Shuffle algebras and integrability

Alexandr Garbali

in collaborations with J. de Gier, A. Negut and P. Zinn-Justin

School of Mathematics and Statistics, The University of Melbourne

GGI, Florence May 2022

THE UNIVERSITY OF MELBOURNE

Overview

- Algebras of currents and shuffle algebras
- The trigonometric Feigin–Odesskii shuffle algebra
- Commuting family of differential operators and their eigensystem
- *R*-matrices from shuffle algebras
- Transfer matrices and the elliptic Feigin–Odesskii shuffle algebra

- Let $\zeta(z)$ be a meromorphic function. Consider an algebra $\mathcal{E}^- = \langle e_n, n \in \mathbb{Z} \rangle$

$$e(z) = \sum_{n \in \mathbb{Z}} z^{-n} e_n, \qquad e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w)$$

- Let $\zeta(z)$ be a meromorphic function. Consider an algebra $\mathcal{E}^- = \langle e_n, n \in \mathbb{Z} \rangle$

$$e(z) = \sum_{n\in\mathbb{Z}} z^{-n} e_n, \qquad e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w)$$

- Consider a map $e_n \to x_1^n$:

$$e_n = \mathcal{O}(x_1^n), \qquad e_n = \oint \frac{\mathrm{d}\, x_1}{x_1} x_1^n e(x_1)$$

- Let $\zeta(z)$ be a meromorphic function. Consider an algebra $\mathcal{E}^- = \langle e_n, n \in \mathbb{Z} \rangle$

$$e(z) = \sum_{n \in \mathbb{Z}} z^{-n} e_n, \qquad e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w)$$

- Consider a map $e_n \to x_1^n$:

$$e_n = \mathcal{O}(x_1^n), \qquad e_n = \oint \frac{\mathrm{d}\, x_1}{x_1} x_1^n e(x_1)$$

- Write the product $e_n e_m$ using the above integral

$$\begin{aligned} e_n e_m = \mathcal{O}(x_1^n)\mathcal{O}(x_1^m) &= \oint \frac{\mathrm{d}\, x_1 \,\mathrm{d}\, x_2}{x_1 x_2} x_1^n x_2^m e(x_1) e(x_2) \\ &= \frac{1}{2} \oint \frac{\mathrm{d}\, x_1 \,\mathrm{d}\, x_2}{x_1 x_2} \left(x_1^n x_2^m \zeta(x_1/x_2) + x_2^n x_1^m \zeta(x_2/x_1) \right) \frac{e(x_1)e(x_2)}{\zeta(x_1/x_2)} \end{aligned}$$

- Let $\zeta(z)$ be a meromorphic function. Consider an algebra $\mathcal{E}^- = \langle e_n, n \in \mathbb{Z} \rangle$

$$e(z) = \sum_{n \in \mathbb{Z}} z^{-n} e_n, \qquad e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w)$$

- Consider a map $e_n \to x_1^n$:

$$e_n = \mathcal{O}(x_1^n), \qquad e_n = \oint \frac{\mathrm{d}\, x_1}{x_1} x_1^n e(x_1)$$

- Write the product $e_n e_m$ using the above integral

$$e_n e_m = \mathcal{O}(x_1^n)\mathcal{O}(x_1^m) = \oint \frac{\mathrm{d}\, x_1 \,\mathrm{d}\, x_2}{x_1 x_2} x_1^n x_2^m e(x_1) e(x_2)$$
$$= \frac{1}{2} \oint \frac{\mathrm{d}\, x_1 \,\mathrm{d}\, x_2}{x_1 x_2} \left(x_1^n x_2^m \zeta(x_1/x_2) + x_2^n x_1^m \zeta(x_2/x_1)\right) \frac{e(x_1)e(x_2)}{\zeta(x_1/x_2)}$$

- The integrand becomes a special kind of product of kernels x_1^n and x_1^m

$$x_1^n * x_1^m = x_1^n x_2^m \zeta(x_1/x_2) + x_2^n x_1^m \zeta(x_2/x_1)$$

  We call $*$ the *shuffle product*. We have a map $e_n e_m \to x_1^n * x_1^m$

$$e_n e_m = \mathcal{O}(x_1^n * x_1^m) = \frac{1}{2} \oint \frac{\mathrm{d}\, x_1 \,\mathrm{d}\, x_2}{x_1 x_2} (x_1^n * x_1^m) \frac{e(x_1)e(x_2)}{\zeta(x_1/x_2)}$$

- Generalize this to products of k operators $e_{n_1} \cdots e_{n_k} \to x_1^{n_1} * \cdots * x_1^{n_k}$

$$e_{n_1} \cdots e_{n_k} = \mathcal{O}(x_1^{n_1} * \cdots * x_1^{n_k}) = \frac{1}{k!} \oint \frac{\mathrm{d}\, x_1 \cdots \mathrm{d}\, x_k}{x_1 \cdots x_k} x_1^{n_1} * \cdots * x_1^{n_k} \frac{e(x_1) \cdots e(x_k)}{\prod_{i<j} \zeta(x_i/x_j)}$$

$$\text{where} \qquad x_1^{n_1} * x_1^{n_2} * \cdots * x_1^{n_k} = \mathrm{Sym}\, x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \prod_{1 \le i < j \le k} \zeta(x_i/x_j)$$

- Generalize this to products of k operators $e_{n_1} \cdots e_{n_k} \to x_1^{n_1} * \cdots * x_1^{n_k}$

$$e_{n_1} \cdots e_{n_k} = \mathcal{O}(x_1^{n_1} * \cdots * x_1^{n_k}) = \frac{1}{k!} \oint \frac{\mathrm{d}\, x_1 \cdots \mathrm{d}\, x_k}{x_1 \cdots x_k} x_1^{n_1} * \cdots * x_1^{n_k} \frac{e(x_1) \cdots e(x_k)}{\prod_{i<j} \zeta(x_i/x_j)}$$

$$\text{where} \qquad x_1^{n_1} * x_1^{n_2} * \cdots * x_1^{n_k} = \mathrm{Sym}\, x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \prod_{1 \le i < j \le k} \zeta(x_i/x_j)$$

- The next step is to take linear combinations:

$$F(x_1, \ldots, x_k) = \sum_{n_1 \ldots n_k} c_{n_1 \ldots n_k} x_1^{n_1} * \cdots * x_1^{n_k}$$

This produces a *graded vector space* of functions with properties determined by ζ. Recover the corresponding element of $\mathcal{E}^-$ by applying $\mathcal{O}$

$$\mathcal{E}^- \ni \mathcal{O}(F) := \frac{1}{k!} \oint \frac{\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k}{x_1 \ldots x_k} F(x_1, \ldots, x_k) \frac{e(x_1) \ldots e(x_k)}{\prod_{1 \le i < j \le k} \zeta(x_i/x_j)}$$

and by construction we have $\mathcal{O}(F * G) = \mathcal{O}(F)\mathcal{O}(G)$.

- Generalize this to products of k operators $e_{n_1} \cdots e_{n_k} \to x_1^{n_1} * \cdots * x_1^{n_k}$

$$e_{n_1} \cdots e_{n_k} = \mathcal{O}(x_1^{n_1} * \cdots * x_1^{n_k}) = \frac{1}{k!} \oint \frac{\mathrm{d}\, x_1 \cdots \mathrm{d}\, x_k}{x_1 \cdots x_k} x_1^{n_1} * \cdots * x_1^{n_k} \frac{e(x_1) \cdots e(x_k)}{\prod_{i<j} \zeta(x_i/x_j)}$$

$$\text{where} \qquad x_1^{n_1} * x_1^{n_2} * \cdots * x_1^{n_k} = \mathrm{Sym}\, x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \prod_{1 \le i < j \le k} \zeta(x_i/x_j)$$

- The next step is to take linear combinations:

$$F(x_1, \ldots, x_k) = \sum_{n_1 \ldots n_k} c_{n_1 \ldots n_k} x_1^{n_1} * \cdots * x_1^{n_k}$$

This produces a *graded vector space* of functions with properties determined by ζ. Recover the corresponding element of $\mathcal{E}^-$ by applying $\mathcal{O}$

$$\mathcal{E}^- \ni \mathcal{O}(F) := \frac{1}{k!} \oint \frac{\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k}{x_1 \ldots x_k} F(x_1, \ldots, x_k) \frac{e(x_1) \ldots e(x_k)}{\prod_{1 \le i < j \le k} \zeta(x_i/x_j)}$$

and by construction we have $\mathcal{O}(F * G) = \mathcal{O}(F)\mathcal{O}(G)$.

Application of the idea: assume we found $F(x_1, \ldots, x_k)$ and $G(x_1, \ldots, x_l)$ s.t.:

$$[F(x_1, \ldots, x_k), G(x_1, \ldots, x_l)]_* = 0 \qquad \Rightarrow \qquad [\mathcal{O}(F), \mathcal{O}(G)] = 0$$

The trigonometric Feigin–Odesskii algebra[1] $\mathcal{A}^-$

- Fix the base field $\mathbb{F} = \mathbb{Q}(q, t)$. Take the algebra

$$e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w), \qquad \zeta(x) = \frac{(x-1)(x-qt^{-1})}{(x-q)(x-t^{-1})}$$

[1] Feigin and Odesskii '97; FHHSY '09; Feigin and Tsymbaliuk '11; Schiffmann and Vasserot '13; Negut '12

The trigonometric Feigin–Odesskii algebra[1] $\mathcal{A}^-$

- Fix the base field $\mathbb{F} = \mathbb{Q}(q, t)$. Take the algebra

$$e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w), \qquad \zeta(x) = \frac{(x-1)(x-qt^{-1})}{(x-q)(x-t^{-1})}$$

- The above construction gives rise to a graded algebra $\mathcal{A}^-$ which is generated by the elements x_1^n and has the product:

[1] Feigin and Odesskii '97; FHHSY '09; Feigin and Tsymbaliuk '11; Schiffmann and Vasserot '13; Negut '12

The trigonometric Feigin–Odesskii algebra[1] $\mathcal{A}^-$

- Fix the base field $\mathbb{F} = \mathbb{Q}(q,t)$. Take the algebra

$$e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w), \qquad \zeta(x) = \frac{(x-1)(x-qt^{-1})}{(x-q)(x-t^{-1})}$$

- The above construction gives rise to a graded algebra $\mathcal{A}^-$ which is generated by the elements x_1^n and has the product:

For $F \in \mathcal{A}_k^-$ and $G \in \mathcal{A}_l^-$ the product $F * G \in \mathcal{A}_{k+l}^+$

$$(F * G)(x_1, \ldots, x_{k+l}) = \frac{1}{k!l!} \operatorname{Sym} F(x_1, \ldots, x_k) G(x_{k+1}, \ldots, x_{k+l}) \prod_{\substack{i \in \{1,\ldots,k\} \\ j \in \{k+1,\ldots,k+l\}}} \zeta(\frac{x_i}{x_j})$$

[1] Feigin and Odesskii '97; FHHSY '09; Feigin and Tsymbaliuk '11; Schiffmann and Vasserot '13; Negut '12

The trigonometric Feigin–Odesskii algebra[1] $\mathcal{A}^-$

- Fix the base field $\mathbb{F} = \mathbb{Q}(q, t)$. Take the algebra

$$e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w), \qquad \zeta(x) = \frac{(x-1)(x-qt^{-1})}{(x-q)(x-t^{-1})}$$

- The above construction gives rise to a graded algebra $\mathcal{A}^-$ which is generated by the elements x_1^n and has the product:

For $F \in \mathcal{A}_k^-$ and $G \in \mathcal{A}_l^-$ the product $F * G \in \mathcal{A}_{k+l}^+$

$$(F * G)(x_1, \ldots, x_{k+l}) = \frac{1}{k!l!} \operatorname{Sym} F(x_1, \ldots, x_k) G(x_{k+1}, \ldots, x_{k+l}) \prod_{\substack{i \in \{1,\ldots,k\} \\ j \in \{k+1,\ldots,k+l\}}} \zeta(\frac{x_i}{x_j})$$

- Examples:

$$\mathcal{A}_0^- = \mathbb{F}, \qquad \mathcal{A}_1^- = \mathbb{F}[x_1^{\pm}]$$

$$x_1^m * x_1^n = x_1^m x_2^n \frac{(x_1 - x_2)(qx_1 - tx_2)}{(qx_1 - x_2)(x_1 - tx_2)} + x_2^m x_1^n \frac{(x_1 - x_2)(tx_1 - qx_2)}{(x_1 - qx_2)(tx_1 - x_2)} \in \mathcal{A}_2^-$$

---

[1] Feigin and Odesskii '97; FHHSY '09; Feigin and Tsymbaliuk '11; Schiffmann and Vasserot '13; Negut '12

Properties of the elements of $\mathcal{A}^-$

The elements of $\mathcal{A}_n^-$ are rational functions of the from

$$F(x_1,\dots,x_n) = \frac{f(x_1,\dots,x_n)\prod_{i\neq j}(x_i - x_j)}{\prod_{i\neq j}(x_i - qx_j)(x_i - t^{-1}x_j)}, \qquad f(x_1,\dots,x_n) \in \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]^{\mathcal{S}_n}$$

The elements of $\mathcal{A}^-$ satisfy the *wheel condition*

$$f(x_1,\dots,x_n) = 0 \quad \text{if} \quad (x_i,x_j,x_k) = (x, \frac{q}{t}x, qx) \quad \text{and} \quad (x_i,x_j,x_k) = (x, \frac{q}{t}x, \frac{1}{t}x)$$

Recursive proof: assume $F \in \mathcal{A}_k^-$ and $G \in \mathcal{A}_l^-$ satisfy the wheel condition, write

$$F * G = \frac{1}{k!l!}\frac{\prod_{i\neq j}(x_i - x_j)}{\prod_{i\neq j}(x_i - qx_j)(x_i - t^{-1}x_j)}\times$$

$$\times \mathrm{Sym}\, f(x_1,\dots,x_k)g(x_{k+1},\dots,x_{k+l}) \prod_{\substack{i\in\{1,\dots,k\} \\ j\in\{k+1,\dots,k+l\}}} \frac{(x_i - qx_j)(x_i - t^{-1}x_j)(x_i - qt^{-1}x_j)}{x_i - x_j}$$

The second line is a symmetric Laurent polynomial satisfying the wheel condition because so do f, g and the product.

The commutative subalgebra $\mathcal{A}^0$

- Consider a subalgebra $\mathcal{A}^0 \subset \mathcal{A}^-$ of the elements $F \in \mathcal{A}^0$ for which

$$\lim_{\xi \to 0} F(\xi x_1, \ldots, \xi x_r, x_{r+1}, \ldots, x_n) = \lim_{\xi \to \infty} F(\xi x_1, \ldots, \xi x_r, x_{r+1}, \ldots, x_n), \qquad \forall r$$

This condition says that the rational function F has degree 0.

The commutative subalgebra $\mathcal{A}^0$

- Consider a subalgebra $\mathcal{A}^0 \subset \mathcal{A}^-$ of the elements $F \in \mathcal{A}^0$ for which

$$\lim_{\xi \to 0} F(\xi x_1, \ldots, \xi x_r, x_{r+1}, \ldots, x_n) = \lim_{\xi \to \infty} F(\xi x_1, \ldots, \xi x_r, x_{r+1}, \ldots, x_n), \qquad \forall r$$

  This condition says that the rational function F has degree 0.

Proposition [FHHSY]. The algebra $(\mathcal{A}^0, *)$ is commutative and the dimension of the graded subspace $\mathcal{A}^0_n$ is equal to the number of partitions of n.

The commutative subalgebra $\mathcal{A}^0$

- Consider a subalgebra $\mathcal{A}^0 \subset \mathcal{A}^-$ of the elements $F \in \mathcal{A}^0$ for which

$$\lim_{\xi \to 0} F(\xi x_1, \ldots, \xi x_r, x_{r+1}, \ldots, x_n) = \lim_{\xi \to \infty} F(\xi x_1, \ldots, \xi x_r, x_{r+1}, \ldots, x_n), \qquad \forall r$$

This condition says that the rational function F has degree 0.

Proposition [FHHSY]. The algebra $(\mathcal{A}^0, *)$ is commutative and the dimension of the graded subspace $\mathcal{A}^0_n$ is equal to the number of partitions of n.

- Simple elements in $\mathcal{A}^0$:

$$\epsilon_n(p) = \prod_{1 \leq i \neq j \leq n} \frac{(x_i - p x_j)(x_i - x_j)}{(x_i - q x_j)(x_i - t^{-1} x_j)}, \qquad \text{for } p = q, t^{-1}, q^{-1} t$$

Due to commutativity the products $\epsilon_i(p) * \epsilon_j(p)$ can be ordered, therefore

$$\epsilon_\lambda(p) = \epsilon_{\lambda_1}(p) * \epsilon_{\lambda_2}(p) * \cdots * \epsilon_{\lambda_{\ell(\lambda)}}(p), \qquad \lambda \in \mathcal{P}$$

give three bases.

Elements of $\mathcal{A}^0$ as shuffle p-commutators

- Define the following element recursively:

$$K_1(p) := 1, \qquad K_n(p) = \frac{1}{x_1 + \cdots + x_n} \left(K_{n-1}(p) * x_1 - p\, x_1 * K_{n-1}(p) \right)$$

One can show that the denominator $x_1 + \cdots + x_n$ cancels for any p, so $K_n(p) \in \mathcal{A}^0$.

[2] AG and P. Zinn-Justin'21

Elements of $\mathcal{A}^0$ as shuffle p-commutators

- Define the following element recursively:

$$K_1(p) := 1, \qquad K_n(p) = \frac{1}{x_1 + \cdots + x_n} \left(K_{n-1}(p) * x_1 - p\, x_1 * K_{n-1}(p) \right)$$

  One can show that the denominator $x_1 + \cdots + x_n$ cancels for any p, so $K_n(p) \in \mathcal{A}^0$.

- The solution can be written using a shuffle e.g.f. $K(v;p) = \sum_{n \geq 0} v^n K_n(p)$

$$K(v;p) = \frac{1}{1-p} \exp_* \left(\sum_{n>0} \frac{(-1)^{n+1}}{n} \frac{1-p^n}{1-q^n} v^n P_n \right) - \frac{p}{1-p},$$

  where $P_n = \text{const}\, K_n(1)$.

[2] AG and P. Zinn-Justin'21

Elements of $\mathcal{A}^0$ as shuffle p-commutators

- Define the following element recursively:

$$K_1(p) := 1, \qquad K_n(p) = \frac{1}{x_1 + \cdots + x_n} \left(K_{n-1}(p) * x_1 - p\, x_1 * K_{n-1}(p) \right)$$

One can show that the denominator $x_1 + \cdots + x_n$ cancels for any p, so $K_n(p) \in \mathcal{A}^0$.

- The solution can be written using a shuffle e.g.f. $K(v;p) = \sum_{n\geq 0} v^n K_n(p)$

$$K(v;p) = \frac{1}{1-p} \exp_* \left(\sum_{n>0} \frac{(-1)^{n+1}}{n} \frac{1-p^n}{1-q^n} v^n P_n \right) - \frac{p}{1-p},$$

where $P_n = \text{const}\, K_n(1)$.

- Special cases of $K_n(p)$:

$$K_n(p) \propto \prod_{1\leq i\neq j\leq n} \frac{(x_i - px_j)(x_i - x_j)}{(x_i - qx_j)(x_i - t^{-1}x_j)}, \qquad \text{for } p = q, t^{-1}, q^{-1}t$$

$$K_n(p) \propto \det_{1\leq i,j\leq n} \frac{1}{(x_i - p_1 x_j)(x_j - p_2 x_i)}, \qquad \text{for } p = q^{-1}, t, qt^{-1}, \quad p_1 \neq p_2 \neq p$$

The Izergin determinants suggest a connection with lattice partition functions[2].

[2] AG and P. Zinn-Justin'21

Heisenberg algebra and Fock space

Define the Heisenberg algebra $H = \{a_n, a_{-n}\}_{n>0}$

$$[a_m, a_n] = \delta_{m,-n}\, m \frac{1-q^{|m|}}{1-t^{|m|}}$$

Note $[a_n, a_m] = [a_{-n}, a_{-m}] = 0$, $m, n > 0$, so we have bases

$$a_\lambda = a_{\lambda_1} \dots a_{\lambda_{\ell(\lambda)}}.$$

This algebra acts on the vector space $\mathcal{F}$ with the lowest vector $|\varnothing\rangle$

$$a_n\,|\varnothing\rangle = 0, \qquad a_{-n}\,|\varnothing\rangle = \text{higher states}, \qquad n > 0$$

The Fock space is spanned by $|a_\lambda\rangle = a_{-\lambda}\,|\varnothing\rangle$ and the action of H is:

$$a_{-\nu}\,|a_\mu\rangle = |a_{\mu\cup\nu}\rangle\,, \qquad a_\nu\,|a_\mu\rangle = z_\nu \begin{bmatrix} m(\mu) \\ m(\nu) \end{bmatrix} |a_{\mu/\nu}\rangle$$

Analogously one defines the dual Fock space $\mathcal{F}^*$ spanned by $\langle a_\lambda|$, then

$$\langle a_\lambda | a_\mu \rangle = \delta_{\lambda,\mu} z_\lambda$$

Realization of $\mathcal{E}^-$ on the Fock space

- Recall the ring of symmetric functions $\Lambda_{\mathbb{F}}$ (use alphabet $(y) = (y_1, y_2, \dots)$). It is isomorphic to the Fock space. The Heisenberg algebra acts on $\Lambda_{\mathbb{F}}$

$$a_{-r}f(y) = p_r(y)f(y), \qquad a_r f(y) = r\frac{1-q^r}{1-t^r}\frac{\partial}{\partial p_r}f(y), \qquad f(y) \in \Lambda_{\mathbb{F}}$$

where $p_r(y) = y_1^r + y_2^r + \dots$ is the power sum and $f(y) = \sum_\mu c_\mu p_\mu(y)$.

[3] Shiraishi '06

Realization of $\mathcal{E}^-$ on the Fock space

- Recall the ring of symmetric functions $\Lambda_{\mathbb{F}}$ (use alphabet $(y) = (y_1, y_2, \dots)$). It is isomorphic to the Fock space. The Heisenberg algebra acts on $\Lambda_{\mathbb{F}}$

$$a_{-r} f(y) = p_r(y) f(y), \qquad a_r f(y) = r \frac{1-q^r}{1-t^r} \frac{\partial}{\partial p_r} f(y), \qquad f(y) \in \Lambda_{\mathbb{F}}$$

  where $p_r(y) = y_1^r + y_2^r + \dots$ is the power sum and $f(y) = \sum_\mu c_\mu p_\mu(y)$.

- The elements of $\mathcal{E}^-$ act on $\Lambda_{\mathbb{F}}$ by vertex operators[3]

$$e(z) \mapsto \eta(z), \qquad \eta(z) := \exp\left(-\sum_r \frac{1-t^r}{r} t^{-r} a_{-r} z^r \right) \exp\left(-\sum_r \frac{1-t^r}{r} a_r z^{-r} \right)$$

[3] Shiraishi '06

Realization of $\mathcal{E}^-$ on the Fock space

- Recall the ring of symmetric functions $\Lambda_{\mathbb{F}}$ (use alphabet $(y) = (y_1, y_2, \dots)$). It is isomorphic to the Fock space. The Heisenberg algebra acts on $\Lambda_{\mathbb{F}}$

$$a_{-r} f(y) = p_r(y) f(y), \qquad a_r f(y) = r \frac{1-q^r}{1-t^r} \frac{\partial}{\partial p_r} f(y), \qquad f(y) \in \Lambda_{\mathbb{F}}$$

  where $p_r(y) = y_1^r + y_2^r + \dots$ is the power sum and $f(y) = \sum_\mu c_\mu p_\mu(y)$.

- The elements of $\mathcal{E}^-$ act on $\Lambda_{\mathbb{F}}$ by vertex operators[3]

$$e(z) \mapsto \eta(z), \qquad \eta(z) := \exp\left(-\sum_r \frac{1-t^r}{r} t^{-r} a_{-r} z^r \right) \exp\left(-\sum_r \frac{1-t^r}{r} a_r z^{-r} \right)$$

- Define the normal ordering $: a_{-\lambda} a_\mu :=: a_\mu a_{-\lambda} := a_{-\lambda} a_\mu$, then

$$: \eta(z) \eta(w) := \frac{\eta(z)\eta(w)}{\zeta(z/w)}$$

[3] Shiraishi '06

Realization of $\mathcal{E}^-$ on the Fock space

- Recall the ring of symmetric functions $\Lambda_{\mathbb{F}}$ (use alphabet $(y) = (y_1, y_2, \dots)$). It is isomorphic to the Fock space. The Heisenberg algebra acts on $\Lambda_{\mathbb{F}}$

$$a_{-r} f(y) = p_r(y) f(y), \qquad a_r f(y) = r \frac{1-q^r}{1-t^r} \frac{\partial}{\partial p_r} f(y), \qquad f(y) \in \Lambda_{\mathbb{F}}$$

  where $p_r(y) = y_1^r + y_2^r + \dots$ is the power sum and $f(y) = \sum_\mu c_\mu p_\mu(y)$.

- The elements of $\mathcal{E}^-$ act on $\Lambda_{\mathbb{F}}$ by vertex operators[3]

$$e(z) \mapsto \eta(z), \qquad \eta(z) := \exp\left(- \sum_r \frac{1-t^r}{r} t^{-r} a_{-r} z^r \right) \exp\left(- \sum_r \frac{1-t^r}{r} a_r z^{-r} \right)$$

- Define the normal ordering $: a_{-\lambda} a_\mu :=: a_\mu a_{-\lambda} := a_{-\lambda} a_\mu$, then

$$: \eta(z) \eta(w) := \frac{\eta(z)\eta(w)}{\zeta(z/w)}$$

Proposition [FHHSY]. The following operators form a commutative set

$$\mathcal{O}(F) = \frac{1}{k!} \oint \frac{\mathrm{d}\, x_1 \dots \mathrm{d}\, x_k}{x_1 \dots x_k} F(x_1, \dots, x_k) : \eta(x_1) \cdots \eta(x_k) : \qquad F \in \mathcal{A}_k^0$$

[3] Shiraishi '06

Diagonalization of $\mathcal{O}(F)$

- The identification of $\mathcal{O}(F)$ with Macdonald operators is based on[4]

$$\widehat{E}_n := \frac{1}{n!}\frac{t^{-n(n+1)/2}}{(1-t^{-1})^n}\oint\prod_{i=1}^{n}\frac{\mathrm{d}\,x_i}{x_i}\prod_{1\le i\neq j\le n}\frac{(x_i-x_j)}{(x_i-t^{-1}x_j)} :\eta(x_1)\ldots\eta(x_n):$$

This is the n-th Macdonald operator which is diagonalized by Macdonald functions

$$\widehat{E}_n P_\lambda(y) = e_n(x)\big|_{x_i=q^{\lambda_i}t^{-i}} P_\lambda(y)$$

[4] FHHSY'09

Diagonalization of $\mathcal{O}(F)$

- The identification of $\mathcal{O}(F)$ with Macdonald operators is based on[4]

$$\widehat{E}_n := \frac{1}{n!}\frac{t^{-n(n+1)/2}}{(1-t^{-1})^n}\oint \prod_{i=1}^{n}\frac{\mathrm{d}\,x_i}{x_i}\prod_{1\le i\neq j\le n}\frac{(x_i-x_j)}{(x_i-t^{-1}x_j)}:\eta(x_1)\ldots\eta(x_n):$$

  This is the n-th Macdonald operator which is diagonalized by Macdonald functions

$$\widehat{E}_n P_\lambda(y) = e_n(x)\big|_{x_i=q^{\lambda_i}t^{-i}} P_\lambda(y)$$

- The identification reads

$$\mathcal{O}(\epsilon_n(q)) = \text{const}\,\widehat{E}_n$$

---

[4] FHHSY'09

Diagonalization of $\mathcal{O}(F)$

- The identification of $\mathcal{O}(F)$ with Macdonald operators is based on[4]

$$\widehat{E}_n := \frac{1}{n!}\frac{t^{-n(n+1)/2}}{(1-t^{-1})^n}\oint\prod_{i=1}^{n}\frac{\mathrm{d}\,x_i}{x_i}\prod_{1\le i\neq j\le n}\frac{(x_i-x_j)}{(x_i-t^{-1}x_j)} :\eta(x_1)\dots\eta(x_n):$$

  This is the n-th Macdonald operator which is diagonalized by Macdonald functions

$$\widehat{E}_n P_\lambda(y) = e_n(x)\big|_{x_i=q^{\lambda_i}t^{-i}} P_\lambda(y)$$

- The identification reads

$$\mathcal{O}(\epsilon_n(q)) = \text{const}\,\widehat{E}_n$$

Proposition. Given $F\in\mathcal{A}_n^0$, there exists an $S\in\Lambda_n$ such that for any partition λ

$$\mathcal{O}(F)P_\lambda(y) = S(x)\big|_{x_i=q^{\lambda_i}t^{-i}} P_\lambda(y)$$

Using this relation we are able to match known symmetric functions with shuffle algebra elements.

[4] FHHSY'09

Dual algebra $\mathcal{E}^+$

- Let $\zeta(z)$ be the same function as before. Consider the algebra $\mathcal{E}^+ = \langle f_n, n \in \mathbb{Z} \rangle$

$$f(z) = \sum_{n\in\mathbb{Z}} z^{-n} f_n, \qquad f(z)f(w)\zeta(z/w) = f(w)f(z)\zeta(w/z)$$

  Perviously we had $e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w)$.

Dual algebra $\mathcal{E}^+$

- Let $\zeta(z)$ be the same function as before. Consider the algebra $\mathcal{E}^+ = \langle f_n, n \in \mathbb{Z} \rangle$

$$f(z) = \sum_{n \in \mathbb{Z}} z^{-n} f_n, \qquad f(z)f(w)\zeta(z/w) = f(w)f(z)\zeta(w/z)$$

  Perviously we had $e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w)$.

- Repeat the same logic as before with $\mathcal{E}^+$ and find the algebra $\mathcal{A}^+ \simeq \mathcal{A}^-$.

Dual algebra $\mathcal{E}^+$

- Let $\zeta(z)$ be the same function as before. Consider the algebra $\mathcal{E}^+ = \langle f_n, n \in \mathbb{Z} \rangle$

$$f(z) = \sum_{n\in\mathbb{Z}} z^{-n} f_n, \qquad f(z)f(w)\zeta(z/w) = f(w)f(z)\zeta(w/z)$$

  Perviously we had $e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w)$.

- Repeat the same logic as before with $\mathcal{E}^+$ and find the algebra $\mathcal{A}^+ \simeq \mathcal{A}^-$.
- The operators of $\mathcal{E}^+$ act on $\Lambda_{\mathbb{F}}$ by vertex operators: $f(z) \mapsto \xi(z)$

$$\xi(z) := \exp\left(\sum_{r=1}^{\infty} \frac{1-t^r}{r} (tq)^{-r/2} a_{-r} z^r\right) \exp\left(\sum_{r=1}^{\infty} \frac{1-t^r}{r} (t/q)^{r/2} a_r z^{-r}\right)$$

Dual algebra $\mathcal{E}^+$

- Let $\zeta(z)$ be the same function as before. Consider the algebra $\mathcal{E}^+ = \langle f_n, n \in \mathbb{Z} \rangle$

$$f(z) = \sum_{n\in\mathbb{Z}} z^{-n} f_n, \qquad f(z)f(w)\zeta(z/w) = f(w)f(z)\zeta(w/z)$$

  Perviously we had $e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w)$.

- Repeat the same logic as before with $\mathcal{E}^+$ and find the algebra $\mathcal{A}^+ \simeq \mathcal{A}^-$.
- The operators of $\mathcal{E}^+$ act on $\Lambda_{\mathbb{F}}$ by vertex operators: $f(z) \mapsto \xi(z)$

$$\xi(z) := \exp\left(\sum_{r=1}^{\infty} \frac{1-t^r}{r} (tq)^{-r/2} a_{-r} z^r\right) \exp\left(\sum_{r=1}^{\infty} \frac{1-t^r}{r} (t/q)^{r/2} a_r z^{-r}\right)$$

- The resulting commutative algebra $\mathcal{O}(F)$ gives dual Macdonald operators

$$\mathcal{O}(F) = \frac{1}{k!} \oint \frac{\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_k}{x_1 \ldots x_k} F(x_1, \ldots, x_k) : \xi(x_1) \cdots \xi(x_k) :$$

  The eigenvalues are the same as before but with $q, t \to q^{-1}, t^{-1}$.

Element $\mathcal{R} \in \mathcal{E}^- \widehat{\otimes} \mathcal{E}^+$

- We can do something interesting with both $\mathcal{E}^{\pm}$. Define $\mathcal{R}_k \in \mathcal{E}^- \widehat{\otimes} \mathcal{E}^+$

$$\mathcal{R}_k := \frac{1}{k!} \oint \frac{\mathrm{d}\, x_1 \dots \mathrm{d}\, x_k}{x_1 \dots x_k} e(x_1) \dots e(x_k) \otimes f(x_1) \dots f(x_k)$$

[5] AG and A. Negut'21

Element $\mathcal{R} \in \mathcal{E}^- \widehat{\otimes} \mathcal{E}^+$

- We can do something interesting with both $\mathcal{E}^{\pm}$. Define $\mathcal{R}_k \in \mathcal{E}^- \widehat{\otimes} \mathcal{E}^+$

$$\mathcal{R}_k := \frac{1}{k!} \oint \frac{\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k}{x_1 \ldots x_k} e(x_1) \ldots e(x_k) \otimes f(x_1) \ldots f(x_k)$$

- The generating function of $\mathcal{R}_k$ can be written as a path ordered exponential

$$\mathcal{R}(u) := \sum_{k \geq 0} u^{-k} \mathcal{R}_k = \mathrm{Pexp} \left(u^{-1} \oint \frac{\mathrm{d}\, x}{x} e(x) \otimes f(x) \right)$$

[5] AG and A. Negut'21

Element $\mathcal{R} \in \mathcal{E}^- \widehat{\otimes} \mathcal{E}^+$

- We can do something interesting with both $\mathcal{E}^\pm$. Define $\mathcal{R}_k \in \mathcal{E}^- \widehat{\otimes} \mathcal{E}^+$

$$\mathcal{R}_k := \frac{1}{k!} \oint \frac{\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k}{x_1 \ldots x_k} e(x_1) \ldots e(x_k) \otimes f(x_1) \ldots f(x_k)$$

- The generating function of $\mathcal{R}_k$ can be written as a path ordered exponential

$$\mathcal{R}(u) := \sum_{k \geq 0} u^{-k} \mathcal{R}_k = \mathrm{Pexp}\left(u^{-1} \oint \frac{\mathrm{d}\, x}{x} e(x) \otimes f(x) \right)$$

- Recall the action on $\Lambda_{\mathbb{F}}$: $(e(x), f(x)) \mapsto (\eta(x), \xi(x))$. Let $c = (q/t)^{1/2}$, and

$$K := \exp\left(\sum_{r>0} \frac{1}{r} \frac{1-t^r}{1-q^r} (c^{-r} - c^r) a_r \otimes a_{-r} \right) c^d \otimes c^d$$

[5] AG and A. Negut'21

Element $\mathcal{R} \in \mathcal{E}^- \widehat{\otimes} \mathcal{E}^+$

- We can do something interesting with both $\mathcal{E}^\pm$. Define $\mathcal{R}_k \in \mathcal{E}^- \widehat{\otimes} \mathcal{E}^+$

$$\mathcal{R}_k := \frac{1}{k!} \oint \frac{\mathrm{d}\, x_1 \dots \mathrm{d}\, x_k}{x_1 \dots x_k} e(x_1) \dots e(x_k) \otimes f(x_1) \dots f(x_k)$$

- The generating function of $\mathcal{R}_k$ can be written as a path ordered exponential

$$\mathcal{R}(u) := \sum_{k \geq 0} u^{-k} \mathcal{R}_k = \mathrm{Pexp} \left(u^{-1} \oint \frac{\mathrm{d}\, x}{x} e(x) \otimes f(x) \right)$$

- Recall the action on $\Lambda_{\mathbb{F}}$: $(e(x), f(x)) \mapsto (\eta(x), \xi(x))$. Let $c = (q/t)^{1/2}$, and

$$K := \exp \left(\sum_{r > 0} \frac{1}{r} \frac{1 - t^r}{1 - q^r} (c^{-r} - c^r) a_r \otimes a_{-r} \right) c^d \otimes c^d$$

[5]The following operator satisfies the Yang–Baxter equation

$$R(u) = \sum_{k \geq 0} u^{-k} \mathcal{R}_k K = \mathrm{Pexp} \left(u^{-1} \oint \frac{\mathrm{d}\, x}{x} \eta(x) \otimes \xi(x) \right) \cdot K$$

The functional dependence on u is completely determined by the eigenvalues of the Macdonald operators $\sum_{n \geq 0} u^{-n} \mathcal{O}(\epsilon_n(q/t))$.

[5] AG and A. Negut'21

The integrable model of $R(u)$

- The matrix $R(u)$ is the R-matrix of the quantum toroidal algebra $U_{q,t}(\ddot{gl}_1)$ with the representation on the Fock space.

The integrable model of $R(u)$

- The matrix $R(u)$ is the R-matrix of the quantum toroidal algebra $U_{q,t}(\ddot{gl}_1)$ with the representation on the Fock space.
- The R-matrix $R(u)$ defines a vertex model whose edge states are labelled by the Young diagrams.

The integrable model of $R(u)$

- The matrix $R(u)$ is the R-matrix of the quantum toroidal algebra $U_{q,t}(\ddot{gl}_1)$ with the representation on the Fock space.
- The R-matrix $R(u)$ defines a vertex model whose edge states are labelled by the Young diagrams.
- The matrix $R(u)$ contains the six vertex model R-matrix as a sub-matrix in the limit $q, t \to 0$ and q/t fixed. The relevant edge labels are $(\emptyset, \Box)$. The integrable model in this limit is related to the Heisenberg spin chain.

The integrable model of $R(u)$

- The matrix $R(u)$ is the R-matrix of the quantum toroidal algebra $U_{q,t}(\ddot{gl}_1)$ with the representation on the Fock space.
- The R-matrix $R(u)$ defines a vertex model whose edge states are labelled by the Young diagrams.
- The matrix $R(u)$ contains the six vertex model R-matrix as a sub-matrix in the limit $q, t \to 0$ and q/t fixed. The relevant edge labels are $(\emptyset, \square)$. The integrable model in this limit is related to the Heisenberg spin chain.
- The integrable model of $R(u)$ generalizes the qKdV model studies by Bazhanov, Lukyanov and Zamolodchikov and the Intermediate Long Wave (ILW) equation of Litvinov.

Commuting operators from $R(u)$

- It is easy to compute the twisted trace of $R(u)$. Define the transfer matrix

$$T(u) = \mathrm{Tr}_2\left(R(u)p^{1\otimes d}\right) = \sum_{n\geq 0} T_n u^{-n}$$

The coefficients form a commuting set of operators

$$[T_n, T_m] = 0$$

[6] FKSW'07, FJM'17

Commuting operators from $R(u)$

- It is easy to compute the twisted trace of $R(u)$. Define the transfer matrix

$$T(u) = \mathrm{Tr}_2\left(R(u)p^{1\otimes d}\right) = \sum_{n\geq 0} T_n u^{-n}$$

The coefficients form a commuting set of operators

$$[T_n, T_m] = 0$$

- Set $\tilde{p} = p(q/t)^{1/2}$ and use explicit form of $R(u)$

$$T_n = \mathrm{Tr}\left(R_n\, K\, \tilde{p}^{1\otimes d}\right) c^d$$

[6] FKSW'07, FJM'17

Commuting operators from $R(u)$

- It is easy to compute the twisted trace of $R(u)$. Define the transfer matrix

$$T(u) = \mathrm{Tr}_2 \left(R(u) p^{1 \otimes d} \right) = \sum_{n \geq 0} T_n u^{-n}$$

The coefficients form a commuting set of operators

$$[T_n, T_m] = 0$$

- Set $\tilde{p} = p(q/t)^{1/2}$ and use explicit form of $R(u)$

$$T_n = \mathrm{Tr} \left(R_n \, K \, \tilde{p}^{1 \otimes d} \right) c^d$$

The operators T_n are given explicitly by[6]

$$T_n = \mathrm{const} \oint \frac{\mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n}{x_1 \cdots x_n} \prod_{1 \leq i \neq j \leq n} \frac{\Theta_{\tilde{p}}(x_j/x_i) \Theta_{\tilde{p}}(q/t x_j/x_i)}{\Theta_{\tilde{p}}(q x_j/x_i) \Theta_{\tilde{p}}(t^{-1} x_j/x_i)} : \widehat{\xi}(x_1) \cdots \widehat{\xi}(x_n) : c^d$$

where $\Theta_p(x) = (p;p)_\infty (x;p)_\infty (p/x;p)_\infty$

$$\widehat{\xi}(z) := \xi(z) \cdot \exp \left(\sum_{r=1}^{\infty} \frac{1}{r} \frac{1 - t^r}{1 - \tilde{p}} (c^r - c^{-r}) a_r z^{-r} \right)$$

[6] FKSW'07, FJM'17

Elliptic shuffle algebra

- The integral formulas for T_n are of the same type as $\mathcal{O}(F)$ but with an elliptic kernel. More precisely, start with the algebra:

$$e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w), \qquad \zeta(x) = \frac{\Theta_p(x)\Theta_p(q/tx)}{\Theta_p(qx)\Theta_p(t^{-1}x)}$$

Elliptic shuffle algebra

- The integral formulas for T_n are of the same type as $\mathcal{O}(F)$ but with an elliptic kernel. More precisely, start with the algebra:

$$e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w), \qquad \zeta(x) = \frac{\Theta_p(x)\Theta_p(q/tx)}{\Theta_p(qx)\Theta_p(t^{-1}x)}$$

- Repeat the procedure as in the trigonometric case. This produces a commutative subalgebra $\mathcal{A}^0(p)$. Distinguished set of elements of $\mathcal{A}^0(p)$:

$$\epsilon_n(p;s) = \prod_{1\leq i\neq j\leq n} \frac{\Theta_p(x_j/x_i)\Theta_p(sx_j/x_i)}{\Theta_p(qx_j/x_i)\Theta_p(t^{-1}x_j/x_i)}, \qquad \text{for } s = q^{-1}, t, qt^{-1}$$

Elliptic shuffle algebra

- The integral formulas for T_n are of the same type as $\mathcal{O}(F)$ but with an elliptic kernel. More precisely, start with the algebra:

$$e(z)e(w)\zeta(w/z) = e(w)e(z)\zeta(z/w), \qquad \zeta(x) = \frac{\Theta_p(x)\Theta_p(q/tx)}{\Theta_p(qx)\Theta_p(t^{-1}x)}$$

- Repeat the procedure as in the trigonometric case. This produces a commutative subalgebra $\mathcal{A}^0(p)$. Distinguished set of elements of $\mathcal{A}^0(p)$:

$$\epsilon_n(p;s) = \prod_{1\leq i\neq j\leq n} \frac{\Theta_p(x_j/x_i)\Theta_p(sx_j/x_i)}{\Theta_p(qx_j/x_i)\Theta_p(t^{-1}x_j/x_i)}, \qquad \text{for } s = q^{-1}, t, qt^{-1}$$

- The commuting transfer matrices T_n correspond to the elements

$$\mathcal{O}(\epsilon_n(p;q/t)) = \frac{1}{k!}\oint \frac{\mathrm{d}\,x_1\cdots \mathrm{d}\,x_k}{x_1\cdots x_k}\epsilon_n(p;q/t)\frac{e(x_1)\cdots e(x_k)}{\prod_{i<j}\zeta(x_i/x_j)}$$

where $e(x)$ acts by the dressed vertex operator $\widehat{\xi}(x)$.

Summary

- Unlike the abstract elements of $\mathcal{E}^-$, the shuffle algebra elements are explicit rational functions which can be studied using specialization techniques.
- In the shuffle algebras setup it is easier to describe commutative subalgebras. Mapping these subalgebras to $\mathcal{E}^-$ immediately produce integral formulas for commuting sets of differential operators.
- Shuffle algebras can also be used to diagonalize the commuting operators.

Future directions

- Computation of non-local integrals of motion (Q-operators) using representations of $e(x)$ on plane partitions.
- Systematic study of the elliptic shuffle algebras.
- Higher rank analogues of shuffle algebras and their commutative subalgebras.
- BC-type shuffle algebras and their connection to coideal subalgebras for quantum toroidal gl_n.
- Bethe ansatz via shuffle algebras