# Emptiness formation in polytropic quantum liquids

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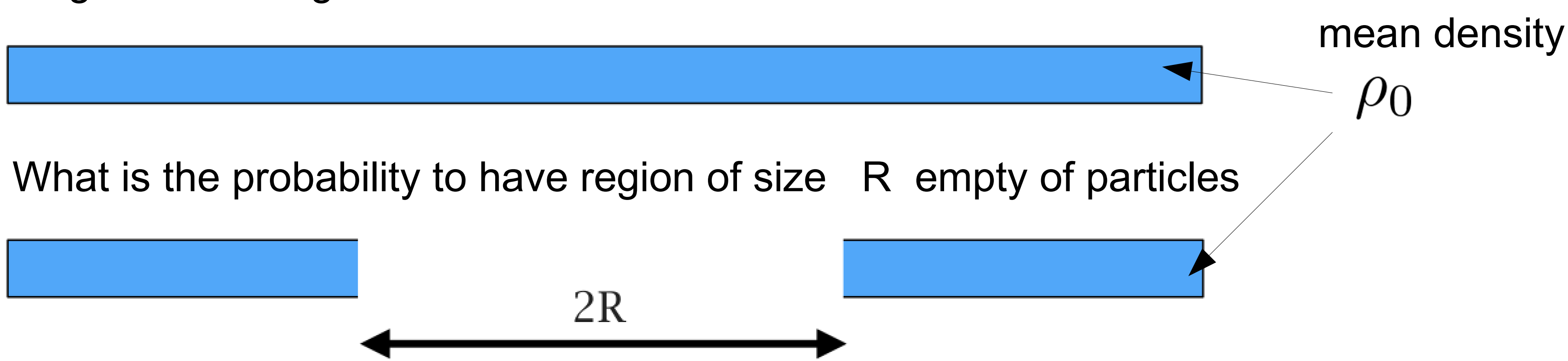
# Large deviations
# - emptiness formation probability

*Abanov 2003*

Imagine a Fermi gas on a line

What is the probability to have region of size R empty of particles




in the ground state?

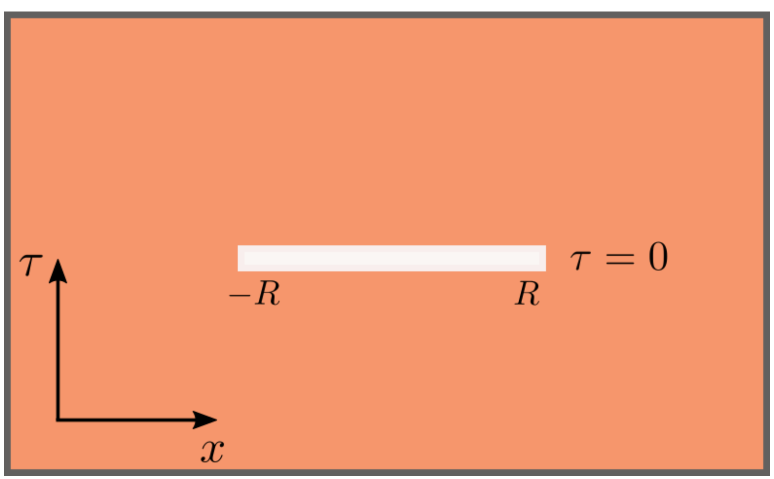
$$P_{\mathrm{EFP}}(R) = \int_{|x_i|>R} \mathrm{d}^N x |\Psi_{\mathrm{GS}}(x_1, x_2, \ldots, x_N)|^2$$

# Path integral

$$P_{\mathrm{EFP}}(R) = \lim_{\beta\to\infty} \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H} |2R\rangle\langle 2R| e^{-\beta H}\right)$$

Hydrodynamic description $\quad R \gg \xi, \rho_0^{-1}$

$$P_{\mathrm{EFP}}(R) = \frac{1}{Z} \int \mathcal{D}[\rho, j] e^{-S[\rho, j]}$$




$$S[\rho, j] = \frac{\rho_0 R^2}{\xi} \int \mathrm{d}t \mathrm{d}x \left[\frac{j^2}{2\rho} + V(\rho)\right]$$

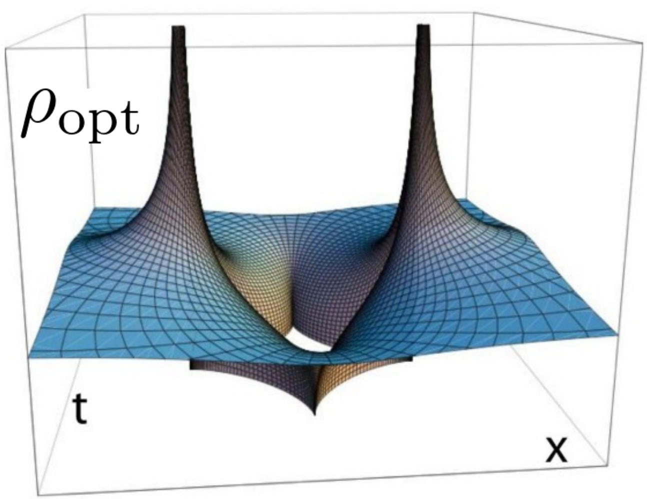
Eq. of state: pressure-density relation at $T = 0$

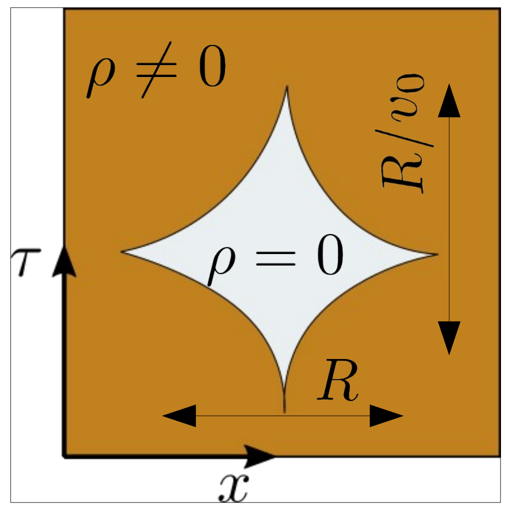
$$P(\rho) = \rho \partial_\rho V(\rho) - V(\rho)$$

# Optimal fluctuation – instanton of hydrodynamical fields

*Abanov 2003*

$$P_{\mathrm{EFP}}(R) \sim e^{-S_{\mathrm{opt}}}$$




Boundary is an *astroid*

$$x^{2/3} + \tau^{2/3} = R^{2/3}$$

$$S_{\mathrm{opt}} = \frac{\pi \rho_0^3}{2} \times R \times R/v_0 \gg 1$$

rare event

$\sim$ area in space-time, # of missing particles

*Courtesy of chabad.org*

# Previous results

- Random Matrices
- Spin chains
- Weakly interacting Bose gas

In all cases

$$S_{\rm opt} \sim \frac{\rho_0 R^2}{\xi}$$

for $R \gg \xi, \rho_0^{-1}$

This talk – Polytropic Quantum Liquid $V(\rho) \sim \rho^\gamma$

$$S_{\rm opt} = f(\gamma)\frac{\rho_0 R^2}{\xi}$$

known values

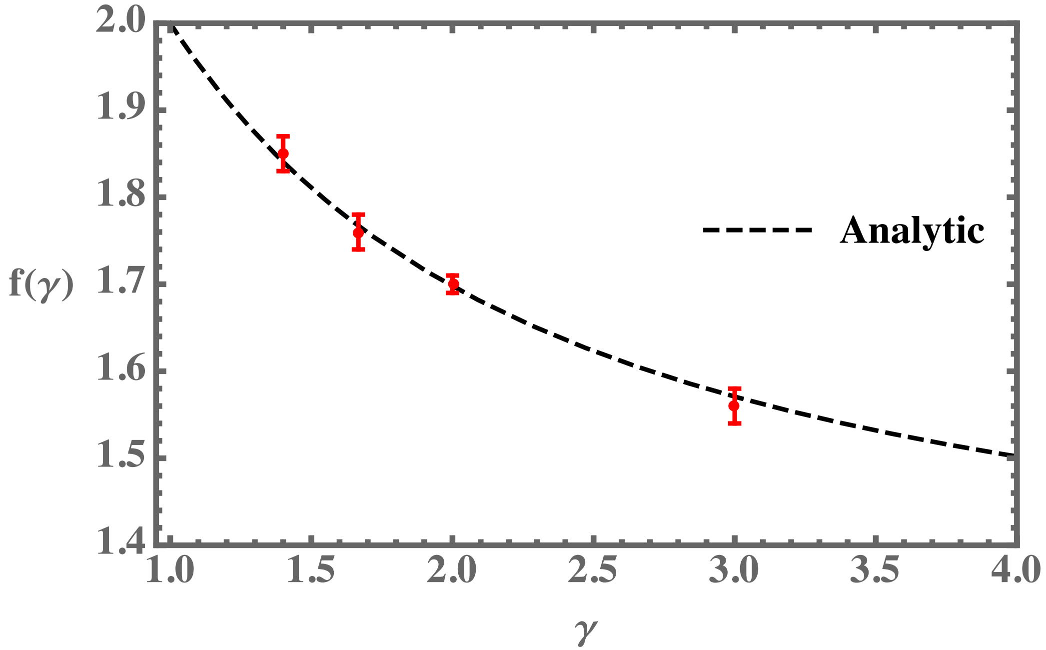
$f(3) = \frac{\pi}{2}$ free fermions, RMT

$f(2) = 1.70(1)$ weakly interacting bosons

*Yeh&Kamenev 2202*

# Main Result

$$f(\gamma) = \frac{\pi \; 2^{\frac{\gamma-5}{\gamma-1}} \; \left[\Gamma\left(\frac{\gamma+1}{\gamma-1}\right)\right]^2}{\Gamma\left(\frac{3\gamma-1}{2\gamma-2}\right) \left[\Gamma\left(\frac{\gamma+1}{2\gamma-2}\right)\right]^3}$$



# Looking for optimal solution

Hydrodynamical equations of motion in imaginary time

continuity $\partial_\tau \rho + \partial_x(\rho v) = 0$

Euler $\partial_\tau v + v\partial_x v = \rho^{\gamma-2}\partial_x \rho$

+ boundary conditions imposed in distant past and future

Riemann invariant $\lambda(x,\tau) = v + \mathrm{i}\frac{2}{\gamma-1}\rho^{\frac{\gamma-1}{2}}$

Complex velocity $w(x,\tau) = v + \mathrm{i}\rho^{\frac{\gamma-1}{2}}$

$$\partial_\tau \lambda + w(\lambda,\bar\lambda)\partial_x \lambda = 0$$
$$\partial_\tau \bar\lambda + \bar w(\lambda,\bar\lambda)\partial_x \bar\lambda = 0$$

coupled *nonlinear* equations

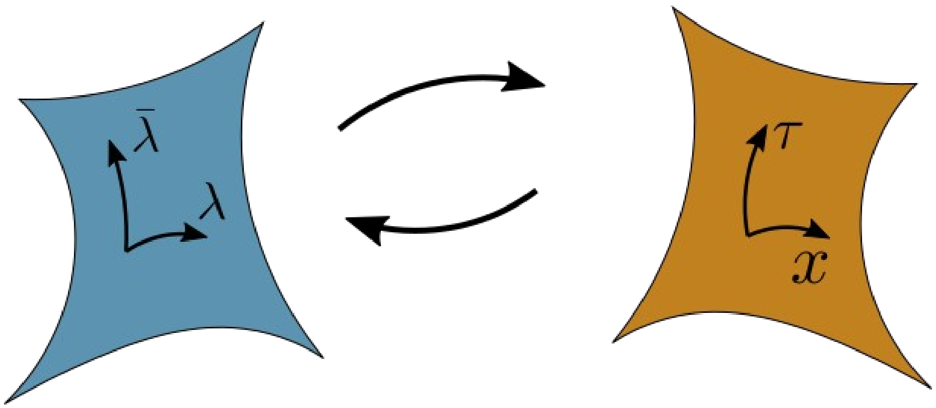
Free fermions $\gamma = 3$ $\quad w = \lambda$ $\quad$ > Complex Burgers $\partial_\tau \lambda + \lambda \partial_x \lambda = 0$

# Hodograph Transform

$$\lambda(x,\tau), \bar{\lambda}(x,\tau) \longrightarrow x(\lambda,\bar{\lambda}), \tau(\lambda,\bar{\lambda})$$

$$\partial_{\bar{\lambda}} x - w(\lambda,\bar{\lambda})\partial_{\bar{\lambda}}\tau = 0$$
$$\partial_{\lambda} x - \bar{w}(\lambda,\bar{\lambda})\partial_{\lambda}\tau = 0.$$

coupled *linear*




Free fermions $w = \lambda$ characteristics $x - \lambda\tau = \partial_\lambda \mathcal{V}$

solve the equations with $\mathcal{V}(\lambda,\bar{\lambda}) = F_0(\lambda) + G_0(\bar{\lambda})$

$$F_0(\lambda) = \overline{G_0(\bar{\lambda})} = \sqrt{\lambda^2+1}$$

found from an electrostatic (RH) problem corresponding to the emptiness boundary condition at $\tau = 0$

# Ballistic Ansatz for general $\gamma$

$$x - w\tau = \partial_\lambda \mathcal{V}$$

$$x - \bar{w}\tau = \partial_{\overline{\lambda}} \mathcal{V}$$

Consistency condition (Euler-Poisson eq)

$$\partial_\lambda \partial_{\overline{\lambda}} \mathcal{V} = \frac{n}{\lambda - \overline{\lambda}} \left( \partial_\lambda \mathcal{V} - \partial_{\overline{\lambda}} \mathcal{V} \right) \qquad \gamma = \frac{2n+3}{2n+1}$$

For $n = 0$ - Laplace equation (free fermions)

For $n =$ integer >0 a closed form solution can be found (*Kamchatnov's book*)

Strategy: solve for infinite sequence of $\gamma$ corresponding to integer $n$
and continue analytically for any value of $\gamma$

# Solution for $n = 0, 1, 2, \ldots$

$$\mathcal{V} = \frac{F_0(\lambda) + G_0(\bar{\lambda})}{(\lambda - \bar{\lambda})^n} + \sum_{m=1}^{n-1} a_m \frac{F_m(\lambda) + (-1)^m G_m(\bar{\lambda})}{(\lambda - \bar{\lambda})^{n+m}}$$

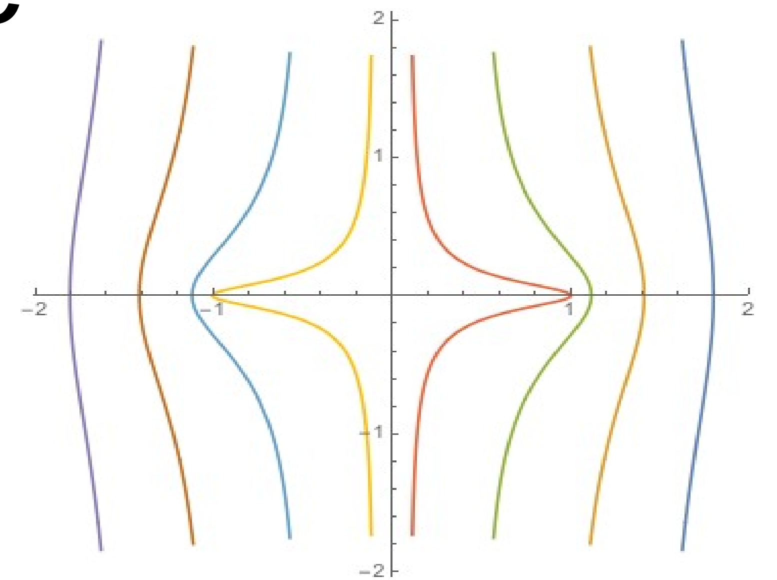
$$F_{m-1} = \partial_\lambda F_m \qquad G_{m-1} = \partial_{\bar{\lambda}} G_m$$

$$a_m = -\frac{(n+m-1)(n-m)}{m} a_{m-1} \qquad a_0 = 1$$

$$n = 0 \qquad \mathcal{V} = F_0(\lambda) + G_0(\bar{\lambda})$$

$$n = 1 \qquad \mathcal{V} = \frac{F_0(\lambda) + G_0(\bar{\lambda})}{\lambda - \bar{\lambda}}$$

# Boundary conditions in $(\lambda, \bar{\lambda})$ plane




$$x - w\tau = \partial_\lambda \mathcal{V}$$

1. Particles accumulate avoiding emptiness region

$$\partial_\lambda \mathcal{V}\big|_{|\lambda| \to \infty} = \pm 1$$

2. Density decays as $\rho \to 1 + 1/x^2$ for $x \to \infty$

$$\partial_\lambda \mathcal{V}\big|_{\lambda \to \mathrm{i}(2n+1)} \sim \frac{1}{\sqrt{\lambda^2 + (2n+1)^2}}$$

Branch points at $x = \pm 1$ for complex functions

$$\lambda(x, \tau = 0), \bar{\lambda}(x, \tau = 0)$$

# Dynamic density profile

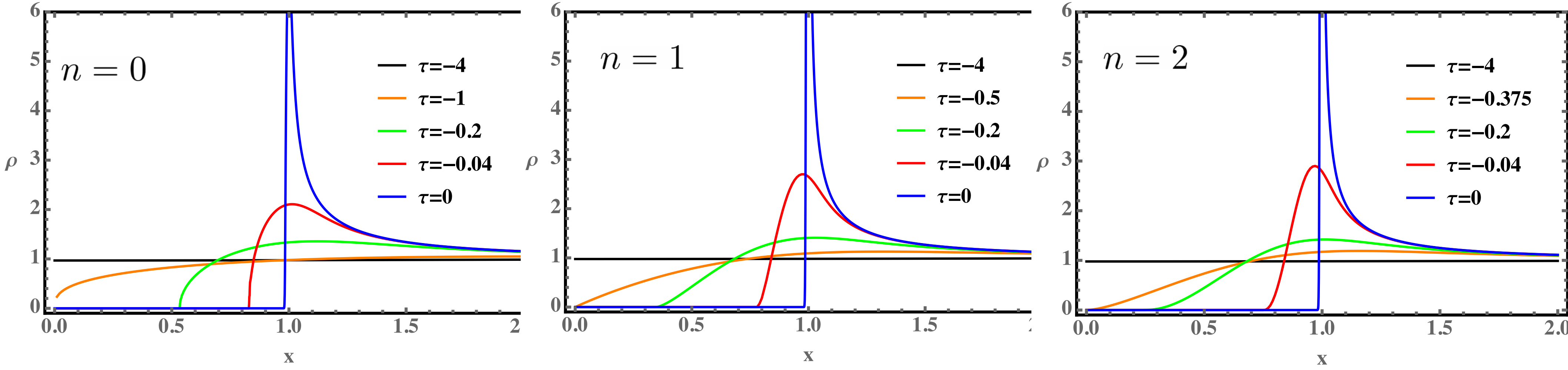
$$F_{n-1} = \frac{\lambda}{n!}\left[\lambda^2 + (2n+1)^2\right]^{\frac{2n-1}{2}} \qquad G_{n-1} = (-1)^n \overline{F}_{n-1}$$

$n = 0$ $\quad \mathcal{V} = F_0 + G_0 \quad F_0 = \overline{G_0(\bar{\lambda})} = \sqrt{\lambda^2 + 1}$

$n = 1$ $\quad \mathcal{V} = \dfrac{F_0 + G_0}{\lambda - \bar{\lambda}} \quad F_0 = -\overline{G_0(\bar{\lambda})} = \lambda\sqrt{\lambda^2 + 9}$

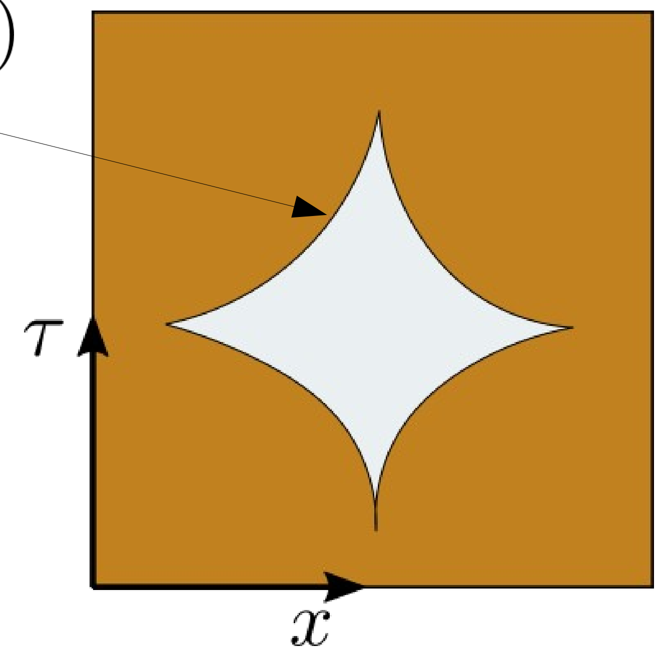
$n = 2$ $\quad \mathcal{V} = \dfrac{F_0 + G_0}{(\lambda - \bar{\lambda})^2} - 2\dfrac{F_1 + G_1}{(\lambda - \bar{\lambda})^3} \quad F_1 = \overline{G_1(\bar{\lambda})} = \dfrac{\lambda}{2}(\lambda^2 + 25)^{3/2} \quad F_0 = \overline{G_0(\bar{\lambda})} = \partial_\lambda F_1$

Solving $x - w\tau = \partial_\lambda \mathcal{V}$ for $\lambda, \bar{\lambda}$ and extracting $\rho(x, \tau)$



# Inside Emptiness

$$\rho = 0 \quad \Rightarrow \quad \lambda = \bar{\lambda} = w = \bar{w} = v(x,\tau)$$




Ballistic evolution $\quad x - v\tau = X_n(v)$

$$X_0(v) = \frac{v}{\sqrt{v^2+1}}$$

$$X_1(v) = \frac{3v}{\sqrt{v^2+9}} - \frac{v^3}{2(v^2+9)^{3/2}}$$

$$X_2(v) = \frac{15v}{8\sqrt{v^2+25}} - \frac{5v^3}{4(v^2+25)^{3/2}} + \frac{3v^5}{8(v^2+25)^{5/2}}$$

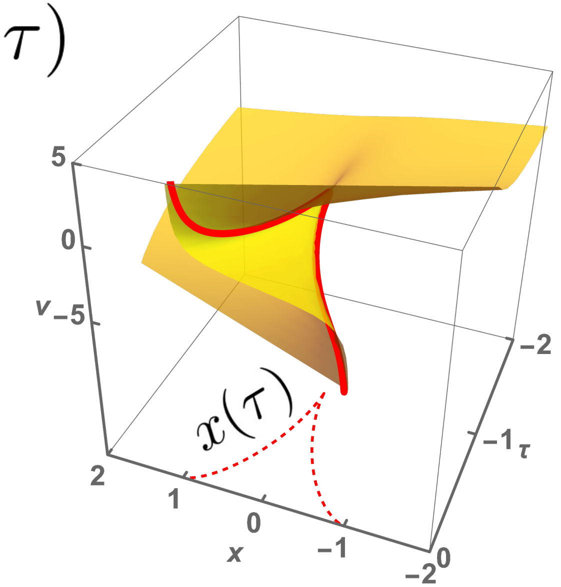
$$X_n(v) \sim 1 + \frac{1}{v^{2n+2}} \qquad \text{as} \qquad v \to \infty$$

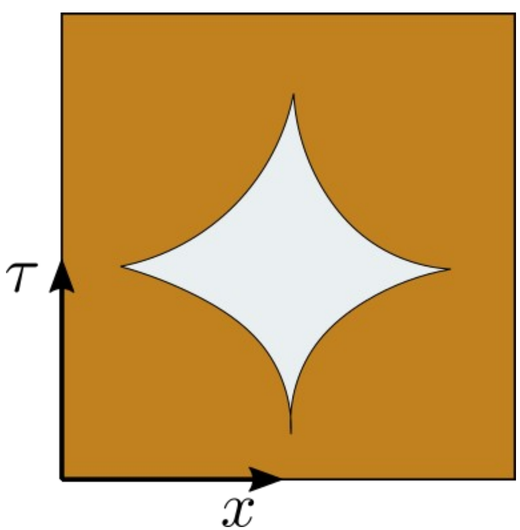
# Boundary of emptiness region (tangent method)

$x - v\tau = X(v)$ defines a surface $v(x, \tau)$

The surface has *folds when* $-\tau = \partial_v X(v)$

Emptiness boundary $x(\tau)$ is Legendre Transform of $X(v)$




Higher singularities – *cusps* – appear when two folds coalesce

„A transparent torus is rarely seen. Let us consider a different transparent body – a bottle (preferably milk). In Fig. 5 two cusp points are visible. By moving the bottle a little we may satisfy ourselves that the cusp singularity is stable. So we have convincing experimental confirmation of Whitney's theorem."

*Vladimir Arnold, "Catastrophe Theory"*

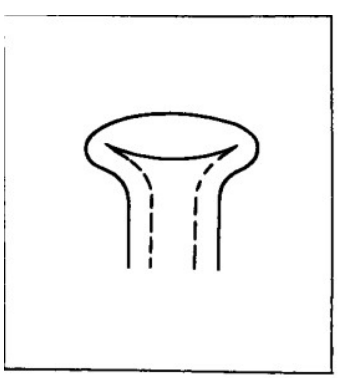


*Fig. 5*

# Universal behaviour near cusps

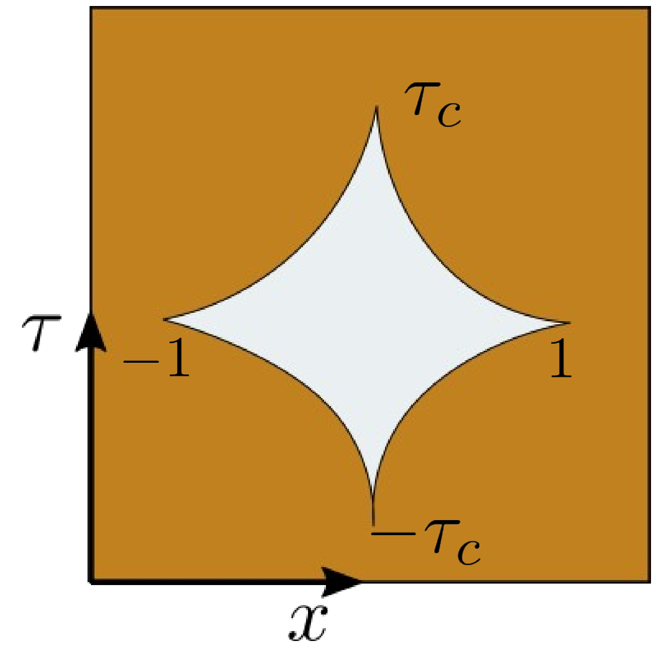
We have two types of cusps

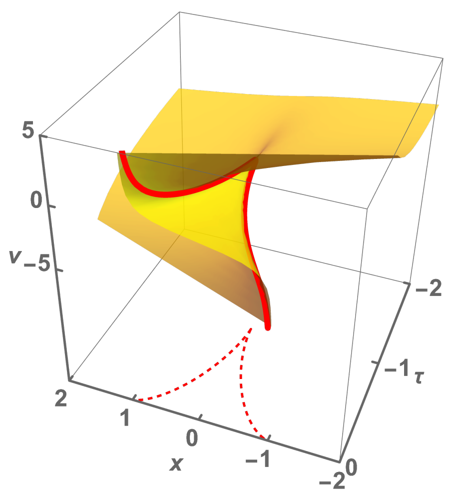
<u>Soft</u> $v \to 0$ $(x,\tau) \to (0, \pm\tau_c)$

$$x = (\tau - \tau_c)v - b_n v^3 \Rightarrow \tau - \tau_c \propto |x|^{2/3}$$

<u>Hard</u> $v \to \infty$ $(x,\tau) \to (\pm 1, 0)$

$$\tau = \frac{1}{v}(x-1) + \frac{c_n}{v^{2n+3}} \Rightarrow \quad 1 - x \propto |\tau|^{\frac{2n+2}{2n+3}}$$




For free fermions: symmetry between soft and hard cusps

$$x \to \tau$$

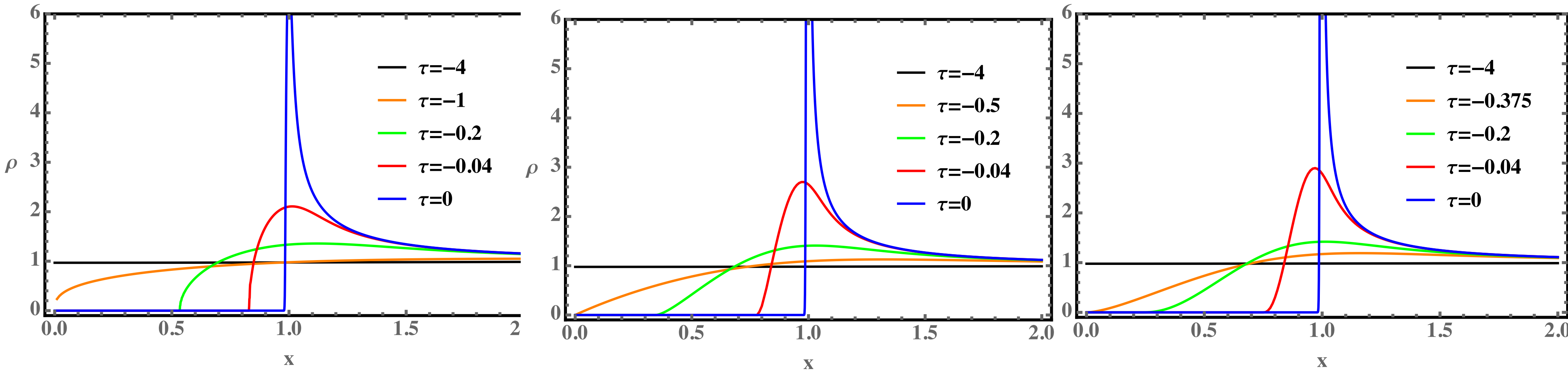
$$v \to 1/v$$

# Universal density profiles near emptiness boundary

$$\tau \neq 0 \qquad \rho \propto (x - x(\tau))^{1/(\gamma-1)}, \qquad x > x(\tau),$$

– same as exponent predicted by Thomas-Fermi at $\tau = 0$

$$\tau = 0 \qquad \rho \propto (x - x(0))^{-2/(\gamma+1)}, \qquad x > 1$$




NB: Don't trust the polytropic law down to zero density: square root density profile near the boundary

# Calculation of Emptiness Probability – Instanton action

From density asymptotics $$\rho(x,0) = 1 + \frac{\alpha}{x^2} + \mathcal{O}\left(\frac{1}{x^4}\right)$$

and correlation length $$\frac{1}{\xi} = \rho_0^{1/(2n+1)}$$

$$\partial_{\rho_0} S_{\mathrm{opt}} = \frac{\pi R^2}{\xi}\alpha$$

$$\alpha = \frac{1}{2(2n+1)}\left[\frac{(2n+1)!!}{2^n n!}\right]^2$$

is extracted from the divergence

$$x \sim \frac{(2n-1)!!}{n!}\frac{\lambda^{n+1}}{(\lambda-\bar{\lambda})^n\sqrt{\lambda^2+(2n+1)^2}}$$

as $\lambda \to \mathrm{i}(2n+1)$

# Result for Polytropic Emptiness

$$S_{\rm opt} = \frac{\rho_0 R^2}{\xi} f(n)$$

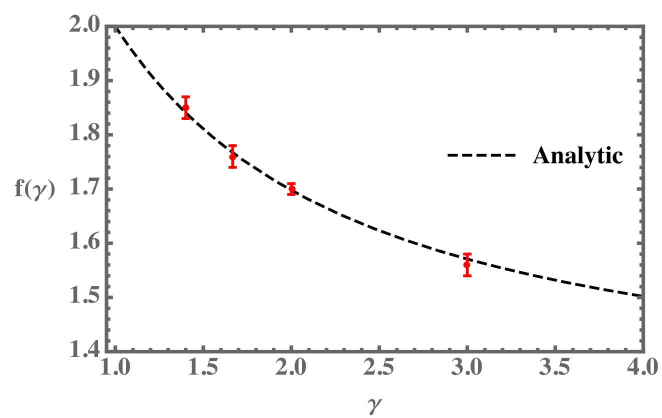
$$f(n) = \frac{\pi \Gamma^2(2n+2)}{2^{4n+1}\Gamma(n+2)\Gamma^3(n+1)}$$

The result can be analytically continued to any real $n$

| | $n=0$ $(\gamma=3)$ | $n=1$ $(\gamma=5/3)$ | $n=2$ $(\gamma=7/5)$ |
|---|---|---|---|
| $f(n)$ | $1.56 \pm 0.02$ | $1.76 \pm 0.02$ | $1.85 \pm 0.02$ |
| Eq.(52) | $\pi/2 \approx 1.571$ | $9\pi/16 \approx 1.767$ | $75\pi/128 \approx 1.841$ |

– numerical estimate

$$f(\gamma) = \frac{\pi\, 2^{\frac{\gamma-5}{\gamma-1}} \left[\Gamma\left(\frac{\gamma+1}{\gamma-1}\right)\right]^2}{\Gamma\left(\frac{3\gamma-1}{2\gamma-2}\right)\left[\Gamma\left(\frac{\gamma+1}{2\gamma-2}\right)\right]^3}$$



# Conclusions

- First calculation of EFP in polytropic quantum liquid as a function of polytropic index.
- Example of interacting system, beyond free fermionic models
- Universal features, including shape of Emptiness Boundary singularities

**Outlook**

- Calculation of subleading terms (logarithmic for both fermions and weakly interacting bosons)
- Statistical models with polytropic coarse grained e.o.s?
- Real time dynamics from $\tau \to it$? Loschmidt echo, return probability
- Other physical models: magnetic impurities in SC

Cheers!

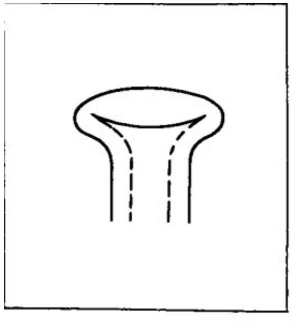


*Fig. 5*