Crosscap States in Integrable Theories

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Based on 2111.09901 with Shota Komatsu

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Crosscap states in 2D
In this section, we generalize the derivation of exact g-functions in integrable field theories to overlaps between the crosscap state and arbitrary excited states. In subsection 2.1, we discuss general properties of the crosscap state and the partition function on the Klein bottle. We also give a definition of the crosscap entropy and explain its relation to the crosscap overlap. In subsection 2.2, we compute the crosscap overlap in integrable field theories. Throughout this section, we assume that the theory is parity-invariant and excitations are scalar.

To define crosscaps, we cut out a disk from a two-dimensional surface and identify antipodal points on the boundary of the disk (see figure 1-(a)). This manipulation makes the surface non-orientable and the state created by this procedure is called the crosscap state. Two commonly-studied closed non-orientable surfaces are \( \mathbb{RP}^2 \) and the Klein bottle. They can be obtained by inserting one or two crosscap states on \( S^2 \) respectively. The crosscap states were studied extensively in 2d CFT, where part of the motivation came from the analysis of string theory in orientifold spacetimes \([39-42]\).

To compute the crosscap overlaps, we consider a cylinder of length \( R \) and circumference \( L \) and contract the two ends with the crosscap states (see figure 1-(b)). This makes the surface topologically equivalent to the Klein bottle. As mentioned above, the Klein bottle can also be obtained by inserting two crosscaps on \( S^2 \), but here it is important to start with the cylinder, which is locally flat, since our interest is in massive QFT, not CFT.

The partition function of this Klein bottle \( Z_{K}(R, L) \) can be expanded in different channels, depending on whether we view \( R \) or \( L \) as the (imaginary) time direction. If we take \( R \) as the time direction, we obtain an expansion

\[
Z_{K}(R, L) = \sum_{L} e^{E_{L}R} |hC|L^{2} = e^{E_{\infty}L} |hC|_{\infty}^{2} + \cdots
\]

(2.1)

Here \( ` \) is the state defined on the spatial length \( ` \), \( |C\rangle \) is the crosscap state, and \( |\infty\rangle \) is the ground state. In the literature, this channel is often called the tree channel. The expansion in the other channel (called the loop channel) is slightly more complicated (see figure 2). Owing to the antipodal identification at the boundary of the cylinder, the Hilbert space in the other channel is defined on a circle of length 2\( R \) not \( R \).

Crosscap states in 2D

\[ z \sim -1/\bar{z} \]
Crosscap states in 2D

\[ z \sim -1/\bar{z} \]

- Cut out a disk from a 2d surface + identify points at the boundary of the disk

- The state created by this procedure is the **crosscap state**
Crosscap states in 2D
Crosscap states in 2D

- Insert one crosscap state on $S^2$: $\mathbb{RP}^2$
Crosscap states in 2D

• Insert one crosscap state on $S^2$: $\mathbb{RP}^2$

• Insert two crosscap states on $S^2$: Klein bottle

Non-orientable manifolds
Boundaries & defects are great

• Wilson/’t Hooft loops in gauge theories: order parameter for confinement

• In 2D, boundaries and interfaces appear naturally as low energy description of lattice systems with impurities (e.g. Kondo effect)

• Strings and holography

D-brane = boundary state on the worldsheet
Boundary states
Boundary states

In 1+1 D QFTs:

• Fixed points of RG and use 2D CFT techniques
Boundary states

In 1+1 D QFTs:

• Fixed points of RG and use 2D CFT techniques

Systematic construction of conformal boundary conditions:

THE BOUNDARY AND CROSSCAP STATES IN CONFORMAL FIELD THEORIES

NOBUYUKI ISHIBASHI

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Received 20 June 1988

A method to obtain the boundary states and the crosscap states explicitly in various conformal field theories, is presented. This makes it possible to construct and analyse open string theories in several closed string backgrounds. We discuss the construction of such theories in the case of the backgrounds corresponding to the conformal field theories with SU(2) current algebra symmetry.

BOUNDARY CONDITIONS, FUSION RULES AND THE VERLINDE FORMULA

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Boundary operators in conformal field theory are considered as arising from the juxtaposition of different types of boundary conditions. From this point of view, the operator content of the theory in an annulus may be related to the fusion rules. By considering the partition function in such a geometry, we give a simple derivation of the Verlinde formula.
Boundary states

In 1+1 D QFTs:

- Use integrable models ($\infty$ conserved charges)
Boundary states

In 1+1 D QFTs:

• Use integrable models (∞ conserved charges)

For special boundaries, called **integrable boundaries**, one can follow their RG flow.
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Crosscap states

• Crosscap states on the worldsheet = orientifolds.

• Common in string compactifications, e.g. de Sitter vacua construction
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- Bootstrap with crosscap states: more restricted structure than boundary states [Giombi, Khanchandani, Zhou’20]
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- Never studied in integrable models!
Outline

• Exact crosscap overlaps & p-function in Integrable Field Theories
• RG flow for the p-function
• Crosscap States in Spin Chain
• Outlook
Crosscap overlaps  \( \langle C | \Psi \rangle \)
Crosscap overlaps $\langle C \mid \Psi \rangle$

- Klein bottle partition function in two channels
Crosscap overlaps $\langle C | \Psi \rangle$

- Klein bottle partition function in two channels

**Tree channel (closed string)**

\[ \langle C | \Psi \rangle \]

\[ Z_K(R, L) = \sum_{L} e^{E_L R} |_{C}^{L} |_{2 R}^{1} = e^{E_{\bar{1}} L} |_{C}^{L} |_{2 R}^{1} + \cdots \]  (2.1)
Crosscap overlaps $\langle C | \Psi \rangle$

- Klein bottle partition function in two channels

Tree channel (closed string)
Crosscap overlaps $\langle C | \Psi \rangle$

- Klein bottle partition function in two channels

Tree channel (closed string)
Crosscap overlaps $\langle C \mid \Psi \rangle$

- Klein bottle partition function in two channels

Tree channel (closed string)

$$Z_K(R, L) = \sum_{\psi_L} e^{-E_{\psi_L}R} \left| \langle C \mid \psi_L \rangle \right|^2 R \to \infty e^{-E_{\Omega_L}R} \left| \langle C \mid \Omega_L \rangle \right|^2 + \cdots$$
Crosscap overlaps $\langle \mathcal{C} | \Psi \rangle$

- Klein bottle partition function in two channels

Tree channel (closed string)

$$Z_\mathcal{K}(R, L) = \sum_{\psi_L} e^{-E_{\psi L} R} \left| \langle \mathcal{C} | \psi_L \rangle \right|^2 \mathcal{R} \rightarrow \infty e^{-E_\Omega R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2 + \cdots$$

Ground state
Loop channel (open string)

Figure 2: Expansion of the Klein bottle partition function in the loop channel. Because of the antipodal identification, the Hilbert space is defined on a union of two red lines, which together form a circle of length $2R$. After the time evolution of $L/2$, a state represented by the dashed line gets identified with its parity image represented by the top red line. In the figure, the states defined on this circle get identified with their parity images after the time evolution for a period $L/2$. This leads to an expression

$$Z_K(R, L) = \text{Tr} e^{H \tau} = \sum_2^R e^{E_2 R L/2} |\uparrow|_2^R.$$  \hfill (2.2)

Here $\uparrow$ is the parity operator while $H$ is the Hamiltonian. Re-organizing the sum in terms of eigenstates of the parity, we can rewrite it as

$$Z_K(R, L) = \sum_2^R \varepsilon_{2R} e^{E_2 R L/2}.$$  \hfill (2.3)

The equality of the two expressions (2.1) and (2.3) in the large $R$ limit gives

$$\lim_{R \to \infty} Z_K(R, L) = \lim_{R \to \infty} \sum_2^R \varepsilon_{2R} e^{E_2 R L/2}.$$  \hfill (2.4)

This shows that the overlap $\langle C | \varepsilon \rangle_L$ controls the density of states weighted by the parity $\varepsilon$.

To make this statement more precise, we consider the parity-weighted free energy

$$F_K = \lim_{R \to \infty} \log Z_K(R, L).$$  \hfill (2.5)

Without the parity weight $\varepsilon$, this would give a definition of a thermal free energy in the infinite volume limit ($R \to \infty$). Now, using the relation (2.4), we find that $F_K$ behaves as

$$F_K = R \varepsilon_{2R} \sum_2^L \log |\varepsilon_{2R} \rangle_\varepsilon \langle \varepsilon | + O(1/R).$$  \hfill (2.6)

This shows that the parity-weighted free energy contains an $O(1)$ term in addition to the usual extensive contribution proportional to the volume $2R$. The structure is reminiscent of the thermal free energy of a system with boundaries, for which the boundary entropy, also known as the $g$-function, gives an $O(1)$ contribution. The boundary entropy is defined in terms of the overlap with the boundary state $|B\rangle$ as

$$s_B = (1/L) \log |\langle B | \varepsilon \rangle_L \rangle.$$  \hfill (2.7)

Based on the similarity, we call the following quantity the crosscap entropy:

$$s_C = \log |\langle C | \varepsilon \rangle_L \rangle.$$  \hfill (2.8)
Loop channel (open string)

\[ Z_K(R, L) = \text{Tr}_{2R} \left[ \Pi e^{-HL/2} \right] = \sum_{\psi_{2R}} e^{-E_{\psi_{2R}}L/2} \langle \psi_{2R} | \Pi | \psi_{2R} \rangle \]
Loop channel (open string)

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where $\varepsilon$ is the eigenvalue of the parity for the state, which takes either +1 or -1. The equality of the two expressions (2.1) and (2.3) in the large $R$ limit gives

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$$s_B = \frac{1}{L} \log \langle B | \Pi | B \rangle$$

Based on the similarity, we call the following quantity the crosscap entropy:

$$s_C = \log \langle p | p \Pi | \psi \rangle$$

(2.7)
Loop channel (open string)

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This shows that the overlap $\langle \psi_{2R} | \Pi | \psi_{2R} \rangle$ controls the density of states weighted by the parity $\epsilon_{\psi_{2R}}$.

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Based on the similarity, we call the following quantity the crosscap entropy:

$$s_C = \log \langle \chi_{\Pi} | \Pi | \chi_{\Pi} \rangle$$
Loop channel (open string) = Tree channel (closed string)
Loop channel (open string) = Tree channel (closed string)

\[
\lim_{R \to \infty} Z_{\mathbb{K}}(R, L) = \lim_{R \to \infty} \left[ \sum_{\psi_{2R}} e^{\psi_{2R} L/2} e^{-E_{\psi_{2R}} R} \right] \simeq e^{-E_{\Omega L} R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2
\]

\[\langle \mathcal{C} | \Omega_L \rangle \] controls the density of states weighted by the parity \( e_\psi \)
Parity-weighted free energy

Loop channel (open string) = Tree channel (closed string)

\[ F_K \equiv - \lim_{R \to \infty} \log Z_K(R, L) \]  

*Parity-weighted free energy*
Loop channel (open string) = Tree channel (closed string)

\[
F_{\mathbb{K}} \equiv - \lim_{R \to \infty} \log Z_{\mathbb{K}}(R, L) \quad \text{Parity-weighted free energy}
\]

\[
= R E_{\Omega_L} - \log \left[ |\langle \mathcal{C} | \Omega_L \rangle|^2 \right] + O(1/R)
\]

\[
\lim_{R \to \infty} Z_{\mathbb{K}}(R, L) \approx e^{-E_{\Omega_L} R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2
\]
Loop channel (open string) = Tree channel (closed string)

\[ F_K \equiv - \lim_{R \to \infty} \log Z_K(R, L) \quad \text{Parity-weighted free energy} \]

\[ = RE_{\Omega_L} - \log \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2 + O(1/R) \]
Loop channel (open string) = Tree channel (closed string)

\[ F_{\mathbb{K}} \equiv - \lim_{R \to \infty} \log Z_{\mathbb{K}}(R, L) \quad \text{Parity-weighted free energy} \]

\[ = RE_{\Omega_L} - \log \left[ |\langle \mathcal{C} | \Omega_L \rangle|^2 \right] + O(1/R) \]

extensive piece \quad \mathcal{O}(1) \text{ piece}
Loop channel (open string) = Tree channel (closed string)

\[ F_{\mathbb{K}} \equiv - \lim_{R \to \infty} \log Z_{\mathbb{K}}(R, L) \quad \text{Parity-weighted free energy} \]

\[ = RE_{\Omega_L} - \log \left[ |\langle \mathcal{C} | \Omega_L \rangle|^2 \right] + O(1/R) \]

- extensive piece
- \( \mathcal{O}(1) \) piece

- Same structure as the thermal free energy of a system with boundaries
- In that case, \( \mathcal{O}(1) \) piece defines the boundary entropy or g-function
Similarly, we define **crosscap entropy** or **p-function**:

$$s_C = \log |p| \quad p \equiv \langle C | \Omega_L \rangle$$

We will study this quantity in integrable models.
p-function in Integrable models

$$\lim_{R \to \infty} \text{Tr}_{2R} \left[ \prod e^{-\hat{H}_L/2} \right] \simeq e^{-E_\omega R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2$$

Large volume partition function

↔ (in integrable models)

**Thermodynamic Bethe Ansatz** + $\mathcal{O}(1)$ fluctuation
p-function in Integrable models

$$\lim_{R \to \infty} \text{Tr}_{2R} \left[ \Pi e^{-\hat{H}L/2} \right] \approx e^{-E_\Omega R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2$$
p-function in Integrable models

\[
\lim_{R \to \infty} \text{Tr}_{2R} \left[ \Pi e^{-\hat{H}_L/2} \right] \simeq e^{-E_\Omega R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2
\]

- **Single type** of particle (massive) (e.g. sinh-Gordon model)
**p-function in Integrable models**

\[
\lim_{R \to \infty} \text{Tr}_{2R} \left[ \prod e^{-\hat{H}/2} \right] \simeq e^{-E_\Omega R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2
\]

- **Single type** of particle (massive) (e.g sinh-Gordon model)
- Energy eigenstates for \( R \to \infty \leftrightarrow M \) excitations labelled by \( | \{ p_j \} \rangle \)

\[
1 = e^{2ip_jR} \prod_{k \neq j} S(p_j, p_k)
\]
\textbf{p-function in Integrable models}

\[
\lim_{R \to \infty} \text{Tr}_{2R} \left[ \Pi e^{-\hat{H}L/2} \right] \approx e^{-E_\Omega R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2
\]

- **Single type** of particle (massive) (e.g sinh-Gordon model)
- Energy eigenstates for \( R \to \infty \leftrightarrow M \) excitations labelled by \( \{p_j\} \)

\[
1 = e^{2ip_jR} \prod_{k \neq j} S(p_j, p_k)
\]

- \( \Pi \{p_j\} \propto \{-p_j\} \)
p-function in Integrable models

\[
\lim_{R \to \infty} \text{Tr}_{2R} \left[ \Pi \, e^{-\hat{H}L/2} \right] \approx e^{-E_\Omega R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2
\]

- **Single type** of particle (massive) (e.g. sinh-Gordon model)
- Energy eigenstates for \( R \to \infty \leftrightarrow M \) excitations labelled by \( | \{ p_j \} \rangle \)
  \[
  1 = e^{2ip_jR} \prod_{k \neq j} S(p_j, p_k)
  \]
- \( \Pi | \{ p_j \} \rangle \propto | \{-p_j\} \rangle \)
- For Bethe states with standard normalization: \( \Pi | \{ p_j \} \rangle = 1 | \{-p_j\} \rangle \)
p-function in Integrable models

$$\lim_{R \to \infty} \text{Tr}_{2R} \left[ \Pi e^{-\hat{H}L/2} \right] \simeq e^{-E_\Omega R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2$$
p-function in Integrable models

\[
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\]

- States whose momenta are **not invariant** under the sign flip **do not** contribute in the parity-weighted trace:

\[
\langle \{p_j\} | \Pi | \{p_j\} \rangle = \langle \{p_j\} | \{-p_j\} \rangle = 0 \quad \text{if} \quad \{p_j\} \neq \{-p_j\}
\]
p-function in Integrable models

\[
\lim_{R \to \infty} \text{Tr}_{2R} \left[ \Pi e^{-\hat{H}L/2} \right] \simeq e^{-E_{\Omega R}} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2
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\[
\langle \{p_j\} | \Pi | \{p_j\} \rangle = \langle \{p_j\} | \{-p_j\} \rangle = 0 \quad \text{if } \{p_j\} \neq \{-p_j\}
\]

- So only states with the set of momenta

\[
\{p_1, \ldots, p_M, -p_M, \ldots, -p_1\} \quad \text{or} \quad \{p_1, \ldots, p_M, 0, -p_M, \ldots, -p_1\}
\]
p-function in Integrable models

\[
\text{Tr}_{2R} \left[ \prod e^{-\hat{H}L/2} \right] = \sum_{\{p_j\}=\{-p_j\}} \exp\left(-\frac{L}{2} \sum_j E(p_j)\right) \approx e^{-E_\Omega R} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2
\]
p-function in Integrable models

\[
\text{Tr}_{2R} \left[ \prod e^{-\hat{H}L/2} \right] = \sum_{\{p_j\} = \{-p_j\}} e^{-\frac{L}{2} \sum_j E(p_j)} \approx e^{-E_{\Omega R}} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2
\]

Standard thermal sum
with the constraint \( \{p_j\} = \{-p_j\} \)
p-function in Integrable models

\[ \text{Tr}_{2R} \left[ \prod e^{-\hat{H}L/2} \right] = \sum_{\{p_j\} = \{-p_j\}} e^{-\frac{L}{2} \sum_j E(p_j)} \sim e^{-E_{\Omega R}} \left| \langle \mathcal{C} | \Omega_L \rangle \right|^2 \]

Standard thermal sum
with the constraint \( \{p_j\} = \{-p_j\} \)

Apply standard TBA techniques to compute the saddle point and its fluctuations
Particles on a circle of size $2R$

\[
\sum_{\{p_j\} = \{-p_j\}} e^{-\frac{L}{2} \sum_j E(p_j)}
\]
Particles on a circle of size 2R

\[ \sum_{\{p_j\}\equiv\{-p_j\}} e^{-\frac{L}{2} \sum_{j} E(p_j)} \]

\[ \{p_1, \ldots, p_M, -p_M, \ldots, -p_1\} \quad \text{or} \quad \{p_1, \ldots, p_M, 0, -p_M, \ldots, -p_1\} \]

**S:** \[ 1 = e^{2ip_jR} S(p_j, -p_j) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k), \]

**T:** \[ 1 = e^{2ip_jR} S(p_j, -p_j) S(p_j, 0) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k). \]

Zero momentum particle
Particles on a circle of size 2R

\[
\sum_{\{p_j\} = \{-p_j\}} \sum_j e^{-\frac{L}{2} \sum_j E(p_j)}
\]

\[
= \sum_{S} e^{-L \sum_{p \gt 0} E(p)} + e^{-\frac{m}{2}} \sum_{T} e^{-L \sum_{p \gt 0} E(p)}
\]

\[
S : \quad 1 = e^{2ip_j R} S(p_j, -p_j) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k),
\]

\[
T : \quad 1 = e^{2ip_j R} S(p_j, -p_j) S(p_j, 0) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k).
\]

Zero momentum particle
Particles on a circle of size $2R$

Formally similar to a system with 2 identical boundaries:

\[
\sum_{\{p_j\} = \{-p_j\}} e^{-\frac{L}{2} \sum_j E(p_j)} = \sum_S e^{-L \sum_{p \geq 0} E(p_j)} + e^{-\frac{mL}{2}} \sum_T e^{-L \sum_{p > 0} E(p_j)}
\]

\[
\{p_1, \ldots, p_M, -p_M, \ldots, -p_1\} \quad \text{or} \quad \{p_1, \ldots, p_M, 0, -p_M, \ldots, -p_1\}
\]

\[
S : \quad 1 = e^{2ip_j R} S(p_j, -p_j) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k),
\]

\[
T : \quad 1 = e^{2ip_j R} S(p_j, -p_j) S(p_j, 0) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k).
\]

Zero momentum particle
Formally similar to a system with 2 identical boundaries:

\[
\sum_{\{p_j\} = \{-p_j\}} e^{-\frac{L}{2} \sum_j E(p_j)}
\]

\[
= \sum_s e^{-L \sum_{p \neq 0} E(p)} + e^{-\frac{mL}{2}} \sum_t e^{-L \sum_{p \neq 0} E(p)}
\]

Particles on a circle of size \(2R\)

\[
\{p_1, \ldots, p_M, -p_M, \ldots, -p_1\} \quad \text{or} \quad \{p_1, \ldots, p_M, 0, -p_M, \ldots, -p_1\}
\]

\[
S : \quad 1 = e^{2ip_j R} S(p_j, -p_j) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k),
\]

\[
T : \quad 1 = e^{2ip_j R} S(p_j, -p_j) S(p_j, 0) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k).
\]

Zero momentum particle

\[
1 = e^{2ip_j R} \left( R(p_j) \right)^2 \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k)
\]
Formally similar to a system with 2 identical boundaries:

\[
\sum e^{-L \sum_j E(p_j)}_{p_j = \{-p_j\}} = \sum_s e^{-L \sum_{p \neq 0} E(p)} + e^{-m \frac{1}{2L}} \sum_t e^{-L \sum_{p > 0} E(p)}
\]

Particles on a circle of size 2R

\[
\{p_1, \ldots, p_M, -p_M, \ldots, -p_1\} \quad \text{or} \quad \{p_1, \ldots, p_M, 0, -p_M, \ldots, -p_1\}
\]

\[
S : \quad 1 = e^{2ip_j R} S(p_j, -p_j) \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k),
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Zero momentum particle

\[
1 = e^{2ip_j R} \left( R(p_j) \right)^2 \prod_{k \neq j} S(p_j, p_k) S(p_j, -p_k)
\]

\[
\left( R(p_j) \right)^2 \leftrightarrow \begin{cases} S(p_j, -p_j) & : S \\ S(p_j, -p_j) S(p_j, 0) & : T \end{cases}
\]
p-function in Integrable models

Result: "Simplest" g-function

\[ |p| = |\langle \mathcal{C} | \Omega_L \rangle| = \sqrt{1 + \sqrt{\frac{Y(0)}{1 + Y(0)}}} \frac{\det[1 - \hat{G}_-]}{\det[1 - \hat{G}_+]} \]

S-sector \hspace{2cm} T-sector
**Result:** “Simplest” g-function

\[
|p| = \left| \langle \mathcal{C} | \Omega_L \rangle \right| = \sqrt{1 + \sqrt{\frac{Y(0)}{1 + Y(0)}} \frac{\det [1 - \hat{G}_-]}{\det [1 - \hat{G}_+]}}
\]

- **S-sector**
- **T-sector**
\[|p| = \left| \langle \mathcal{C} | \Omega_L \rangle \right| = \sqrt{\left( 1 + \sqrt{\frac{Y(0)}{1 + Y(0)}} \right) \frac{\det \left[ 1 - \hat{G}_- \right]}{\det \left[ 1 - \hat{G}_+ \right]}}\]
\[ |p| = |\langle C | \Omega_L \rangle| = \sqrt{1 + \sqrt{\frac{Y(0)}{1 + Y(0)}}} \frac{\det [1 - \hat{G}_-]}{\det [1 - \hat{G}_+]} \]
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**Y-function** \[ 0 = LE(u) + \log Y(u) - \log(1 + Y) \star \mathcal{K}_+(u) \]

**Dispersion relation** \[ \mathcal{K}_\pm(u, v) = \frac{1}{i} \partial_u \left[ \log S(u, v) \pm \log S(u, -v) \right] \]
\begin{align*}
|p| &= \left| \langle \mathcal{C} | \Omega_L \rangle \right| = \sqrt{1 + \frac{Y(0)}{1 + Y(0)}} \frac{\det [1 - \hat{G}_-]}{\det [1 - \hat{G}_+]}
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\textbf{Y-function} \quad 0 = LE(u) + \log Y(u) - \log(1 + Y) \star \mathcal{K}_+(u)

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**Fredholm determinants:**

\[ \hat{G}_\pm \cdot f(u) = \int_0^\infty \frac{dv}{2\pi} \frac{\mathcal{K}_\pm(u,v)}{1 + 1/Y(v)} f(v) \]

**Y-function**

\[ 0 = LE(u) + \log Y(u) - \log(1 + Y) \star \mathcal{K}_+(u) \]

**Dispersion relation**

\[ \mathcal{K}_\pm(u, v) = \frac{1}{i \partial_u} \left[ \log S(u, v) \pm \log S(u, -v) \right] \]
Can be generalized for any excited state $|\langle C | \Psi_L \rangle|$ using analytic continuation of this formula, similar to Dorey-Tateo trick.

$$|\langle C | \Psi_L \rangle| = \sqrt{\left( 1 + \sqrt{\frac{Y(0)}{1 + Y(0)}} \right) \frac{\det \left[ 1 - \hat{G}_-^* \right]}{\det \left[ 1 - \hat{G}_+^* \right]}}$$

$$\hat{G}_+^* \cdot f(u) = \sum_k \frac{i \mathcal{H}_\pm(u, u_k)}{\partial_u \log Y(\tilde{u}_k)} f(\tilde{u}_k) + \int_0^\infty dv \frac{\mathcal{H}_\pm(u, v)}{2\pi \frac{1}{1 + 1/Y(v)}} f(v)$$
• Can be generalized for any **excited state** $|\langle \mathcal{C} | \Psi_L \rangle |$ using analytic continuation of this formula, similar to Dorey-Tateo trick.

$$
|\langle \mathcal{C} | \Psi_L \rangle | = \sqrt{1 + \sqrt{\frac{Y(0)}{1 + Y(0)}} \det \left[ 1 - \hat{G}_-^{\star} \right] \frac{\det \left[ 1 - \hat{G}_+^{\star} \right]}{\det \left[ 1 - \hat{G}_- \right]}}
$$

$$
\hat{G}_\pm \cdot f(u) = \sum_k \frac{i \mathcal{H}_\pm(u, u_k)}{\partial_u \log Y(\tilde{u}_k)} f(\tilde{u}_k) + \int_0^\infty dv \frac{\mathcal{H}_\pm(u, v)}{2\pi 1 + 1/Y(v)} f(v)
$$

• Asymptotic limit

$$
|\langle \mathcal{C} | \Psi_L \rangle | \xrightarrow{L \to \infty} \sqrt{\frac{\det G_+}{\det G_-}}
$$

$$(G_\pm)_{1 \leq i, j \leq \frac{M}{2}} = \left[ L \partial_u p(u_i) + \sum_{k=1}^{M} \mathcal{H}_+(u_i, u_k) \right] \delta_{ij} - \mathcal{H}_\pm(u_i, u_j)$$
RG flow of p-function
RG flow of p-function

- **Goal**: use previous result to study how p-function evolves under RG
RG flow of p-function

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- Use *staircase model of Al. Zamolodchikov*
RG flow of p-function

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- Use **staircase model of Al. Zamolodchikov**
- Start with sinh-Gordon model (integrable)

\[
\mathcal{L}_{\text{shG}} = \frac{1}{2} (\partial \Phi)^2 - \frac{m^2}{b^2} \cosh(b\Phi)
\]
**RG flow of p-function**

- **Goal:** use previous result to study how p-function evolves under RG
- **Use** staircase model of Al. Zamolodchikov
- **Start with** sinh-Gordon model (integrable)

\[ \mathcal{L}_{\text{shG}} = \frac{1}{2} (\partial \Phi)^2 - \frac{m^2}{b^2} \cosh(b\Phi) \]

- **Exact** S-matrix

\[ S(u - v) = \frac{\sinh(u - v) - i \sin \gamma}{\sinh(u - v) + i \sin \gamma} \]

\[ \gamma = \frac{\pi b^2}{8\pi + b^2} \]
RG flow of $p$-function

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S-matrix invariant under: \( \gamma \rightarrow \pi - \gamma \Leftrightarrow \text{weak-strong coupling} \) duality
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Al. Zamolodchikov said:

\[ \gamma = \frac{\pi}{2} \pm i\theta_0 \]

Real parameter
RG flow of p-function

\[ S(u - v) = \frac{\sinh(u - v) - i \sin \gamma}{\sinh(u - v) + i \sin \gamma} \]

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S-matrix invariant under: \( \gamma \rightarrow \pi - \gamma \iff \text{weak-strong coupling duality} \)

Al. Zamolodchikov said:

\[ \gamma = \frac{\pi}{2} \pm i\theta_0 \]

Real parameter

• Resulting S-matrix still physical (Real analytic, unitary, crossing symmetric)

• Lagrangian description not so clear
Staircase model
Staircase model

\[ \gamma = \frac{\pi}{2} \pm i\theta_0 \]
Staircase model

\[ \gamma = \frac{\pi}{2} \pm i\theta_0 \]

Take \( \theta_0 \) to infinity and compute the effective central charge (i.e. ground state energy)
Staircase model

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Take \( \theta_0 \) to infinity and compute the effective central charge (i.e. ground state energy)
Staircase model

$\gamma = \frac{\pi}{2} \pm i \theta_0$

Take $\theta_0$ to infinity and compute the effective central charge (i.e. ground state energy)

$c_{\text{eff}}(L) \sim c_m = 1 - \frac{6}{m(m+1)}$  \quad (\log L \sim -(m - 3)\theta_0/2)$

Central charge of A-series minimal models $\mathcal{M}^{(A)}_m$

$\mathcal{M}^{(A)}_m \to \mathcal{M}^{(A)}_{m-1}$ induced by the least relevant operator $\phi_{13}$
p-function for staircase model
Figure 4: The $p$-function for the staircase model as a function of the volume $L$. Similarly to the effective central charge, it develops plateaux (orange horizontal lines) at the values of the $p$-function for the minimal models in the $A$-series. Enjoy an infinite sequence of benchmark points provided by the unitary minimal models for which we can analytically compute a large amount of information purely from CFT.

Result. The result of the numerical evaluation of \[
\text{(2.36)}
\]
is given in the figure 4. The crosscap entropy (log $|p|$) monotonically decreases with the RG flow and decays to zero in the deep infrared. Along the way, it develops plateaux whose values correspond to the $p$-functions of the minimal models in the $A$-series. These plateaux values can be calculated analytically. Referring to the formula \[
\text{(2.36)},
\]
the (ratio of) Fredholm determinants and the value of $Y(0)$ at the plateau corresponding to $M(A)^m$ were computed in [11]. The results read (see [11] for derivations)

\[
\begin{align*}
\det \hat{G}_1 & > \det \hat{G}_1 + i \det \hat{G}_1 > \det \hat{G}_1 > \det \hat{G}_1 > 0, \\
\sin \frac{\pi}{2} & > \sin \frac{\pi}{2} > \sin \frac{\pi}{2} > \sin \frac{\pi}{2} + 1 > 0, \\
\end{align*}
\]

\[
\begin{align*}
Y(0) & = \cot \frac{\pi}{2} > \cot \frac{\pi}{2} > \cot \frac{\pi}{2} > \cot \frac{\pi}{2} + 1 > 0. \\
\end{align*}
\]

Plugging these values into \[
\text{(2.36)},
\]
we obtain a closed-form expression for the $p$-function of the $A$-series minimal models:

\[
\begin{align*}
M(A)^m & : |p| = |h_C| \frac{1}{R} & : \text{odd} \\
\end{align*}
\]

\[
\begin{align*}
\frac{1}{4} \sin \frac{\pi}{2} > \frac{1}{4} \sin \frac{\pi}{2} > \frac{1}{4} \sin \frac{\pi}{2} > \frac{1}{4} \sin \frac{\pi}{2} + 1 > 0, \\
\end{align*}
\]

\[
\begin{align*}
Y(0) & = \cot \frac{\pi}{2} > \cot \frac{\pi}{2} > \cot \frac{\pi}{2} > \cot \frac{\pi}{2} + 1 > 0. \\
\end{align*}
\]

(3.6)
Figure 4: The $p$-function for the staircase model as a function of the volume $L$.

Similarly to the effective central charge, it develops plateaux (orange horizontal lines) at the values of the $p$-function for the minimal models in the $A$-series.

We enjoy an infinite sequence of benchmark points provided by the unitary minimal models for which we can analytically compute a large amount of information purely from CFT.

The result of the numerical evaluation of $\left(\frac{2.36}{\ldots}\right)$ is given in the figure 4.

The crosscap entropy ($\log|p|$) monotonically decreases with the RG flow and decays to zero in the deep infrared. Along the way, it develops plateaux whose values correspond to the $p$-functions of the minimal models in the $A$-series. These plateaux values can be calculated analytically.

Referring to the formula $\left(\frac{2.36}{\ldots}\right)$, the (ratio of) Fredholm determinants and the value of $Y(0)$ at the plateau corresponding to $M(A)^m$ were computed in [11].

The results read (see [11] for derivations)

$$Y(0) = \begin{cases} \cot \frac{\pi}{m+1} & \text{for odd } m \\ \cot \frac{\pi}{m} & \text{for even } m \end{cases}$$

Plugging these values into $\left(\frac{2.36}{\ldots}\right)$, we obtain a closed-form expression for the $p$-function of the $A$-series minimal models:

$$|p| = \frac{1}{4} \sin \frac{\pi}{m+1} \sin \frac{\pi}{m} : \text{odd}$$

$$|p| = \frac{1}{4} \sin \frac{\pi}{2(m+1)} \sin \frac{\pi}{m} \sin \frac{\pi}{m+1} : \text{even}$$
Figure 4: The $p$-function for the staircase model as a function the volume $L$. Similarly to the effective central charge, it develops plateaux (orange horizontal lines) at the values of the $p$-function for the minimal models in the $A$-series.

Enjoy an infinite sequence of benchmark points provided by the unitary minimal models for which we can analytically compute a large amount of information purely from CFT.

Result. The result of the numerical evaluation of (2.36) is given in the plot figure 4.

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\[
\begin{align*}
\det h_1 \hat{G}^i & = 8 > 8 > 8 > < 8 > < 8 > < 8 > < \\
\det h_1 \hat{G}^{i+1} & = 14 \sin \frac{m \pi}{2} \sin \frac{m \pi + 1}{m+1} \\
\hat{Y}(0) & = \cot \frac{m \pi + 1}{2} m: \text{odd} \\
\hat{Y}(0) & = \cot \frac{m \pi}{2} m: \text{even}.
\end{align*}
\]

Plugging these values into (2.36), we obtain a closed-form expression for the $p$-function of the $A$-series minimal models:

\[
|p| = |h_C| \mathcal{I} \mathcal{I} \mathcal{I} = \left( \frac{\cot \frac{m \pi + 1}{2} m}{\cot \frac{m \pi}{2} m} \right) m: \text{odd},
\]

(3.8)

Orange lines determined from the CFT
p-function for staircase model

Orange lines determined from the CFT

\[ |\langle \mathcal{C} | \Omega \rangle| = \left( \sum_{a} n_{a,a} S_{a,1} \right)^{\frac{1}{2}} \]

irreducible representation of the Virasoro algebra
modular S-matrix \( a \) and identity representation
degeneracy of states in the representation \( a \) for the chiral and anti-chiral part
**p-function for staircase model**

Orange lines determined from the CFT

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irreducible representation of the Virasoro algebra

degeneracy of states in the representation \( a \) for the chiral and anti-chiral part

modular \( S \)-matrix \( a \) and identity representation

Specifying for \( \mathcal{M}^{(A)}_m \):

\[ | p | = | \langle \mathcal{C} | \Omega \rangle | = \left( \frac{2}{m(m+1)} \right)^{\frac{1}{4}} \sqrt{\cot \frac{\pi}{2m} \cot \frac{\pi}{2(m+1)}} \]
Staircase for D-series minimal models
Staircase for D-series minimal models

- A-series contain $\mathbb{Z}_2$ global symmetry
Staircase for D-series minimal models

- A-series contain $\mathbb{Z}_2$ global symmetry
- D-series = gauging $\mathbb{Z}_2$ ($\mathbb{Z}_2$ orbifold of A-series)
- Non diagonal, also contains an emergent $\mathbb{Z}_2$
Staircase for D-series minimal models

- A-series contain $\mathbb{Z}_2$ global symmetry
- D-series = gauging $\mathbb{Z}_2$ (orbifold of A-series)
- Non diagonal, also contains an emergent $\mathbb{Z}_2$

- This amounts to:
  1. Add twisted sector, e.g. $\phi(\sigma + 2\pi) = - \phi(\sigma)$
  2. Restrict Hilbert space to $\mathbb{Z}_2$ invariant states
Staircase for D-series minimal models

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- **Bethe Ansatz** counterpart
  1. Allow particles to be also anti-periodic (twisted sector)
  2. States with even number of particles
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Redoing TBA, with this Bethe Ansatz as a starting point:

\[
\mathbb{Z}_2\text{-orbifold} : \quad |p| = |\langle \mathcal{C} | \Omega_L \rangle| = \sqrt{\left( 1 + \frac{1}{\sqrt{1 + Y(0)}} \right) \frac{\det \left[ 1 - \hat{G}_- \right]}{\det \left[ 1 - \hat{G}_+ \right]} }
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Staircase for D-series minimal models

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\]

\[
A\text{-series} \quad |p| = \sqrt{\left(1 + \sqrt{\frac{Y(0)}{1 + Y(0)}}\right) \frac{\det \left[ 1 - \hat{G}_- \right]}{\det \left[ 1 - \hat{G}_+ \right]}}
\]
RG flow of p-function (D-series)
Figure 5: The $p$-function for the generalized staircase model as a function the volume $L$.

Similarly to the effective central charge and the previous example of the $p$-function, this also develops plateaux (orange horizontal lines) at the values of the $p$-function entropy for the minimal models in the $D$-series.

Numerical computation. As can be seen in figure 5, the result of the numerical computation exhibits two important differences from that of the non-orbifolded theory. First, although the $p$-function decreases monotonically for most part of the flow, it starts to increase in the deep infrared, after the plateau corresponding to the Ising model. Second, the infrared value of the crosscap entropy is $\frac{1}{2} \log 2$ while that of the original staircase model is 0. This difference translates to the following behaviors of the Klein bottle partition function in the infrared:

$$Z_K(R, L) \sim (\text{original staircase})^2 \text{orbifolded staircase}. \quad (3.18)$$

To understand the physical meaning of this difference, it is useful to analyze $Z_K$ in the loop channel. In the original staircase model, there is a unique vacuum state in the $S$-sector and its contribution dominates in the infrared ($L > 1$). This is because all the states in the $T$-sector contain massive excitations. If we further take the infinite $R$ limit, the ground state energy asymptotes to zero and therefore we get $Z_K \sim e^{\log L / 2} \sim 1$. On the other hand in the orbifolded theory, both of the two sectors $S$ and $U$ have a state without excitations and their energies both asymptote to zero in the infinite $R$ limit. This is why we get $Z_K \sim 2$. Now, the crucial point is that these two states are oppositely charged under the emergent $\mathbb{Z}_2$-symmetry, which assigns $+1$ to the untwisted sector and $-1$ to the twisted sector. Physically, this means that the emergent $\mathbb{Z}_2$-symmetry is spontaneously broken in the orbifolded theory in the infrared.

Although we need to study more examples in order to draw any conclusion, it is tempting to speculate that these two features—the sudden increase of the $p$-function in the infrared and the emergent $\mathbb{Z}_2$-symmetry—are characteristic of orbifolded theories.
Figure 5: The $p$-function for the generalized staircase model as a function of the volume $L$.

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Although we need to study more examples in order to draw any conclusion, it is tempting to speculate that these two features—the sudden increase of the $p$-function in the infrared—

As discussed in Appendix A of [66], one can argue more generally that, if the $\mathbb{Z}_2$-symmetry of the original theory is unbroken, the emergent $\mathbb{Z}_2$-symmetry of the orbifolded theory must be broken and vice versa.
Similarly to the effective central charge and the previous example of the $p$-function, this also develops plateaux (orange horizontal lines) at the values of the $p$-function entropy for the minimal models in the $D$-series.

As can be seen in figure 5, the result of the numerical computation exhibits two important differences from that of the non-orbifolded theory. First, although the $p$-function decreases monotonically for most part of the flow, it starts to increase in the deep infrared, after the plateau corresponding to the Ising model. Second, the infrared value of the crosscap entropy is $\log 2$ while that of the original staircase model is 0. This difference translates to the following behaviors of the Klein bottle partition function in the infrared:

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$$Z_K \sim e^{LE/2R} \sim 1.$$  

On the other hand in the orbifolded theory, both of the two sectors $S$ and $U$ have a state without excitations and their energies both asymptote to zero in the infinite $R$-limit. This is why we get

$$Z_K \sim 2.$$  

Now, the crucial point is that these two states are oppositely charged under the emergent $Z_2$-symmetry, which assigns +1 to the untwisted sector and -1 to the twisted sector. Physically, this means that the emergent $Z_2$-symmetry is spontaneously broken in the orbifolded theory in the infrared.

Although we need to study more examples in order to draw any conclusion, it is tempting to speculate that these two features—the sudden increase of the $p$-function in the infrared and the spontaneous breaking—may be common to other orbifolded theories.
RG flow of $p$-function (D-series)

Figure 5: The $p$-function for the generalized staircase model as a function the volume $L$.

Similarly to the effective central charge and the previous example of the $p$-function, this also develops plateaux (orange horizontal lines) at the values of the $p$-function entropy for the minimal models in the D-series.

Numerical computation. As can be seen in figure 5, the result of the numerical computation exhibits two important differences from that of the non-orbifolded theory. First, although the $p$-function decreases monotonically for most part of the flow, it starts to increase in the deep infrared, after the plateau corresponding to the Ising model.

Second, the infrared value of the crosscap entropy is $1/2 \log 2$ while that of the original staircase model is 0. This difference translates to the following behaviors of the Klein bottle partition function in the infrared:

$$Z_K(R, L) \approx \begin{cases} 1 & \text{original staircase} \\ 2 & \text{orbifolded staircase} \end{cases}$$

(3.18)

To understand the physical meaning of this difference, it is useful to analyze $Z_K$ in the loop channel. In the original staircase model, there is a unique vacuum state in the $S$-sector and its contribution dominates in the infrared ($L \to 1$). This is because all the states in the $T$-sector contain massive excitations. If we further take the infinite $R$ limit, the ground state energy asymptotes to zero and therefore we get $Z_K \approx e^{L \log 2} \approx 1$. On the other hand in the orbifolded theory, both of the two sectors $S$ and $U$ have a state without excitations and their energies both asymptote to zero in the infinite $R$ limit. This is why we get $Z_K \approx 2$. Now, the crucial point is that these two states are oppositely charged under the emergent $\mathbb{Z}_2$-symmetry, which assigns +1 to the untwisted sector and 1 to the twisted sector. Physically, this means that the emergent $\mathbb{Z}_2$-symmetry is spontaneously broken in the infrared.

Although we need to study more examples in order to draw any conclusion, it is tempting to speculate that these two features—the sudden increase of the $p$-function in the infrared and the spontaneous breaking of the emergent $\mathbb{Z}_2$-symmetry—can be understood in terms of the original $\mathbb{Z}_2$-symmetry of the theory. As discussed in Appendix A of [66], one can argue more generally that, if the $\mathbb{Z}_2$-symmetry of the original theory is unbroken, the emergent $\mathbb{Z}_2$-symmetry of the orbifolded theory must be broken and vice versa.
The theory is unbroken, the emergent orbifolded theory in the infrared.

Physically, this means that the emergent excitations and their energies both asymptote to zero in the infinite ground state energy asymptotes to zero and therefore we get its contribution dominates in the infrared (\( \sim 1/2 \log 2 \)).

In the original staircase model, there is a unique vacuum state in the channel. In the deep infrared, after the plateau corresponding to the Ising model, although the \( p \)-function decreases monotonically for most part of the flow, it starts to increase.

We get the emergent shifts from that of the non-orbifolded theory. First, the infrared value of the crosscap entropy is 0. This difference comes from the fact that the orbifolded theory in the deep infrared, after the plateau corresponding to the Ising model.

The RG flow of the \( p \)-function for the generalized staircase model as a function the volume.

Similarly to the minimal models, the orbifolded staircase theory is 0. This difference comes from the fact that the orbifolded theory must be broken and vice versa.

As discussed in Appendix A of \([1] \), one can argue more generally that, if the theory is unbroken, the emergent orbifolded theory in the infrared.

Third, the infrared value of the crosscap entropy is 0. This difference comes from the fact that the orbifolded theory must be broken and vice versa.

In the deep infrared, after the plateau corresponding to the Ising model, although the \( p \)-function decreases monotonically for most part of the flow, it starts to increase.
RG flow of p-function (D-series)

\[ Z_K(R, L)^{R, L \gg 1} \begin{cases} 
1 & \text{original staircase} \\
2 & \text{orbifolded staircase}
\end{cases} \]
To understand the physical meaning of this difference, it is useful to analyze partition functions in the infrared: 

\[ Z_{\mathbb{K}}(R, L) \sim 1 \begin{cases} 
1 & \text{original staircase} \\
2 & \text{orbifolded staircase} 
\end{cases} \]

Deep IR: unique vacuum

Deep IR: 2 vacua,

\[ Z_2 \] symmetry spontaneously broken
The theory is unbroken, the emergent excitations and their energies both asymptote to zero in the infinite limit. On the other hand in the orbifolded theory, both of the two sectors have a state without entropy translates to the following behaviors of the Klein bottle partition function in the infrared:

\[ Z_{K}(R, L) \overset{R,L\geq 1}{\sim} \begin{cases} 1 & \text{original staircase} \\ 2 & \text{orbifolded staircase} \end{cases} \]

- **Deep IR**: unique vacuum
- **Deep IR**: 2 vacua, \( Z_2 \) symmetry spontaneously broken

Along the D-series, \( p \)-function is monotonically decreasing. It increases in the deep IR in a symmetry breaking phase.

To understand the physical meaning of this difference, one can argue more generally that, if the orbifolded theory of an effective central charge and the previous example of the minimal models in the orbifolded staircase model is 0. This difference translates to the following behaviors of the Klein bottle partition function in the infrared:

\[ Z_{K}(R, L) \overset{R,L\geq 1}{\sim} \begin{cases} 1 & \text{original staircase} \\ 2 & \text{orbifolded staircase} \end{cases} \]

- **Deep IR**: unique vacuum
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Along the D-series, \( p \)-function is monotonically decreasing. It increases in the deep IR in a symmetry breaking phase.
Crosscap states in spin chains
Crosscap states in spin chains

- XXX SU(2) spin chain

\[ H_{SU(2)} \propto \sum_j \vec{S}_j \cdot \vec{S}_{j+1} \]
Crosscap states in spin chains

- XXX SU(2) spin chain
  \[ H_{SU(2)} \propto \sum_j \vec{S}_j \vec{S}_{j+1} \]

- Mimic the definition in field theory: identify states on antipodal sites of the chain:
  \[ |c\rangle_j \equiv |\uparrow\rangle_j \otimes |\uparrow\rangle_{j+\frac{L}{2}} + |\downarrow\rangle_j \otimes |\downarrow\rangle_{j+\frac{L}{2}} \]
Crosscap states in spin chains

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\[ |c\rangle_j \equiv |\uparrow \rangle_j \otimes |\uparrow \rangle_{j+\frac{L}{2}} + |\downarrow \rangle_j \otimes |\downarrow \rangle_{j+\frac{L}{2}} \]

\[ |C\rangle \equiv \prod_{j=1}^{L/2} \left( |c\rangle_j \right) \otimes \text{Long-range entangled} \]

(As opposed to the short-range entangled in spin chain boundary state)

\[ |b\rangle_j \sim |\uparrow \rangle_j \otimes |\uparrow \rangle_{j+1} + |\downarrow \rangle_j \otimes |\uparrow \rangle_{j+1} + |\uparrow \rangle_j \otimes |\downarrow \rangle_{j+1} \]
Crosscap states in spin chains

- XXX SU(2) spin chain
  \[ H_{SU(2)} \propto \sum_j \vec{S}_j \cdot \vec{S}_{j+1} \]

- Mimic the definition in field theory: identify states on antipodal sites of the chain:
  \[ |c\rangle_j \equiv |\uparrow\rangle_j \otimes |\uparrow\rangle_{j+\frac{L}{2}} + |\downarrow\rangle_j \otimes |\downarrow\rangle_{j+\frac{L}{2}} \]

- One can show:
  \[ (T(u) - T(-u)) |\mathcal{C}\rangle = 0 \iff Q_{2n+1} |\mathcal{C}\rangle = 0 \]
  \((\infty \text{ many conserved charges})\)
Crosscap states in spin chains

\[ |\mathcal{C}\rangle \equiv \prod_{j=1}^{L/2} (|c\rangle_j) \otimes \]

This section generalizes the derivation of exact g-functions in integrable field theories to overlaps between the crosscap state and arbitrary excited states. In subsection 2.1, we discuss general properties of the crosscap state and the partition function on the Klein bottle. We also give a definition of the crosscap entropy and explain its relation to the crosscap overlap. In subsection 2.2, we compute the crosscap overlap in integrable field theories. Throughout this section, we assume that the theory is parity-invariant and excitations are scalar.

2.1 Klein bottle and crosscap entropy

To define crosscaps, we cut out a disk from a two-dimensional surface and identify antipodal points on the boundary of the disk (see figure 1-(a)). This manipulation makes the surface non-orientable and the state created by this procedure is called the crosscap state. Two commonly-studied closed non-orientable surfaces are \( \mathbb{RP}^2 \) and the Klein bottle. They can be obtained by inserting one or two crosscap states on \( S^2 \) respectively. The crosscap states were studied extensively in 2d CFT, where part of the motivation came from the analysis of string theory in orientifold spacetimes [39–42].

To compute the crosscap overlaps, we consider a cylinder of length \( R \) and circumference \( L \) and contract the two ends with the crosscap states (see figure 1-(b)). This makes the surface topologically equivalent to the Klein bottle. As mentioned above, the Klein bottle can also be obtained by inserting two crosscaps on \( S^2 \), but here it is important to start with the cylinder, which is locally flat, since our interest is in massive QFT, not CFT.

The partition function of this Klein bottle \( Z_K(R, L) \) expands in different channels, depending on either we view \( R \) or \( L \) as the (imaginary) time direction. If we take \( R \) as the time direction, we obtain an expansion

\[
Z_K(R, L) = \sum_{\lambda, \mu} e^{\lambda E} L^{\mu} \langle h C | L i 2 R \rangle 1 + \cdots.
\]

Here \( ` \) is the state defined on the spatial length \( ` \), \( |C_i \rangle \) is the crosscap state, and \( \lambda \) is the ground state. In the literature, this channel is often called the tree channel.

The expansion in the other channel (called the loop channel) is slightly more complicated (see figure 2). Owing to the antipodal identification at the boundary of the cylinder, the Hilbert space in the other channel is defined on a circle of length 2 \( R \) not \( R \). As can be seen...
Crosscap states in spin chains

\[ | \mathcal{C} \rangle \equiv \prod_{j=1}^{L/2} (|c\rangle_j) \otimes \]

In this section, we generalize the derivation of exact \( g \)-functions in integrable field theories to overlaps between the crosscap state and arbitrary excited states. In subsection 2.1, we discuss general properties of the crosscap state and the partition function on the Klein bottle. We also give a definition of the crosscap entropy and explain its relation to the crosscap overlap. In subsection 2.2, we compute the crosscap overlap in integrable field theories. Throughout this section, we assume that the theory is parity-invariant and excitations are scalar.
Crosscap states in spin chains

\[ |\mathcal{C}\rangle \equiv \prod_{j=1}^{L/2} (|c\rangle_j)^\otimes \]

\[
\frac{\langle \mathcal{C} | u \rangle}{\sqrt{\langle u | u \rangle}} = \sqrt{\frac{\det G_+}{\det G_-}}
\]

\[
\mathcal{H}_\pm(u, v) = \frac{1}{i} \partial_u [\log S(u, v) \pm \log S(u, -v)]
\]
Crosscap states in spin chains

\[ \frac{\langle \mathcal{C} | u \rangle}{\sqrt{\langle u | u \rangle}} = \sqrt{\frac{\det G_+}{\det G_-}} \]

Boundary overlap:

\[ \frac{\langle \mathcal{B} | u \rangle}{\sqrt{\langle u | u \rangle}} = (\text{non-universal factor}) \times \sqrt{\frac{\det G_+}{\det G_-}} \]
Conclusions

• Studied crosscap states in integrable models: **integrability is preserved**

• Exactly computed crosscap overlaps

• Observed monotonically decrease of p-function under RG for A-series.

• Generalized staircase to the D-series (also discussed in the paper: generalization to fermionic integrable models)

• In the D-series it also decreases, except in the deep IR in a symmetry breaking phase, where the theory becomes massive.
Outlook

- Study further the behaviour of the p-function under RG: is there a p-theorem under certain assumptions?
- Crosscap state as a initial state for a quantum quench?
- Generalize overlap formula to more general theories, such as theories with bound states and theories with non-diagonal scatterings
- Setup in AdS/CFT realizing crosscap states
THANK YOU