(Interlacing) partitions
Vertex models & tilings

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GGI, May 2022

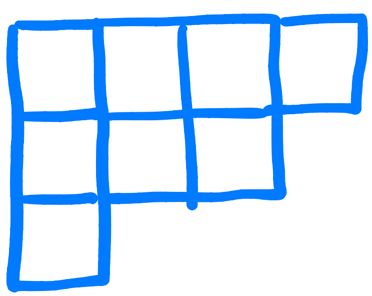
Integer partition

$$\lambda = (\lambda_1, \dots, \lambda_k)$$

$$\lambda_1 \geq \dots \geq \lambda_k$$

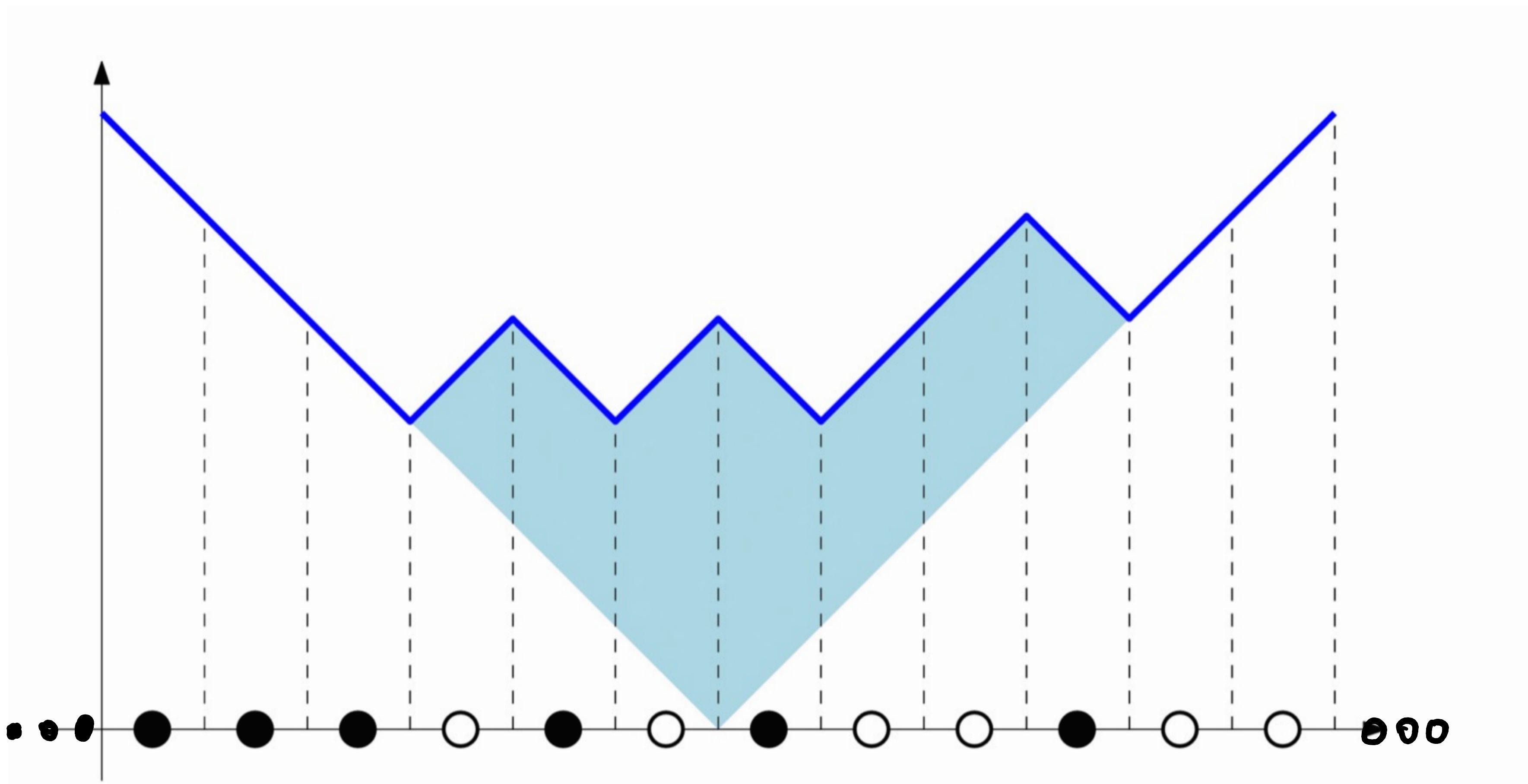
ex $\lambda = (4, 3, 1)$

4
3
1



Ferrers diagram

Maya - diagrams

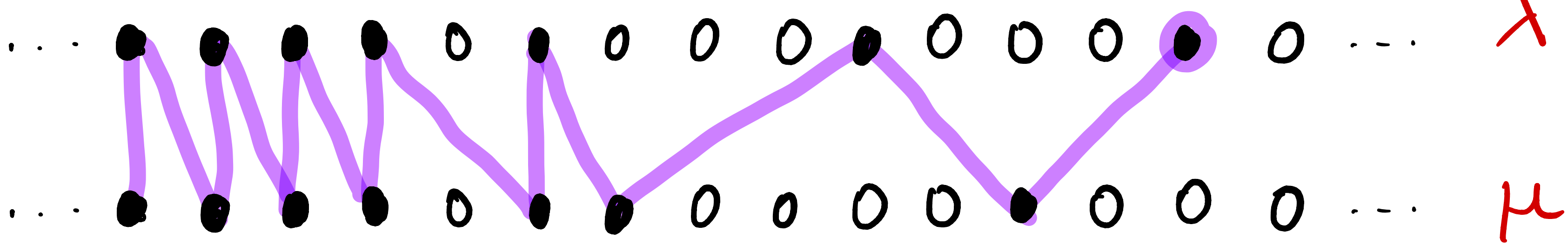


Interlacing partitions

$\lambda \succ \mu$

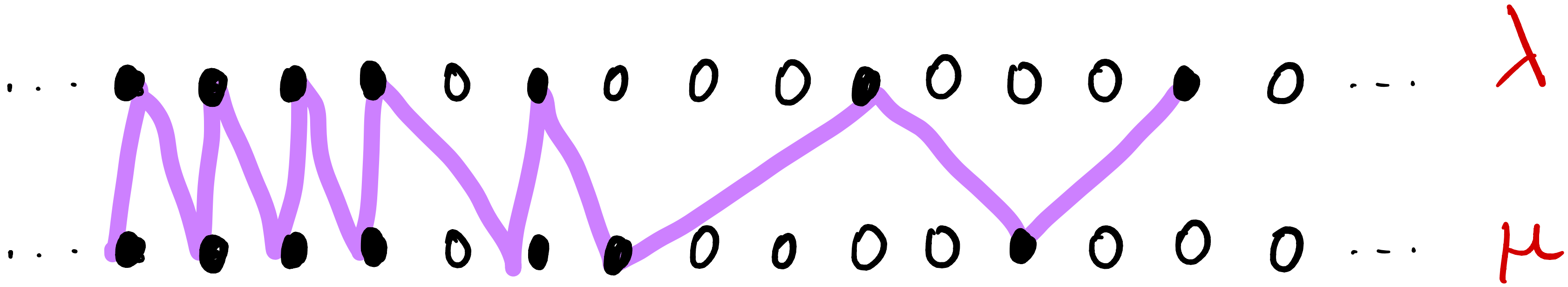
$\lambda = (\lambda_1, \lambda_2, \dots)$ $\mu = (\mu_1, \mu_2, \dots)$

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \cdots$$

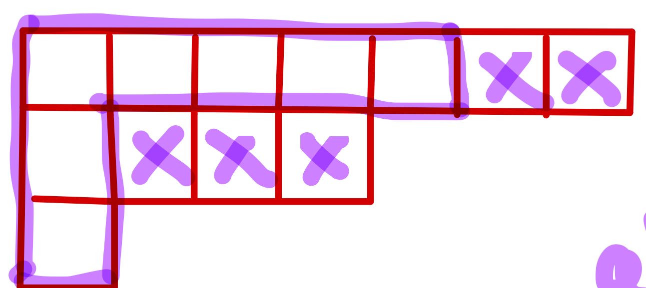


Interlacing partitions

$\lambda \succsim \mu$



$\lambda = (7, 4, 1) \qquad \mu = (5, 1, 1)$



$\lambda \succsim \mu$ interlace iff λ / μ has at most 1 cell in each column.

Proposition

There is a bijection between SSYT of shape λ and entries less or equal to n

and sequences of partitions

$$\lambda^{(0)} \preccurlyeq \lambda^{(1)} \preccurlyeq \lambda^{(2)} \preccurlyeq \cdots \preccurlyeq \lambda^{(n)}$$

such that $\lambda^{(0)} = \emptyset$ and $\lambda^{(n)} = \lambda$

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such that $\lambda^{(0)} = \emptyset$ and $\lambda^{(n)} = \lambda$

$\leqslant$

$\wedge$

1	1	2	4
2	2	4	
3			

$\lambda = (4,3,1)$ $\quad n = 4$

$\lambda^{(0)} = \emptyset$ $\quad \lambda^{(1)} = (2)$

$\lambda^{(2)} = (3,2)$

$\lambda^{(3)} = (3,2,1)$ $\quad \lambda^{(4)} = (4,3,1)$

$$\emptyset \preccurlyeq (2) \preccurlyeq (3,2) \preccurlyeq (3,2,1) \preccurlyeq (4,3,1,0)$$

Gelfand - Tsetlin pattern

$$\emptyset \preccurlyeq (2) \preccurlyeq (3,2) \preccurlyeq (3,2,1) \preccurlyeq (4,3,1,0)$$

Gelfand - Tsetlin pattern

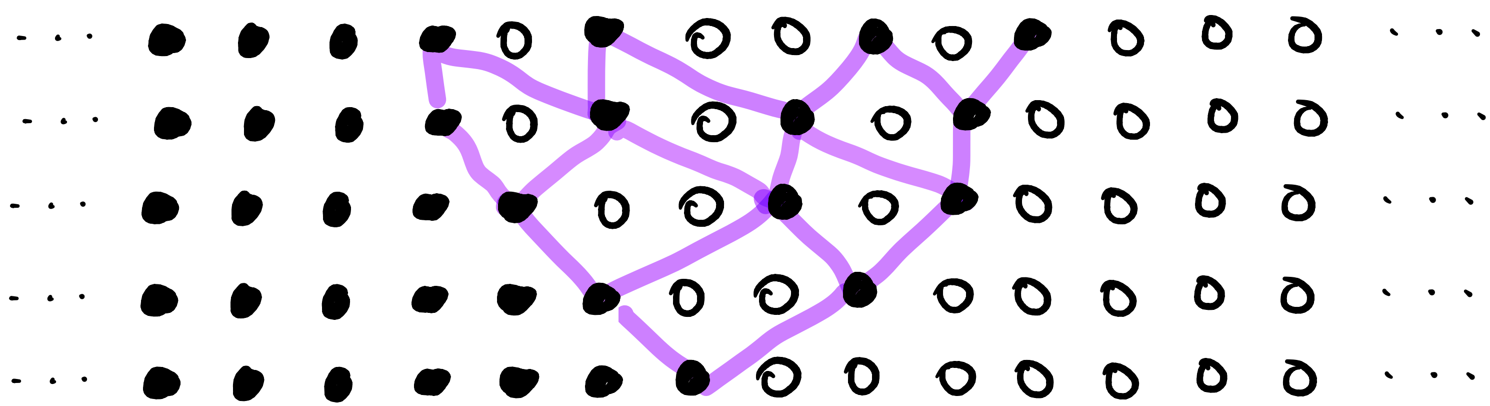
$$\begin{array}{ccccccccccccc}
4 & & \geq & & 3 & & \geq & & 1 & & \geq & & 0 \\
& & 3 & & \geq & & 2 & & \geq & & 1 & & \\
& & & & 3 & & \geq & & 2 & & & & \\
& & & & & & 2 & & & & & &
\end{array}$$

$$\emptyset \precsim (2) \precsim (3,2) \precsim (3,2,1) \precsim (4,3,1,0)$$

Gelfand - Tsetlin pattern

$$\begin{array}{ccccccc} 4 & & 3 & & 1 & & 0 \\ & 3 & & 2 & & 1 & \\ & & 3 & & 2 & & \\ & & & 2 & & & \end{array}$$

Sequence of interlacing maya diagrams



$(4,3,1,0)$

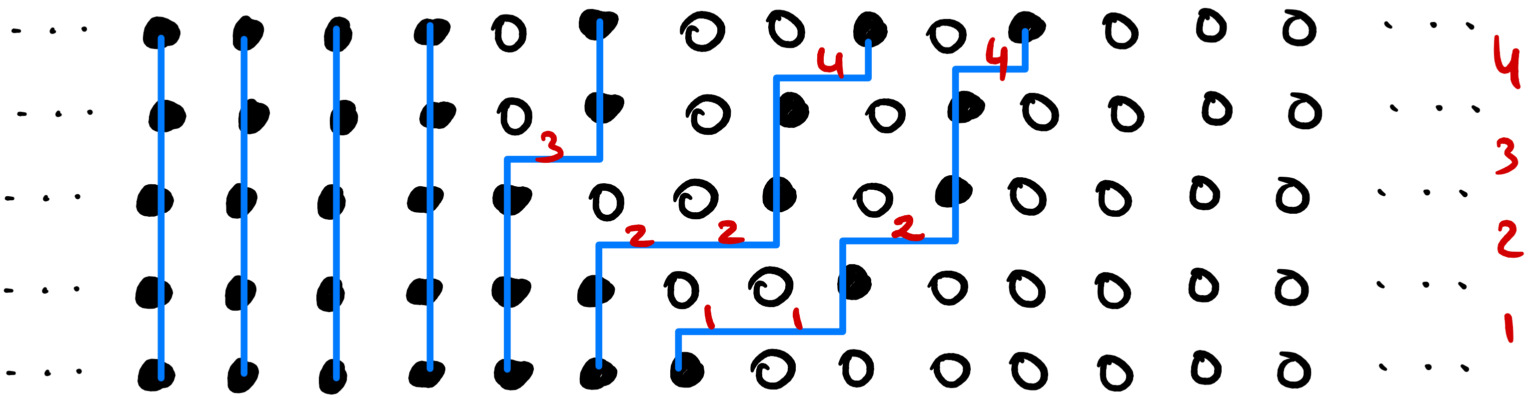
$(3,2,1)$

$(3,2)$

(2)

$\emptyset$

Non intersecting lattice paths



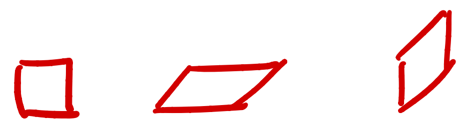
$(4,3,1,0)$
$(3,2,1)$
$(3,2)$
(2)
$\emptyset$

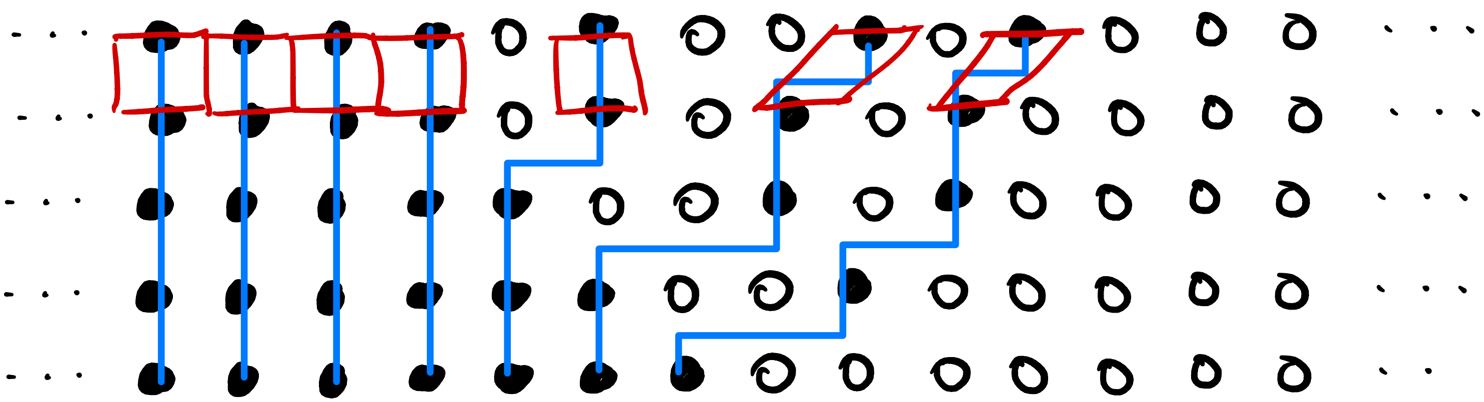
$\leq$

$<$

1	1	2	4
2	2	4	
3			

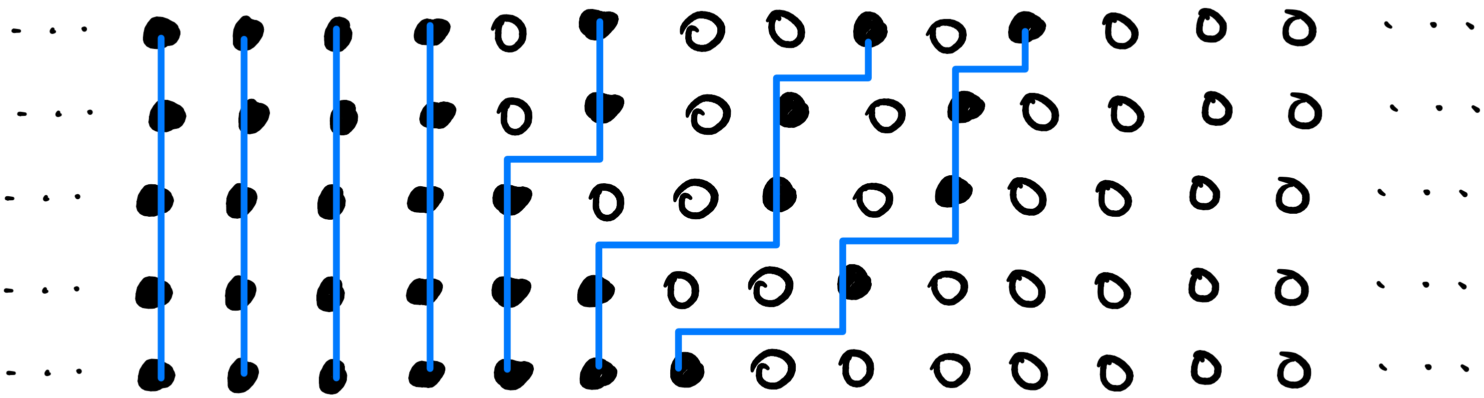
Tilings





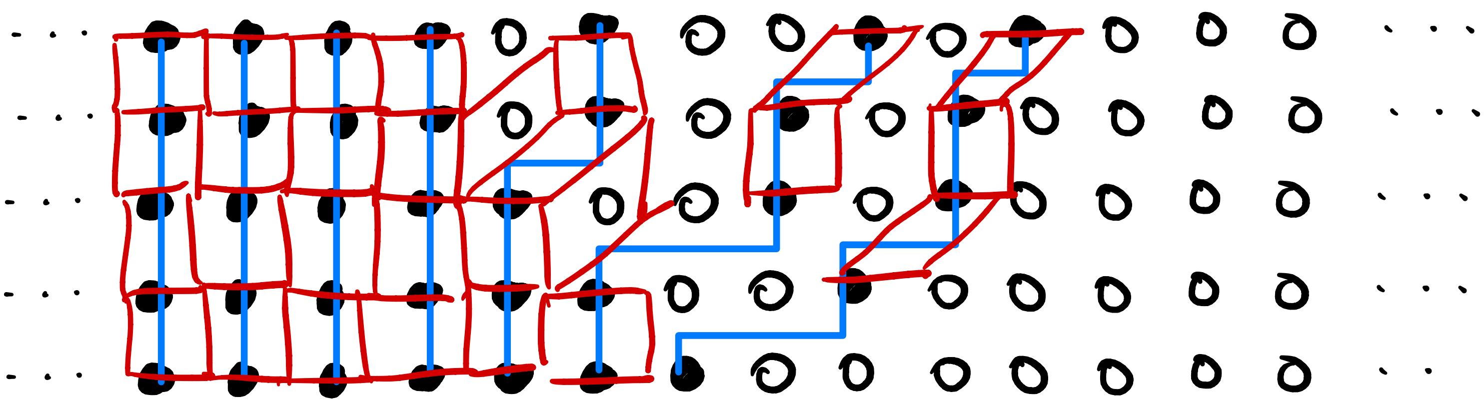
$(4,3,1,0)$
$(3,2,1)$
$(3,2)$
(2)
$\emptyset$

Non intersecting lattice paths

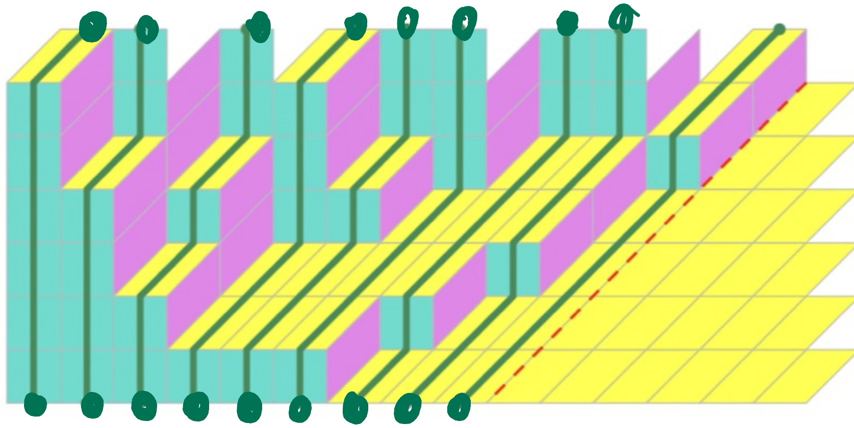


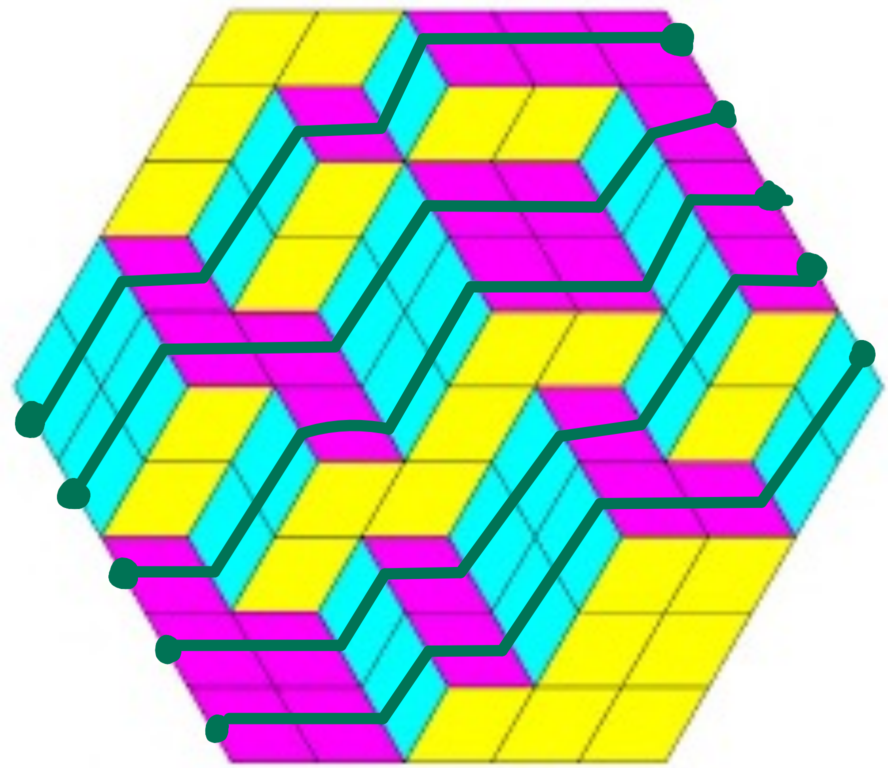
$(4,3,1,0)$
$(3,2,1)$
$(3,2)$
(2)
$\emptyset$

Tilings

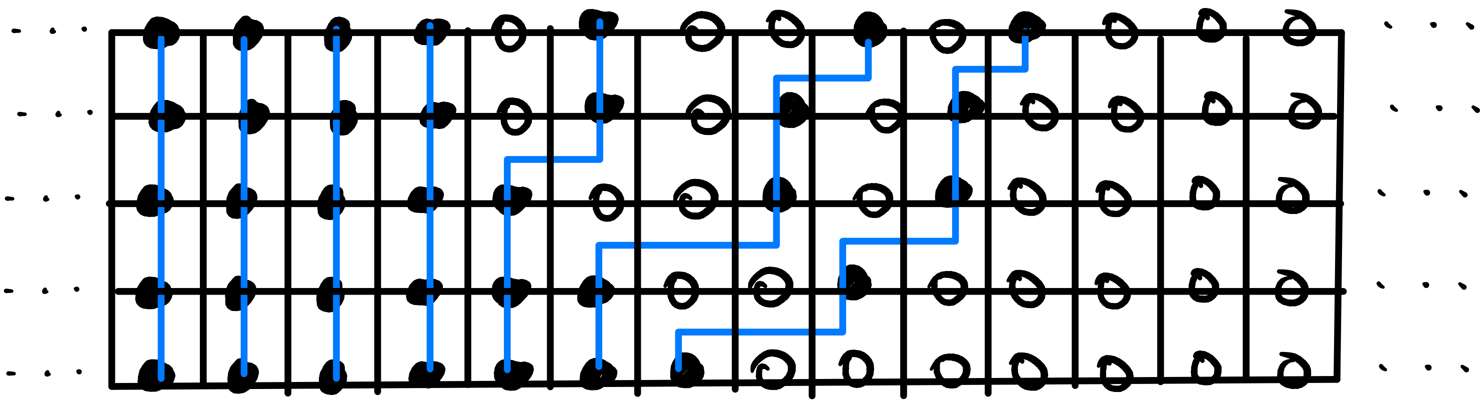


$(4,3,1,0)$
$(3,2,1)$
$(3,2)$
(2)
$\emptyset$

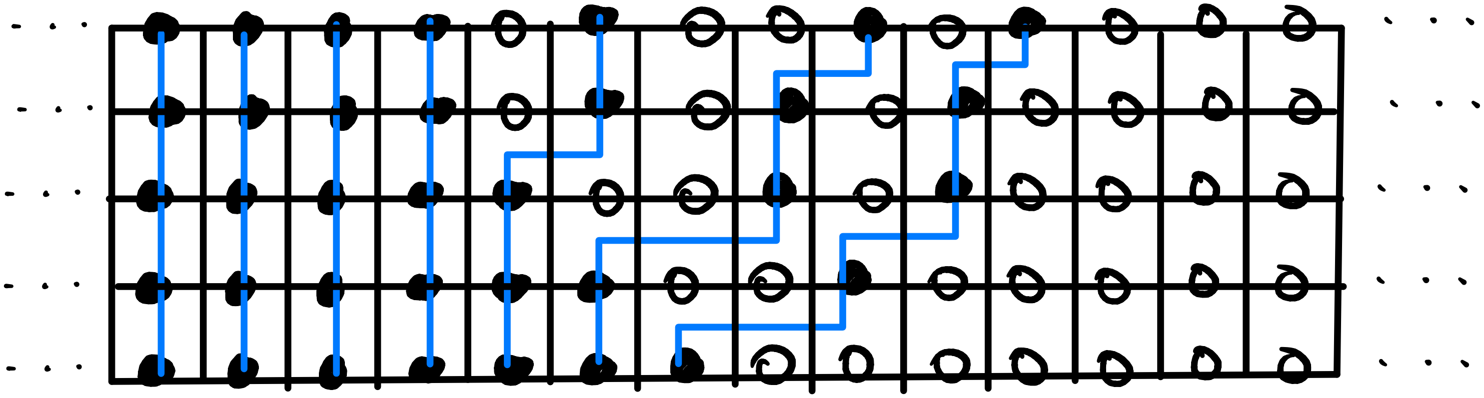


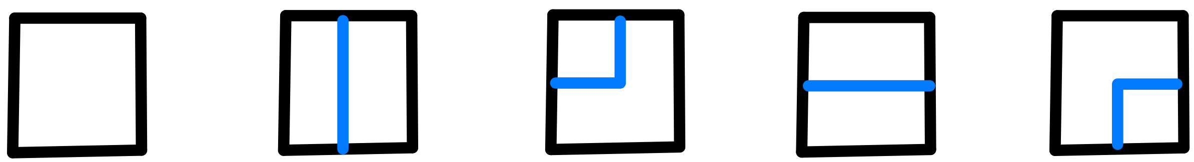


Vertex model

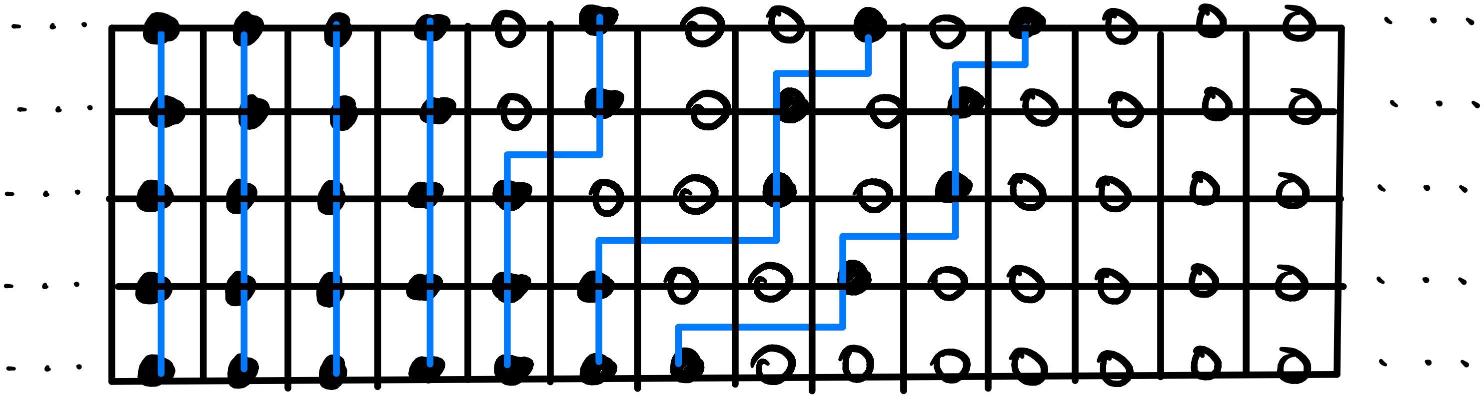


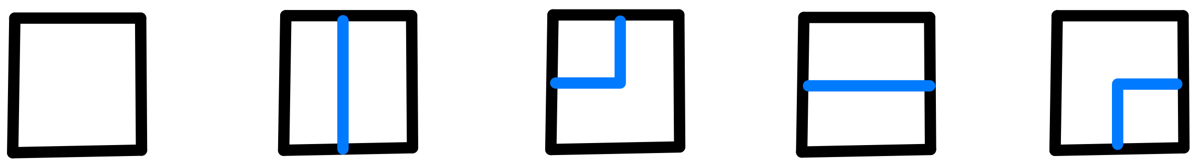
Vertex model

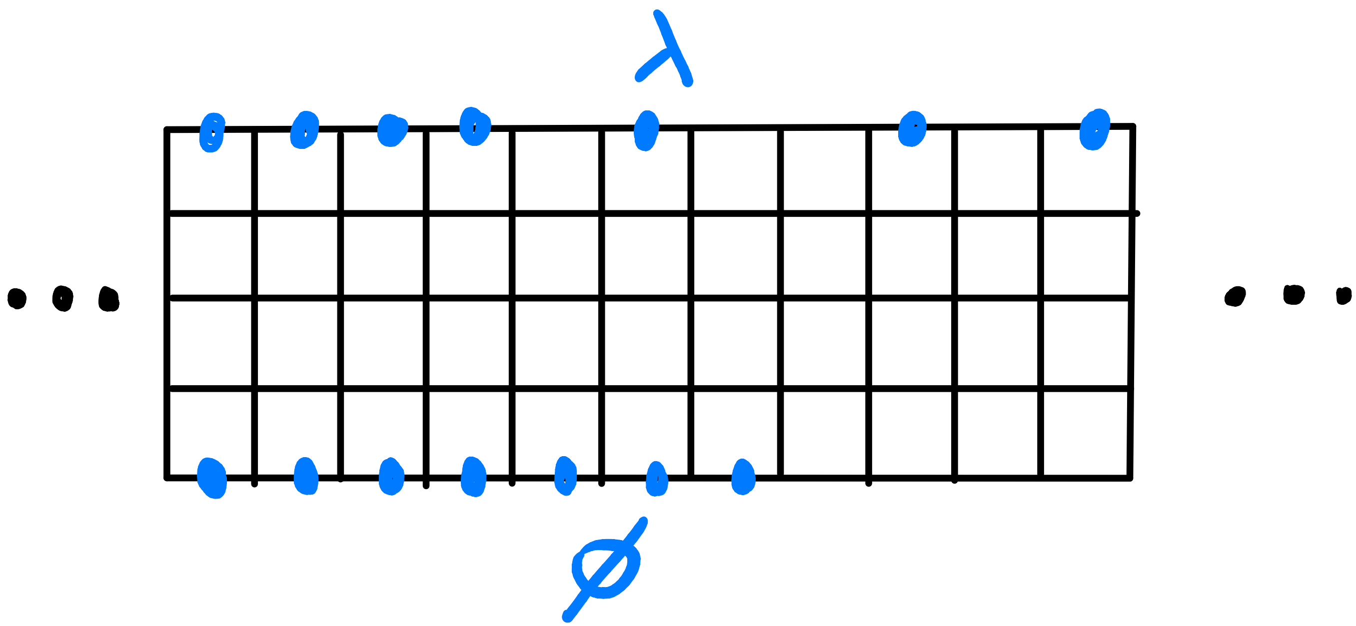




Vertex model







(Reverse) plane partitions

$$\lambda = (4,3,1)$$

$$\leq$$

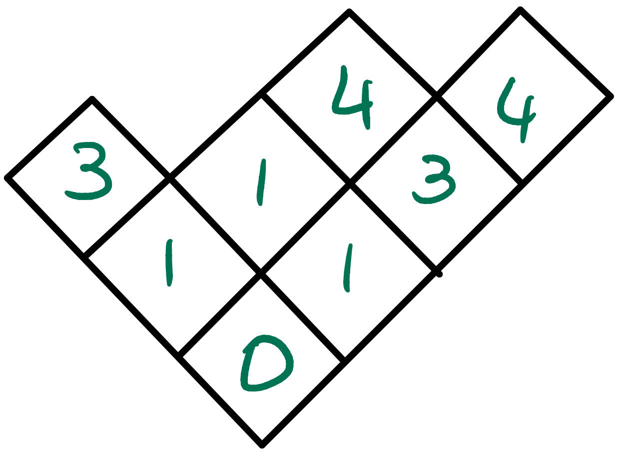
$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 3 & 4 \\ \hline 1 & 1 & 4 & \\ \cline{1-3} 3 & & & \\ \cline{1-1}\end{array}$$

Reverse plane partitions

$$\lambda = (4,3,1)$$

$\leq$

0	1	3	4
1	1	4	
3			



$$\lambda^{(0)} = \lambda^{(7)} = \emptyset$$

$\lambda^{(1)} = (3)$ $\lambda^{(2)} = (1)$ $\lambda^{(3)} = (1,0)$

$\lambda^{(4)} = (4,1)$ $\lambda^{(5)} = (3)$

$\lambda^{(6)} = (4)$

$$\lambda^{(0)} \prec \lambda^{(1)} \succ \lambda^{(2)} \prec \lambda^{(3)} \prec \lambda^{(4)} \succ \lambda^{(5)} \prec \lambda^{(6)} \succ \lambda^{(7)}$$

Reverse plane partitions

$\lambda = (4, 3, 1)$

$\leq$

$$\Lambda \quad \begin{array}{|c|c|c|c|} \hline 0 & 1 & 3 & 4 \\ \hline 1 & 1 & 4 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} \end{array}$$

$|\Lambda| = 8 + 6 + 3$
$= 17$

$$\sum_{\Lambda \in RPP(\lambda)} q^{|\Lambda|} = \prod_{(i,j) \in \lambda} \frac{1}{1 - q^{h_{ij}}}$$

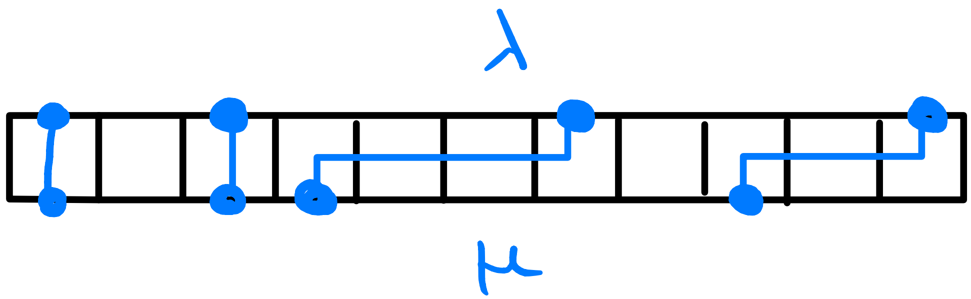
Hooks

$$\begin{array}{|c|c|c|c|} \hline 6 & 4 & 3 & 1 \\ \hline 4 & 2 & 1 & \\ \cline{1-3} 1 & & & \\ \cline{1-1} \end{array}$$

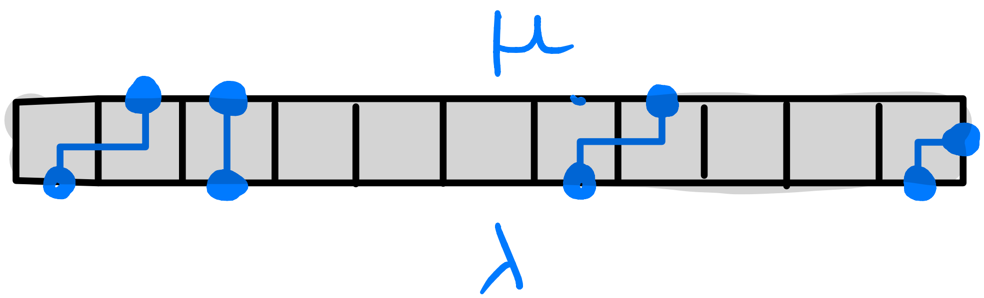
$$\frac{1}{(1-q)^3 (1-q^2)(1-q^3)(1-q^4)^2 (1-q^6)}$$

How to prove such a formula?

using vertex models?

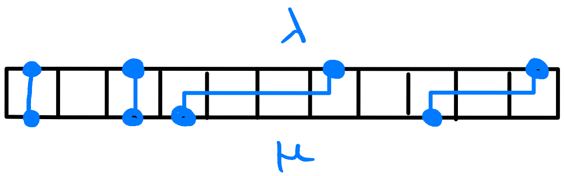


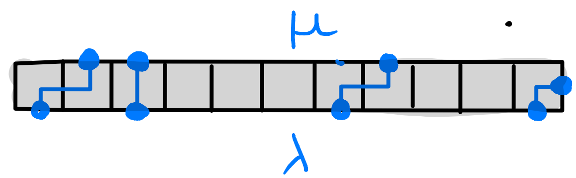
$\lambda \succ \mu$

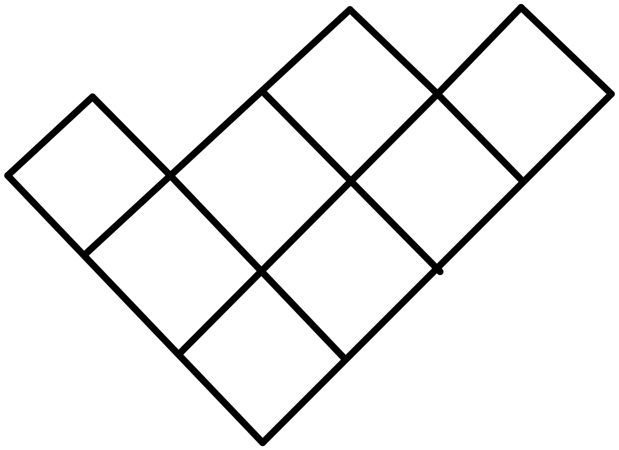


$\mu \prec \lambda$

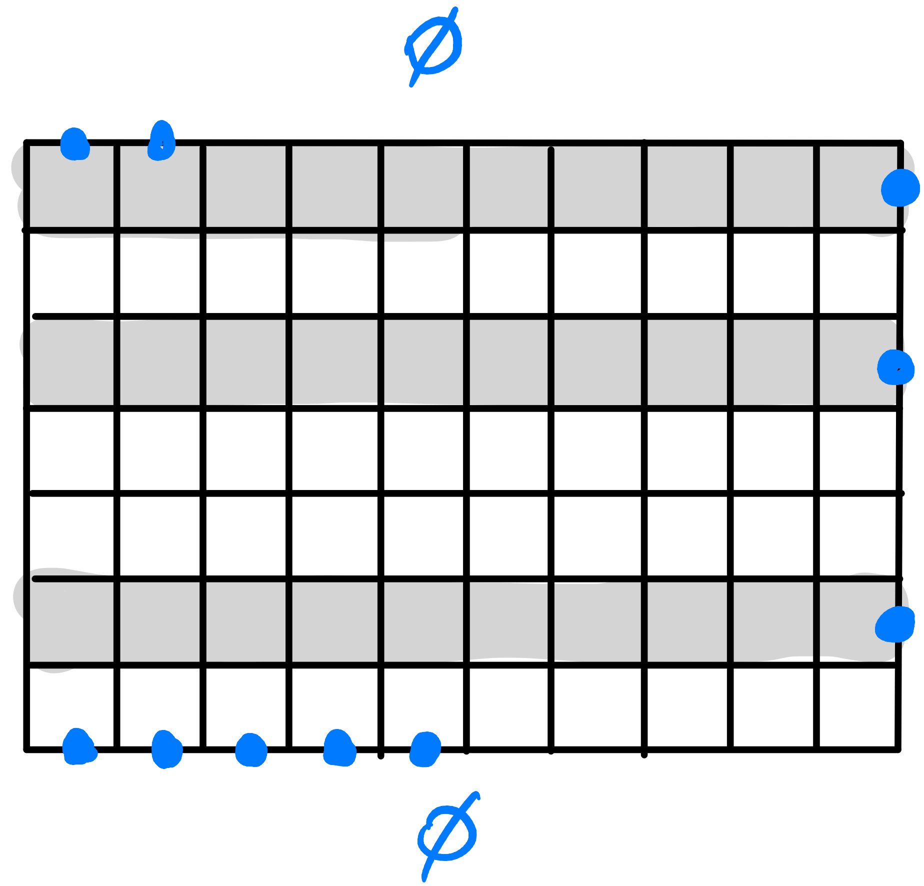
How to prove such a formula ?







RPP of shape $(4,3,1)$



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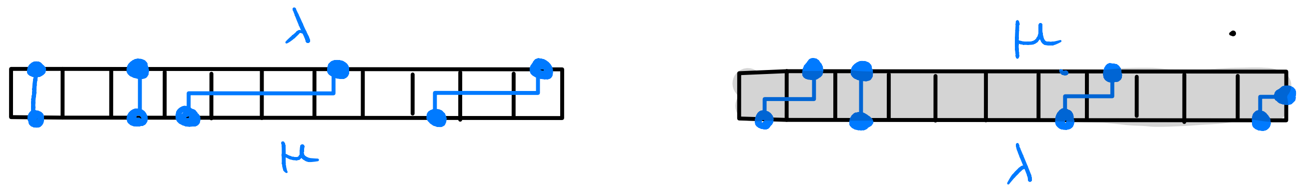
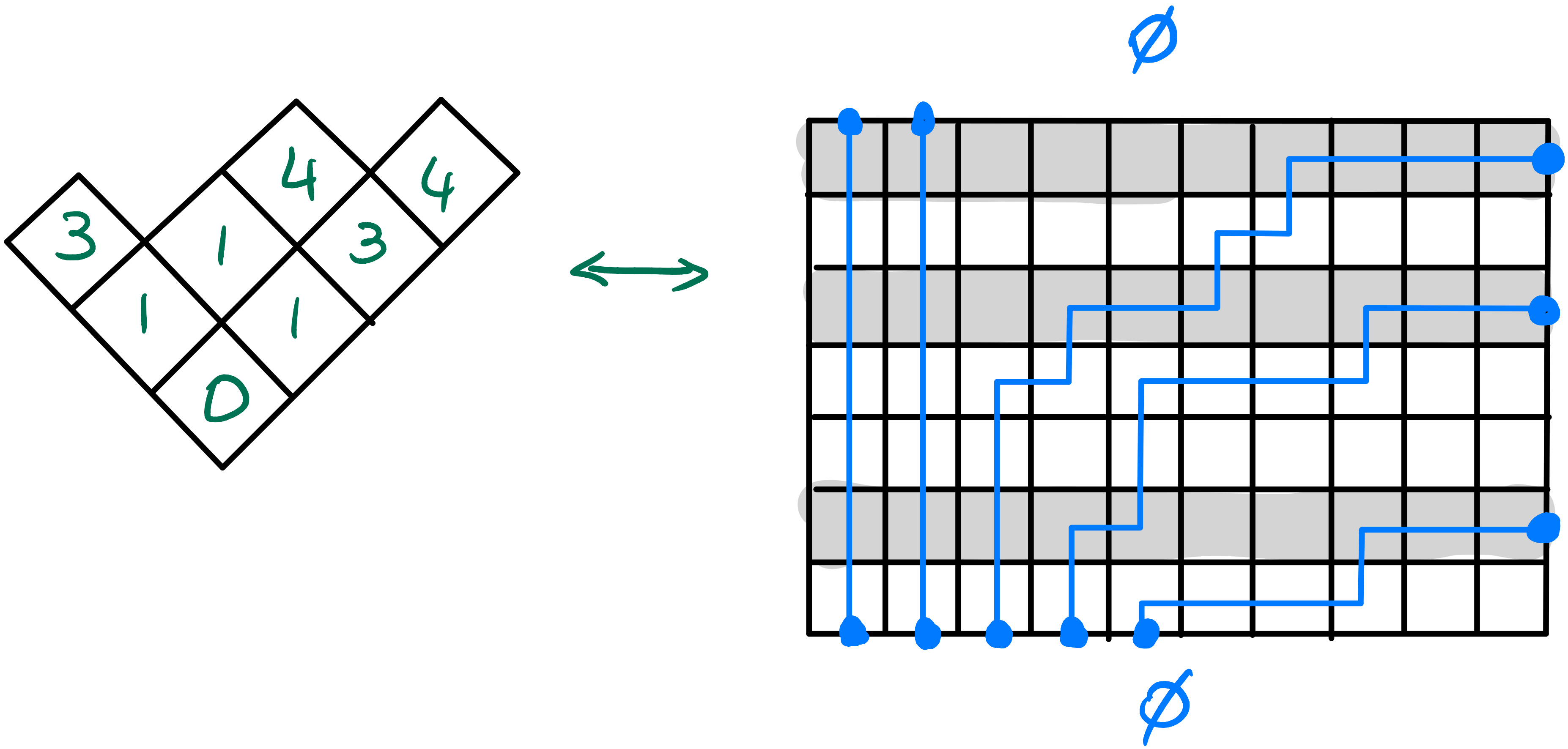
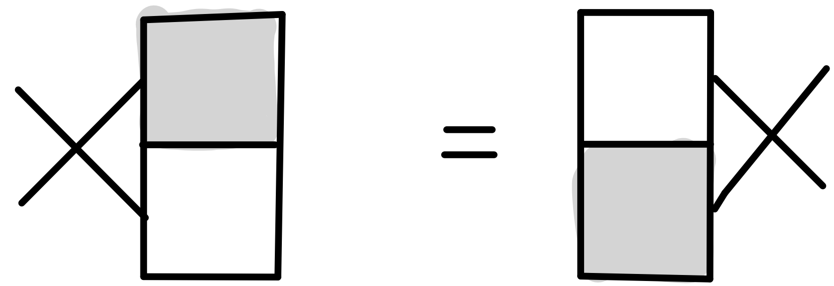
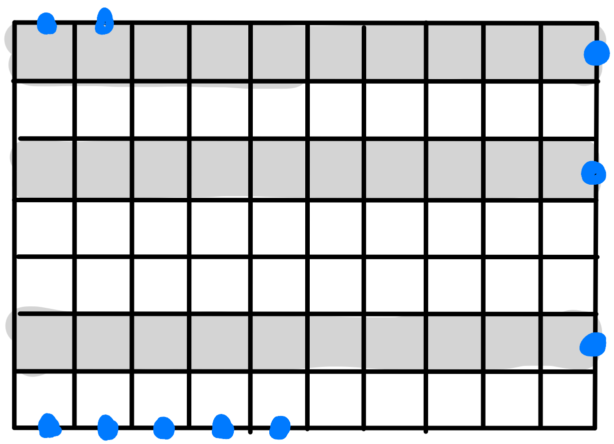


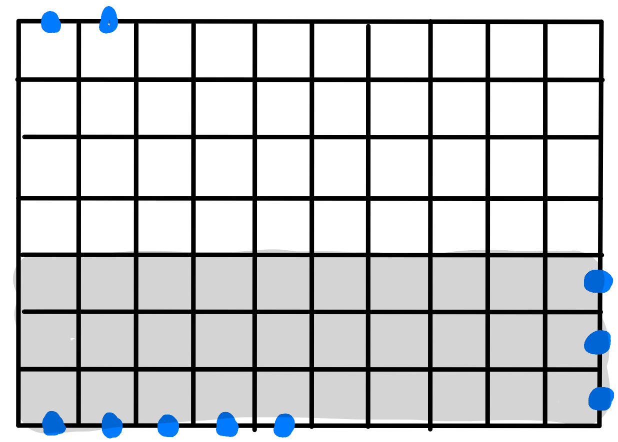


Yang - Baxter equation





$$= \prod_{x\in\lambda} \frac{1}{1-q^{h(\lambda)}} \times$$



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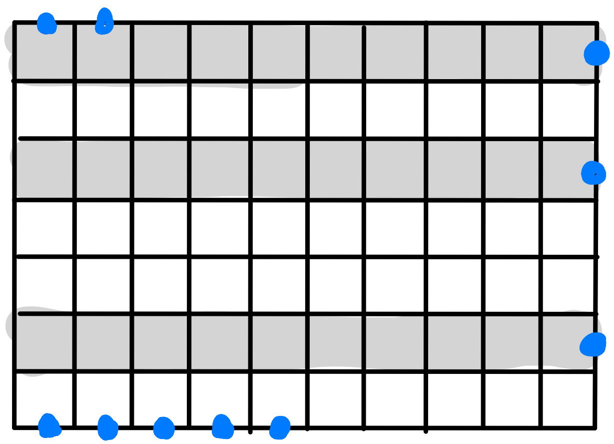
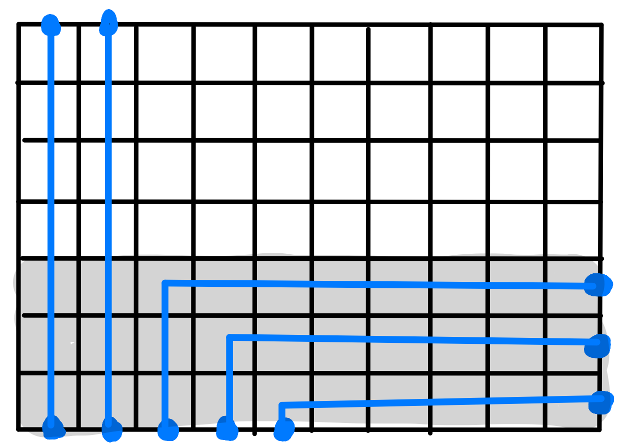
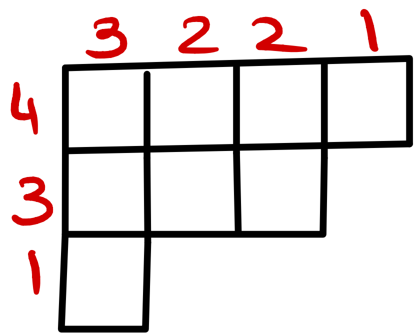


$$= \prod_{x\in\lambda} \frac{1}{1-q^{h(\lambda)}} \times$$

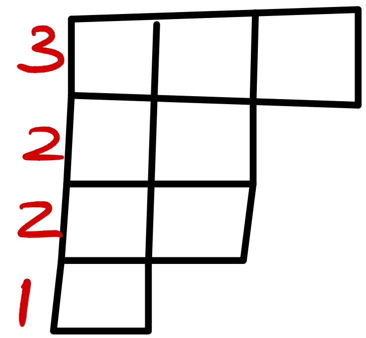


Conjugate partitions

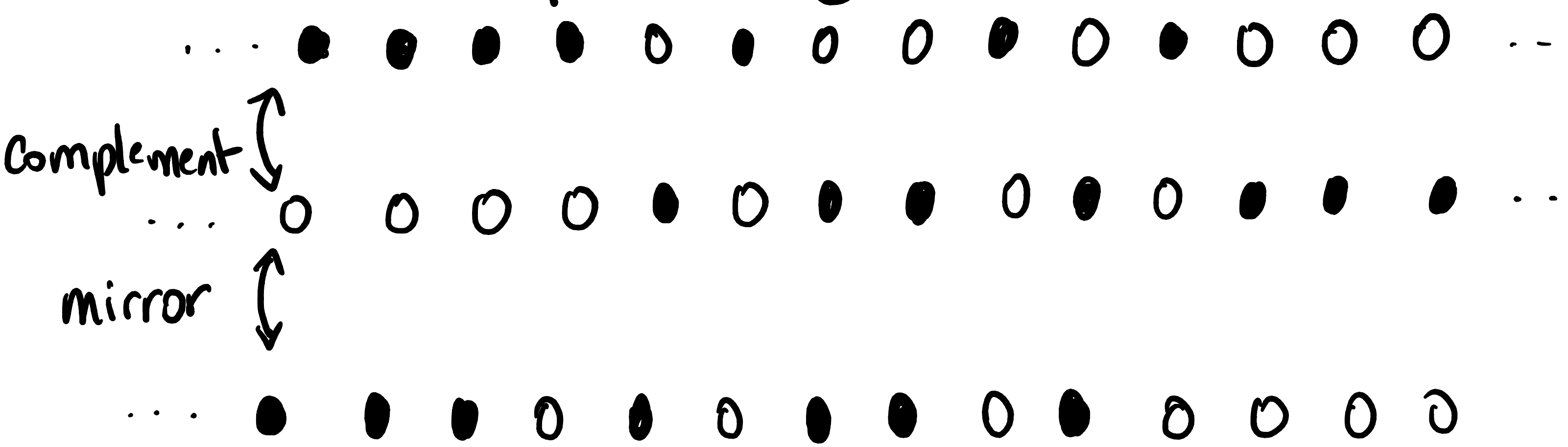
$\lambda = (4,3,1)$



$\lambda' = (3,2,2,1)$



On the maya diagram



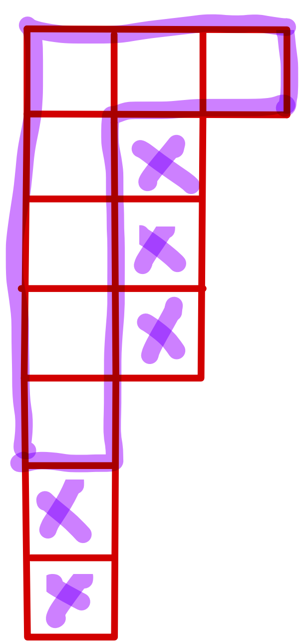
Co interlacing partitions

$$\lambda \succsim' \mu \quad \text{iff} \quad \lambda' \succsim \mu'$$

ex

$$\lambda = (3,2,2,2,1,1,1)$$

$$\mu = (3,1,1,1,1)$$



Co interlacing partitions

$$\lambda \succsim' \mu \quad \text{iff} \quad \lambda' \succsim \mu'$$

ex $\lambda = (3,2,2,2,1,1,1)$

$\mu = (3,1,1,1,1)$

● ● ● ○ ● ● ● ○ ● ● ● ○ ● ○ ○ ○ . . .

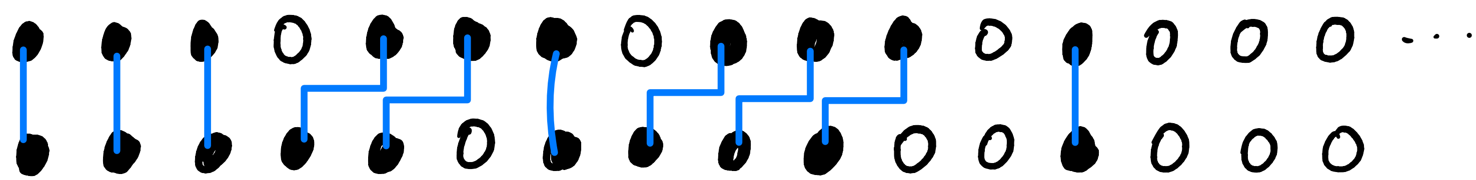
● ● ● ● ● ○ ● ● ● ● ○ ○ ● ○ ○ ○

Co interlacing partitions

$$\lambda \succsim' \mu \quad \text{iff} \quad \lambda' \succsim \mu'$$

ex $\quad \lambda = (3,2,2,2,1,1,1)$

$\mu = (3,1,1,1,1)$



Tableaux

$<$

1	2	3	4
1	3	4	
2			

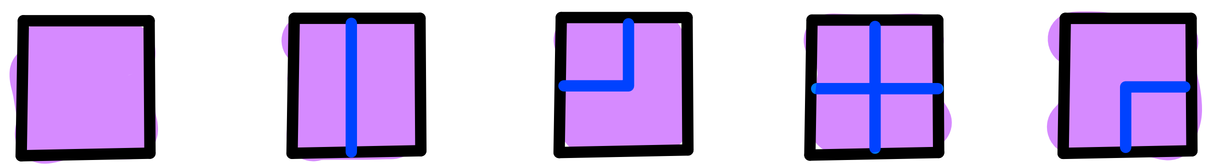
$$\emptyset \precsim' (1,1) \precsim' (2,1,1) \precsim' (3,2,1) \precsim' (4,3,1,0)$$

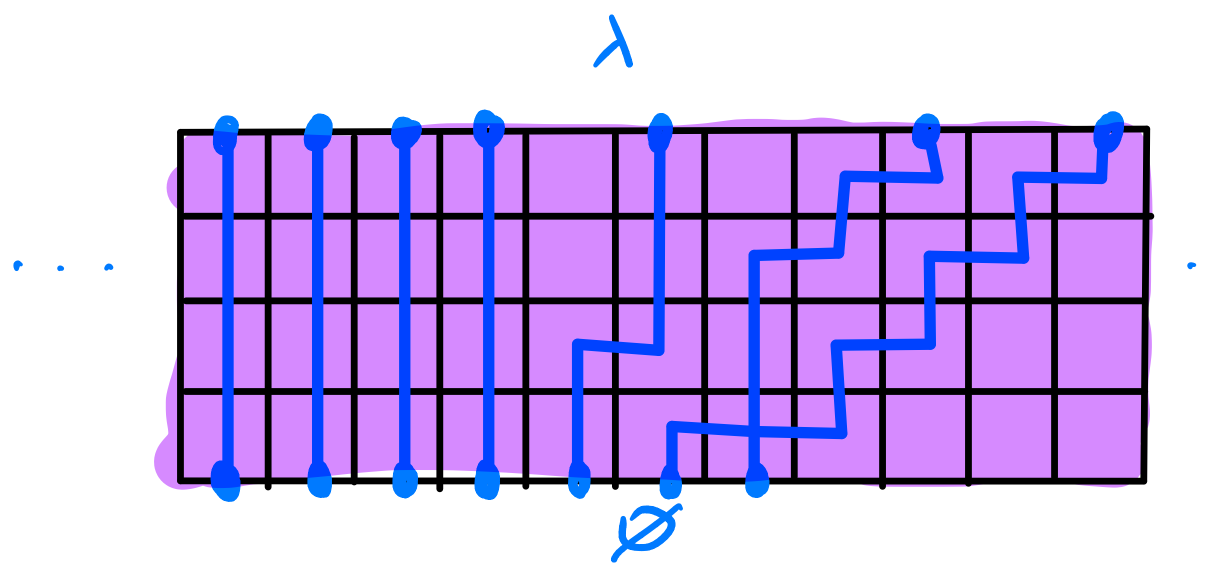
Tableaux

	$<$			
$\wedge\!\mid$	1	2	3	4
	1	3	4	
	2			

$$\emptyset \precsim' (1,1) \precsim' (2,1,1) \precsim' (3,2,1) \precsim' (4,3,1,0)$$

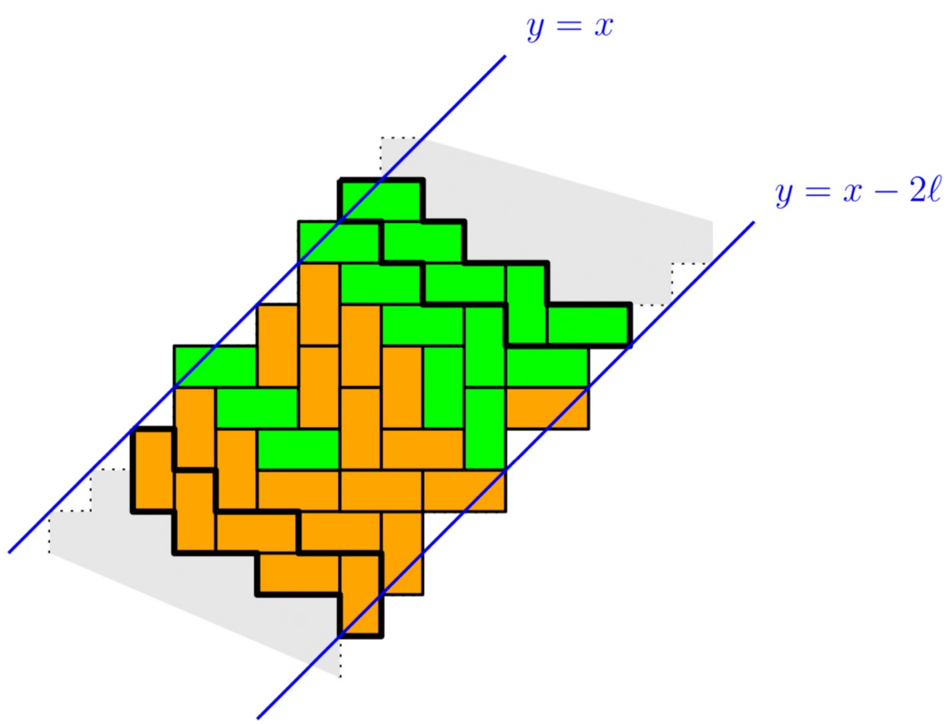
Vertex model



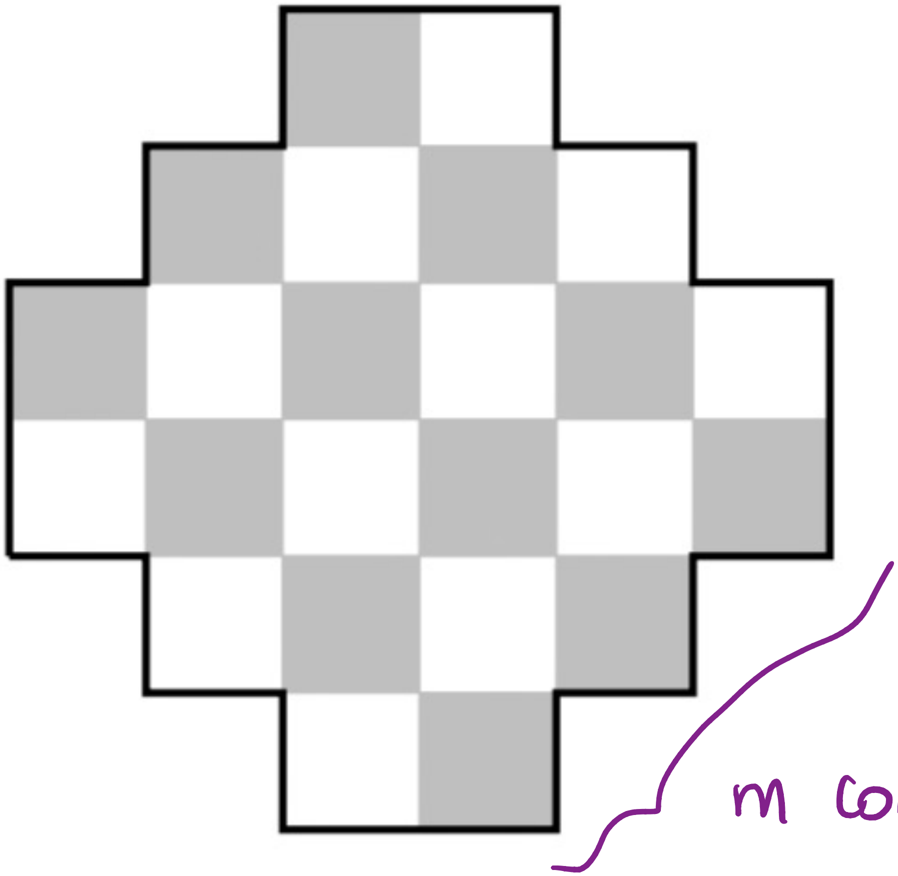


What do we get if we mix interlacing & co-interlacing?

Steep tilings (Boutier, Chapuy, C. 2014)

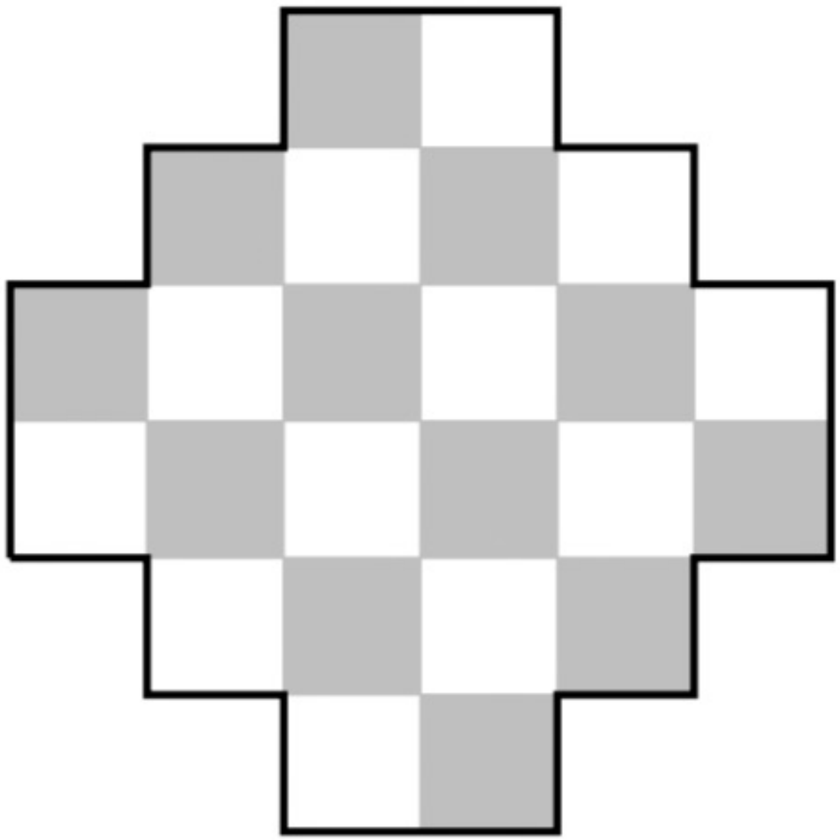


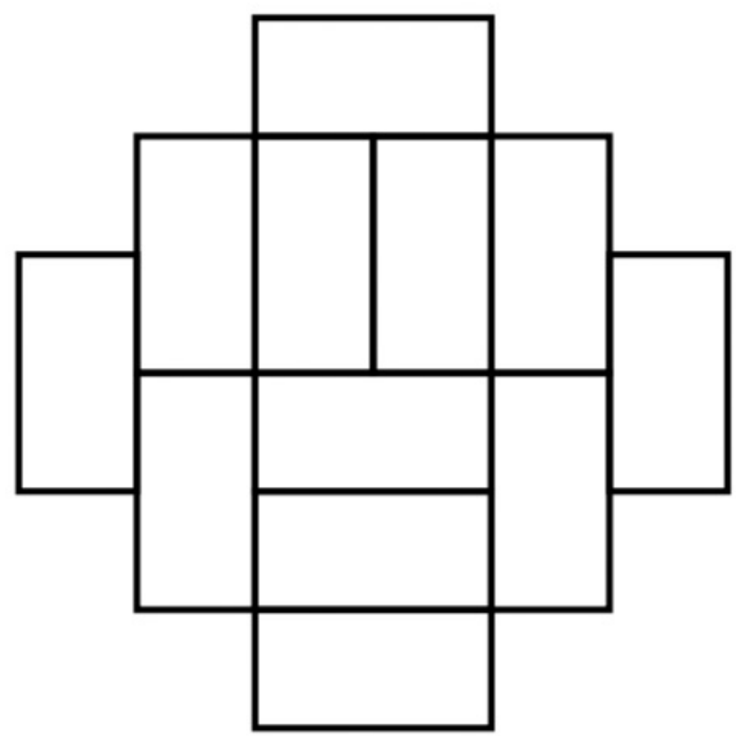
Domino tilings of the Aztec diamond (Propp et al 90s)



m corners

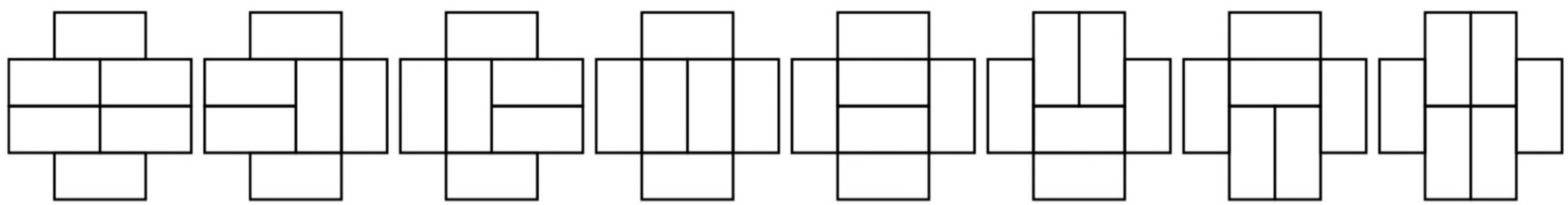
Domino tilings of the Aztec diamond (Propp et al 90s)





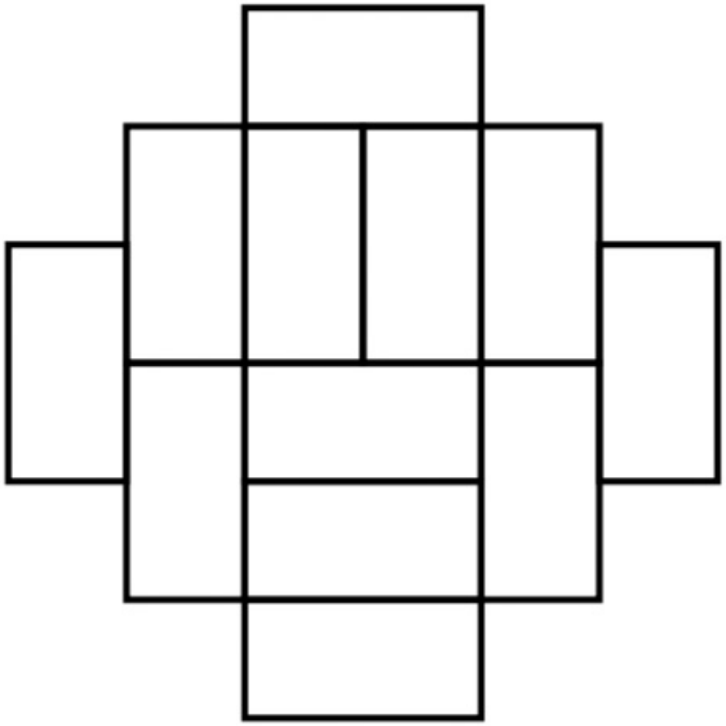
Domino tilings of the Aztec diamond

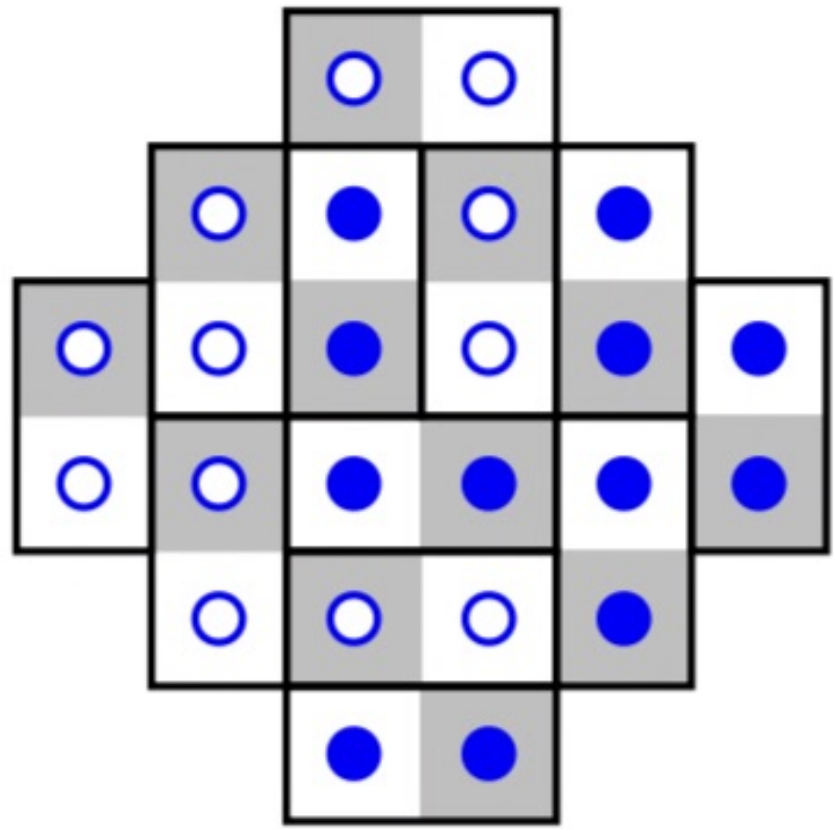
$m=2$



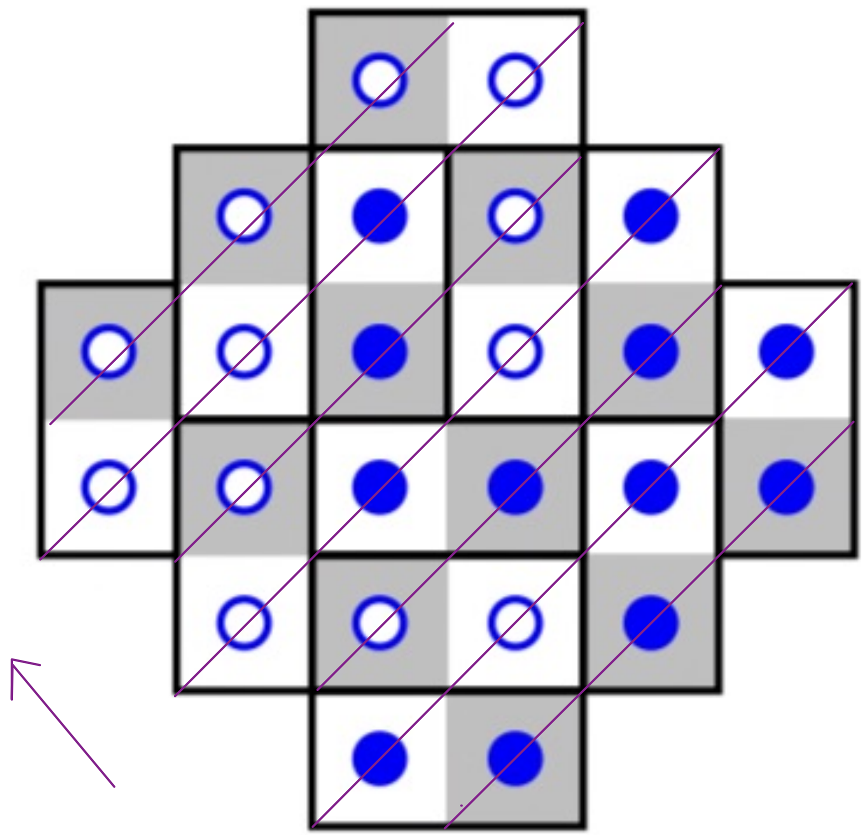
Theorem The number of domino Tilings of the Aztec diamond with m corners is $2^{\binom{m+1}{2}}$

Domino tilings of the Aztec diamond

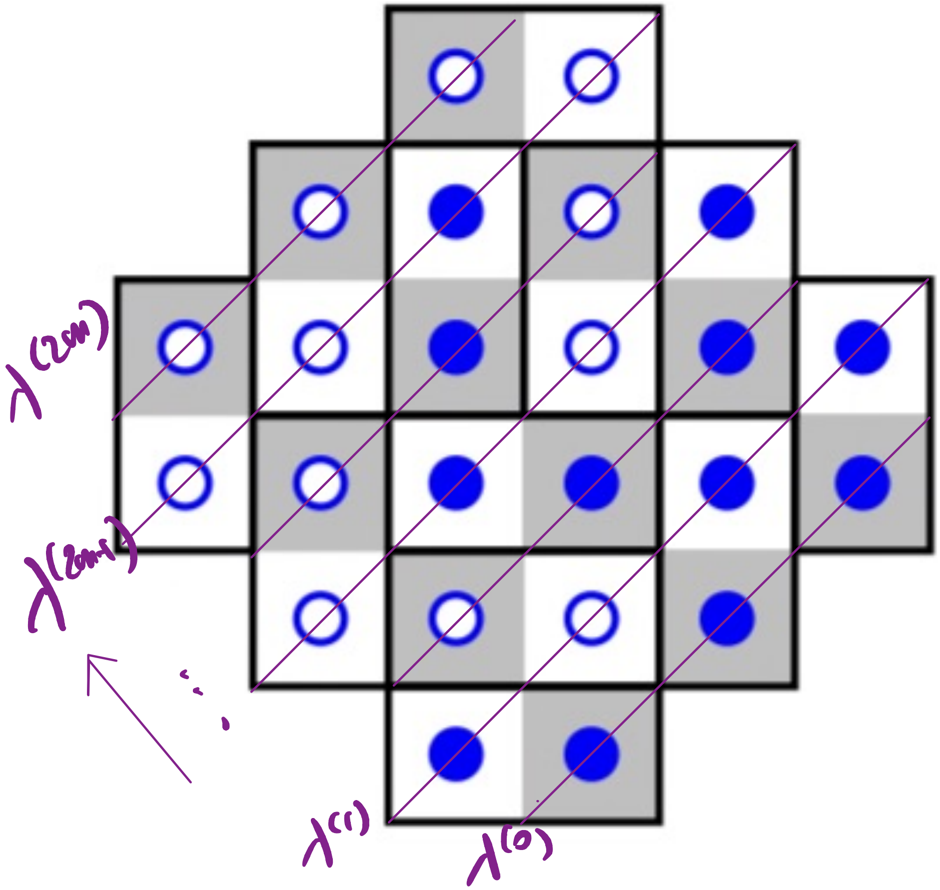




Domino tilings of the Aztec diamond



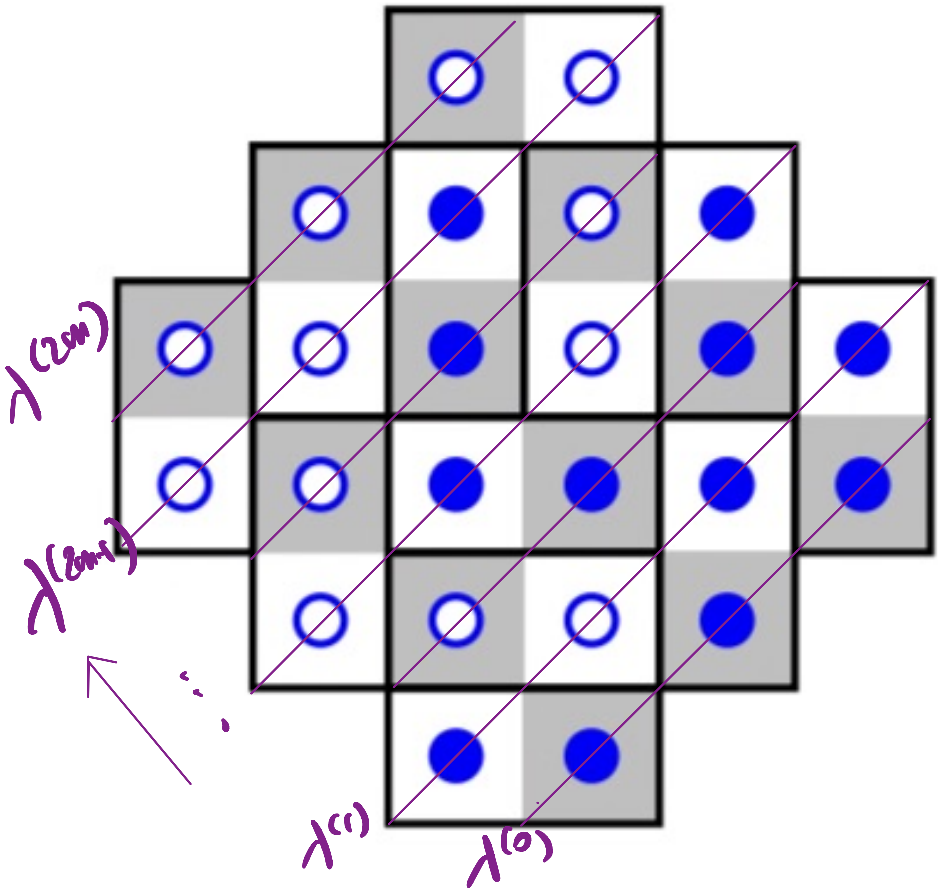
Domino tilings of the Aztec diamond (BCC14)



$$\emptyset = \lambda^{(0)} \prec' \lambda^{(1)} \succ \lambda^{(2)} \cdots \prec' \lambda^{(2m-1)} \succ \lambda^{(2m)} = \emptyset$$

Domino tilings of the Aztec diamond (BCC14)

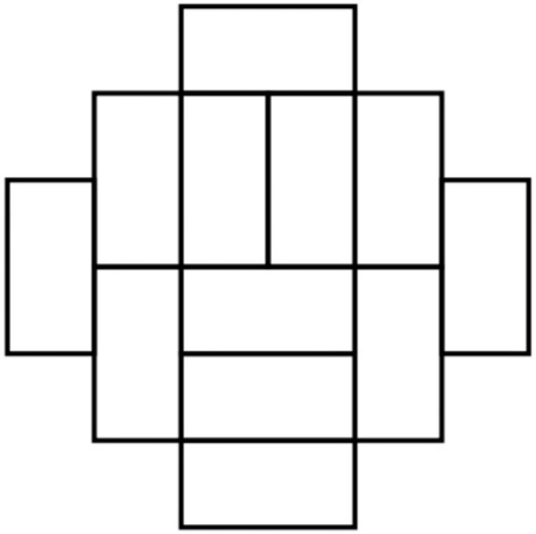
$$\emptyset = \lambda^{(0)} \prec' \lambda^{(1)} \succ \lambda^{(2)} \cdots \prec' \lambda^{(2m-1)} \succ \lambda^{(2m)} = \emptyset$$



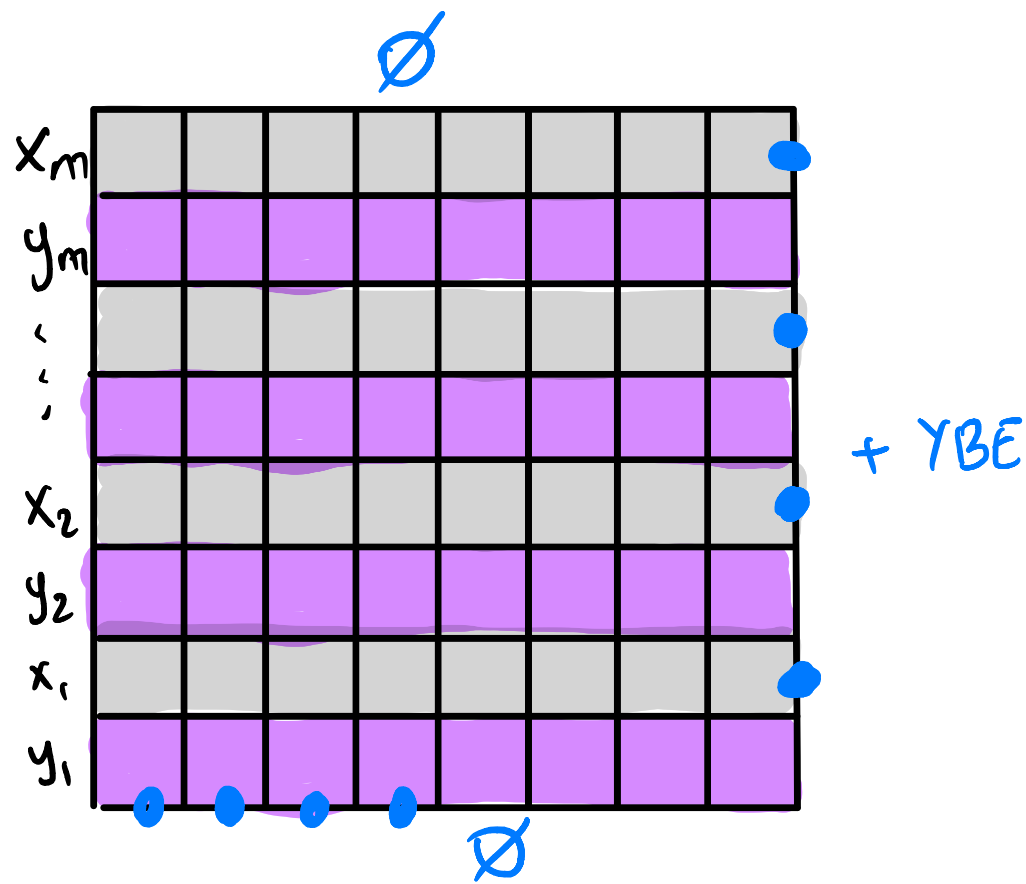
$$Z_m(x,y) = \sum_{\Lambda} \prod_i x_i^{|\lambda^{(2i-1)}/\lambda^{(2i-2)}|}\, y_i^{|\lambda^{(2i-1)}/\lambda^{(2i)}|}$$

$$= \prod_{1 \le i \le j \le m} (1 + x_i y_j)$$

Aztec diamond $\longrightarrow$ Vertex models



$$\prod_{1 \le i < j \le m} (1 + x_i y_j)$$



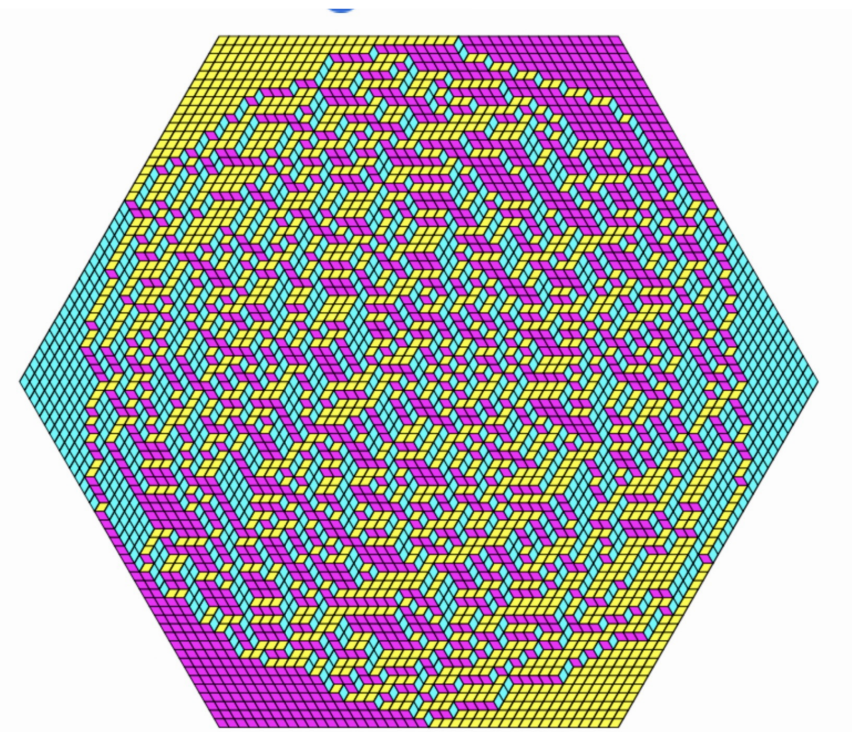
Asymptotics (Schur processes)

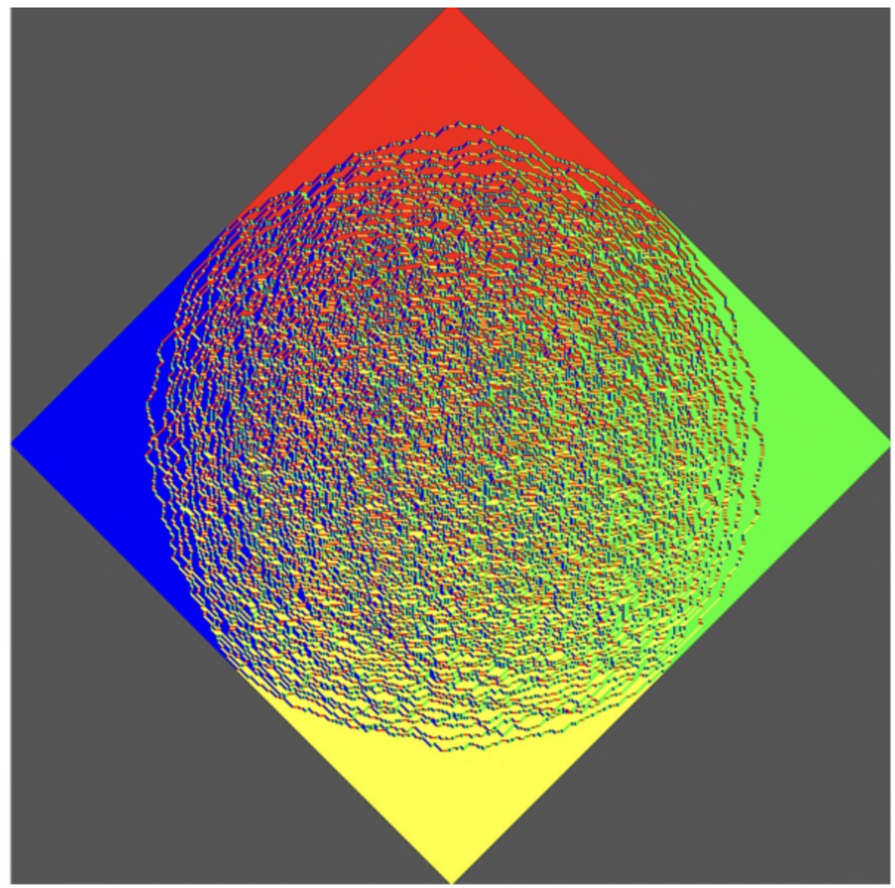
Okounkov, Reshetikhin

Borodin, Petrov, Gorin ...

Integrable probability

Lozenge tiling

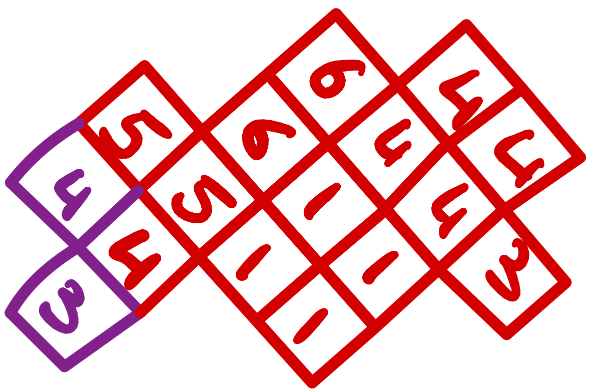




RYG – Z. Li (2021)

Generalisations

① Vertex models on the cylinder



$$\lambda^{(0)} \preceq \lambda^{(1)} \succeq \lambda^{(2)} \preceq \lambda^{(3)} \preceq \lambda^{(4)}$$

$$\succeq \lambda^{(5)} \preceq \lambda^{(6)} \succeq \lambda^{(7)}$$

$$\lambda^{(0)} = \lambda^{(7)}$$

Generalisations

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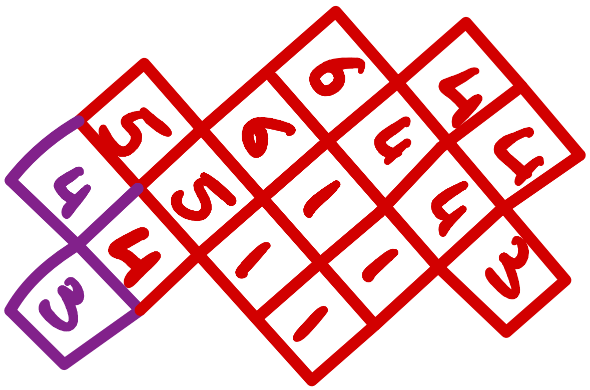
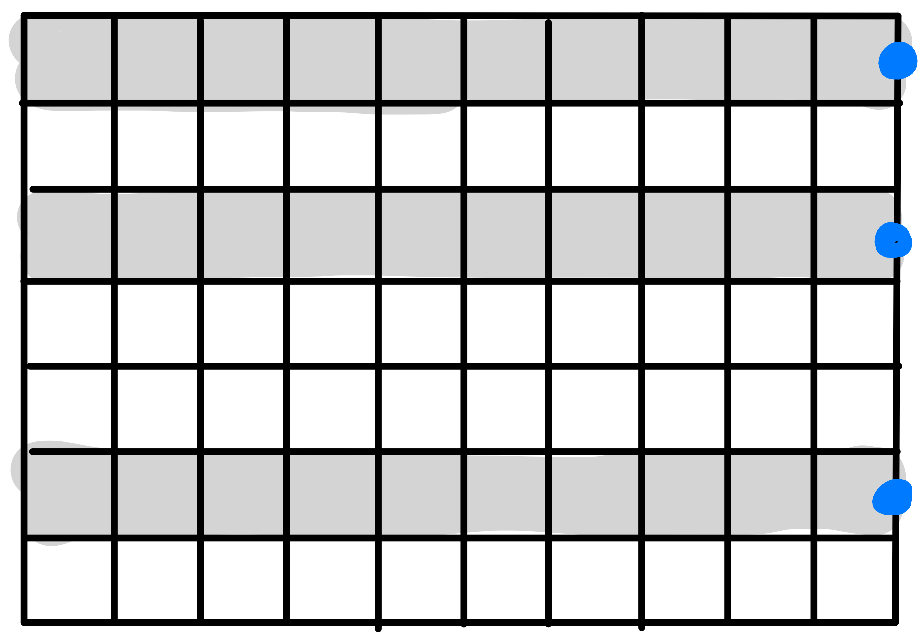


$$\lambda^{(0)} \preccurlyeq \lambda^{(1)} \succcurlyeq \lambda^{(2)} \preccurlyeq \lambda^{(3)} \preccurlyeq \lambda^{(4)}$$

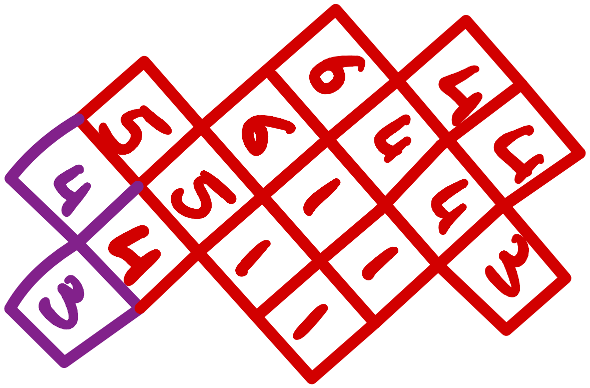
$$\succcurlyeq \lambda^{(5)} \preccurlyeq \lambda^{(6)} \succcurlyeq \lambda^{(7)}$$

$$\lambda^{(0)} = \lambda^{(7)}$$



Generalisations

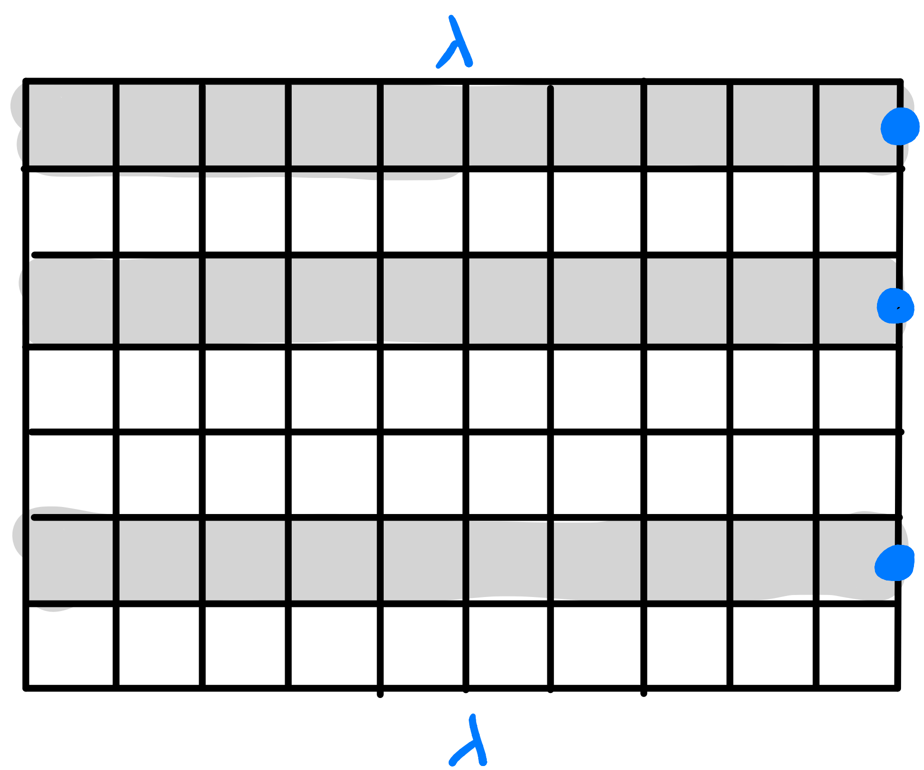
① Vertex models on the cylinder



$$\lambda^{(0)} \preccurlyeq \lambda^{(1)} \succcurlyeq \lambda^{(2)} \preccurlyeq \lambda^{(3)} \preccurlyeq \lambda^{(4)} \succcurlyeq \lambda^{(5)} \preccurlyeq \lambda^{(6)} \succcurlyeq \lambda^{(7)}$$

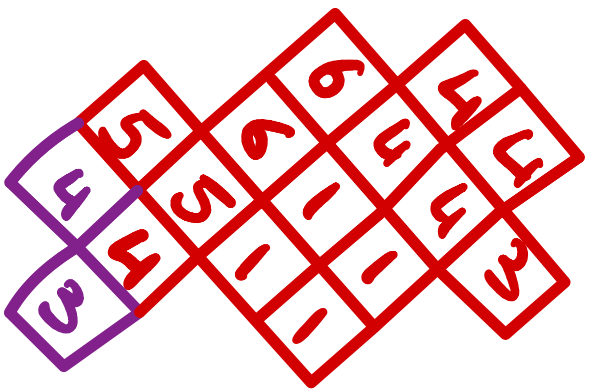
$$\lambda^{(0)} = \lambda^{(7)}$$

$\sum_{\lambda}$



Generalisations

① Vertex models on the cylinder



Gessel & Krattenthaler 93

Borodin 2007

Betea & Bouttier, Ahn et al

Imamura, Mucciconi & Sasamoto

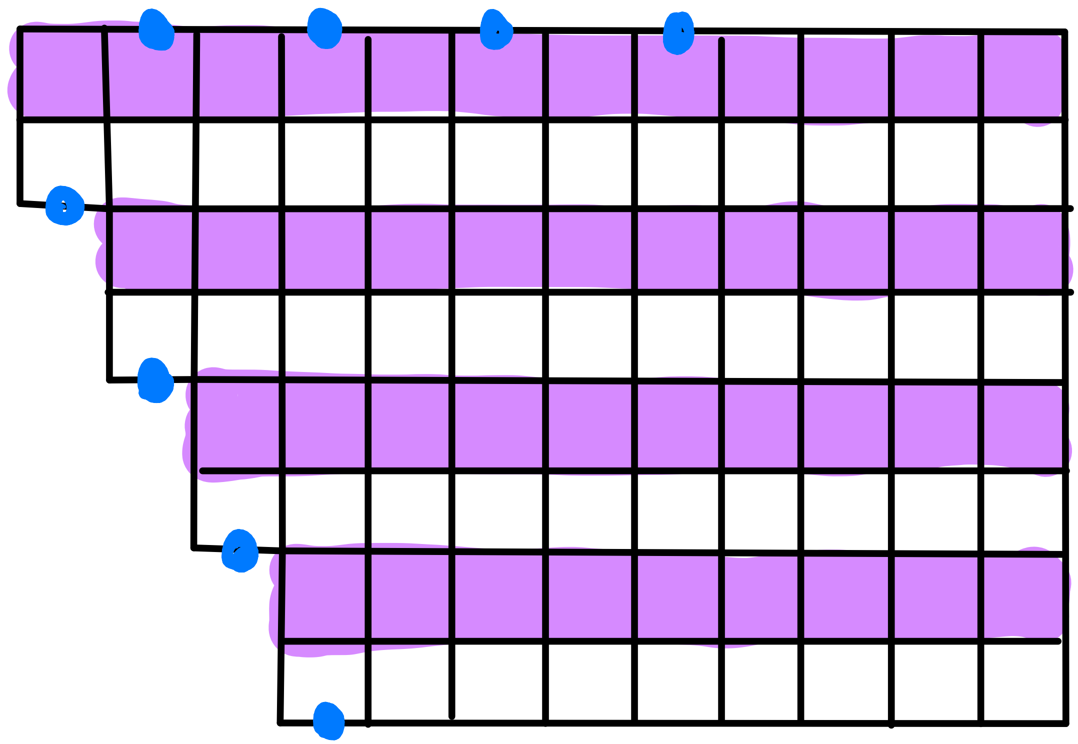
Foda & Welsh

Corteel & Welsh, Warnaar

Corteel, Dousse & Uncu

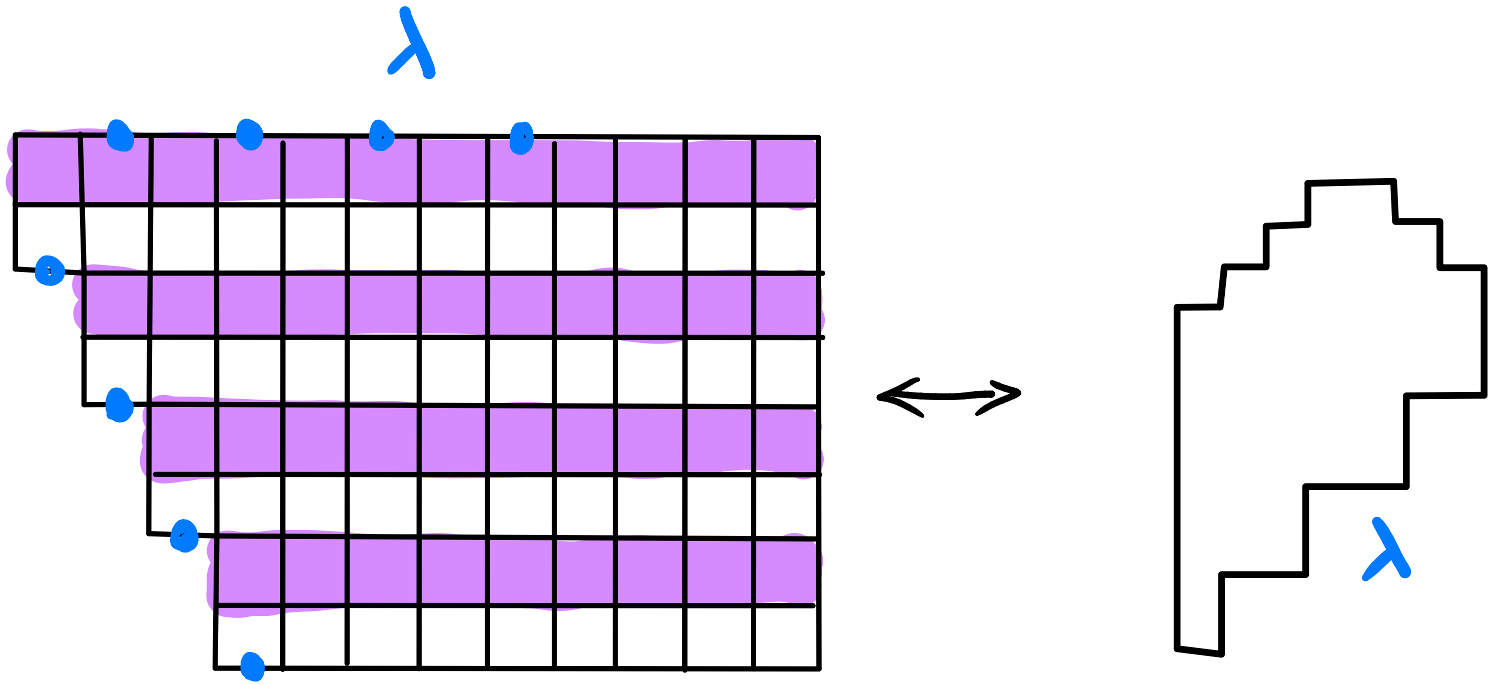
Kanade & Russell

② Vertex model on staircases



Huang 2022+

$\propto$ domino tilings



<u>Conjecture (Di Francesco)</u> $\lambda = (n, n-1, \ldots, 2, 1)$

$$2^{\binom{n}{2}} \prod_{j=0}^{n-1} \frac{(4j+2)!}{(n+2j+1)!}$$

3 Colored vertex models

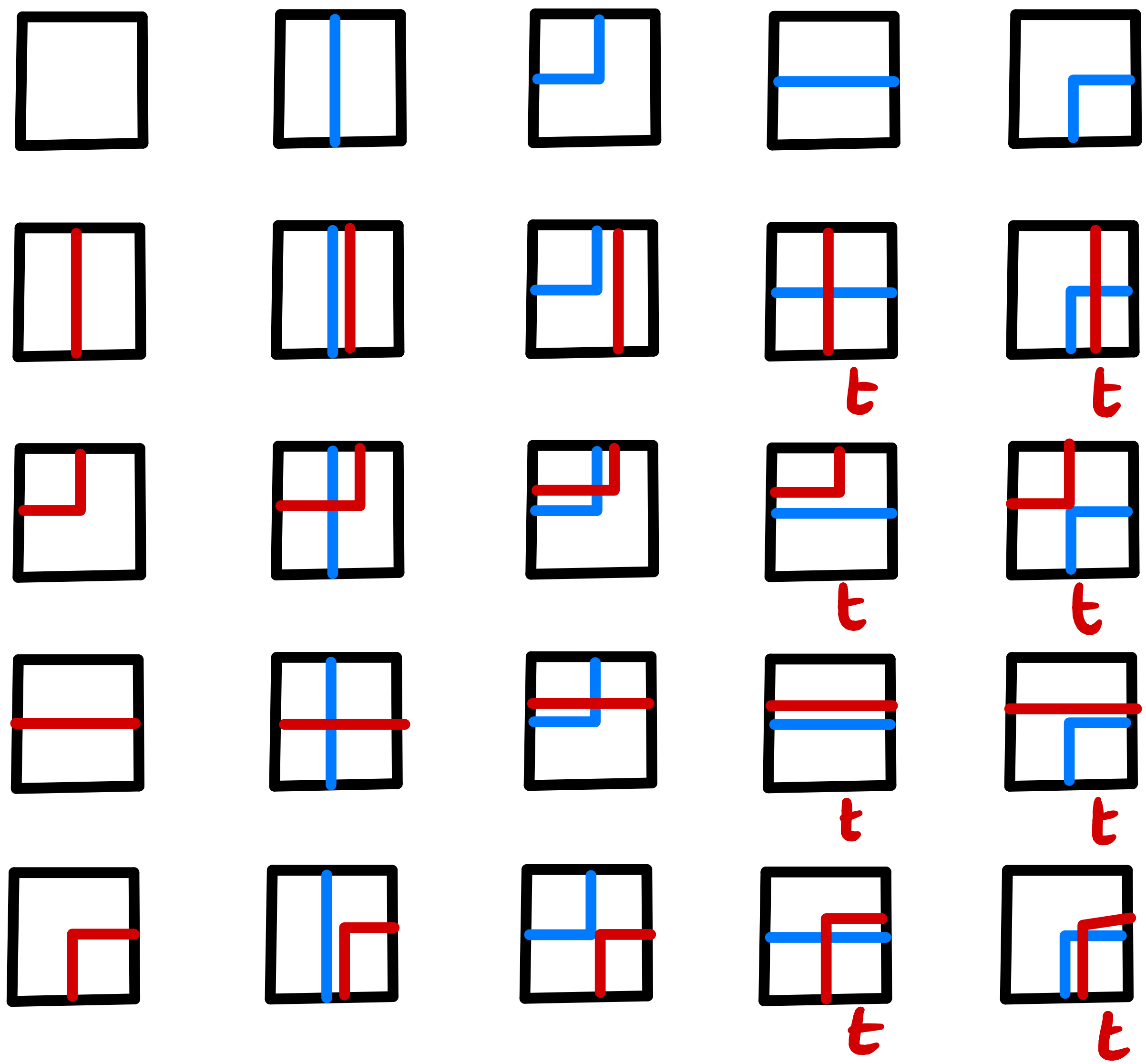
Corteel, Gitlin, Keating & Meza (2020)

Aggarwal, Borodin & Wheeler (2020)

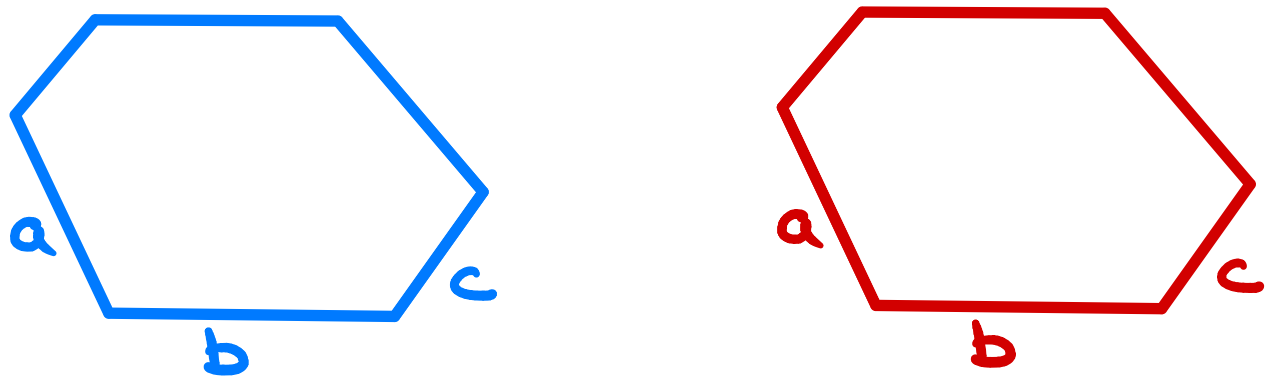
+ many more

③ Colored vertex models

Corteel, Gitlin, Keating & Meza (2020)

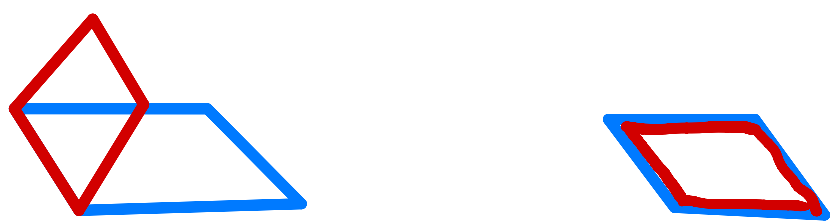


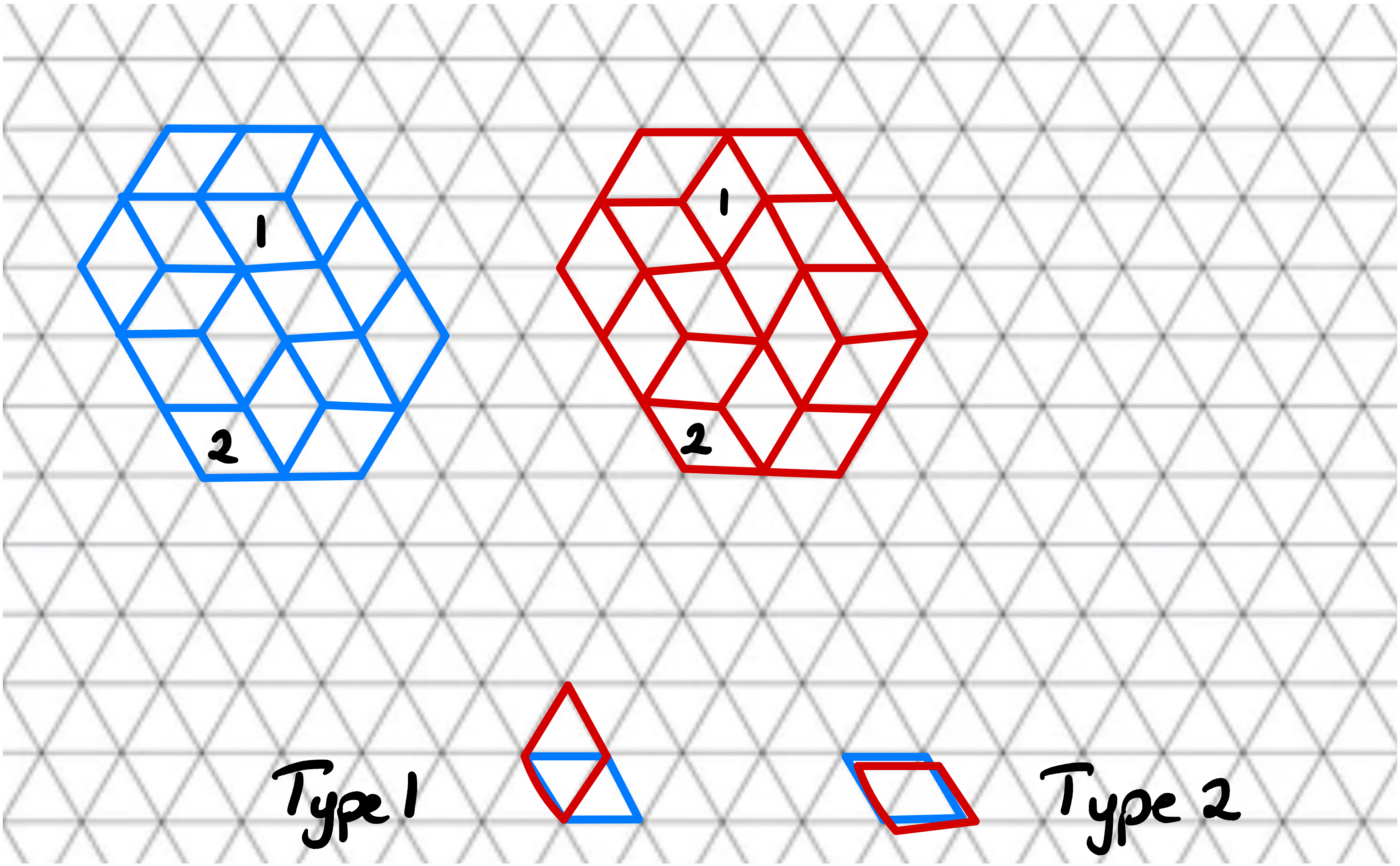
2-tilings (k-tilings for $k \geq 2$)

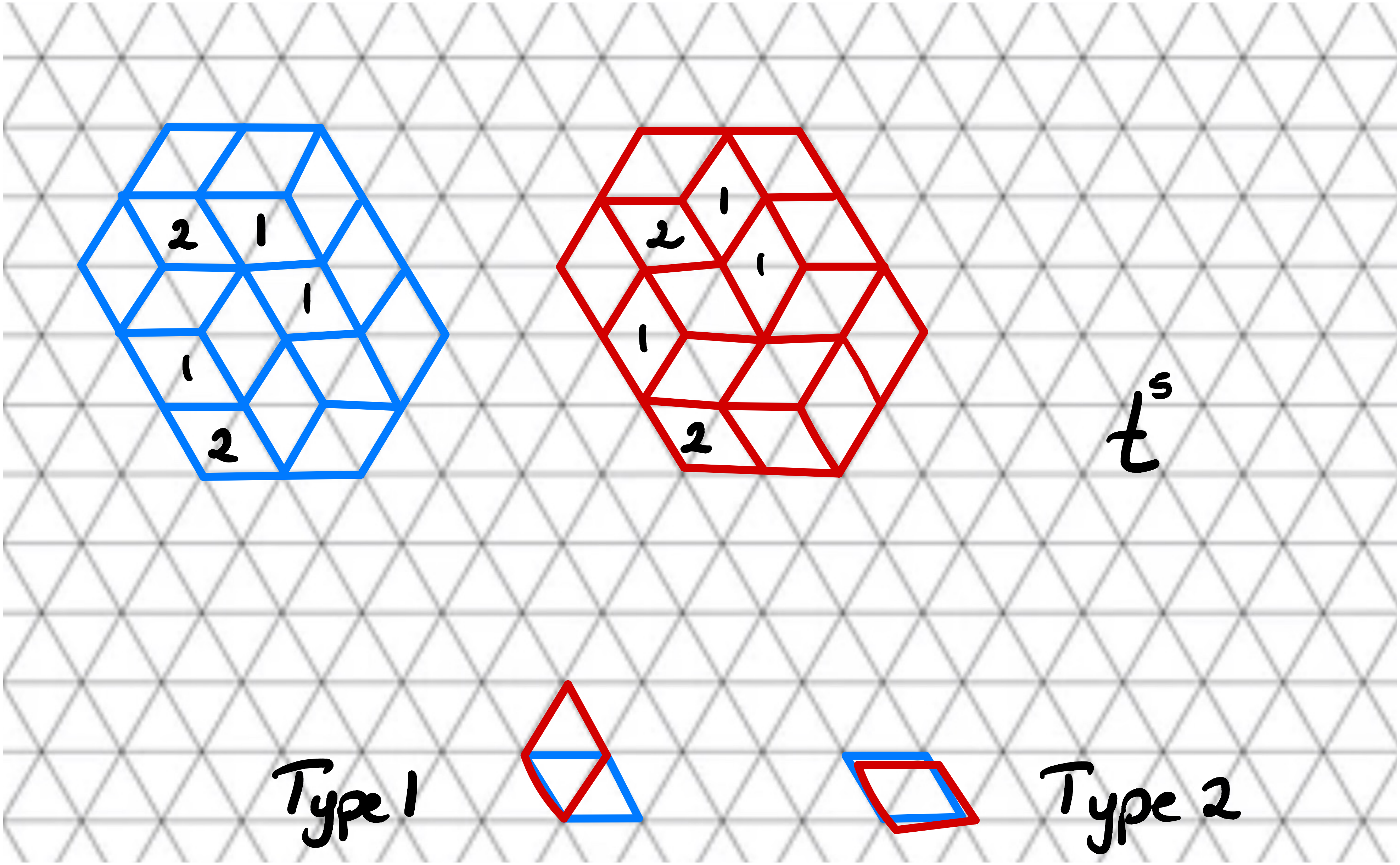


Tile each hexagon, superimpose

$$P^{(k)}_{a,b,c}(t) = \sum_{k\text{-tilings}} t^{\#\text{ interactions}}$$




1
2
1
2
Type 1
Type 2


2
1
1
1
2
1
2
1
1
1
2
t^s
Type 1
Type 2

Polynomials $P^{(2)}_{a,b,c}(t)$

a	b	c	Generating polynomial
1	1	1	$3t + 1$
1	1	2	$6t + 3$
1	2	1	$5t^2 + 3t + 1$
1	2	2	$15t^2 + 15t + 6$
1	2	3	$35t^2 + 45t + 20$
2	1	1	$3t(2t + 1)$
2	1	2	$20t^2 + 15t + 1$
2	2	1	$t^2(15t^2 + 15t + 6)$
2	2	2	$105t^4 + 175t^3 + 104t^2 + 15t + 1$
3	1	1	$t^2(10t + 6)$
3	1	2	$5t(10t^2 + 9t + 1)$
3	2	1	$t^4(35t^2 + 45t + 20)$
3	2	2	$t^2(490t^4 + 1050t^3 + 770t^2 + 175t + 15)$

What do we know?

$$\boxed{\bullet\ P^{(k)}_{a,b,c}(t) = \mathrm{LLT}_{(\underbrace{b,\dots,b}_{a})^k}(\underbrace{1,\dots,1}_{a+c};t)}$$

"co-inversion" Lascoux-Leclerc-Thibon polynomials

What do we know?

- $P^{(k)}_{a,b,c}(t) = LLT_{(\underbrace{b,\dots,b}_{a})^k}(\underbrace{1,\dots,1}_{a+c}\,;t)$

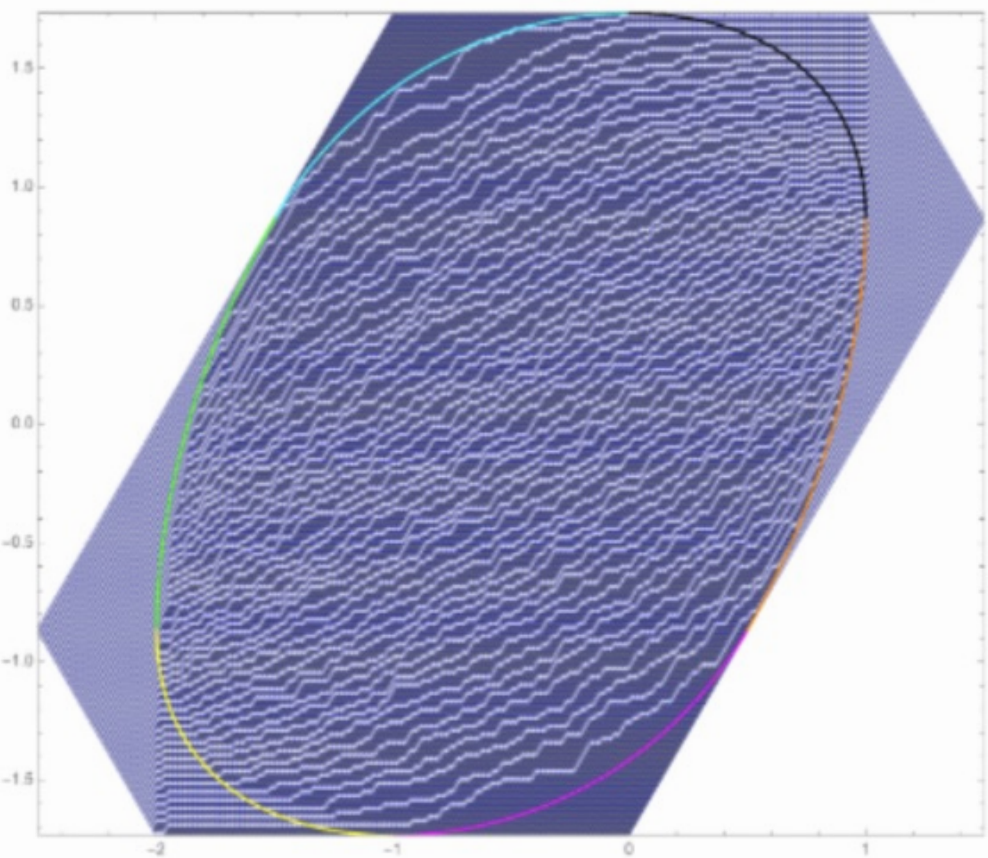
- $P^{(k)}_{a,b,c}(0) = \begin{cases} 0 & \text{if } a < (k-1)c \\ P^{(1)}_{a-(k-1)c,\,b,\,c} & \text{otherwise} \end{cases}$

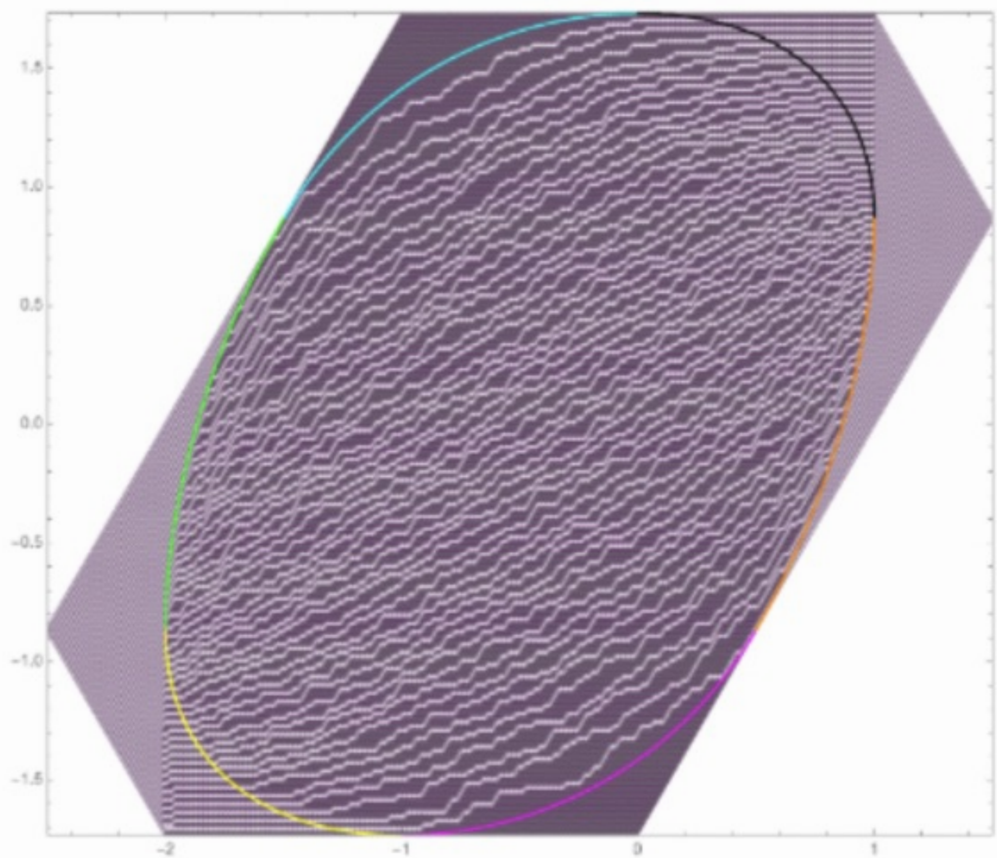
- The coeff. of $t^{\binom{k}{2}ab}$ is $P^{(1)}_{a,kb,c}$

$$\boxed{\bullet\ P^{(k)}_{a,b,c}(t) = t^{\binom{k}{2}b(a-c)}\, P^{(k)}_{c,b,a}(t)}$$

<u>Courbe arctique</u> $b = 2a$ $c = 3a$ $a \to \infty$

$t \to 0$





Q: Limit shape?

Arctic curve for $t \neq 0, 1, \infty$?

k - domino tilings

D. Keating (Next week).