Integrable Field Theories
- RG flow for the p-function
- Crosscap States in Spin Chain
- Outlook

Crosscap overlaps $\langle \mathscr{C} \mid \Psi \rangle$

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- Klein bottle partition function in two channels

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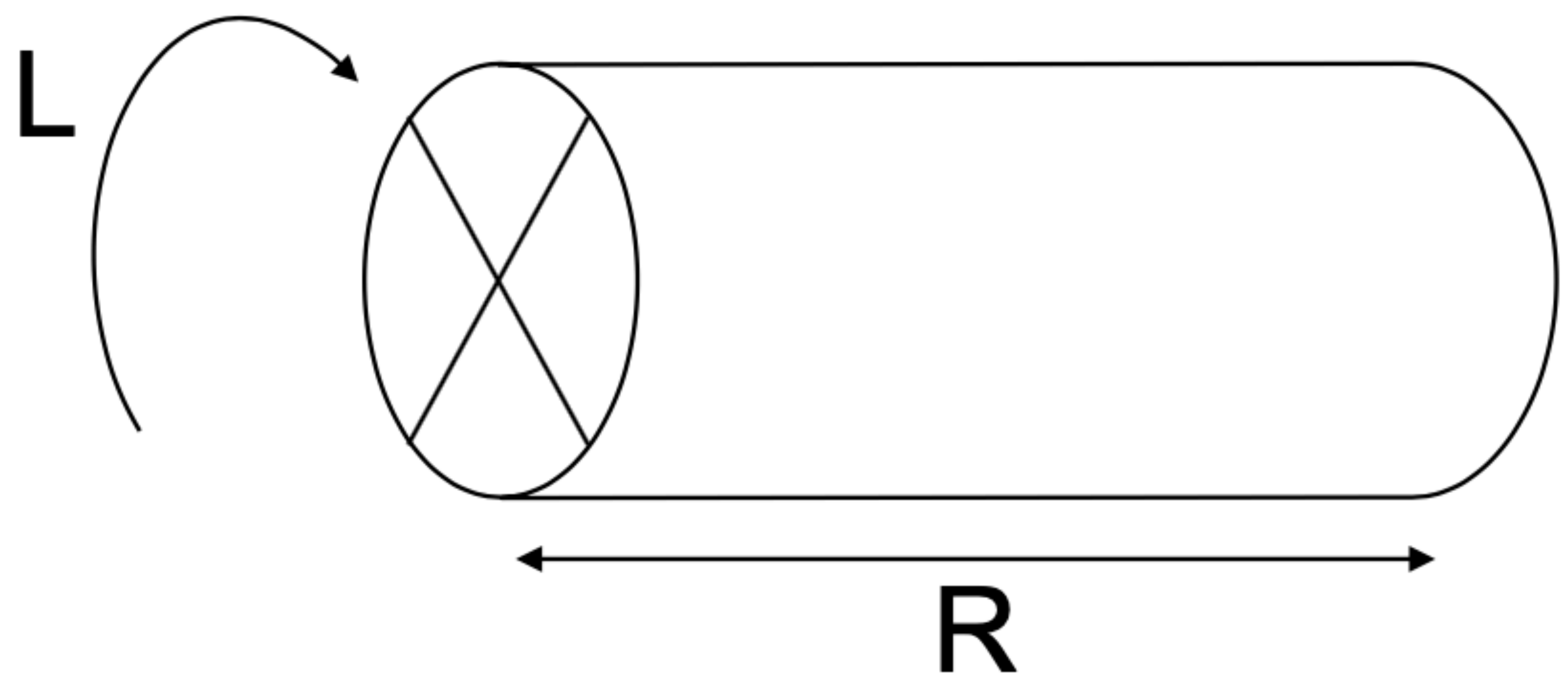
- Klein bottle partition function in two channels

Tree channel (closed string)



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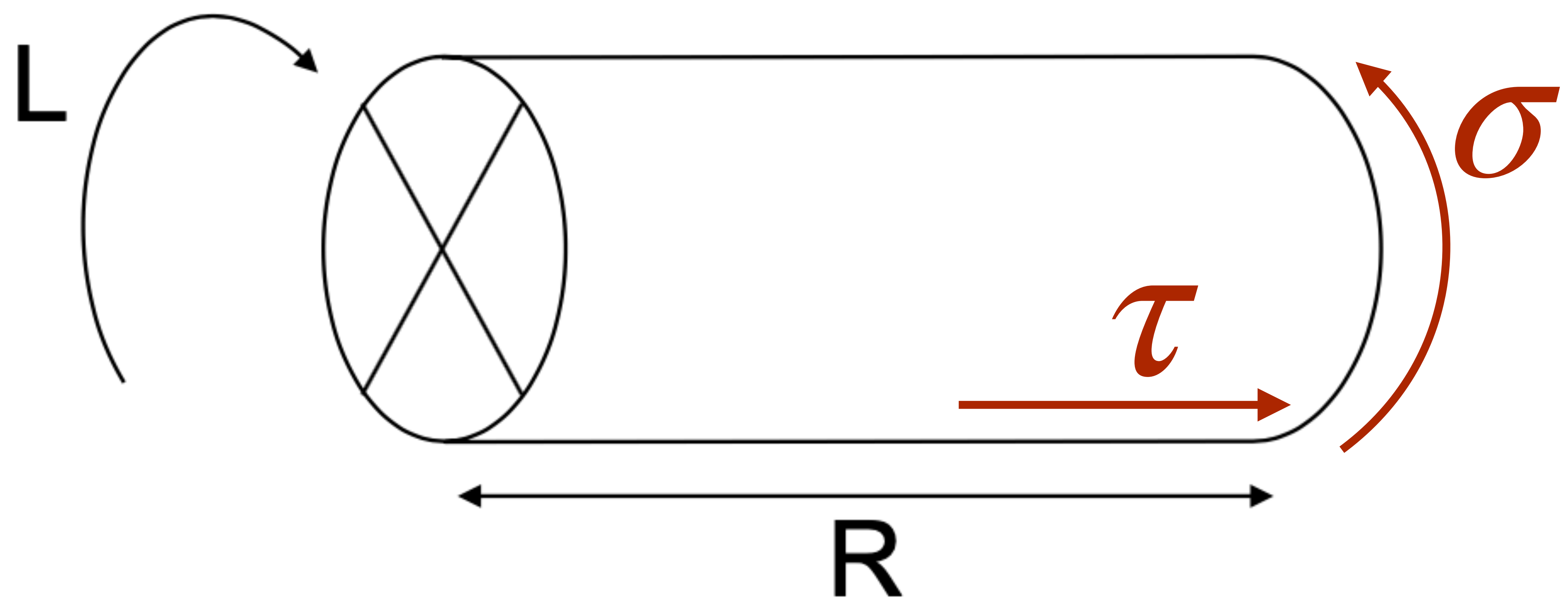

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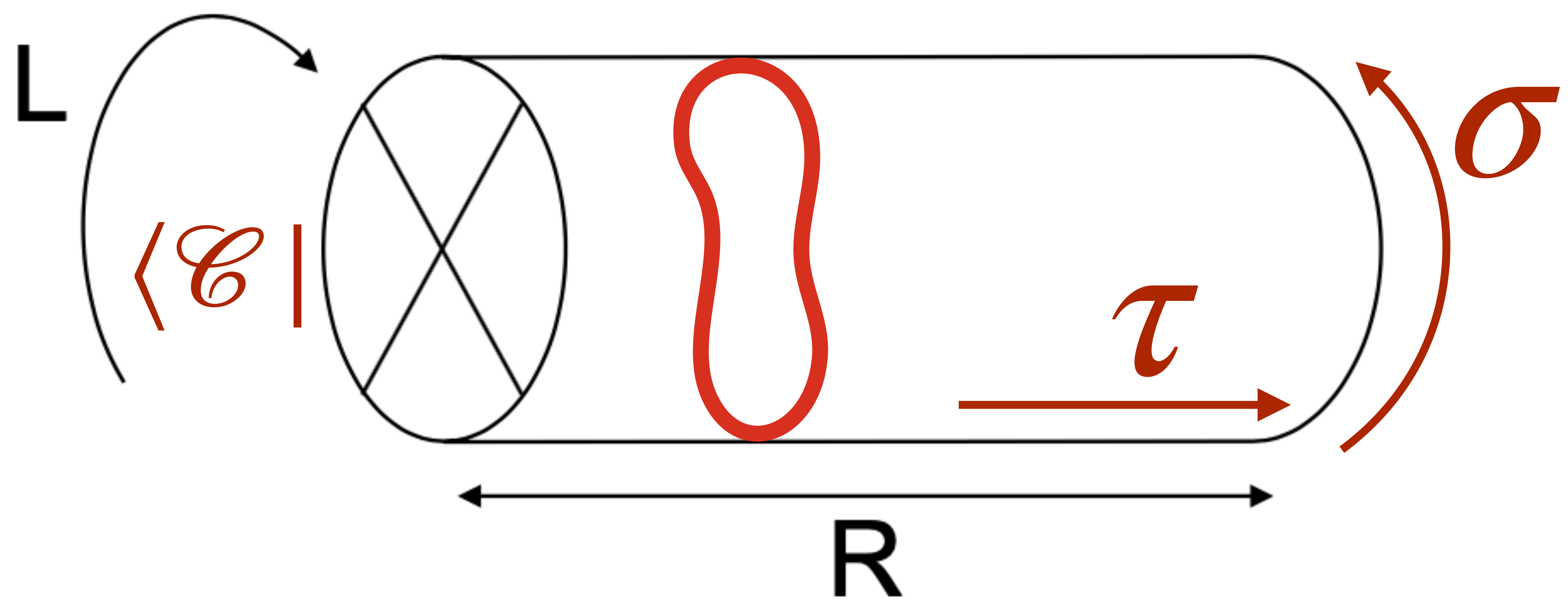

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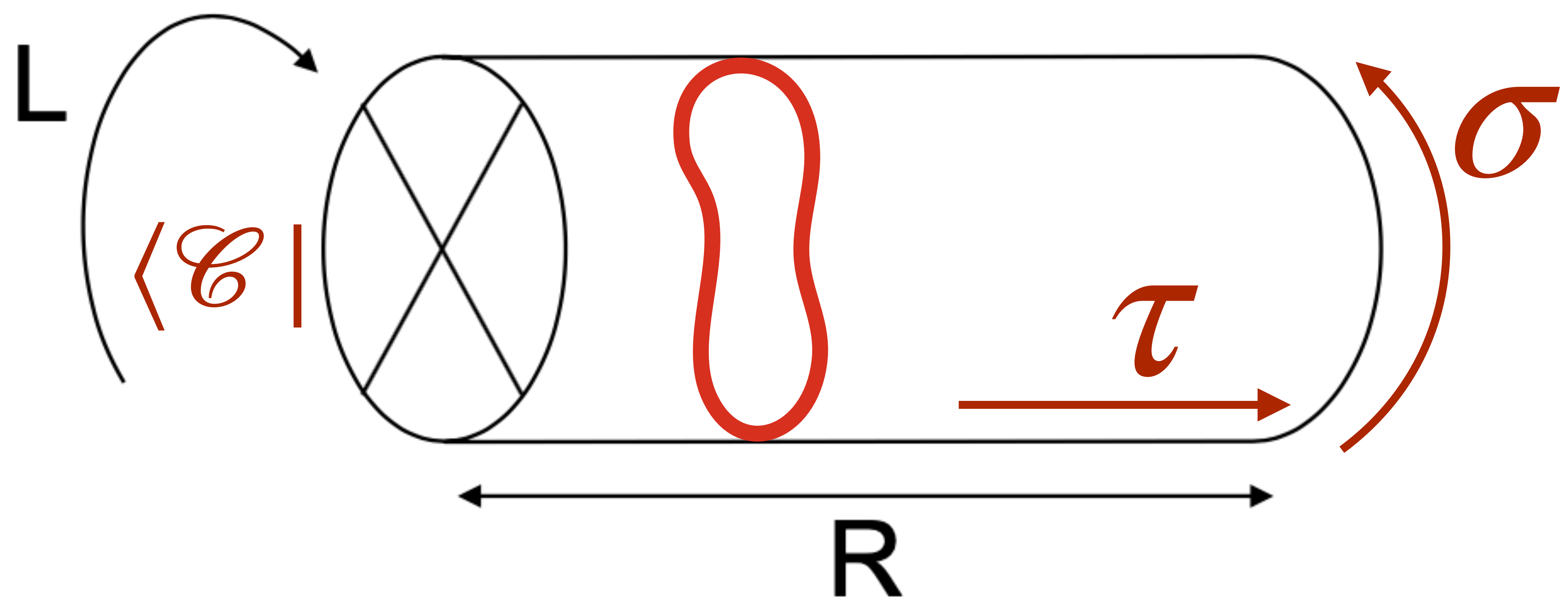

$$Z_{\mathbb{K}}(R, L) = \sum_{\psi_L} e^{-E_{\psi_L} R} \left| \langle \mathscr{C} | \psi_L \rangle \right|^2 \overset{R \to \infty}{=} e^{-E_{\Omega_L} R} \left| \langle \mathscr{C} | \Omega_L \rangle \right|^2 + \cdots$$

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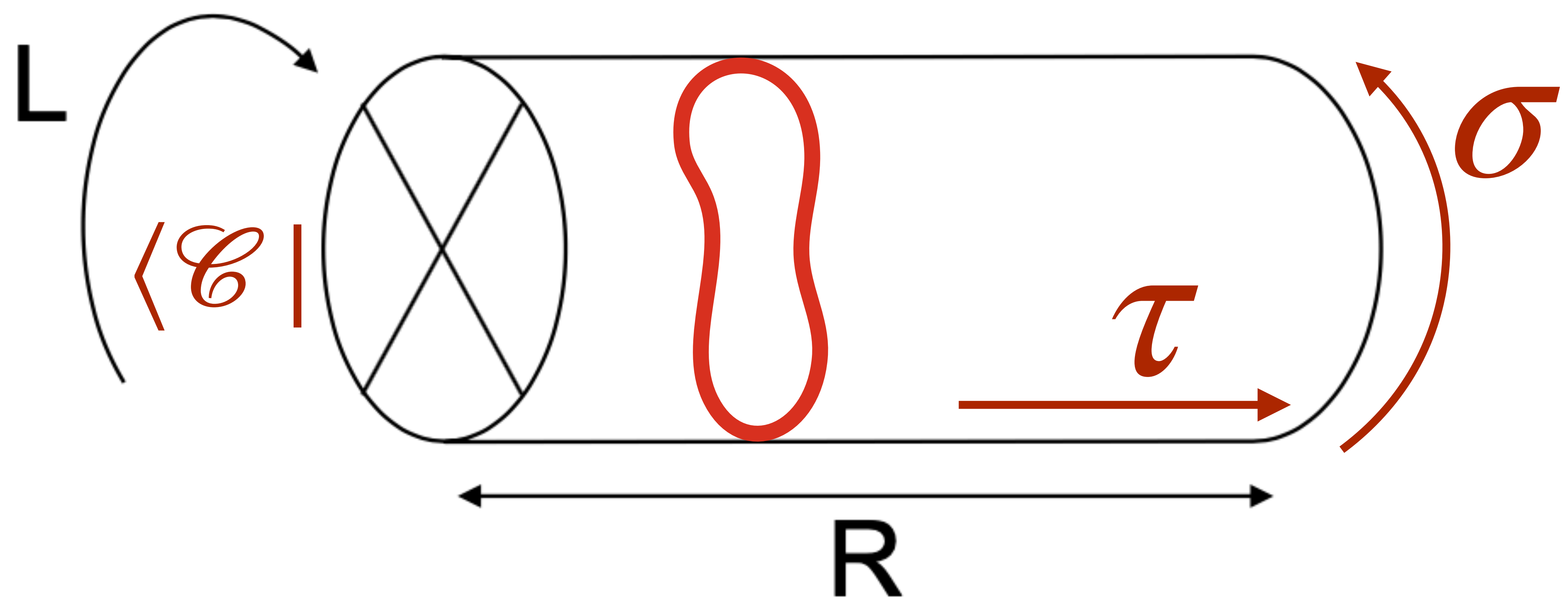

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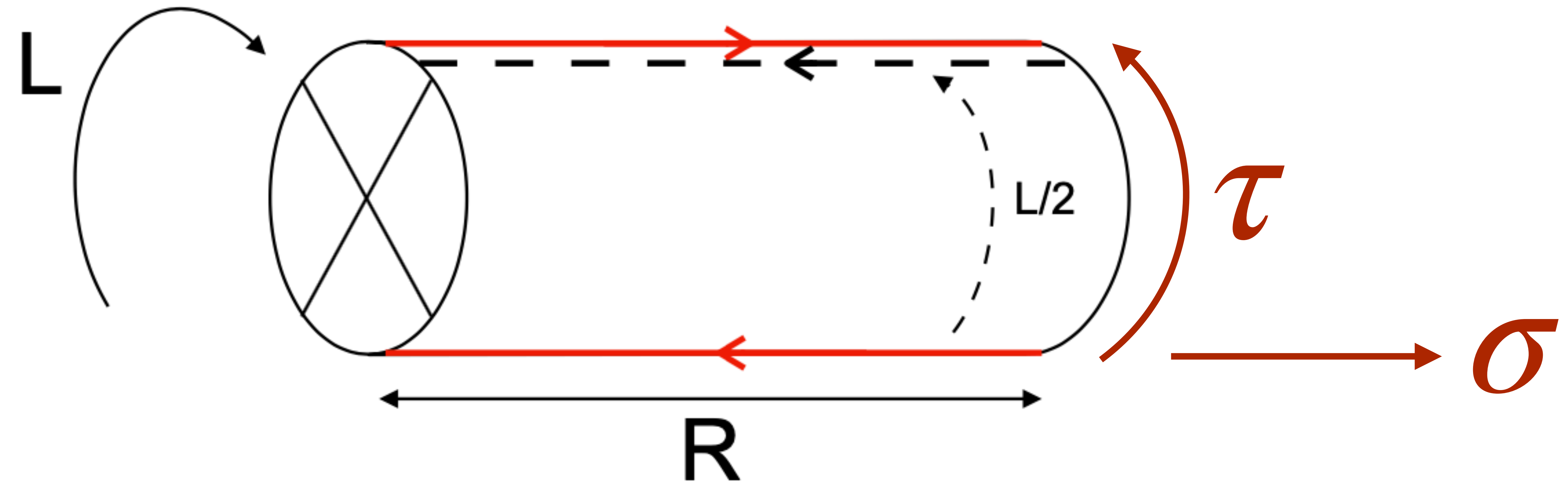
Ground state

Loop channel (open string)



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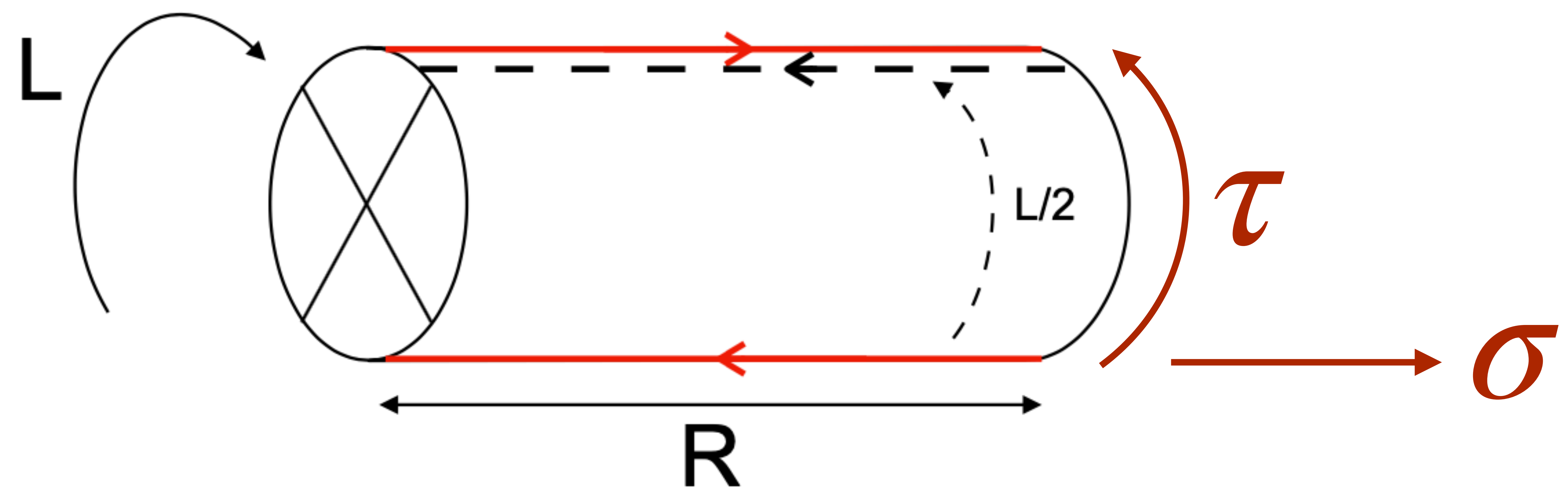

$$Z_{\mathbb{K}}(R,L) = \mathrm{Tr}_{2R}\left[\Pi\, e^{-HL/2}\right] = \sum_{\psi_{2R}} e^{-E_{\psi_{2R}} L/2}\, \langle \psi_{2R} | \Pi | \psi_{2R} \rangle$$

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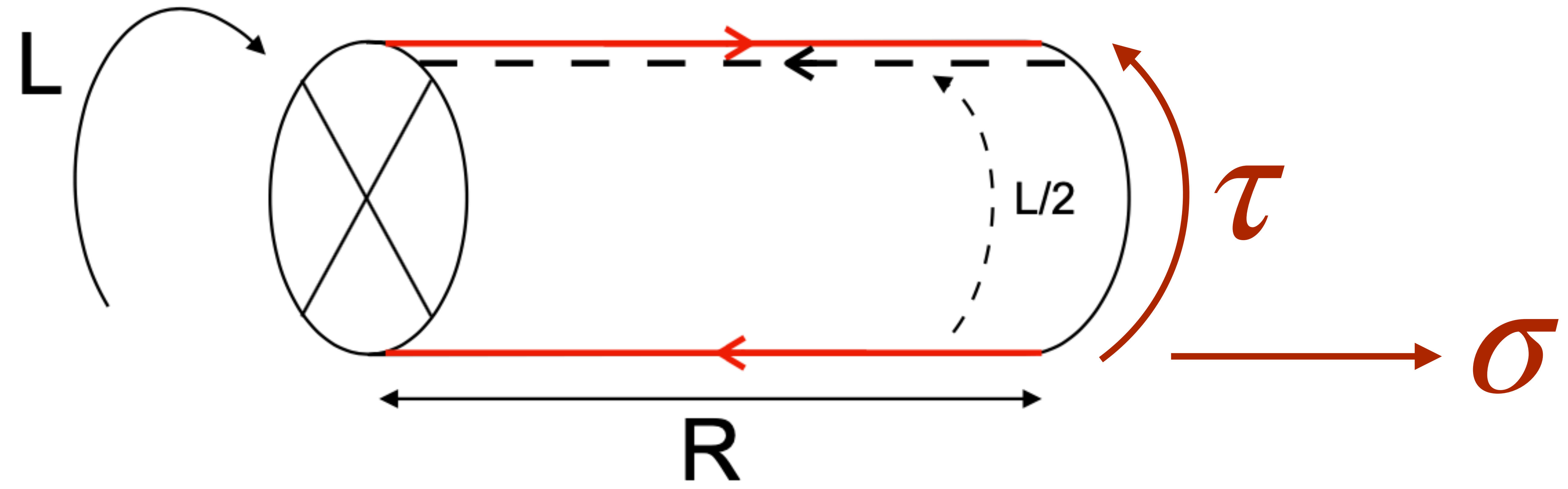

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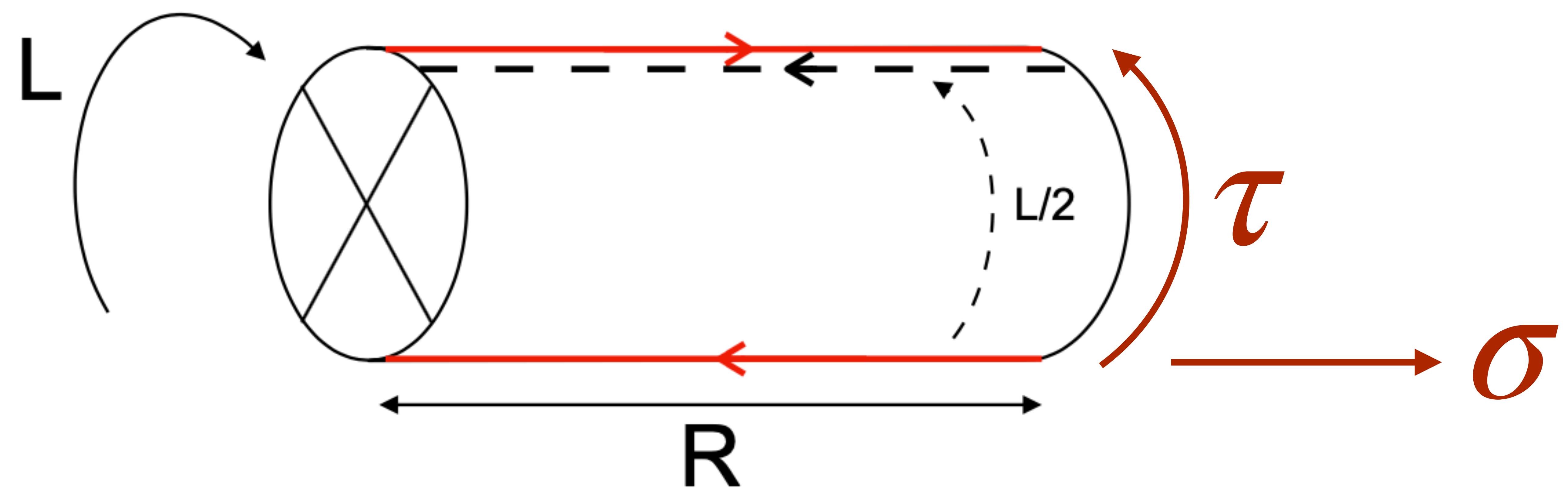
Parity operator

Loop channel (open string)



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Parity operator

$$= \sum_{\psi_{2R}} \epsilon_{\psi_{2R}} e^{-E_{\psi_{2R}}L/2}$$

Parity eigenstates

Parity eigenvalues ±1

Loop channel (open string) = Tree channel (closed string)

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$$\lim_{R\to\infty} Z_{\mathbb{K}}(R,L) = \lim_{R\to\infty}\left[\sum_{\psi_{2R}} \epsilon_{\psi_{2R}} e^{-E_{\psi_{2R}}L/2}\right] \simeq e^{-E_{\Omega_L}R}\left|\langle \mathscr{C} \,|\, \Omega_L\rangle\right|^2$$

$\langle \mathscr{C} \,|\, \Omega_L\rangle$ controls the density of states weighted by the parity ϵ_ψ

Loop channel (open string) = Tree channel (closed string)

$$F_{\mathbb{K}} \equiv -\lim_{R\to\infty} \log Z_{\mathbb{K}}(R, L)$$

***Parity-weighted* free energy**

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$$= RE_{\Omega_L} - \log\left[\,|\langle \mathscr{C} \,|\, \Omega_L \rangle|^2\right] + O(1/R)$$

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extensive piece

$\mathcal{O}(1)$ **piece**

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extensive piece — RE_{Ω_L}

$\mathcal{O}(1)$ **piece** — $\log \left[|\langle \mathscr{C} | \Omega_L \rangle|^2 \right]$

- Same structure as the thermal free energy of a system with boundaries
- In that case, $\mathcal{O}(1)$ piece defines the **boundary entropy** or **g-function**

p-function

- Similarly, we define **crosscap entropy** or **p-function:**

$$s_{\mathscr{C}} = \log |p| \qquad\qquad p \equiv \langle \mathscr{C} | \Omega_L \rangle$$

- We will study this quantity in integrable models

p-function in Integrable models

$$\lim_{R\to\infty} \mathrm{Tr}_{2R}\left[\Pi\, e^{-\hat{H}L/2}\right] \simeq e^{-E_\Omega R}\left|\langle \mathscr{C} \,|\, \Omega_L\rangle\right|^2$$

Large volume partition function

$\leftrightarrow$ (in integrable models)

Thermodynamic Bethe Ansatz + $\mathcal{O}(1)$ fluctuation

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$$1 = e^{2ip_jR}\prod_{k\neq j} S(p_j, p_k)$$

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- $\Pi\,|\{p_j\}\rangle \propto |\{-p_j\}\rangle$
- For Bethe states with standard normalization: $\Pi\,|\{p_j\}\rangle = \mathbf{1}\,|\{-p_j\}\rangle$

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- States whose momenta are **not invariant** under the sign flip **do not** contribute in the parity-weighted trace:

$$\langle \{p_j\}\,|\,\Pi\,|\,\{p_j\}\rangle = \langle \{p_j\}\,|\,\{-p_j\}\rangle = 0 \qquad \text{if}\;\; \{p_j\} \neq \{-p_j\}$$

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- So only states with the set of momenta

$$\{p_1, \ldots, p_M, -p_M, \ldots, -p_1\} \qquad \text{or} \qquad \{p_1, \ldots, p_M, 0, -p_M, \ldots, -p_1\}$$

p-function in Integrable models

$$\mathrm{Tr}_{2R}\left[\Pi\, e^{-\hat{H}L/2}\right] = \sum_{\{p_j\}=\{-p_j\}} e^{-\frac{L}{2}\sum_j E(p_j)} \simeq e^{-E_\Omega R}\left|\langle \mathscr{C} \,|\, \Omega_L\rangle\right|^2$$

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Standard thermal sum

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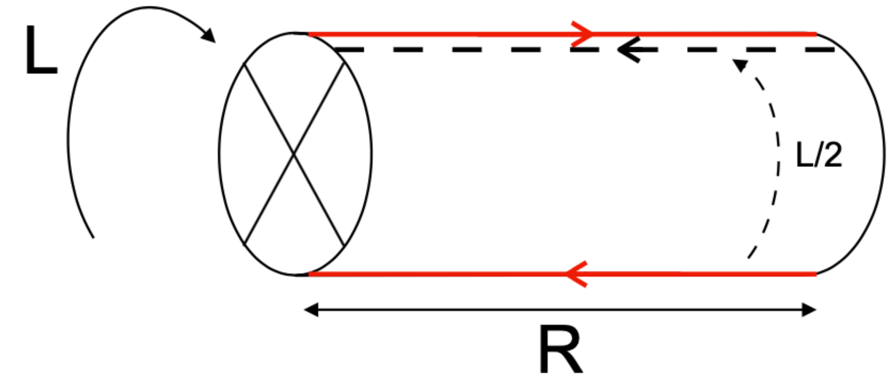
Apply **standard TBA** techniques to compute the **saddle point** and its **fluctuations**

Particles on a circle of size $2R$



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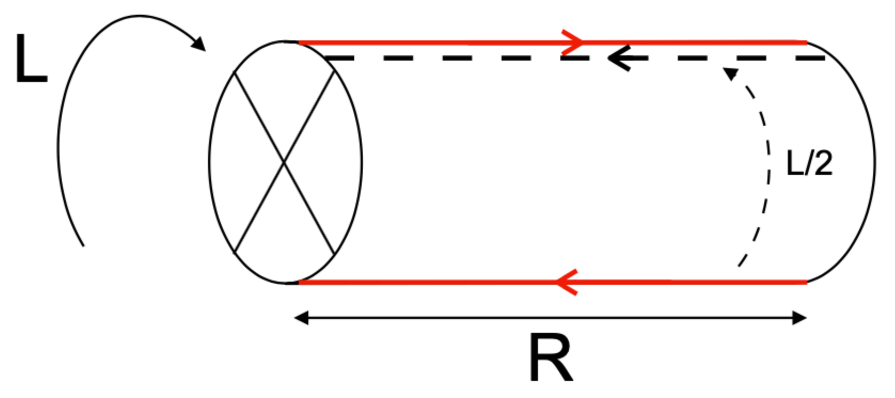

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$\{p_1, \ldots, p_M, -p_M, \ldots, -p_1\}$ or $\{p_1, \ldots, p_M, 0, -p_M, \ldots, -p_1\}$

$$\mathbf{S}: \quad 1 = e^{2ip_jR} S(p_j, -p_j) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k)\,,$$

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Zero momentum particle

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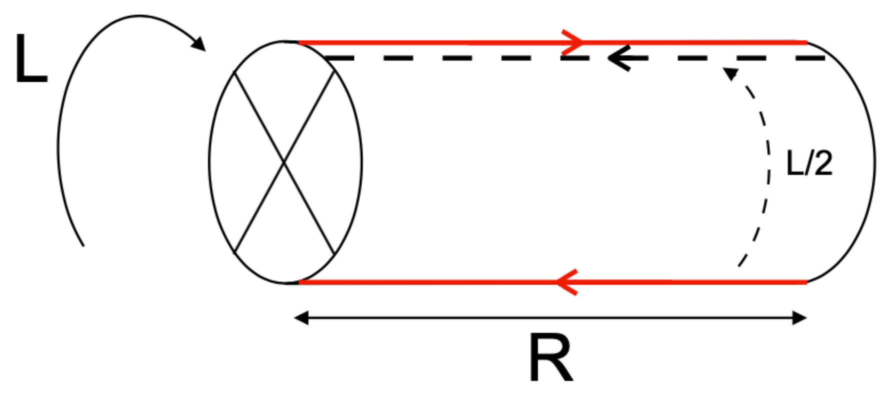

$$\sum_{\{p_j\}=\{-p_j\}} e^{-\frac{L}{2}\sum_j E(p_j)}$$

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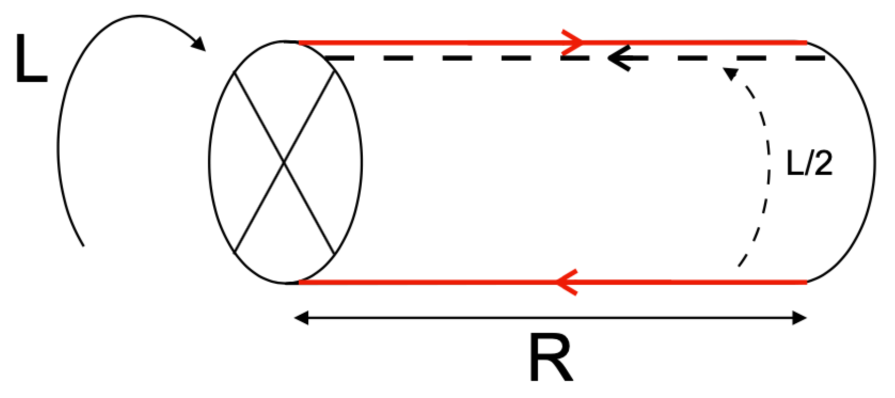

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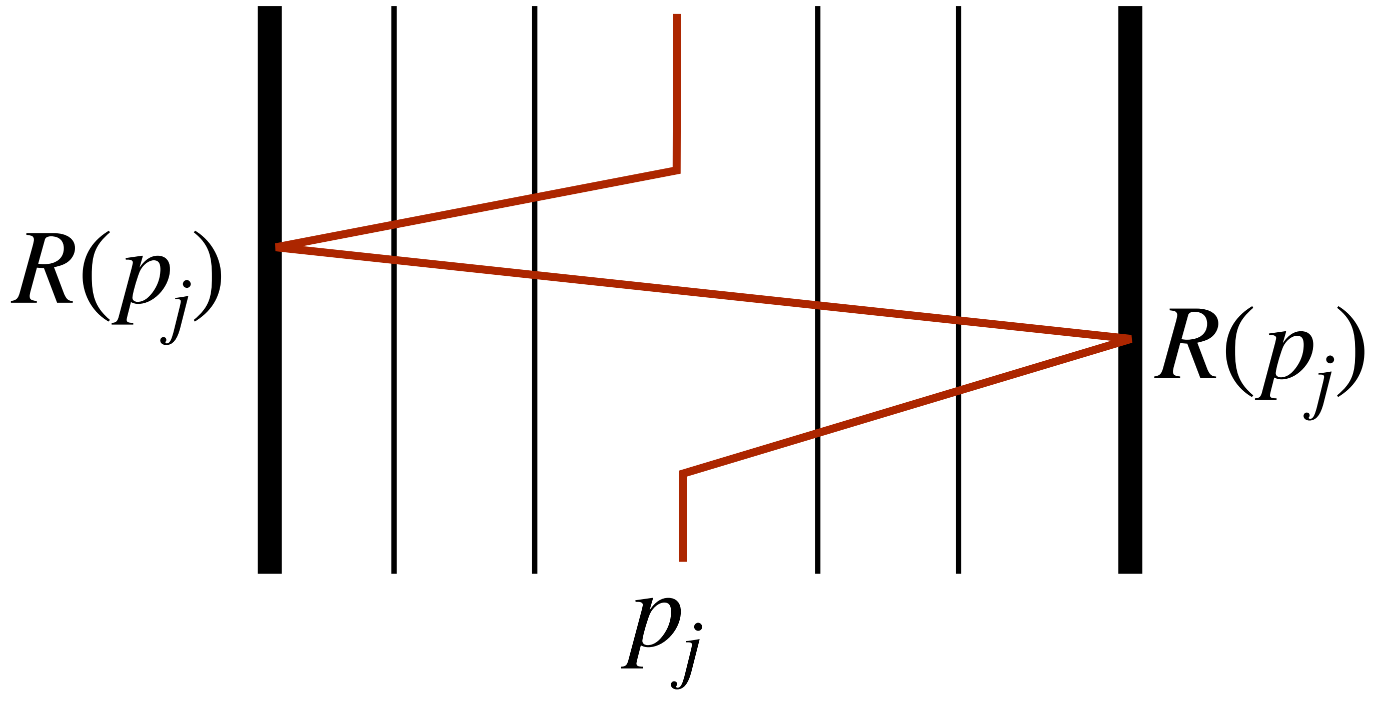
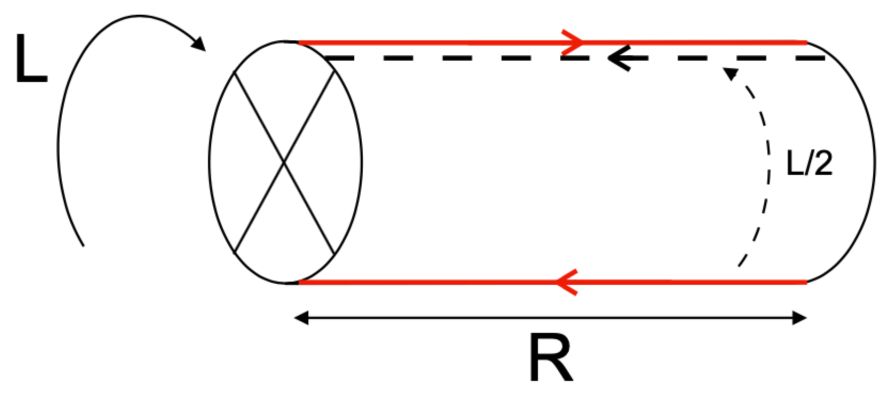
Zero momentum particle

Formally similar to a system with 2 identical boundaries:



Particles on a circle of size $2R$



$$\sum_{\{p_j\}=\{-p_j\}} e^{-\frac{L}{2}\sum_j E(p_j)}$$

$$= \sum_{\mathbf{S}} e^{-L\sum_{p_j>0} E(p_j)} + e^{-\frac{mL}{2}} \sum_{\mathbf{T}} e^{-L\sum_{p_j>0} E(p_j)}$$

$\{p_1, \ldots, p_M, -p_M, \ldots, -p_1\}$ or $\{p_1, \ldots, p_M, 0, -p_M, \ldots, -p_1\}$

$$\mathbf{S}: \quad 1 = e^{2ip_jR} S(p_j, -p_j) \prod_{k\neq j} S(p_j, p_k) S(p_j, -p_k)\,,$$

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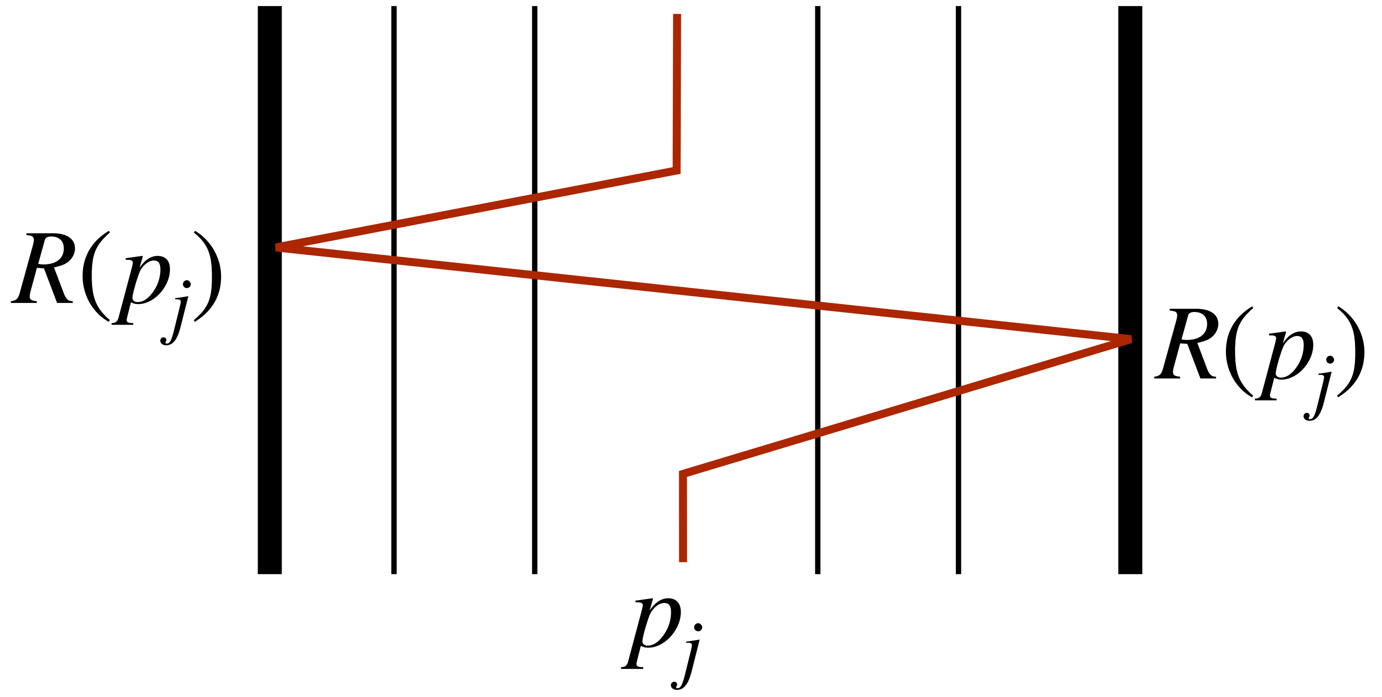
Zero momentum particle

Formally similar to a system with 2 identical boundaries:



$$1 = e^{2ip_jR} \left(R(p_j)\right)^2 \prod_{k\neq j} S(p_j, p_k) S(p_j, -p_k)$$

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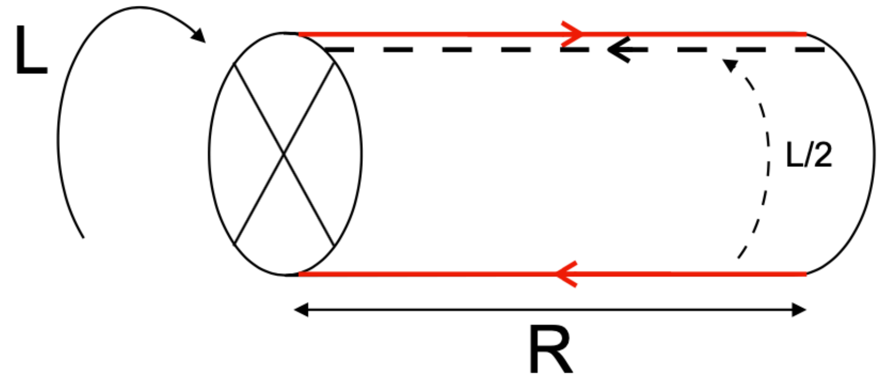

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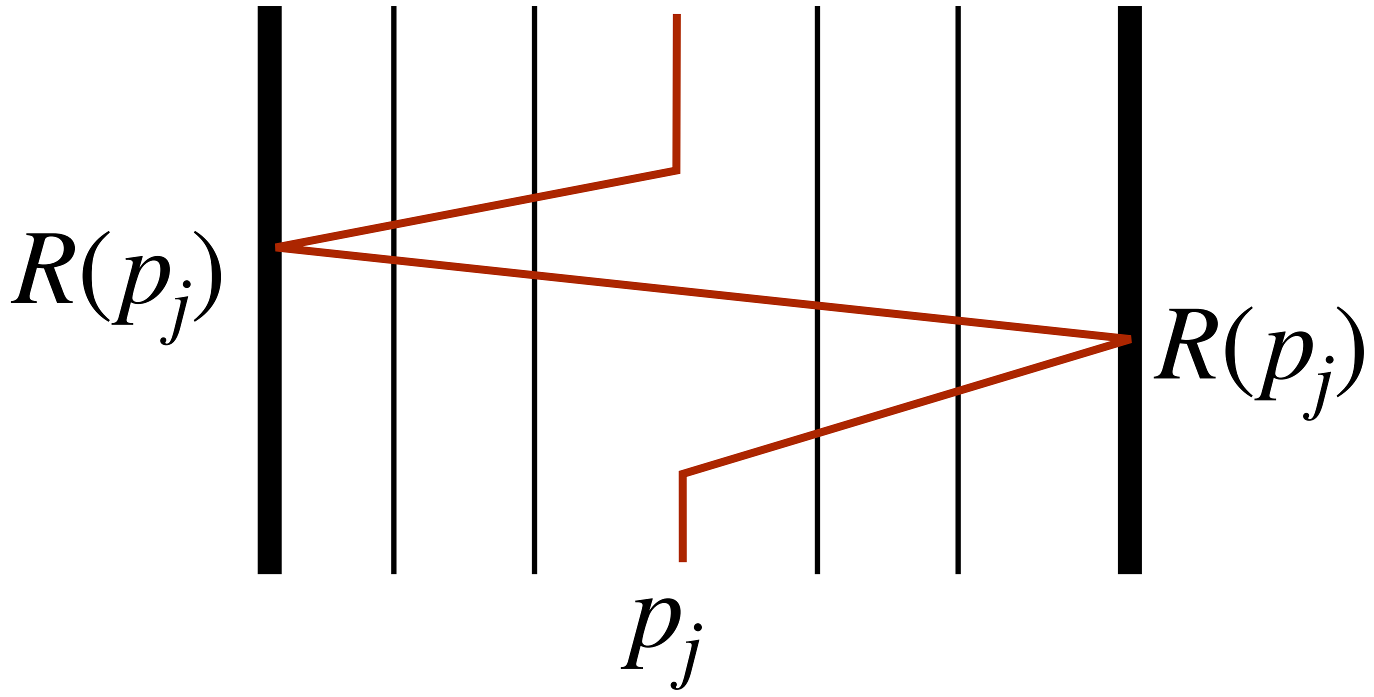

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p-function in Integrable models

Result: "**Simplest**" g-function

$$|p| = \left|\langle \mathscr{C} | \Omega_L \rangle\right| = \sqrt{\left(1 + \sqrt{\frac{Y(0)}{1+Y(0)}}\right) \frac{\det\left[1 - \hat{G}_-\right]}{\det\left[1 - \hat{G}_+\right]}}$$

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$$\mathscr{K}_{\pm}(u,v) = \frac{1}{i}\partial_u \left[\log S(u,v) \pm \log S(u,-v)\right]$$

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- Asymptotic limit

$$|\langle \mathscr{C} | \Psi_L \rangle| \overset{L \to \infty}{=} \sqrt{\frac{\det G_{+}}{\det G_{-}}}$$

$$\left(G_{\pm}\right)_{1 \le i,j \le \frac{M}{2}} = \left[L\partial_u p(u_i) + \sum_{k=1}^{\frac{M}{2}} \mathscr{K}_{+}(u_i, u_k)\right] \delta_{ij} - \mathscr{K}_{\pm}(u_i, u_j)$$

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$$\gamma = \frac{\pi b^2}{8\pi + b^2}$$

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Real parameter

Al. Zamolodchikov said:

- Resulting S-matrix still physical (Real analytic, unitary, crossing symmetric)
- Lagrangian description not so clear

Staircase model

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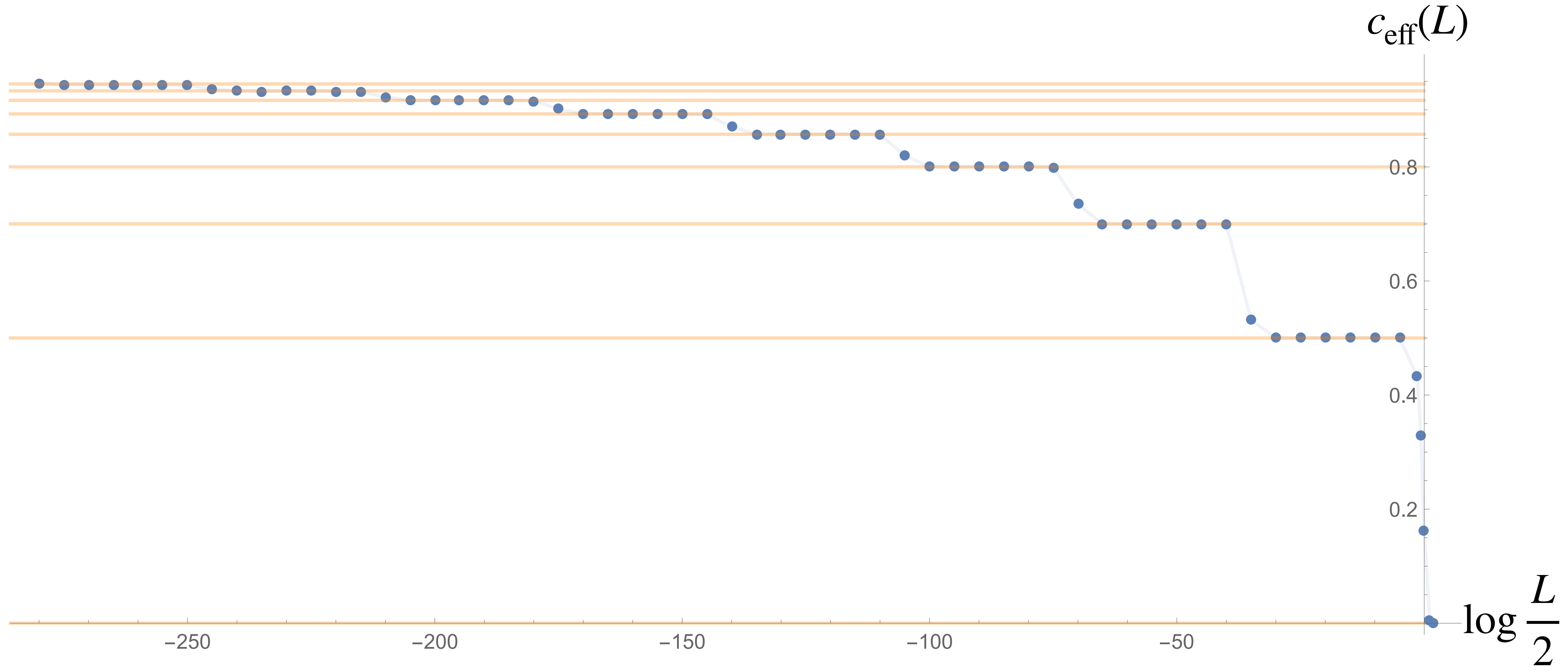

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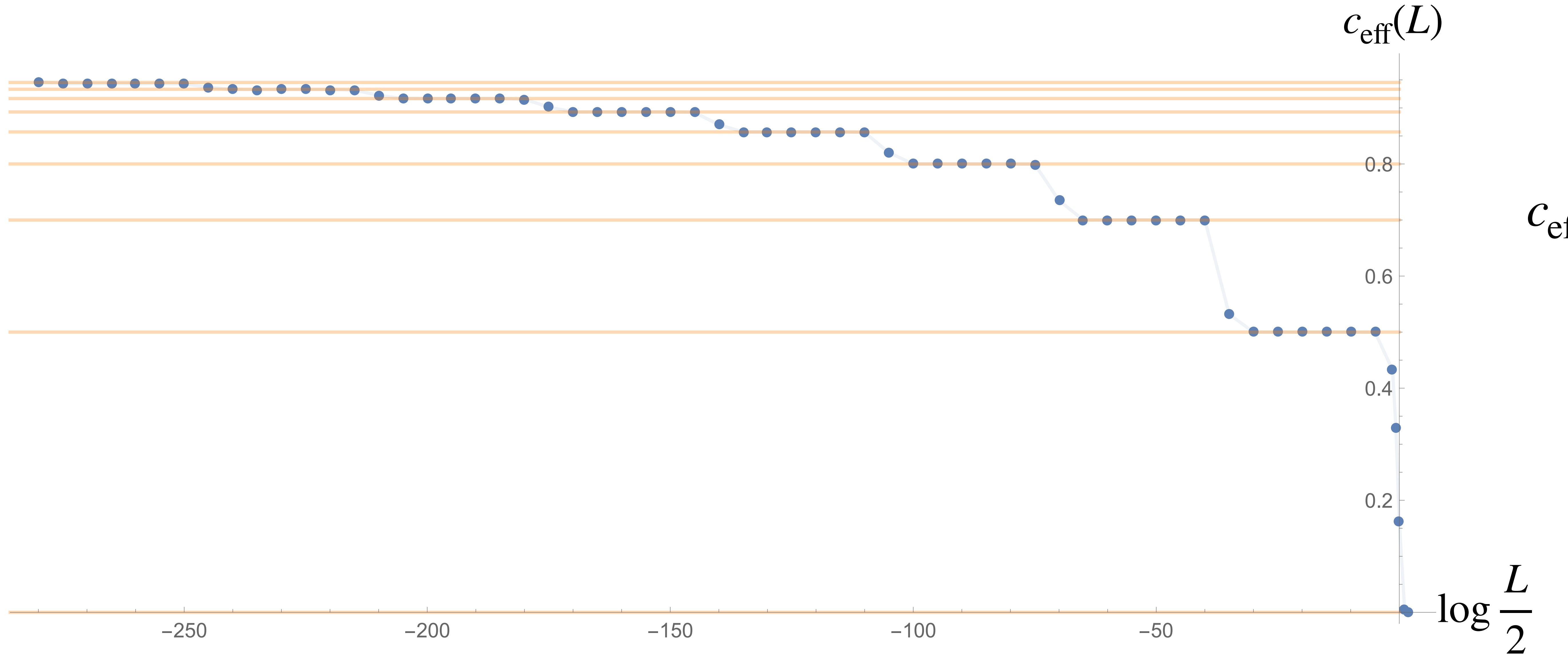

$$c_{\text{eff}}(L) \sim c_m = 1 - \frac{6}{m(m+1)} \qquad \left(\log L \sim -(m-3)\theta_0/2\right)$$

Central charge of A-series minimal models $\mathscr{M}_m^{(A)}$

$$\mathscr{M}_m^{(A)} \to \mathscr{M}_{m-1}^{(A)}$$ induced by the least relevant operator ϕ_{13}

p-function for staircase model

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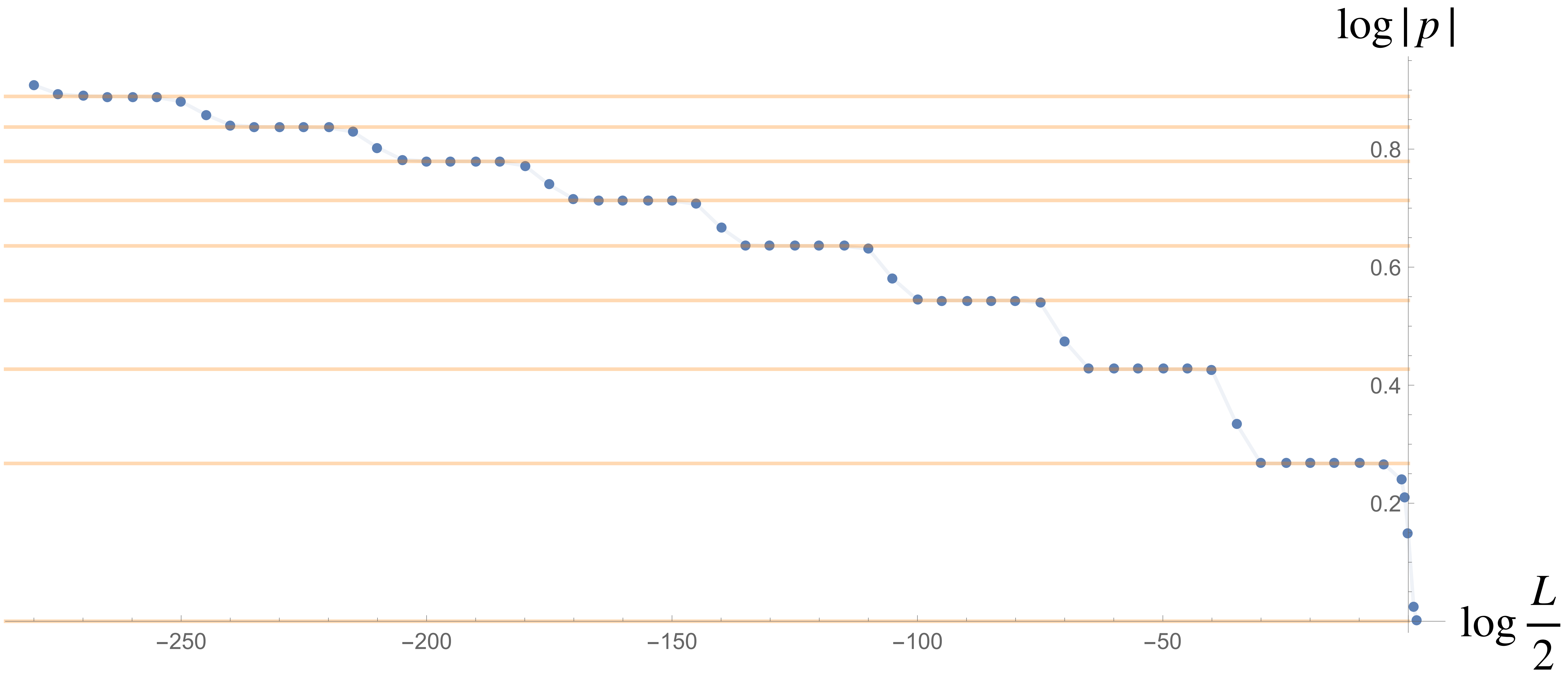

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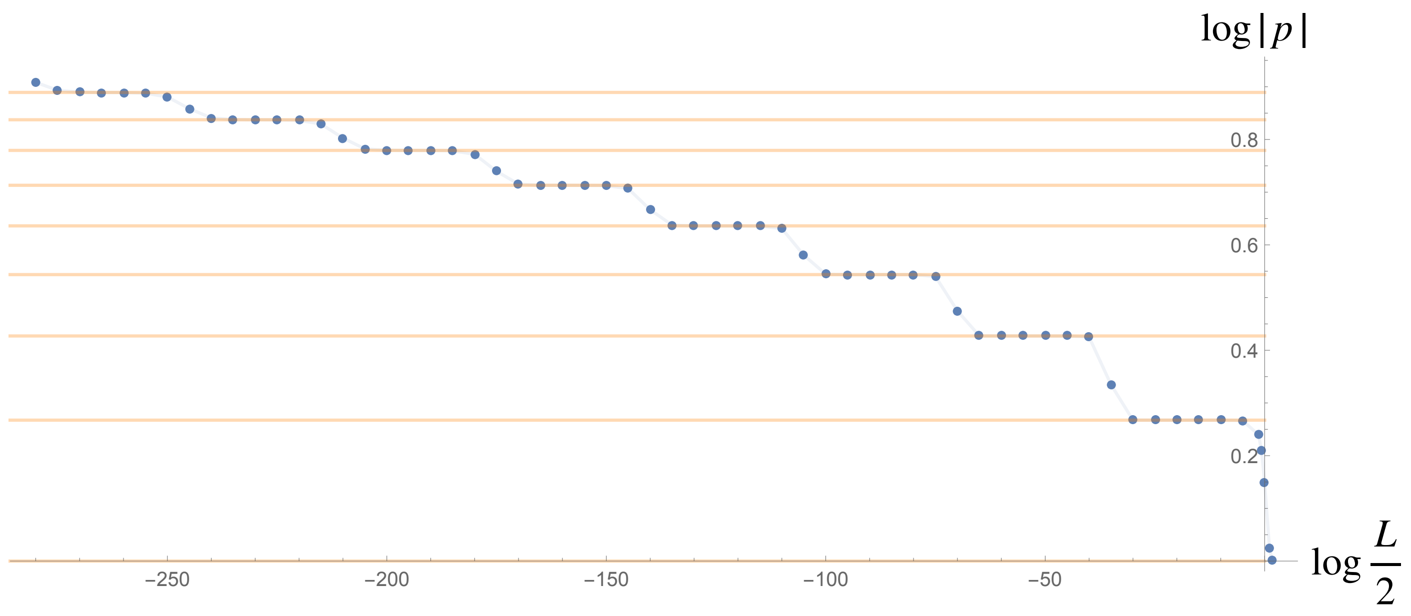

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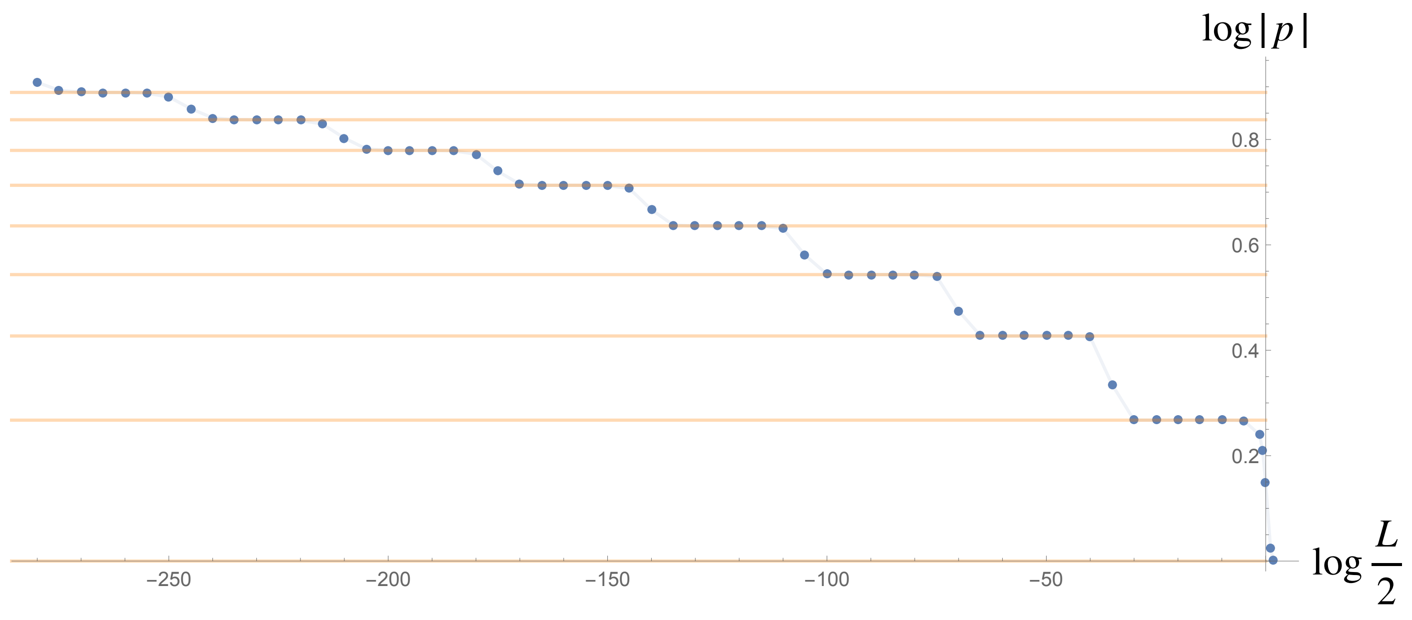

Orange lines determined from the CFT

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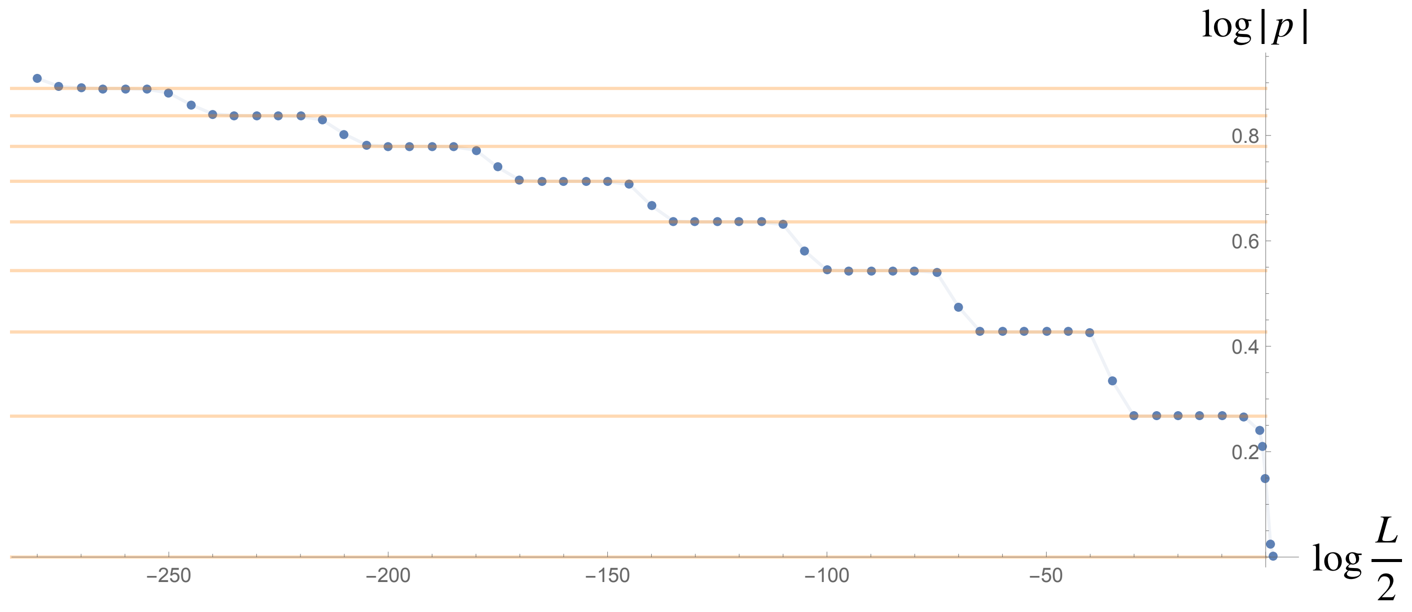

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$$|\langle \mathscr{C} | \Omega \rangle| = \left(\sum_a n_{a,a} S_{a,I} \right)^{\frac{1}{2}}$$

irreducible representation of the Virasoro algebra

degeneracy of states in the representation a for the chiral and anti-chiral part

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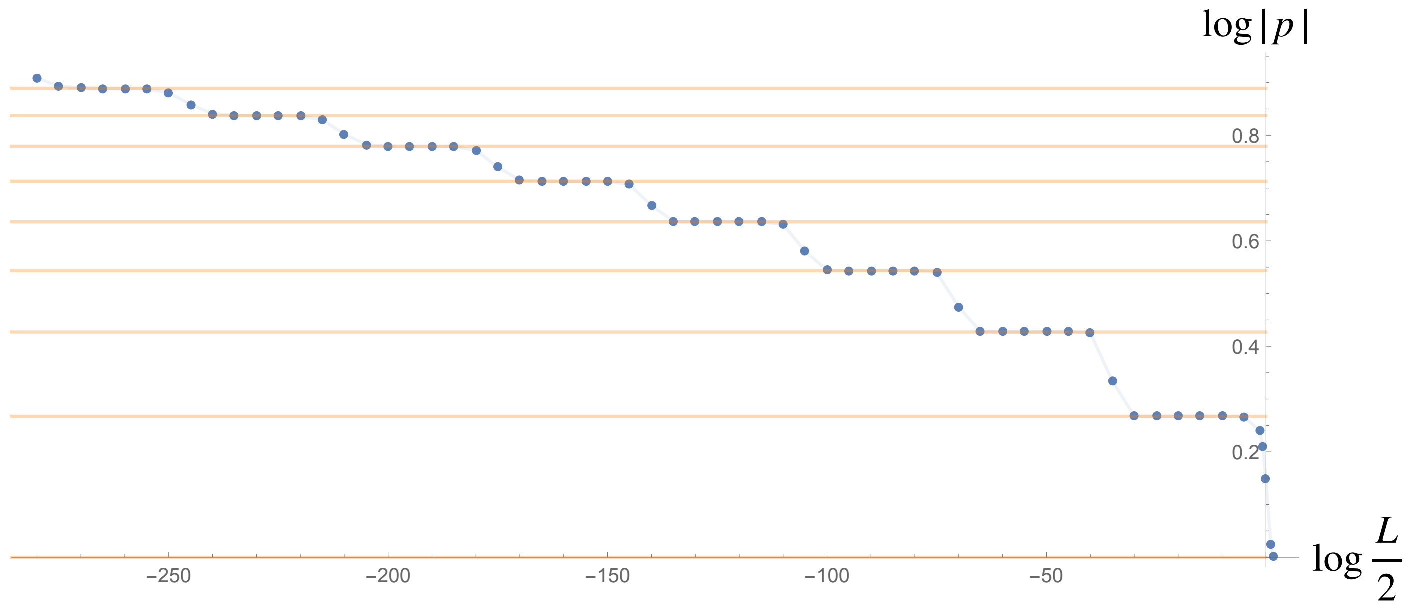

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- $n_{a,a}$: degeneracy of states in the representation a for the chiral and anti-chiral part
- $S_{a,I}$: modular S-matrix a and identity representation

Specifiying for $\mathscr{M}_m^{(A)}$:

$$|p| = |\langle \mathscr{C} | \Omega \rangle| = \left(\frac{2}{m(m+1)} \right)^{\frac{1}{4}} \sqrt{\cot \frac{\pi}{2m} \cot \frac{\pi}{2(m+1)}}$$

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A-series

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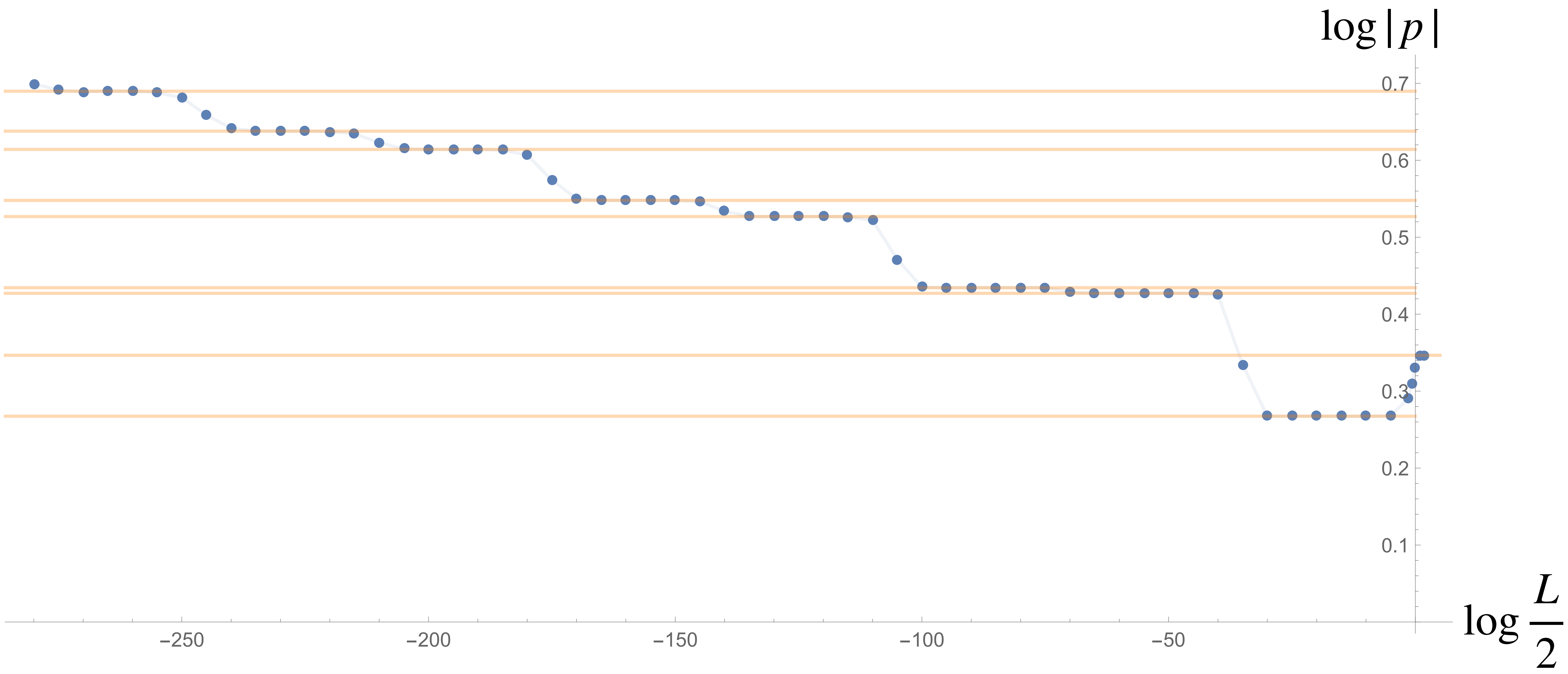

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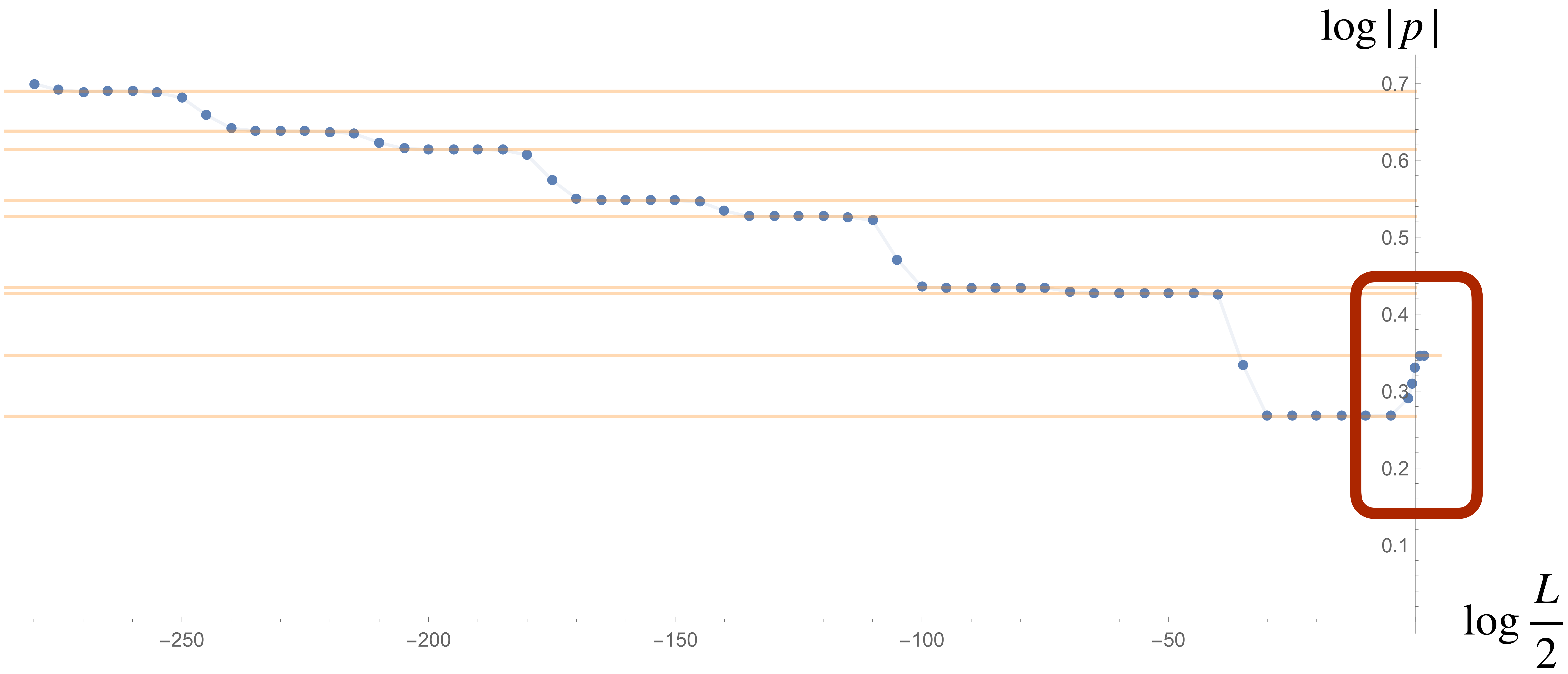

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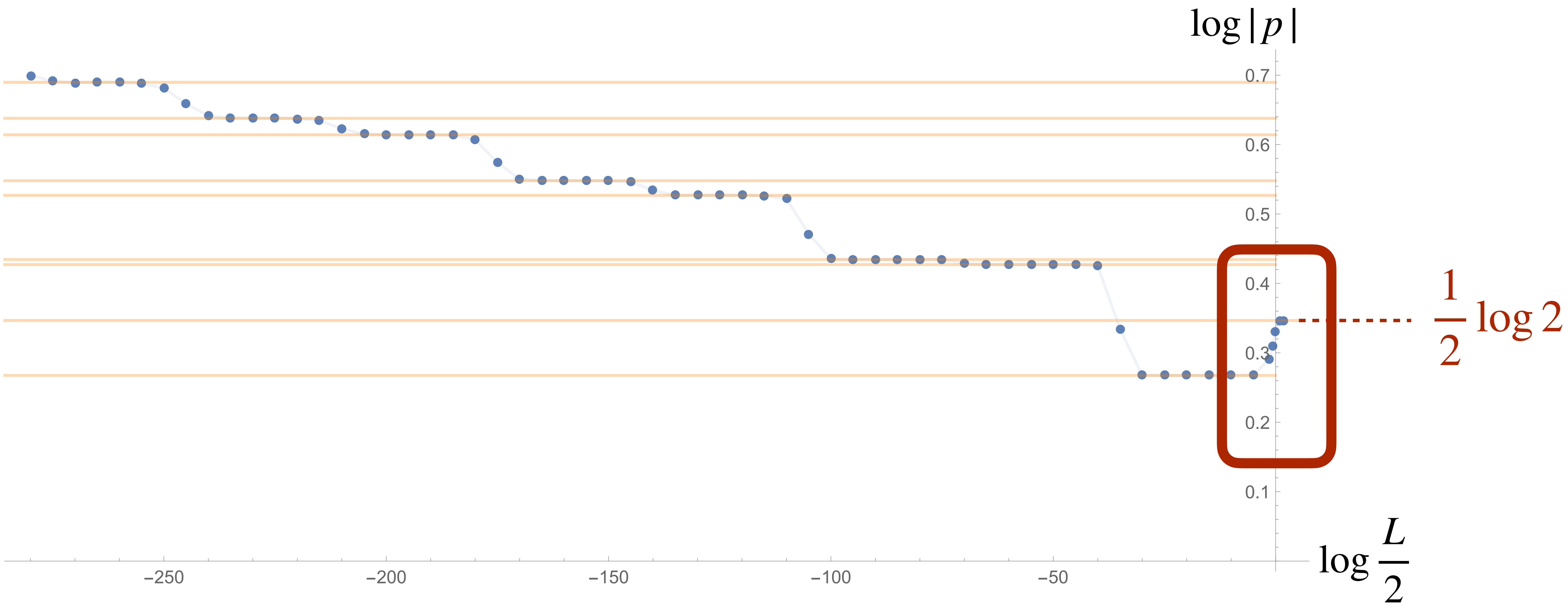

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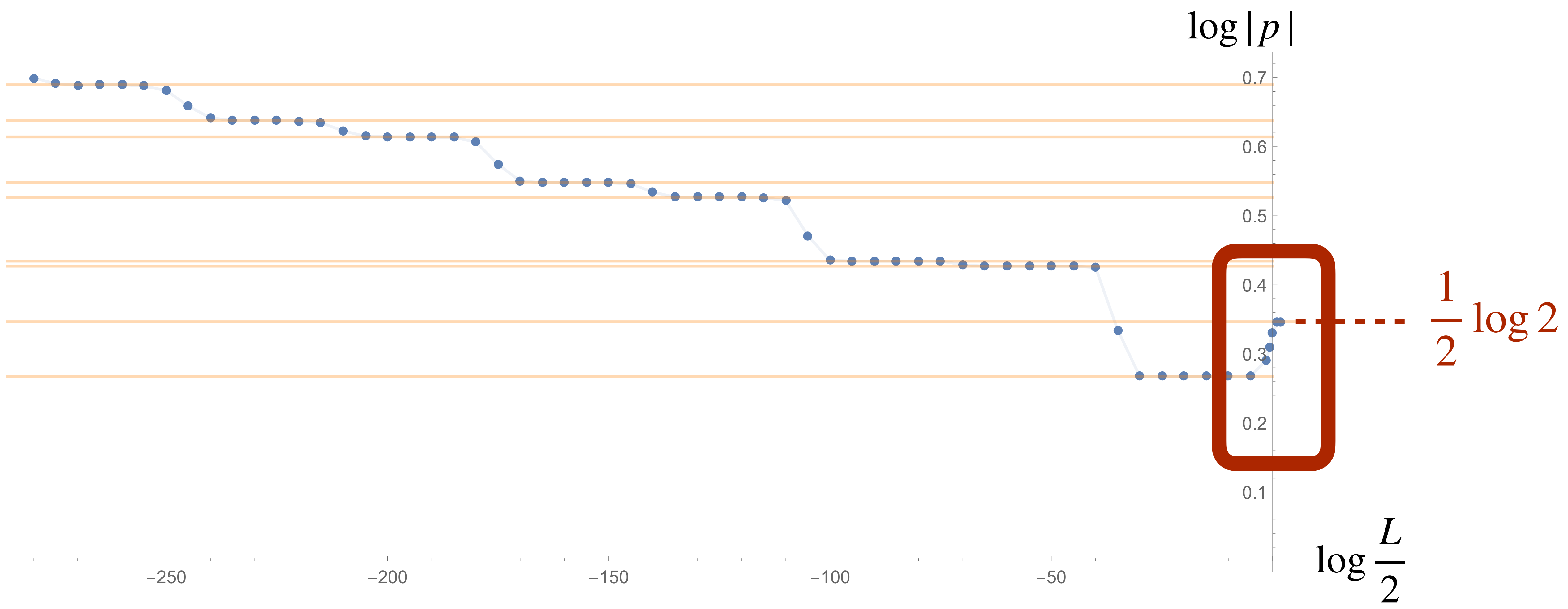

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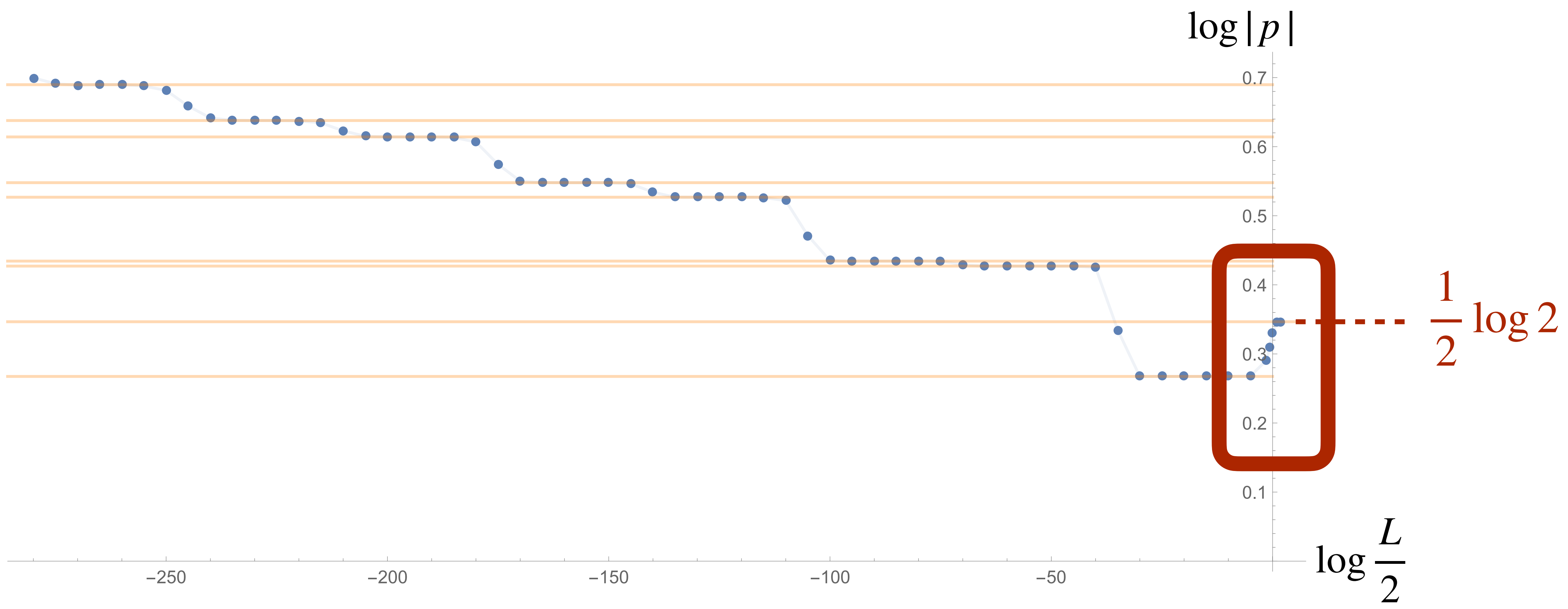

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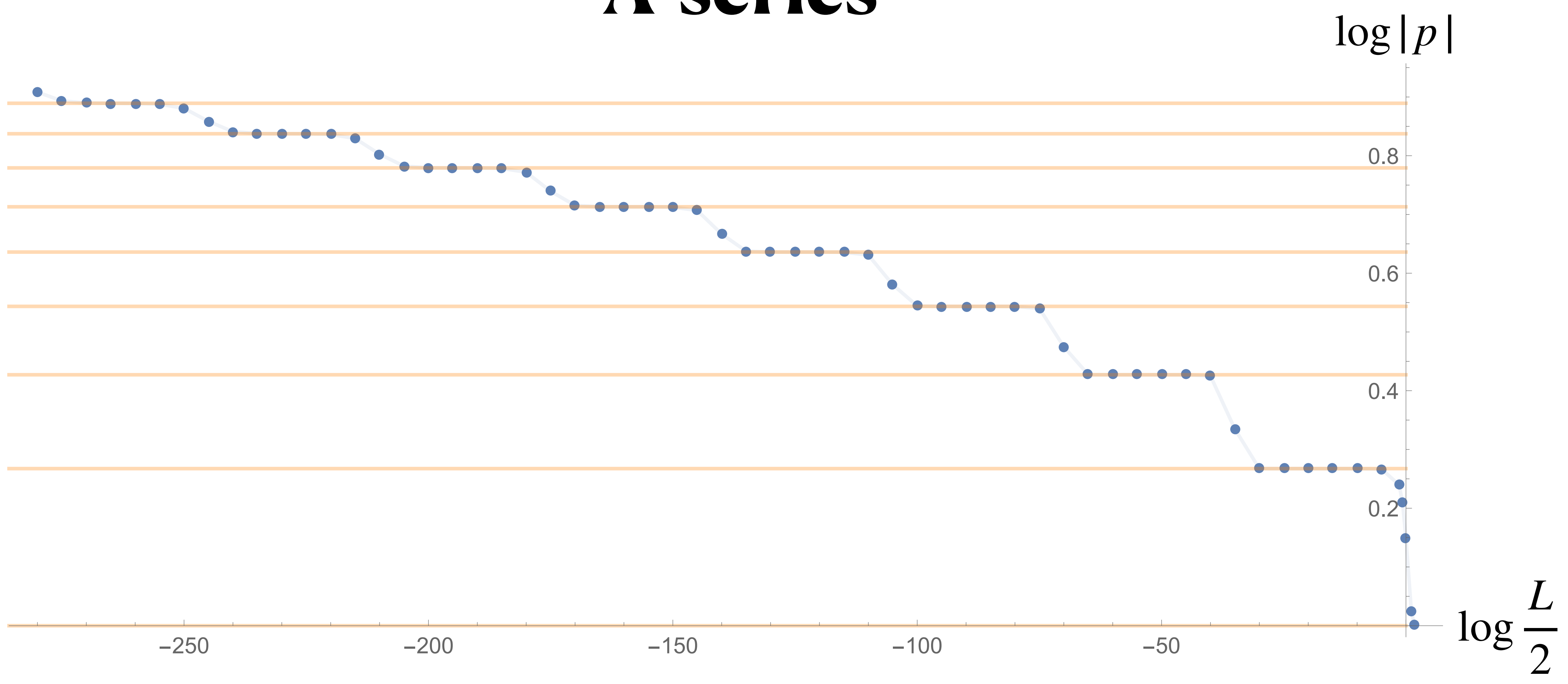

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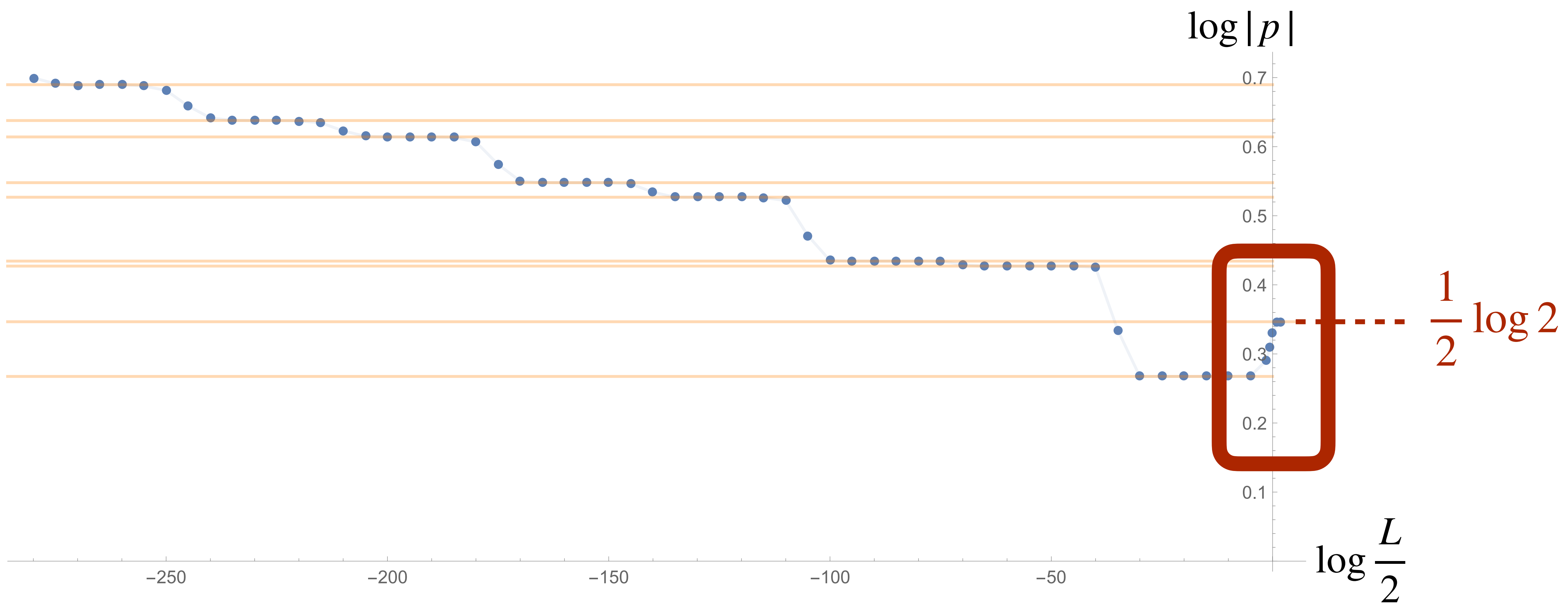

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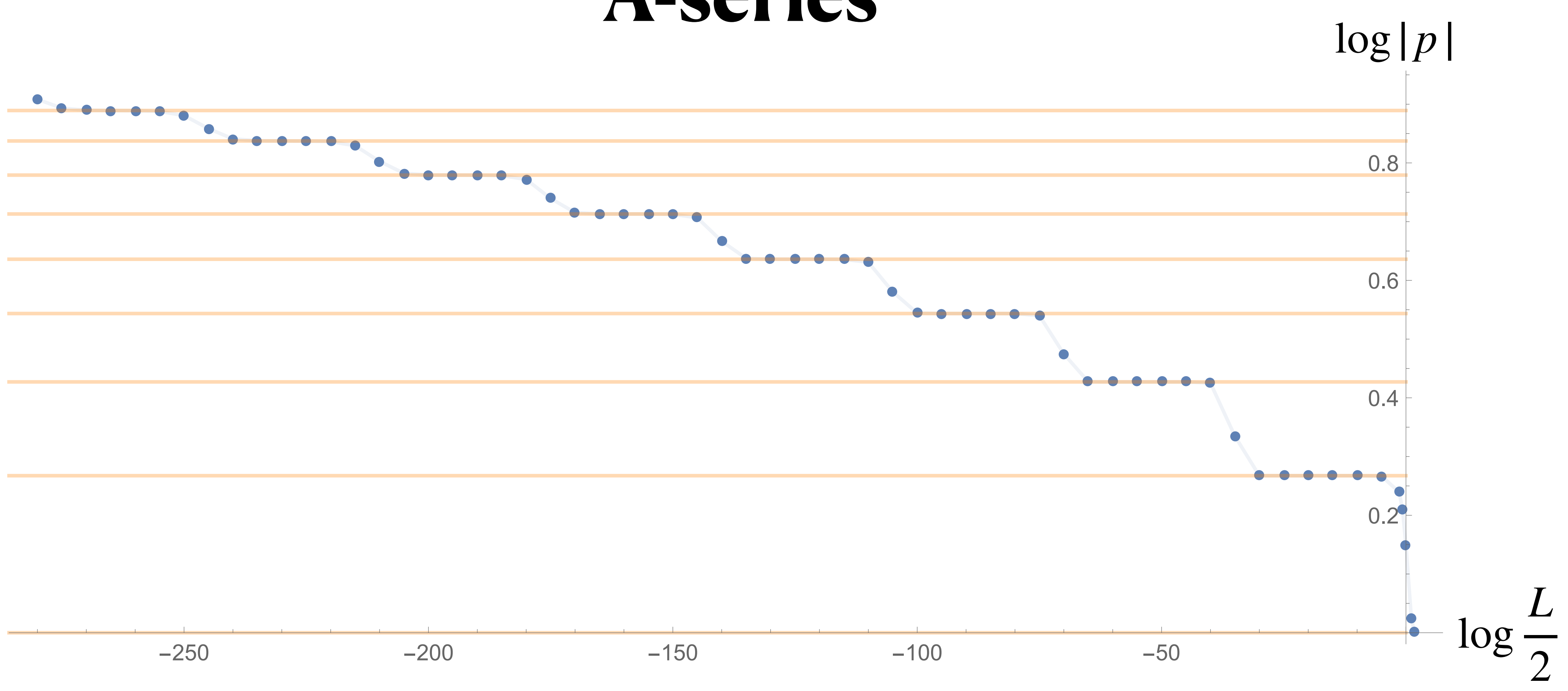

$$Z_{\mathbb{K}}(R,L) \overset{R,L\gg 1}{\sim} \begin{cases} 1 & \text{original staircase} \\ 2 & \text{orbifolded staircase} \end{cases}$$

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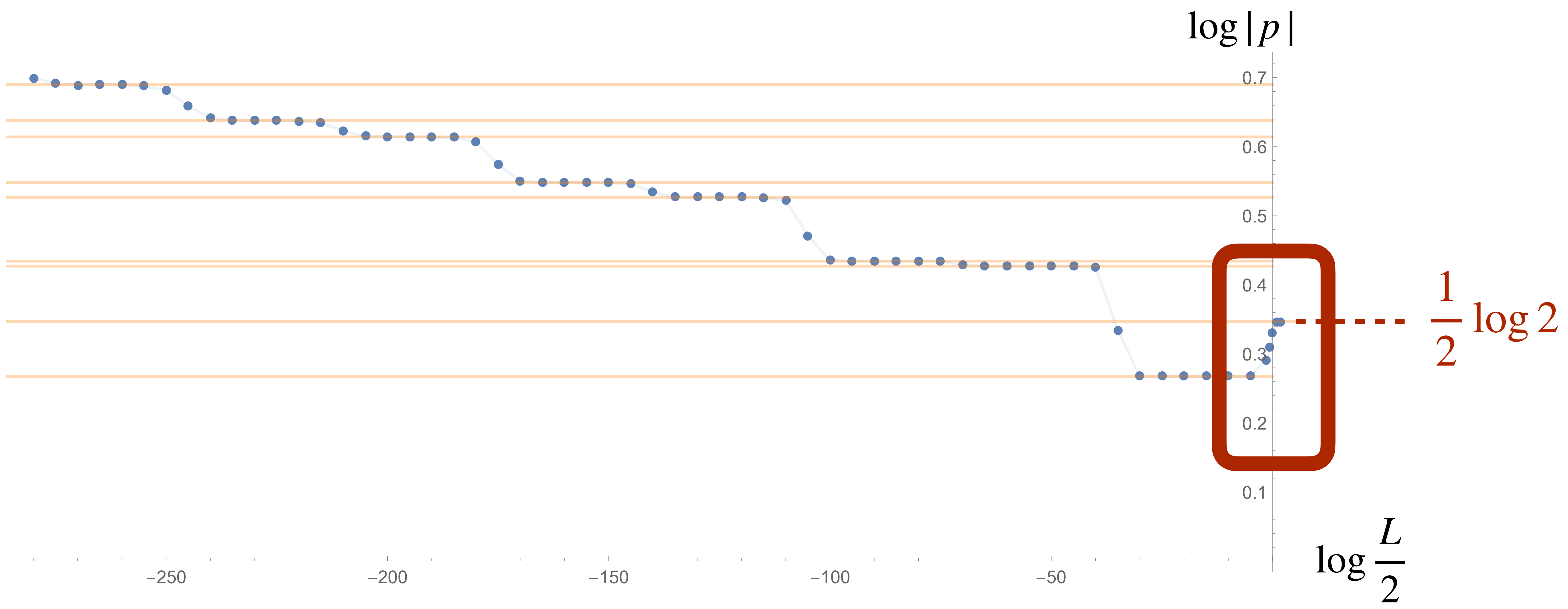

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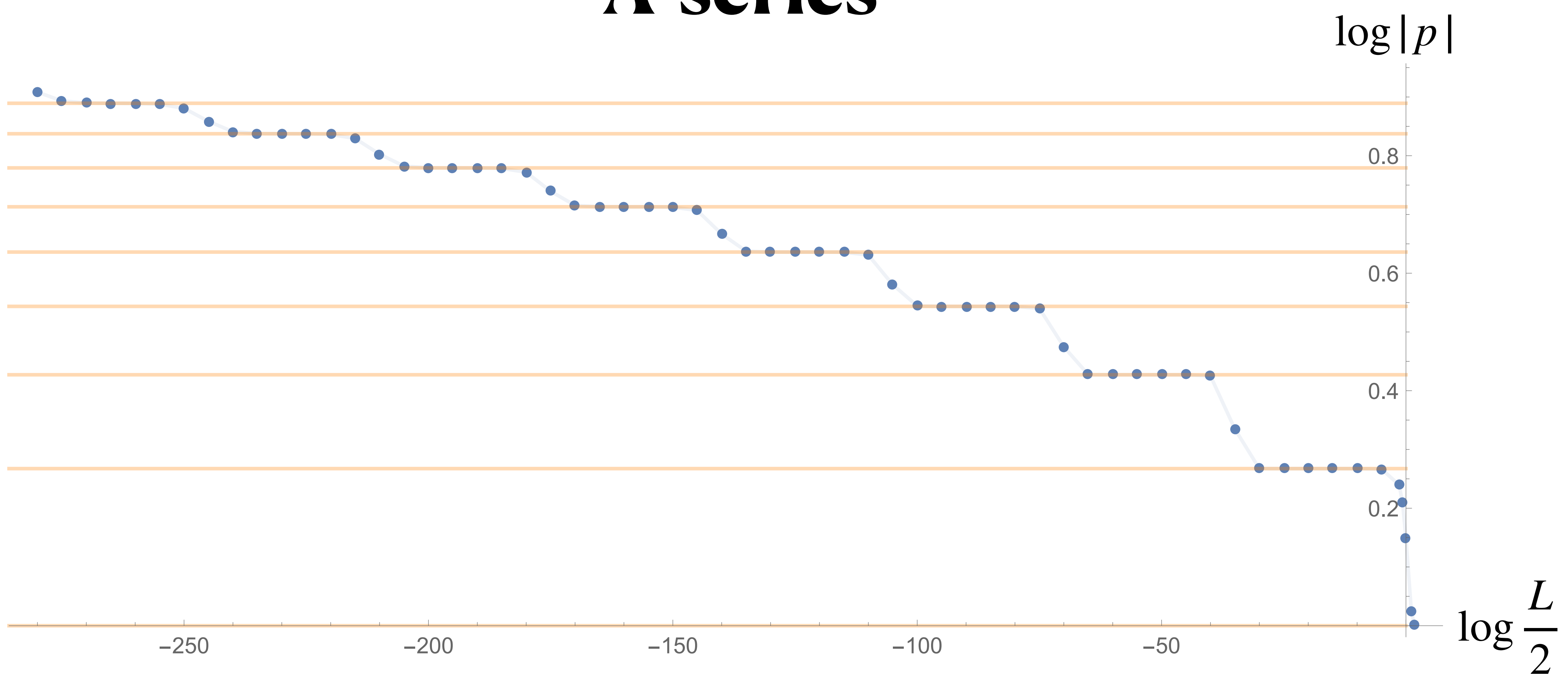

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original staircase → **Deep IR**: unique vacuum

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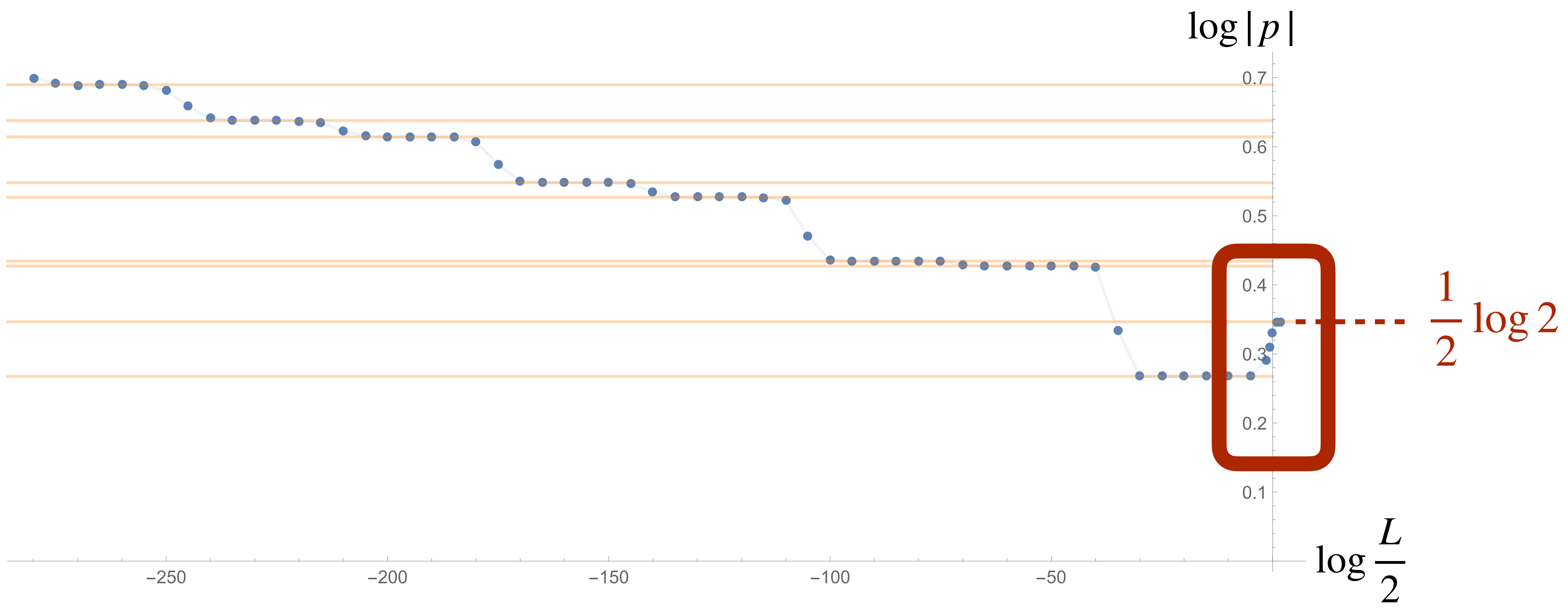

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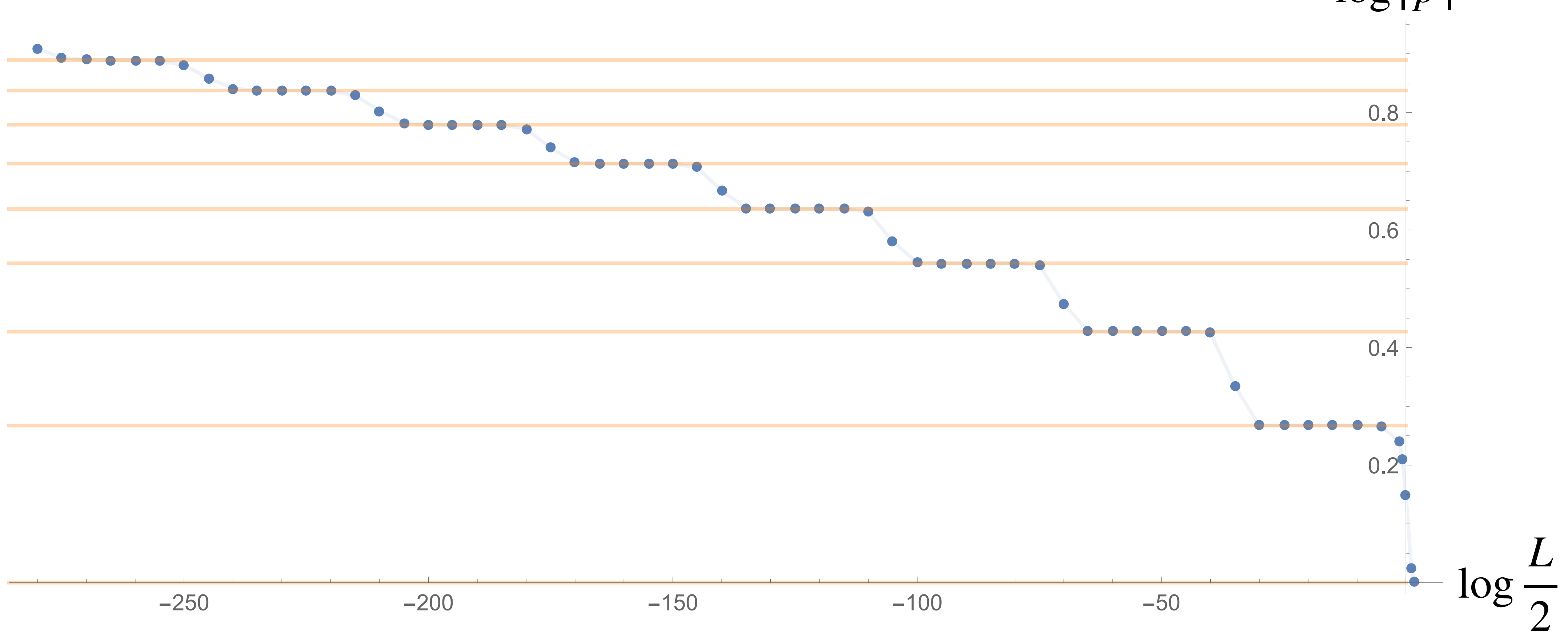

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- Along the D-series, p-function is monotonically decreasing.
- It increases in the deep IR in a symmetry breaking phase.

Crosscap states in spin chains

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- XXX SU(2) spin chain $H_{\mathrm{SU}(2)} \propto \sum_j \vec{S}_j \vec{S}_{j+1}$

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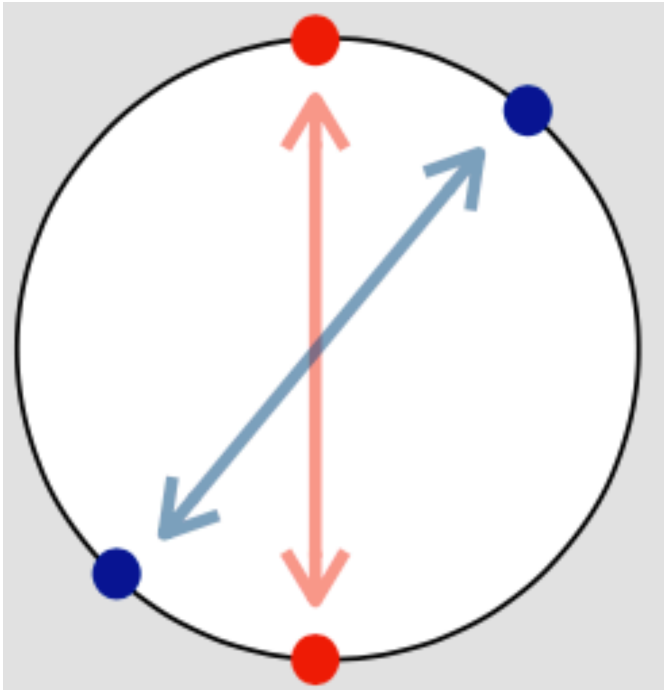

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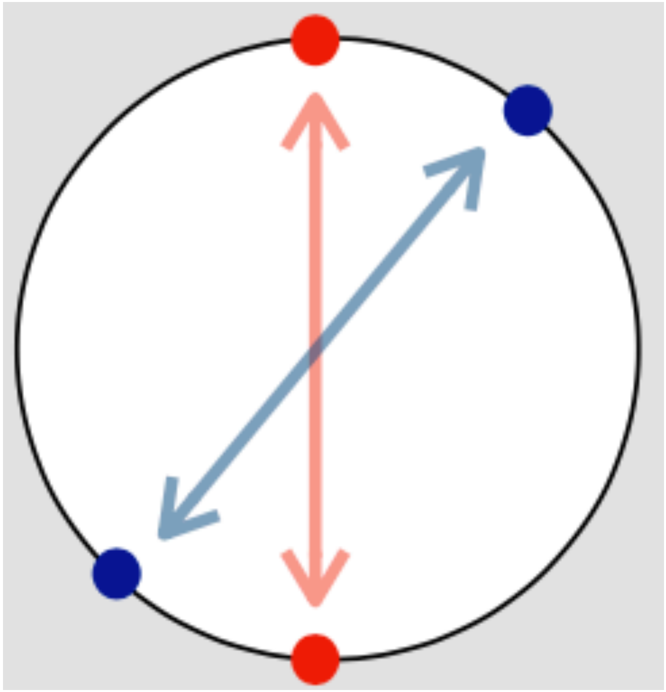
- Mimic the definition in field theory: identify states on antipodal sites of the chain:

$$|c\rangle\rangle_j \equiv |\uparrow\rangle_j \otimes |\uparrow\rangle_{j+\frac{L}{2}} + |\downarrow\rangle_j \otimes |\downarrow\rangle_{j+\frac{L}{2}}$$

site j

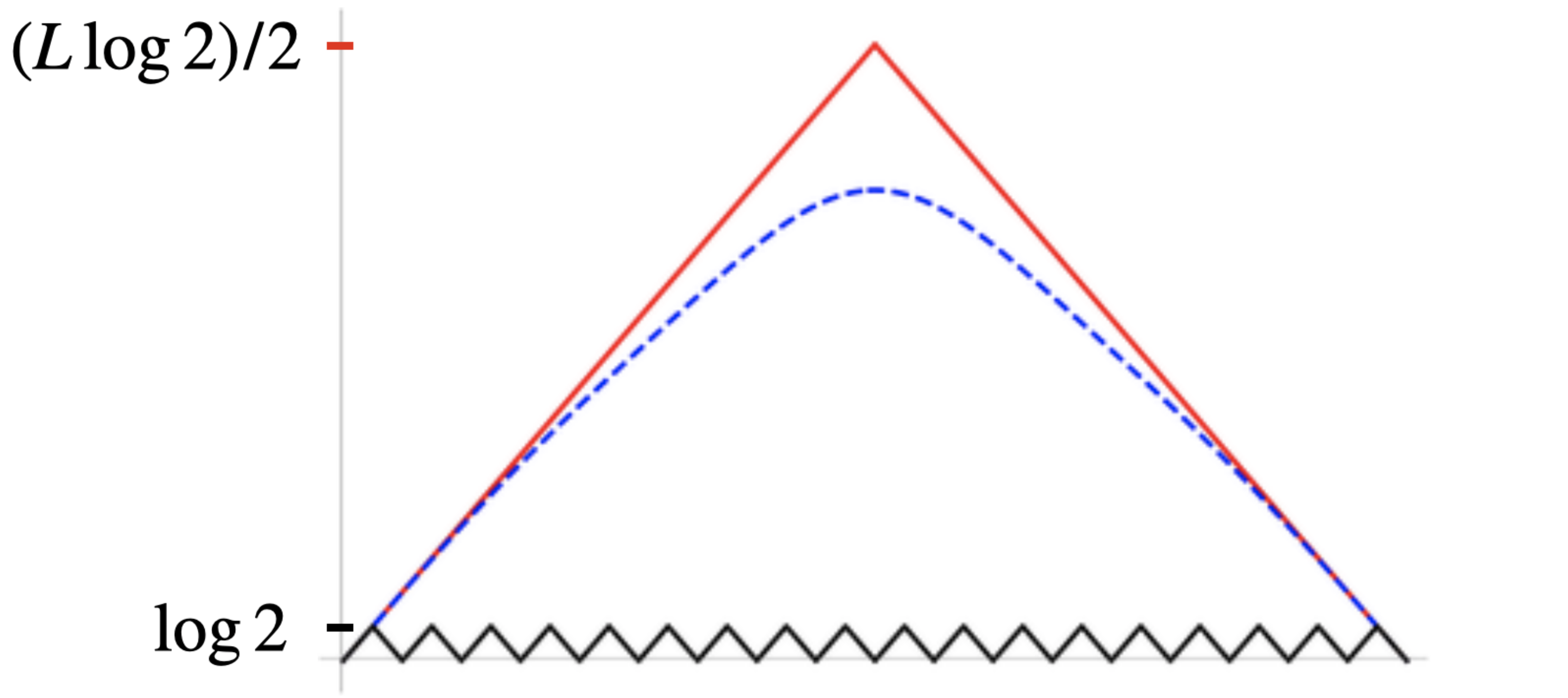
Crosscap states in spin chains



- XXX SU(2) spin chain $\qquad H_{\mathrm{SU(2)}} \propto \sum_j \vec{S}_j \vec{S}_{j+1}$

- Mimic the definition in field theory: identify states on antipodal sites of the chain:

$$|c\rangle\rangle_j \equiv |\uparrow\rangle_j \otimes |\uparrow\rangle_{j+\frac{L}{2}} + |\downarrow\rangle_j \otimes |\downarrow\rangle_{j+\frac{L}{2}}$$

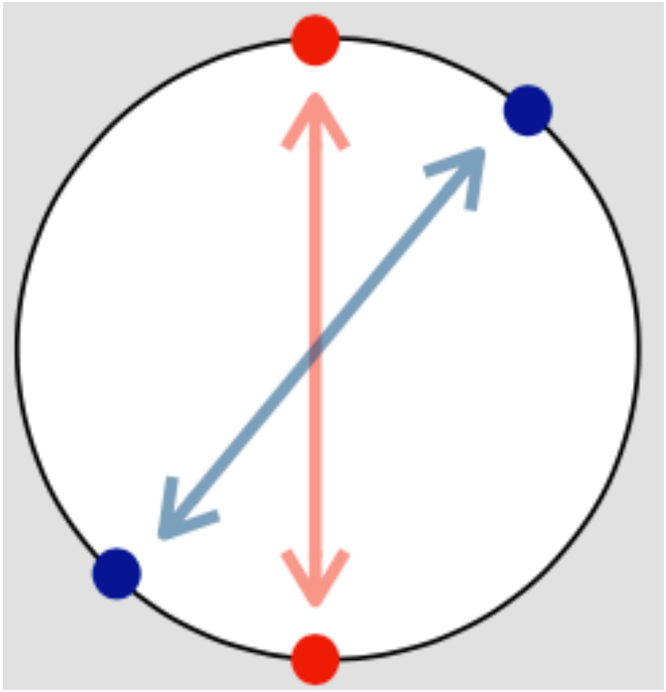


$$|\mathscr{C}\rangle \equiv \prod_{j=1}^{\frac{L}{2}} \left(|c\rangle\rangle_j \right)^{\otimes}$$

Long-range entangled
(As opposed to the short-range entangled in spin chain boundary state)

$$|b\rangle\rangle_j \sim \#|\uparrow\rangle_j \otimes |\uparrow\rangle_{j+1} + \#|\downarrow\rangle_j \otimes |\uparrow\rangle_{j+1} + \#|\uparrow\rangle_j \otimes |\downarrow\rangle_{j+1}$$

Crosscap states in spin chains



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site j

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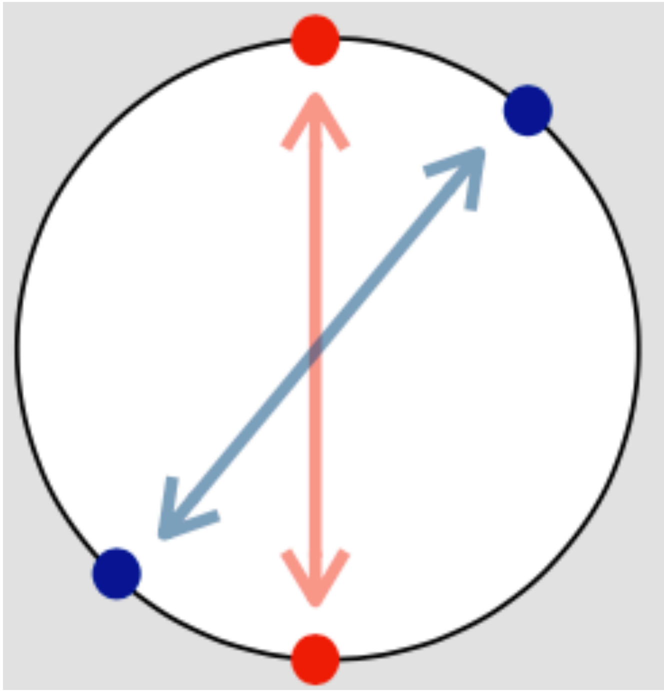
Long-range entangled
(As opposed to the short-range entangled in spin chain boundary state)

- One can show: $$\Big(T(u) - T(-u)\Big)|\mathscr{C}\rangle = 0 \Leftrightarrow Q_{2n+1}|\mathscr{C}\rangle = 0$$

(∞ many conserved charges)

Crosscap states in spin chains



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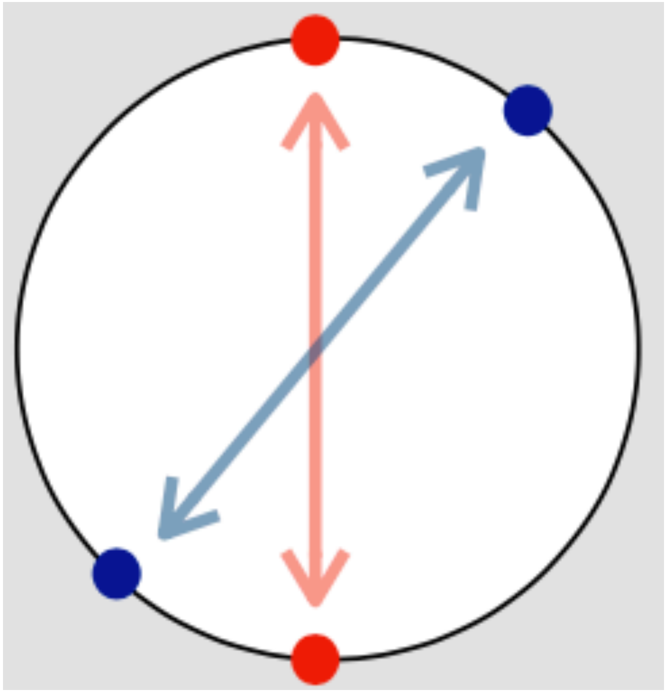
Crosscap states in spin chains



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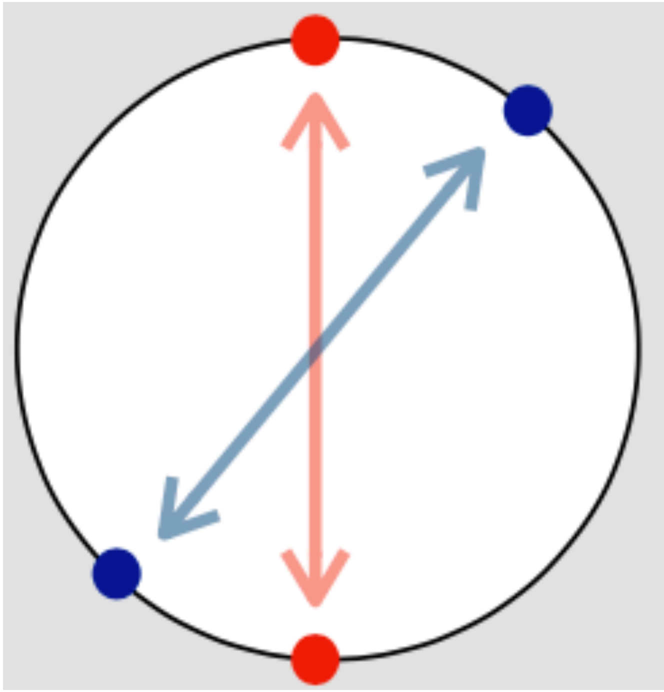
Crosscap states in spin chains



$$|\mathscr{C}\rangle \equiv \prod_{j=1}^{\frac{L}{2}} \Big(|c\rangle\!\rangle_j \Big)^{\otimes}$$

Bethe state

$$\frac{\langle \mathscr{C} | \mathbf{u} \rangle}{\sqrt{\langle \mathbf{u} | \mathbf{u} \rangle}} = \sqrt{\frac{\det G_+}{\det G_-}}$$

$$\mathscr{K}_{\pm}(u,v) = \frac{1}{i}\partial_u \left[\log S(u,v) \pm \log S(u,-v)\right]$$

Crosscap states in spin chains

$$\frac{\langle \mathscr{C} \,|\, \mathbf{u} \rangle}{\sqrt{\langle \mathbf{u} \,|\, \mathbf{u} \rangle}} = \sqrt{\frac{\det G_+}{\det G_-}}$$

Boundary overlap: $$\frac{\langle \mathscr{B} \,|\, \mathbf{u} \rangle}{\sqrt{\langle \mathbf{u} \,|\, \mathbf{u} \rangle}} = (\text{non-universal factor}) \times \sqrt{\frac{\det G_+}{\det G_-}}$$

Conclusions

- Studied crosscap states in integrable models: **integrability is preserved**
- Exactly computed crosscap overlaps
- Observed monotonically decrease of p-function under RG for A-series.
- Generalized staircase to the D-series (also discussed in the paper: generalization to fermionic integrable models)
- In the D-series it also decreases, except in the deep IR in a symmetry breaking phase, where the theory becomes massive.

Outlook

- Study further the behaviour of the p-function under RG: is there a p-theorem under certain assumptions?
- Crosscap state as a initial state for a quantum quench?
- Generalize overlap formula to more general theories, such as theories with bound states and theories with non-diagonal scatterings
- Setup in AdS/CFT realizing crosscap states

THANK YOU