First-passage resetting (and some other first-passage problems)

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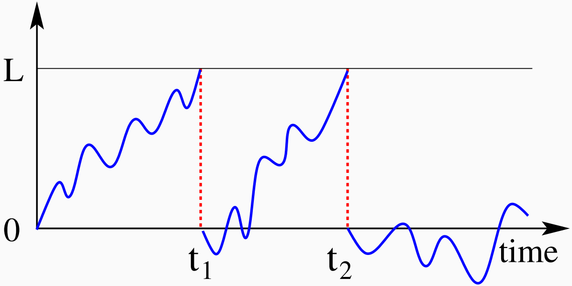
Introduction: Brownian motion, resetting, first passage etc.

Topics in this talk include:

- stochastic processes – specifically Brownian motion
- first-passage problems – [see eg *A Guide to First-Passage Processes*, Redner 2001]
- resetting of a stochastic process:
  - restart at regular fixed times [textbook, see eg Ex. 22 in Grimmett & Stirzaker]
  - restart at random times indep. of the process:
    eg stochastic resetting with exp. dist. times admits optimal rate
    [see Evans & Majumdar 2011, and review by Evans, Majumdar, Schehr 2020]
  - restart at random times dep. on the process:
    elementary/instantaneous return process [Feller 1953, Sherman 1957],
    Brownian motion with rebirth [Grigorescu & Kang 2007]
    first-passage resetting [de Bruyne, R-F, Redner 2020]
- path transformations – i.e. bijections between sets of sample paths

First-passage resetting

First-passage resetting in Brownian motion: semi-infinite geometry



- ▷ Brownian particle starting at the origin, with diffusion constant D
- ▷ when it reaches $L \longrightarrow$ instantaneously reset to the origin
- ▷ and so on — successive "first"-passage/reset times $t_1, t_2, \ldots$

Pdf for n^{th} reset time? Pdf for position at t? Number of resets?

(recall that average time between two resets is infinite)

First-passage resetting in Brownian motion: pdf of the n^{th} reset

- Define: $F_n(L,t)$ = pdf for the n^{th} reset to occur at time t.
- One has the renewal equation:

$$F_n(L,t) = \int_0^t dt'\, F_{n-1}(L,t')\, F_1(L,t-t'), \quad n > 1\,.$$

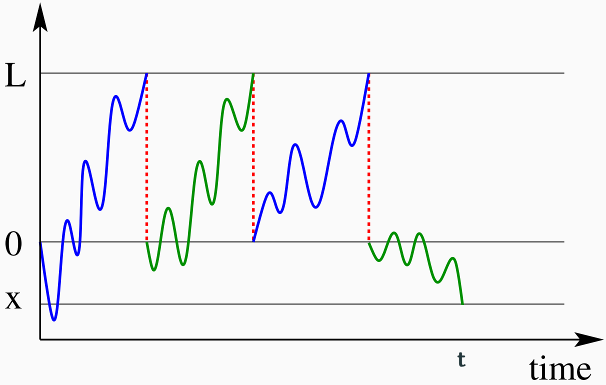
Convolution structure $\longrightarrow$ Laplace transform:

$$\widetilde{F}_n(L,s) = \left[\widetilde{F}_1(L,s)\right]^n = \left[e^{-L\sqrt{s/D}}\right]^n = e^{-n\,L\sqrt{s/D}}$$

$$F_n(L,t) = \frac{n\,L}{\sqrt{4\pi D\,t^3}}\, e^{-n^2\,L^2/4Dt}$$

$\looparrowright$ **pdf for n^{th} reset at t = pdf for 1^{st} passage at $n\,L$ at t**

First-passage resetting in Brownian motion: pdf for the position at t



$$P(x,t)\,\mathrm{d}x \;=\; \mathrm{Prob}\,(\,\text{particle in } [x, x+\mathrm{d}x] \text{ at time } t\,)$$

First-passage resetting in Brownian motion: pdf for the position at t

- Introduce the propagator with absorption at L,

$$G(x, L, t) = \left[e^{-x^2/4Dt} - e^{-(x-2L)^2/4Dt}\right]/\sqrt{4\pi Dt}\,.$$

- Then one has the *forward* renewal equation:

$$P(x, t) = G(x, L, t) + \sum_{n\geq 1} \int_0^t dt'\, F_n(L, t')\, G(x, L, t-t'),$$

- or equivalently the *backward* renewal equation:

$$P(x, t) = G(x, L, t) + \int_0^t dt'\, F_1(L, t') P(x, t - t')\,.$$

First-passage resetting in Brownian motion: pdf for the position at t

Again, convolution structure $\longrightarrow$ Laplace transform:

$$\widetilde{P}(y,s) = \frac{\widetilde{G}(y,y_L,s)}{1-\widetilde{F}_1(y_L,s)} = \frac{1}{\sqrt{4Ds}}\,\frac{\left[e^{-|y|}-e^{-|y-2y_L|}\right]}{1-e^{-y_L}}\,,$$

with reduced variables $y = x\sqrt{s/D}$ and $y_L = L\sqrt{s/D}$.

$$\text{2 distinct cases} \quad — \quad \begin{cases} 0 \le y \le y_L & \text{i.e. } 0 \le x \le L \\ y < 0 & \text{i.e. } x < 0 \end{cases}$$

First-passage resetting in Brownian motion: pdf for the position at t

Computing $P(x,t)$ **– when** $0 \leq x \leq L$ **i.e.** $0 \leq y \leq y_L$

$$\widetilde{P}(y,s) = \frac{1}{\sqrt{4Ds}} \left[e^{-y} - e^{-(2y_L - y)}\right] \sum_{n\geq 0} e^{-ny_L} = \frac{1}{\sqrt{4Ds}} \sum_{n\geq 0} \left[e^{-(y+ny_L)} - e^{-[(n+2)y_L - y]}\right],$$

from which

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \sum_{n\geq 0} \left[e^{-(x+nL)^2/4Dt} - e^{-[x-(n+2)L]^2/4Dt}\right].$$

In the long-time limit, $t \gg 1$

$$P(x,t) \simeq \frac{1}{\sqrt{\pi Dt}} \frac{L-x}{L}$$

↬ balance between diffusive flux exiting at $x = L$ and re-injected at $x = 0$.

First-passage resetting in Brownian motion: pdf for the position at t

Computing $P(x,t)$ **– when** $x<0$ **i.e.** $y<0$

$$\widetilde{P}(y,s)=\frac{1}{\sqrt{4Ds}}\left[\frac{e^{y}-e^{y-2y_L}}{1-e^{-y_L}}\right]=\frac{1}{\sqrt{4Ds}}\left[e^{y}+e^{(y-y_L)}\right],$$

from which

$$P(x,t)=\frac{1}{\sqrt{4\pi Dt}}\left[e^{-x^2/4Dt}+e^{-(x-L)^2/4Dt}\right].$$

$\looparrowright$ i.e. superposition of paths from 0 and paths from L... Interpretation?

First-passage resetting in Brownian motion: pdf for the position at t

Computing $P(x,t)$ **– when** $x<0$ **i.e.** $y<0$

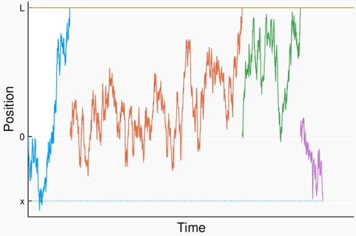
$$\widetilde{P}(y,s)=\frac{1}{\sqrt{4Ds}}\left[\frac{e^{y}-e^{y-2y_L}}{1-e^{-y_L}}\right]=\frac{1}{\sqrt{4Ds}}\left[e^{y}+e^{(y-y_L)}\right],$$

from which

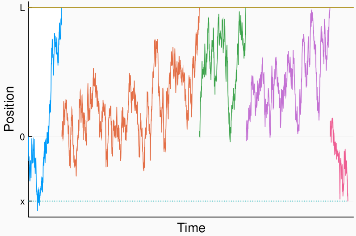
$$P(x,t)=\frac{1}{\sqrt{4\pi Dt}}\left[e^{-x^2/4Dt}+e^{-(x-L)^2/4Dt}\right].$$

↬ i.e. superposition of paths from 0 and paths from L – easy with path transformation

First-passage resetting in Brownian motion: pdf for the position at t



(a)



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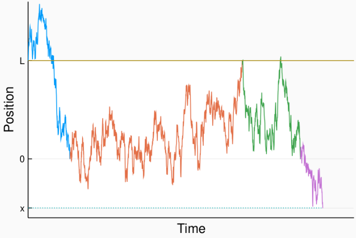

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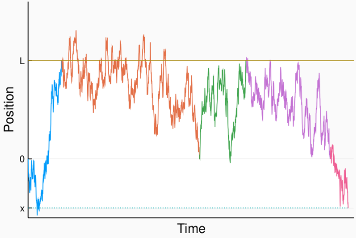


(d)

Odd number of resets

Even number of resets

First-passage resetting in Brownian motion: number of resets

- Define: $Q_n(L,t) = \mathrm{Prob}(\text{exactly } n \text{ resets occur up to time } t)$.

$$Q_n(L,t) = \int_0^t dt' \left[F_n(L,t') - F_{n+1}(L,t')\right] = \mathrm{erf}\left(\frac{(n+1)L}{\sqrt{4Dt}}\right) - \mathrm{erf}\left(\frac{nL}{\sqrt{4Dt}}\right)$$

- Backward renewal equation for the average number of resets $\mathcal{N}(t)$:

$$\mathcal{N}(t) = \int_0^t dt'\, F_1(L,t')\big(1 + \mathcal{N}(t-t')\big)\,.$$

- Laplace transforming:

$$\widetilde{\mathcal{N}}(s) = \frac{\widetilde{F}_1(L,s)}{s\left[1 - \widetilde{F}_1(L,s)\right]} = \frac{e^{-y_L}}{s(1 - e^{-y_L})}$$

$$\mathcal{N}(t) \simeq \sqrt{\frac{4Dt}{\pi L^2}} \qquad \text{when } t \gg 1$$

Optimization in first-passage resetting

First-passage resetting in Brownian motion: optimization

Consider now:

- $x(t)$ models the operating point of a system;
- $x(t) \geq 0$ and if $x(t) = L$ the system breaks down, incurring a cost C;
- control mechanism modelled by a drift v.

And seek to maximize:

$$\mathcal{F} = \lim_{T\to\infty} \frac{1}{T}\left[\frac{1}{L}\int_0^T \langle x(t)\rangle \, dt - C\,\mathcal{N}(T)\right].$$

First-passage resetting in Brownian motion: optimization

Convection diffusion equation for the pdf $P(x,t)$:

$$\partial_t P + v\partial_x P = D\partial_{xx} P + \delta(x)(-D\partial_x P + vP)|_{x=L}\,,$$

subject to the initial and boundary conditions

$$\begin{cases} (D\partial_x P - vP)|_{x=0} = \delta(t)\,, \\ P(L,t) \;=\; P(x,0) = 0\,. \end{cases}$$

It admits a steady-state solution,

$$P(x) \simeq \frac{1}{L} \times \frac{1 \,-\, e^{-2\mathrm{Pe}\,(L-x)/L}}{1 \,-\, \mathrm{Pe}^{-1}\, e^{-\mathrm{Pe}}\, \sinh\left(\mathrm{Pe}\right)}\,,$$

where $\mathrm{Pe} \equiv vL/2D$ is the Péclet number.

First-passage resetting in Brownian motion: optimization

From the steady-state solution, one obtains the normalized first moment:

$$\frac{\langle x \rangle}{L} = \frac{1}{L} \int_0^L x\, P(x)\, \mathrm{d}x = \frac{\left(2\mathrm{Pe}^2 - 2\mathrm{Pe} + 1\right)\, e^{2\mathrm{Pe}} - 1}{2\mathrm{Pe}\left[(2\mathrm{Pe} - 1)\, e^{2\mathrm{Pe}} + 1\right]} \, .$$

Again with a backward equation, one also obtains the average number of breakdowns:

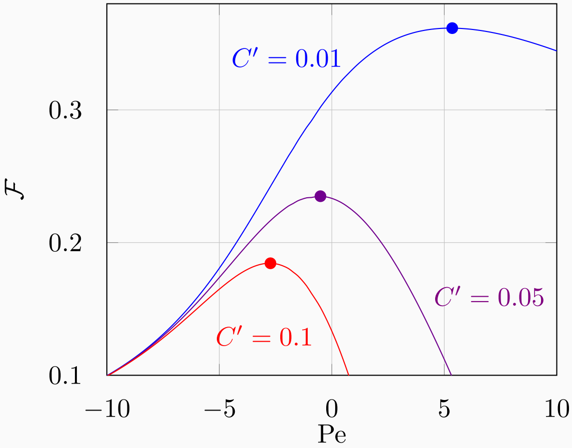
$$\mathcal{N}(T) \simeq \frac{4\mathrm{Pe}^2}{2\mathrm{Pe} - 1 + e^{-2\mathrm{Pe}}} \, \frac{T}{L^2/D} \, .$$

Hence:

$$\mathcal{F} \simeq \frac{\left(2\mathrm{Pe}^2 - 2\mathrm{Pe} + 1\right)\, e^{2\mathrm{Pe}} - 1}{2\mathrm{Pe}\left[(2\mathrm{Pe} - 1)\, e^{2\mathrm{Pe}} + 1\right]} - \frac{4\mathrm{Pe}^2}{2\mathrm{Pe} - 1 + e^{-2\mathrm{Pe}}} \, \frac{C}{L^2/D} \, ,$$

with $\mathrm{Pe} \equiv vL/2D$ the Péclet number.

First-passage resetting in Brownian motion: optimization



Objective function $\mathcal{F}$ in terms of Péclet number $\mathrm{Pe} \equiv v\,L\,/\,2D$
for different values of normalized cost $C' \equiv C\,/\,(L^2/D)$.

First-passage resetting in Brownian motion: variations

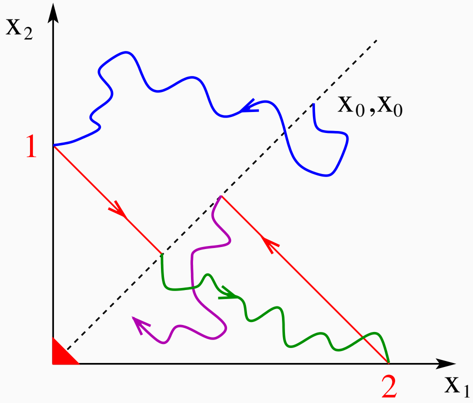
Variations include:

- ▷ delay for "repairs" after breakdown [de Bruyne, R-F, Redner 2020, 2021a]
- ▷ boundary recession [de Bruyne, R-F, Redner 2021a, 2022]
- ▷ higher-dimensional cases [Sherman 1957, de Bruyne, R-F, Redner 2021a]
- ▷ reset at random point [Feller 1953, Sherman 1957, Grigorescu, Kang 2007]
- ▷ multi-particle resetting [de Bruyne, R-F, Redner 2021b]

Multi-particle first-passage resetting

Multi-particle first-passage resetting

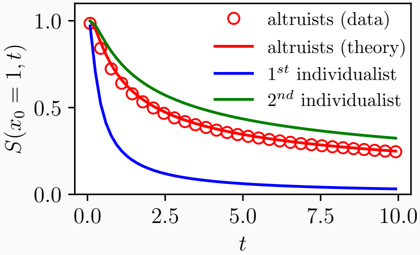
▷ multi-particle resetting: eg two "altruistic" particles



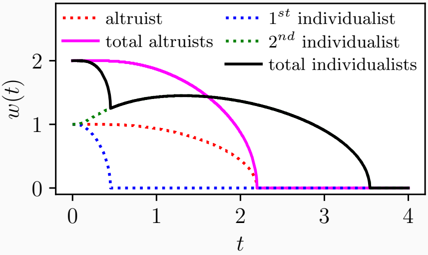
Multi-particle first-passage resetting

- Compare "altruistic" vs "individualistic" systems – eg for $N = 2$ agents

Use order statistics + $X_a(t) = \frac{x_1(t)+x_2(t)}{2}$ follows a BM with diff. cst $D_\parallel = D/2$



(a) Survival probability

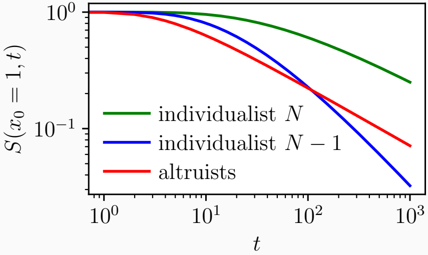


(b) Median "wealth" of agents

Multi-particle first-passage resetting

▷ Compare "altruistic" vs "individualistic" systems – eg for $N = 16$ agents

Use order statistics + $X_a(t) = \frac{x_1(t)+\ldots x_N(t)}{2}$ follows a BM with diff. cst $D_{\parallel} = D/N$



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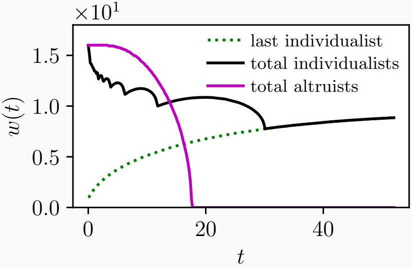
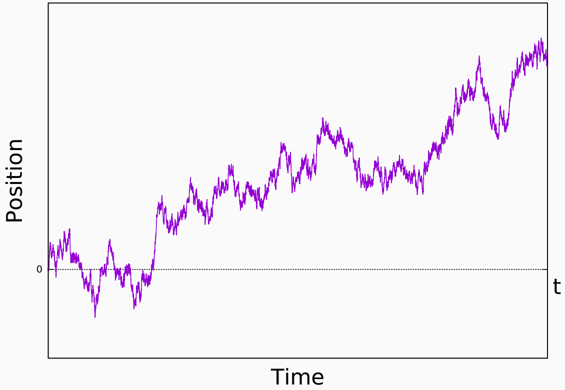


(b) Median "wealth" of agents

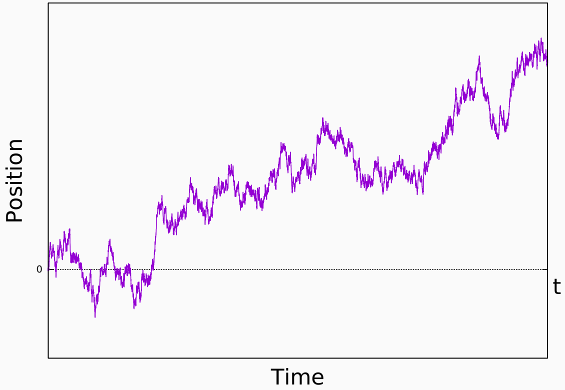
A first-passage problem for the Brownian supremum

A (toy) foraging problem (by P. Krapivsky)



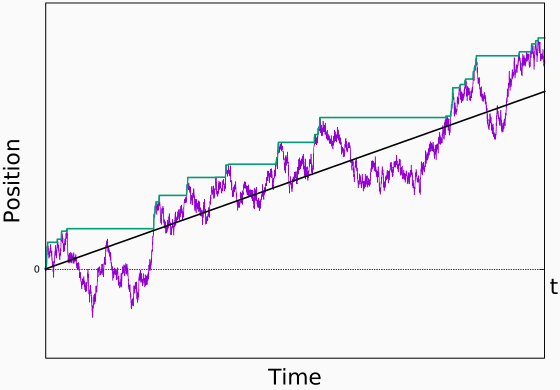
- ▷ "forager" on a line → Brownian walker with position $B(t)$
- ▷ one unit of "food" per unit length, no food replenishment
- ▷ "metabolism": walker stockpiles, needs one unit of food per unit time
- ▷ survival probability?

One-sided version



- ▷ "forager" on a line $\rightarrow$ Brownian walker with position $B(t)$
- ▷ one unit of "food" per unit length, no food replenishment
- ▷ "metabolism": walker needs one unit of food per unit time
- ▷ food on > 0 side only

One-sided version: a hitting-time problem for the supremum



Letting

$$M(s) = \sup_{0 \le \tau \le s} B(\tau)$$

Survival probability is:

$$P(t) = \mathrm{Prob}\left(M(s) > s,\, \forall s \le t\right)$$

A hitting-time problem for the supremum

Survival probability $P(t) = \mathrm{Prob}\,(M(s) > s,\, \forall s \leq t)$

$$\downarrow$$

$$f(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\, P(t)$$

Probability density function (pdf) of extinction time

$$\|$$

Pdf of first hitting time for $M(s)$ on the diagonal, $\inf\limits_{s>0} \{M(s) - s = 0\}$

Idea: look at paths with $M(t) = t$

Path going extinct at $t \Rightarrow M(t) = t$, but...

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$$M(t) = t \not\Rightarrow \text{path going extinct at } t$$

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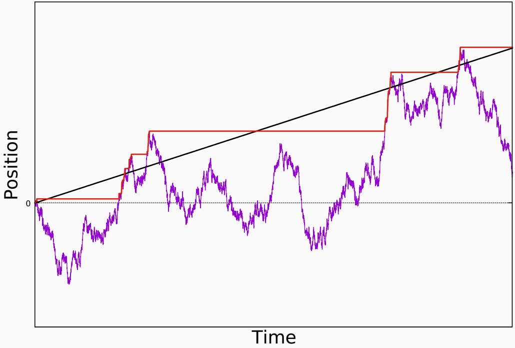
because path could have gone extinct before.



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$$M(t) = t \nRightarrow \text{path going extinct at } t$$

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$$f(t) = \overbrace{\mathrm{pdf}\,(M(t) = t)}^{\text{paths with } M(t)=t} - \underbrace{g(t)}_{\substack{\text{paths with } M(t)=t \\ \textit{and} \\ M(s)=s \text{ for some } s<t}}$$

Idea: look at paths with $M(t) = t$

Path going extinct at $t \Rightarrow M(t) = t$, but...

$M(t) = t \not\Rightarrow$ path going extinct at t

because path could have gone extinct before.

$$f(t) = \overbrace{\sqrt{\frac{2}{\pi t}} \exp\left(-\frac{t}{2}\right)}^{\text{paths with } M(t)=t} - \underbrace{g(t)}_{\substack{\text{paths with } M(t)=t \\ \text{and} \\ M(s)=s \text{ for some } s<t}}$$

A path transformation (1)

Given a path with $M(t) = t$

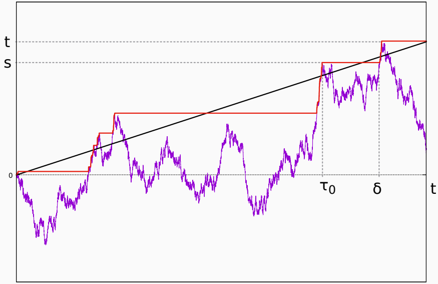
and $M(s) = s$ for some $s < t$, define:

- hitting time of $M(s) = s$:

  $$\tau_0 = \inf \{r > 0, B(r) = M(s) \text{ i.e. } B(r) = s\}$$

- first time level s is hit after τ_0:

  $$\delta = \inf \{r > \tau_0, B(r) \geq M(s) \text{ i.e. } B(r) \geq s\}$$



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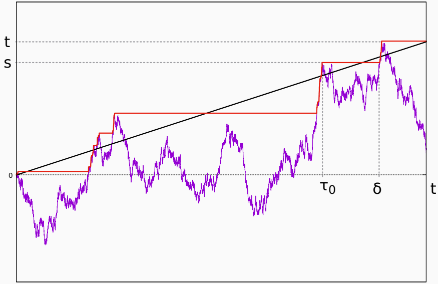

Then

- Note: **path between τ_0 and δ is a (downward) excursion**

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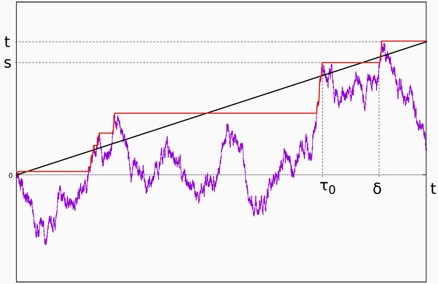
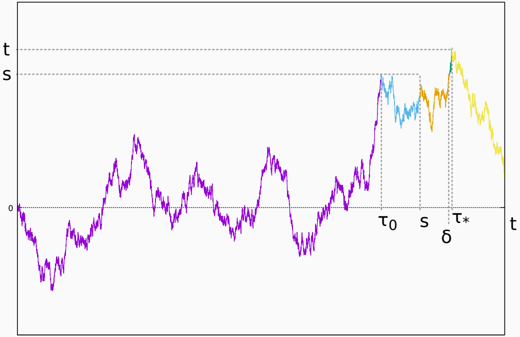


Then

- Note: **path between τ_0 and δ is a (downward) excursion**
- Idea : extract this excursion & use it to hit a new global maximum $> t$

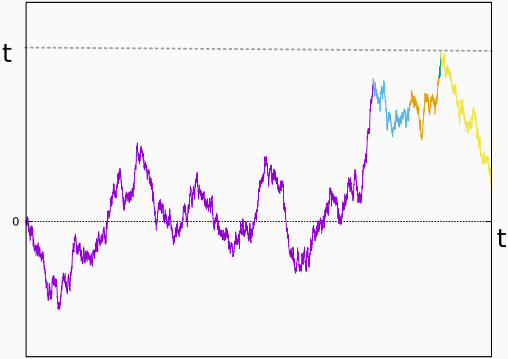
A path transformation (1)



- ▷ Define hitting time of the global maximum,

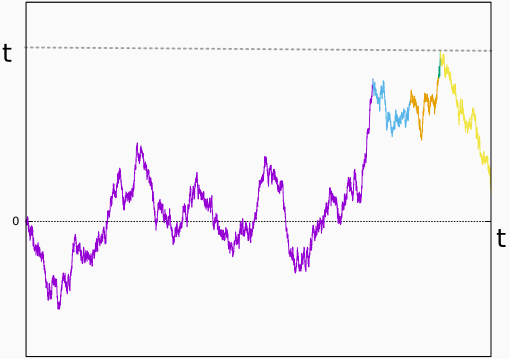
$$\tau_* = \inf\{r > 0, B(r) = M(t) \text{ i.e. } B(r) = t\}$$

A path transformation (1)



- ▷ extract excursion & bring “forward” (to τ_0) the $[\delta, \tau_*]$ part
- ▷ insert then the excursion transformed into an (upward) first passage bridge
- ▷ insert the final, post-τ_* part shifted upward as needed

A path transformation (1)



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- ▷ insert then the excursion transformed into an (upward) first passage bridge
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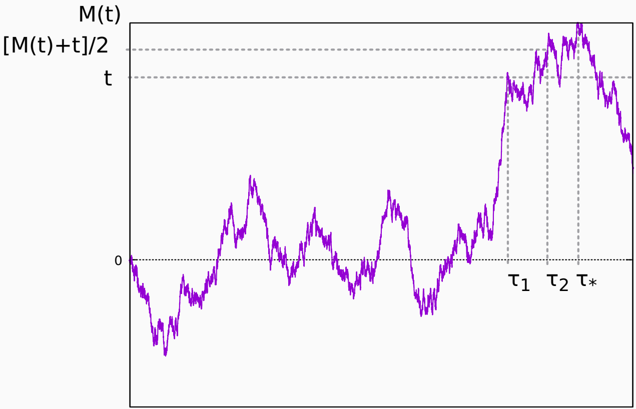
↬ obtain a path with global maximum $> t$

A path transformation (2)

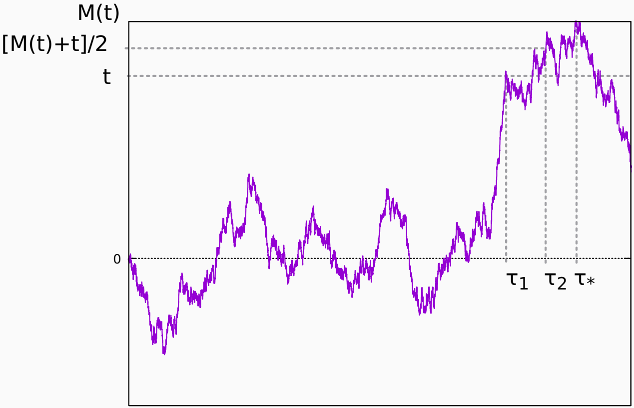


Start with a Brownian path having $M(t) > t$, and set:

$$\tau_1 = \inf\{s > 0, B(s) = t\}, \quad \tau_* = \inf\{s > 0, B(s) = M(t)\},$$

and $$\tau_2 = \inf\{s > 0, B(s) = [M(t) + t]/2\}.$$

A path transformation (2)



Note that $B(0) - 0 = 0$ and $B(\tau_1) - \tau_1 = t - \tau_1 > \tau_2 - \tau_1$, so

$$\exists\, \tau_0 \in \,]0, \tau_1[\ \text{ s.t. } \tau_0 = \inf\{s > 0, B(s) - s = \tau_2 - \tau_1\}\,.$$

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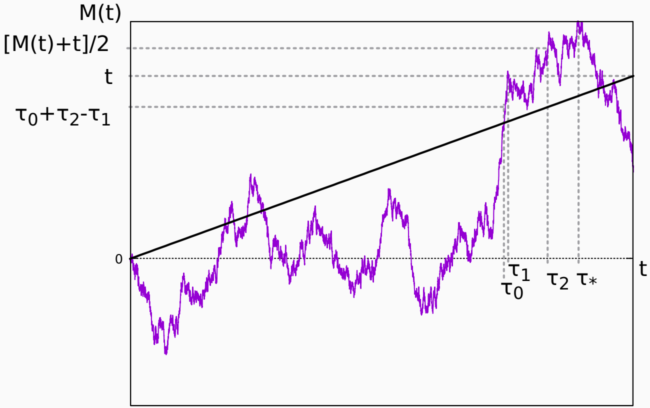

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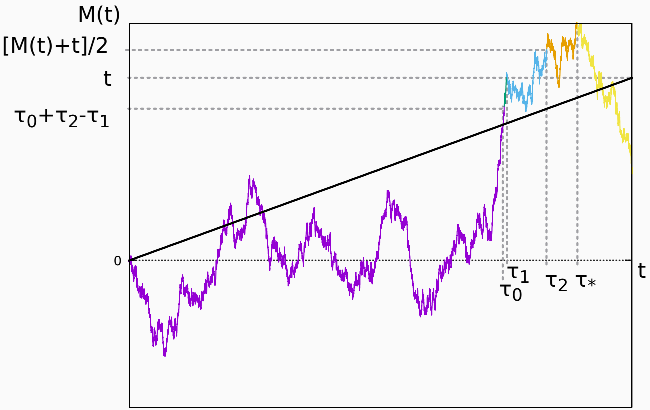
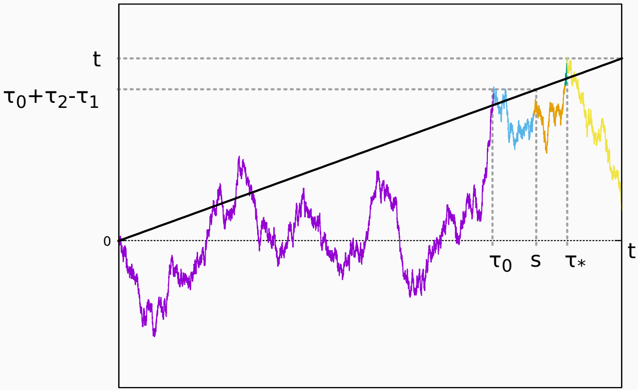


Decompose the Brownian path as follows:

- take the τ_1 to τ_* part out,
- form an excursion of duration $\tau_* - \tau_1$ with subpath $[\tau_1, \tau_*]$
- insert excursion (downward) at time τ_0
- append then the $[\tau_0, \tau_1]$ part and the post-τ_* part

A path transformation (2)



$\looparrowright$ obtain a path with $M(t) = t$, "dying" (for sure) at time $s = \tau_0 + \tau_2 - \tau_1$.

PDF of extinction time in the one-sided case

Recall:

$$f(t) = \overbrace{\sqrt{\frac{2}{\pi t}}\exp\left(-\frac{t}{2}\right)}^{\text{paths with } M(t)=t} - \underbrace{g(t)}_{\substack{\text{paths with } M(t)=t \\ \textit{and} \\ M(s)=s \text{ for some } s<t}}$$

PDF of extinction time in the one-sided case

Now:

$$f(t) = \overbrace{\sqrt{\frac{2}{\pi t}} \exp\left(-\frac{t}{2}\right)}^{\text{paths with } M(t)=t} - \underbrace{g(t)}_{\text{paths with } M(t)>t}$$

PDF of extinction time in the one-sided case

That is,

$$f(t) \;=\; \sqrt{\frac{2}{\pi t}}\,\exp\left(-\frac{t}{2}\right) \;-\; \int_t^\infty \sqrt{\frac{2}{\pi t}}\,\exp\left(-\frac{m^2}{2t}\right)\,dm$$

PDF of extinction time in the one-sided case

Finally:

$$f(t) \;=\; \sqrt{\frac{2}{\pi t}}\,\exp\left(-\frac{t}{2}\right) \;-\; \mathrm{erfc}\left(\sqrt{\frac{t}{2}}\right)$$

PDF of extinction time in the one-sided case

Finally:

$$f(t) = \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{t}{2}\right) - \mathrm{erfc}\left(\sqrt{\frac{t}{2}}\right)$$

Two other approaches:

- look at the reciprocal process of $M(s) - s$

  $\rightarrow$ this is a spectrally positive Lévy process

- show that the first passage time of $M(t) - t$
  is distributed like the sojourn time above 0 of the process $B(t) - t$

R.A. Doney *Hitting probabilities for spectrally positive Lévy processes*, Journal of the LMS, 2(3):566-576 (1991)

J-P Imhof *On the time spent above a level by Brownian motion with negative drift* Adv. in Appl. Prob., 18(4):1017-1018 (1986)

A first-passage problem for the Brownian range

Two-sided version: Brownian range

- "forager" on a line $\rightarrow$ Brownian walker with position $B(t)$
- one unit of "food" per unit length, no food replenishment
- "metabolism": walker needs one unit of food per unit time
- food on both sides

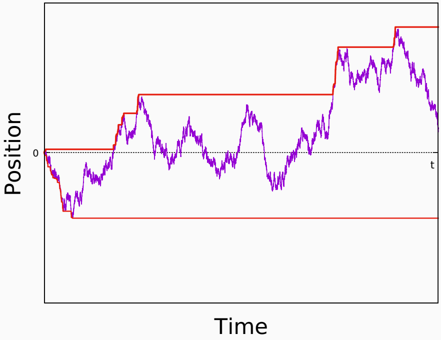
Letting

$$R(s) = \sup_{0\leq\tau\leq s} B(\tau) - \inf_{0\leq\tau\leq s} B(\tau)$$

Survival probability is now:

$$P(t) = \mathrm{Prob}\left(R(s) > s,\ \forall s \leq t\right)$$

Two-sided version: a hitting-time problem for the range



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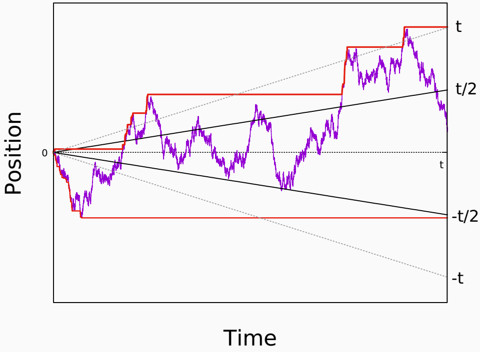
Two-sided version: a hitting-time problem for the range



Letting

$$R(s) = \sup_{0 \le \tau \le s} B(\tau) - \inf_{0 \le \tau \le s} B(\tau)$$

Survival probability is now:

$$P(t) = \mathrm{Prob}\left(R(s) > s,\ \forall s \le t\right)$$

Idea: look at paths with $R(t) = t$

Path going extinct at $t \Rightarrow R(t) = t$, but...

$$R(t) = t \not\Rightarrow \text{path going extinct at } t$$

because path could have gone extinct before.

$$f(t) = \overbrace{\mathrm{pdf}\,(R(t) = t)}^{\text{paths with } R(t)=t} - \underbrace{g(t)}_{\substack{\text{paths with } R(t)=t \\ \textit{and} \\ R(s)=s \text{ for some } s<t}} \tag{1}$$

Idea: look at paths with $R(t) = t$

Path going extinct at $t \Rightarrow R(t) = t$, but...

$R(t) = t \not\Rightarrow$ path going extinct at t

because path could have gone extinct before.

$$f(t) = \overbrace{\frac{8}{\sqrt{2\pi t}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-\frac{k^2 t}{2}}}^{\substack{\text{paths with } R(t)=t \\ \text{W. Feller, Ann. Math. Statist. 22, 427 (1951)}}} - \underbrace{g(t)}_{\substack{\text{paths with } R(t)=t \\ \textit{and} \\ R(s)=s \text{ for some } s<t}} \qquad (2)$$

— ongoing work with P. Salminen, P. Vallois & P. Krapivsky

Many thanks for the invitation and for your attention!