XXX chain: spinons, bound states and form factors.

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**XXZ spin chain**

Defined on a one-dimensional lattice with $M$ sites, with Hamiltonian, $H = H^{(0)} - h S_z$,

$$H^{(0)} = \sum_{m=1}^{M} \left\{ \sigma^x_m \sigma^x_{m+1} + \sigma^y_m \sigma^y_{m+1} + \Delta (\sigma^z_m \sigma^z_{m+1} - 1) \right\},$$

$$S_z = \frac{1}{2} \sum_{m=1}^{M} \sigma^z_m, \quad [H^{(0)}, S_z] = 0.$$

$\sigma^{x,y,z}_m$ are the local spin operators (in the spin-$\frac{1}{2}$ representation) associated with each site $m$ of the chain and $\Delta = \cos(\zeta)$, $\zeta$ real or imaginary, is the anisotropy parameter, $h$ - external magnetic field; $h \geq 0$. We impose the periodic boundary conditions.

If $\Delta = 1$ XXX Heisenberg chain (1928), solved by H. Bethe (1931).

For $h = 0$ if $\Delta > 1$ - massive antiferromagnetic regime, $|\Delta| < 1$ - massless regime.
Form Factors

Form factors: matrix elements of local fields, local spin operators $\sigma^a_m$, $a = x, y, z$

$|\Psi_g\rangle$ the ground state of the model $|\Psi_e\rangle$ - an excited state

The most basic form factors

$$|\mathcal{F}_a(\Psi_e)|^2 = \frac{\langle \Psi_g | \sigma^a_m | \Psi_e \rangle \langle \Psi_e | \sigma^a_m | \Psi_g \rangle}{\langle \Psi_g | \Psi_g \rangle \langle \Psi_e | \Psi_e \rangle}$$

Then more advanced questions can be studied like matrix elements of currents

- Integrable QFT - F. Smirnov 1992 bootstrap approach
- Massive XXZ, M. Jimbo and T. Miwa 1995 $q$-vertex operator approach
- General XXZ, N.K, J.M. Maillet, V. Terras, 1999 Algebraic Bethe ansatz approach
Massless models: computation of the correlation functions through the form factors seems to be a strange idea (at least). However the integrable systems are very special!

- Dynamical correlation functions at zero temperature:

\[ f_a(m, t) = \langle \sigma^a_{m+1}(t)\sigma^a_1(0) \rangle = \sum_{\Psi_e} \exp(it\Delta E_e - im\Delta p_e) |F_a(\Psi_e)|^2 \]

Turns out to be an excellent tool of asymptotic analysis.

- Dynamical structure factors:

\[ S(k, \omega) = \int_{-\infty}^{\infty} dt \sum_{m=-\infty}^{\infty} f_a(m, t) \exp(imk - it\omega) \]

Experimentally mesurable quantity: can be computed numerically from the form factors (J.S. Caux et al.) and asymptotically (edge exponents).
State of the art

Analytic computation of the form factors, two main approaches:

- **q-vertex operator approach**: Multiple integral representations for $h = 0$, $\Delta \geq 1$, infinite chain.
  - Advantages: explicit results for the simplest excited states
  - Disadvantages: No access to the bound states, difficulties to get past 4 spinons (XXX) or 2 spinons (XXZ).

- **Algebraic Bethe ansatz approach**: Determinant representations for the finite chain, all the regimes.
  - Advantages: Access to the asymptotics, all the regimes, possibility to treat bound states
  - Disadvantages: Final results always contain Fredholm determinants, difficulties to take the $h = 0$ limit.

The explicit results from the q-vertex operator approach were never reproduced from the ABA (with one exception: spontaneous magnetisation)
XXX chain: Algebraic Bethe ansatz


\[ T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_a. \]

- Diagonal elements \( \rightarrow \) commuting conserved charges: transfer matrix

\[ \mathcal{T}(\lambda) = \text{tr}_a T_a(\lambda) = A(\lambda) + D(\lambda), \quad [\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0 \]

- Hamiltonian:

\[ H = 2i \frac{\partial}{\partial \lambda} \log \mathcal{T}(\lambda) \bigg|_{\lambda = \frac{i}{2}}, \quad [H, \mathcal{T}(\lambda)] = 0 \]

- Non-diagonal elements \( \rightarrow \) creation/annihilation operators.
Bethe states

Off-shell Bethe states:

\[ |\Psi(\{\lambda_1, \ldots, \lambda_N\})\rangle = B(\lambda_1) \ldots B(\lambda_N) |0\rangle, \quad |0\rangle = |\uparrow \uparrow \ldots \uparrow\rangle \]

For any Bethe state we define **Baxter polynomial** and **exponential counting function**

\[
q(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_j), \quad a(\lambda) = \left( \frac{\lambda - \frac{i}{2}}{\lambda + \frac{i}{2}} \right)^M \frac{q(\lambda + i)}{q(\lambda - i)}.
\]

if the **Bethe equations** are satisfied (on-shell Bethe state)

\[ a(\lambda_j) + 1 = 0, \quad j = 1, \ldots N \]

then it is an eigenstate of the **transfer matrix** and the Hamiltonian

\[
\mathcal{T}(\mu) |\Psi(\{\lambda\})\rangle = \tau(\mu) |\Psi(\{\lambda\})\rangle, \quad \tau(\mu) = (a(\mu) + 1) \frac{q(\mu - i)}{q(\mu)}.
\]
Multiplet structure

XXX chain: additional $\mathfrak{su}(2)$ symmetry:

$$[\mathcal{T}(\lambda), S_a] = 0, \quad a = x, y, z.$$ 

On-shell Bethe vectors are $\mathfrak{su}(2)$ highest weight vectors

$$S_+ |\Psi(\{\lambda\})\rangle = 0, \quad S_+ = \sum_{m=1}^{M} \sigma^+_m.$$ 

For XXX there are solutions of Bethe equations only if $N \leq M/2$. For $N = M/2 - k$ they generate $2k + 1$ multiplets

$$|\Psi_\ell(\{\lambda\})\rangle = S^-_\ell |\Psi(\{\lambda\})\rangle, \quad \ell = 0, \ldots, 2k, \quad \mathcal{T}(\mu) |\Psi_\ell(\{\lambda\})\rangle = \tau(\mu) |\Psi_\ell(\{\lambda\})\rangle.$$ 

Multiplets can be seen as Bethe states with infinite rapidities $\lim_{\lambda \to \infty} \lambda B(\lambda) = S_-$.
The ground state

Ground state solution of the Bethe equations

\[ a(\lambda_j) + 1 = 0, \quad j = 1, \ldots N \]

Yang and Yang 66: \( N = \frac{M}{2} \) (singlet), all the roots are real. There is no holes i.e. all the real zeroes of \( a_g(\lambda) + 1 \) are Bethe roots.
The ground state density

The Bethe roots fill the real line with some density in the thermodynamic limit:

\[
\frac{1}{M} \sum_{j=1}^{M} f(\lambda_j) = \int_{-\infty}^{\infty} f(\lambda) \rho(\lambda) d\lambda + o\left(\frac{1}{M}\right).
\]

The ground state density solves the Lieb equation

\[
\rho_g(\lambda) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} K(\lambda - \mu) \rho_g(\mu) d\mu = \frac{1}{2\pi i} t(\lambda - i/2),
\]

Where \( t(\lambda) = \frac{i}{\lambda(\lambda+i)} \) and \( K(\lambda) = t(\lambda) + t(-\lambda) \).

The ground state solution:

\[
\rho(\lambda) = \frac{1}{2 \cosh(\pi \lambda)}.
\]
Excitations: spinons

**Holes** (spinons) $\mu_h$ is not a **Bethe root** but:

$$a_e(\mu_h) + 1 = 0$$

Number of holes $n_h$ is always even, $N = \frac{M}{2} - \frac{n_h}{2}$ if all roots are real.
Complex roots: strings

Complex roots: if $\mu_+$ is one of the Bethe roots then $\mu_- = \mu_+$ is also a root (bound state). For a finite chain with large $M$ the simplest configuration: 2-string:

$$
\mu_+ = \mu_c + \frac{i}{2} - i\delta, \quad \mu_- = \mu_c - \frac{i}{2} + i\delta,
$$

Where $\mu_c \in \mathbb{R}$ - string center and $\delta = O(M^{-\infty})$ - string deviation.
Complex roots: close pairs and quartets

Destri, Lowenstein 1982; Babelon, de Vega, Viallet 1983.

**Close pair:** $\mu_+, \mu_-, \quad 0 < \Im(\mu_+) < 1$. Note: 2-string is a close pair. Otherwise close pairs form quartets:

$$\mu_+, \quad \mu_- + i - i\delta, \quad \mu_+ - i + i\delta, \quad \mu_-.$$
Complex roots: wide pairs

Wide pair: $\mu_+, \mu_-, \Im(\mu_+) > 1$. 
Excited states

We denote $n_h$ - number of holes (even), $n_s$ - number of 2-strings, $n_q$ - number of quartets, $n_w$ - number of wide pairs. Total number of Bethe roots

$$N = \frac{M}{2} - \frac{n_h}{2} + n_s + 2n_q + 2n_w, \quad N \leq \frac{M}{2}$$

Positions of holes $\mu_{h_a} \in \mathbb{R}$, $a = 1, \ldots n_h$ (arbitrary in the thermodynamic limit). The position of complex roots are defined from the position of holes from the higher level Bethe equations Destri, Lowenstein 1982; Babelon, de Vega, Viallet 1983.

$$\prod_{a=1}^{n_h} \frac{z_j - \mu_{h_a} - \frac{i}{2}}{z_j - \mu_{h_a} + \frac{i}{2}} \prod_{k=1}^{n_c} \frac{z_j - z_k + i}{z_j - z_k - i} + 1 = 0, \quad j = 1, \ldots, n_c.$$

Here $n_c = n_s + 2n_q + 2n_w$.

$z_j = \mu_c$ for a 2 string

$z_j = \mu_+ - \frac{i}{2}, \quad z_{j+1} = \mu_- + \frac{i}{2}$ for a quartet and a wide pair.
Energy and momentum

The complex roots don’t influence energy and momentum, they depend only on hole positions (spinon rapidities).

\[ \Delta E \equiv E_e - E_g = \sum_{a=1}^{n_h} \varepsilon(\mu_{h_a}), \quad \varepsilon(\mu) = \frac{\pi}{2 \cosh \pi \mu}, \]

\[ \Delta P \equiv P_e - P_g = \sum_{a=1}^{n_h} p(\mu_{h_a}), \quad p(\mu) = \frac{\pi}{2} - \arctan(\sinh \pi \mu). \]

With fixed spinon rapidities \(2^{n_h}\)-fold degeneracy.

Example: two-spinon sector, 4-fold degenerate:

- singlet with two holes \(\mu_{h_1}, \mu_{h_2}\) and one 2-string \(\mu_c + \frac{i}{2}, \mu_c - \frac{i}{2}\), higher level Bethe equations have only one solution: \(\mu_c = \frac{1}{2}(\mu_{h_1} + \mu_{h_2})\)

- triplet with two holes \(\mu_{h_1}, \mu_{h_2}\) and no complex roots
Form factors and multiplet structure

We want to compute:

\[ |\mathcal{F}_z|^2 = \frac{\langle \Psi_e | \sigma^z_m | \Psi_g \rangle \langle \Psi_g | \sigma^z_m | \Psi_e \rangle}{\langle \Psi_g | \Psi_g \rangle \langle \Psi_e | \Psi_e \rangle}, \]

for XXX no difference between \( x \), \( y \) or \( z \) form factors.

It is easy to see from \( \sigma^z_m = [S^+, \sigma_m^-] \) that there are non-trivial form factors only for the triplet states \((N = \frac{M}{2} - 1)\) and

\[
\begin{align*}
\langle \Psi_{e_1} | \sigma_m^z | \Psi_g \rangle & = -2 \langle \Psi_{e_0} | \sigma_m^+ | \Psi_g \rangle, \\
\langle \Psi_g | \sigma_m^z | \Psi_{e_1} \rangle & = \langle \Psi_g | \sigma_m^+ | \Psi_{e_2} \rangle, \\
\langle \Psi_{e_1} | \Psi_{e_1} \rangle & = 2 \langle \Psi_{e_0} | \Psi_{e_0} \rangle.
\end{align*}
\]
Form factors and inverse problem

Quantum inverse problem: local operators in terms of the monodromy matrix elements
N.K., J.M. Maillet and V. Terras 1999:

\[ \sigma^z_m = T^{m-1} \left( \frac{i}{2} \right) \left\{ A \left( \frac{i}{2} \right) - D \left( \frac{i}{2} \right) \right\} T^{-m} \left( \frac{i}{2} \right), \]
\[ \sigma^-_m = T^{m-1} \left( \frac{i}{2} \right) B \left( \frac{i}{2} \right) T^{-m} \left( \frac{i}{2} \right), \]
\[ \sigma^+_m = T^{m-1} \left( \frac{i}{2} \right) C \left( \frac{i}{2} \right) T^{-m} \left( \frac{i}{2} \right). \]

Due to the commutation relations of the monodromy matrix elements everything expressed in terms of scalar products of off-shell and on-shell multiplet Bethe states.

\[ \left| \mathcal{F}_z \right|^2 = -\frac{\tau_e \left( \frac{i}{2} \right)}{\tau_g \left( \frac{i}{2} \right)} \frac{\langle \Psi_{e0} | C \left( \frac{i}{2} \right) | \Psi_g \rangle \langle \Psi_g | C \left( \frac{i}{2} \right) | \Psi_{e2} \rangle}{\langle \Psi_g | \Psi_g \rangle \langle \Psi_{e0} | \Psi_{e0} \rangle}, \]
Scalar products and norms

N. Slavnov, 1989: \{\lambda_1, \ldots \lambda_N\} - solution of Bethe equations, \{\mu_1, \ldots \mu_N\} - generic set of parameters.

\[ \langle \Psi(\{\mu\}) | \Psi(\{\lambda\}) \rangle = \frac{\prod_{k=1}^{N} q(\mu_k - i)}{\prod_{j>k} (\lambda_j - \lambda_k)(\mu_k - \mu_j)} \det M(\{\lambda\}|\{\mu\}), \]

\[ M_{j,k}(\{\lambda\}|\{\mu\}) = a(\mu_k)t(\mu_k - \lambda_j) - t(\lambda_j - \mu_k), \quad t(\lambda) = \frac{i}{\lambda(\lambda + i)}. \]

Norms of the on-shell Bethe states are given by the Gaudin formula

\[ \langle \Psi(\{\lambda\}) | \Psi(\{\lambda\}) \rangle = (-1)^{\sum_{j=1}^{N} q(\lambda_j - i)} \frac{\prod_{j=1}^{N} q(\lambda_j - i)}{\prod_{j \neq k} (\lambda_j - \lambda_k)} \det \mathcal{N}(\{\lambda\}), \]

\[ \mathcal{N}_{j,k}(\{\lambda\}) = a'(\lambda_j)\delta_{j,k} - K(\lambda_j - \lambda_k), \quad K(\lambda) = t(\lambda) + t(-\lambda). \]
Slavnov formula can be adapted for the **multiplet states** Foda, Wheeler 2012.

Finite chain determinant representation for the factors form factors corresponding to arbitrary triplet excited state:

\[
|\mathcal{F}_z|^2 = -2 \prod_{j=1}^{M-1} \frac{q_g(\mu_j - i)}{q_e(\mu_j - i)} \prod_{k=1}^{M} \frac{q_e(\lambda_k - i)}{q_g(\lambda_k - i)} 
\]

\[
\times \frac{\det_{M/2} \mathcal{M}(\{\lambda\}|\{\mu_1 \ldots \mu_{M-1}, \frac{i}{2}\}) \det_{M/2+1} \mathcal{M}^{(2)}(\{\mu\}|\{\lambda_1, \ldots \lambda_{M}, \frac{i}{2}\})}{\det_{M/2} \mathcal{N}(\{\lambda\}) \det_{M/2-1} \mathcal{N}(\{\mu\})}.
\]

Here \(\mathcal{M}^{(2)}\) - Foda Wheeler variant of the Slavnov matrix for triplets (two extra rows).

\[
\mathcal{M}_{j,k}^{(\ell)}(\{\lambda\}|\{\mu\}) = a(\mu_k)(\mu_k + i)^{j-N-1} - \mu_k^{j-N-1}, \quad \text{for } j > N.
\]
Computation of determinants

The main idea is extremely simple: we compute the following matrices

\[ F_g = \mathcal{N}^{-1}(\{\lambda\}) \mathcal{M} \left( \{\lambda\} | \{\mu_1 \ldots \mu_{M-1}, \frac{i}{2}\} \right), \]

\[ F_e = \mathcal{N}^{(2)-1}(\{\mu\}) \mathcal{M}^{(2)} \left( \{\mu\} | \{\lambda_1, \ldots \lambda_M, \frac{i}{2}\} \right), \]

For the first (ground state) matrix: system of linear equations

\[ a'_g(\lambda_j) F_{g,j,k} - \sum_{a=1}^{M/2} K(\lambda_j - \lambda_a) F_{g,a,k} = a_g(\mu_k) t(\mu_k - \lambda_j) - t(\lambda_j - \mu_k). \]

We set

\[ a'_g(\lambda_j) F_{g,j,k} = G_g(\lambda_j; \mu_k) \]
Linear equations $\rightarrow$ Contour integral equation for $G_g(\lambda; \mu)$

\[
G_g(\lambda; \mu_k) - \frac{1}{2\pi i} \oint_G d\nu \frac{K(\lambda - \nu)}{G_g(\nu; \mu_k)} = \frac{(a_g(\mu_k) + 1)t(\mu_k - \lambda)}{1 + a_g(\nu)},
\]

We set

\[
G_g(\lambda; \mu) = (1 + a_g(\mu))\rho_g(\lambda; \mu)
\]

Thermodynamic limit $\rightarrow$ Integral equation

\[
\rho_g(\lambda; \mu) - \frac{1}{2\pi i} \int_{\mathbb{R} + i\epsilon} d\nu K(\lambda - \nu)\rho_g(\nu; \mu) = t(\mu - \lambda).
\]

**Lieb equation** for the density of Bethe roots!

Solution:

\[
F_{gj,k} = \frac{a_g(\mu_k) + 1}{a'_{g}(\lambda_j)} \frac{\pi}{\sinh \pi(\mu_k - \lambda_j)}
\]
$F_e$ is slightly more complicated (holes contributions, Foda-Wheeler rows) but also has this basic Cauchy structure.

Example without complex roots:

$$F_{e,j,k} = \frac{a_e(\lambda_k) + 1}{a'_e(\mu_j)} \left( \frac{\pi}{\sinh \pi(\lambda_k - \mu_j)} \right)$$

$$- 2\pi i \sum_{a=1}^{n_h} \frac{\rho_h(\mu_j - \mu_{ha})}{a'_e(\mu_{ha})} \frac{\pi}{\sinh \pi(\lambda_k - \mu_{ha})} \right), \quad j \leq \frac{M}{2} - 1$$

$$F_{e,M/2,k} = a_e(\lambda_k) - 1, \quad F_{e,M/2+1,k} = a_e(\lambda_k)(\lambda_k + i) - \lambda_k.$$ 

Composed only of **Cauchy columns**! Very similar structure with the complex roots.
Structure of results

After this step typically the form factor is written as a product of two Cauchy determinants and determinants of a $n_h \times n_h$ matrix and a $n_c \times n_c$ matrix.

\[
|\mathcal{F}_z|^2 = -2 \prod_{j=1}^{M/2} \frac{\tau_e(\lambda_j)}{\tau_g(\lambda_j)} \prod_{k=1}^{M/2-1} \frac{\tau_g(\mu_k)}{\tau_e(\mu_k)} \frac{\det C_g}{M/2} \frac{\det C_e}{M/2+1} \\
\times \prod_{j=1}^{M/2} \prod_{k=1}^{M/2-1} (\lambda_j - \mu_k)^2 \det \mathcal{Q}_g \det \mathcal{Q}_e \\
\prod_{j \neq k}^{M/2} (\lambda_j - \lambda_k) \prod_{j \neq k}^{M/2-1} (\mu_j - \mu_k),
\]

\[C_{jk} = \frac{\pi}{\sinh \pi (\mu_k - \lambda_j)} - \text{Cauchy determinant} \text{ (can be computed)}\]
**Thermodynamic limit : 2-spinon case**

Two-spinon form factor: no complex roots, the only “extra” matrix

\[
Q_e = \begin{pmatrix}
\frac{1}{a'_e(\mu_{h1})} & \frac{1}{a'_e(\mu_{h2})} \\
\frac{\mu_{h1} + \frac{i}{2}}{a'_e(\mu_{h1})} & \frac{\mu_{h2} + \frac{i}{2}}{a'_e(\mu_{h2})}
\end{pmatrix}, \quad \det Q_e = \frac{\mu_{h1} - \mu_{h2}}{\pi^2 M^2} \prod_{a=1}^{2} \cosh \pi \mu_{ha}
\]

With expected scaling \( \frac{1}{M^2} \). Final result for the scaled form factor:

\[
|\mathcal{Y}(\mu_{h1} - \mu_{h2})|^2 = \lim_{M \to \infty} M^2 |\mathcal{F}_z|^2 = \frac{2}{G^4(\frac{1}{2})} \left| \frac{G \left( \frac{\mu_{h1} - \mu_{h2}}{2i} \right)}{G \left( \frac{1}{2} + \frac{\mu_{h1} - \mu_{h2}}{2i} \right)} \frac{G \left( 1 + \frac{\mu_{h1} - \mu_{h2}}{2i} \right)}{G \left( \frac{3}{2} + \frac{\mu_{h1} - \mu_{h2}}{2i} \right)} \right|^2.
\]

Where \( G(z) \) iz the Barnes \( G \)-function (related to the double \( \Gamma \)-function).

\[
G(z + 1) = \Gamma(z)G(z), \quad G(1) = 1.
\]
Relation with $q$-vertex operator approach

Using integral representations for $\log G(z)$ we obtain

$$|\mathcal{Y}(h_1 - h_2)|^2 = 2e^{-I(h_1 - h_2)},$$

$$I(h_1 - h_2) = \int_0^\infty \frac{dt}{t} e^t \frac{\cos (2(h_1 - h_2)t) \cosh(2t) - 1}{\cosh(t) \sinh(2t)}.$$

This reproduces the result for the two-spinon form factor obtained in the $q$-vertex operator framework from the M. Jimbo and T. Miwa multiple integral formulas by A. H. Bougourzi, M. Couture and M. Kacir

Starting from four-spinon case the things become more interesting as we use a different basis with respect to the Jimbo-Miwa approach.
General case

General excited state with \( n_h \) spinons and \( n_c = n_s + 2n_q + 2n_w \) complex roots

General representation: pre-factor (Cauchy determinants) and finite size determinant

\[
\lim_{M \to \infty} M^{n_h} |\mathcal{F}_z|^2 = \prod_{j > k} |\mathcal{V}(\mu_{h_j} - \mu_{h_k})|^2 \mathcal{R}(\mu_h, \mu_c) \det \hat{Q}_g \det \hat{Q}_e.
\]

Here \( \mathcal{R}(\mu_h, \mu_c) \) - rational function of holes and complex roots
\( \hat{Q}_g \) and \( \hat{Q}_e \) - finite matrices also written in terms of positions of holes and complex roots. Includes also a Higher level Gaudin matrix

Simpler form for finite matrices: work in progress.

Interesting interplay with Jimbo-Miwa-Smirnov fermionic approach.
Conclusion and outlook

Advantages of the new approach:

- Explicit results, no Fredholm determinants.
- We know how to deal with bound states.
- Possibility to apply in a systematic way for all the regimes of the XXZ chain.

Open problems: out-of-equilibrium systems, overlaps instead of form factors

- Second densification (holes, complex roots distributed with some density).
- Can we apply this method far from the ground state?
- Macroscopic changes in the system (quenches).