k-tilings of the Aztec Diamond

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Randomness, Integrability and Universality
Galileo Galilei Institute, Firenze
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Outline

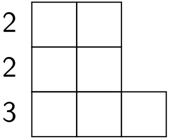
1. Review interlacing partitions (see Sylvie's talk)
2. Use them to construct domino tilings of the Aztec diamond
3. Describe a vertex model formulation of the model, use it to compute the partition fucntion
4. Introduce a interacting k-tilings of the Aztec diamond
5. Relate them to a coupling of k 5-vertex models related to the coinversion LLT polynomials
6. State/prove some combinatorial results

This is joint work with Sylvie Corteel and Andrew Gitlin (arXiv:2202.06020 [math.CO]).

Part 1: Interlacing partitions and the Aztec diamond

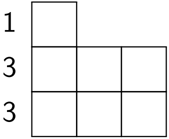
Partitions

Integer partition: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$
Size of the partition: $|\lambda| = \sum \lambda_i$



$\lambda = (3, 2, 2)$, $|\lambda| = 7$

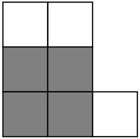
Conjugate partition: ‘swap columns and rows.’



$\lambda' = (3, 1, 1)$, $|\lambda'| = 7$

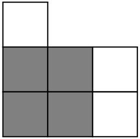
Interlacing conditions

We say that two partitions interlace and write $\mu \preceq \lambda$ if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$.



$$(2,2) \preceq (3,2,2)$$

We say they co-interlace and write $\mu \preceq' \lambda$ if $\mu' \preceq \lambda'$.

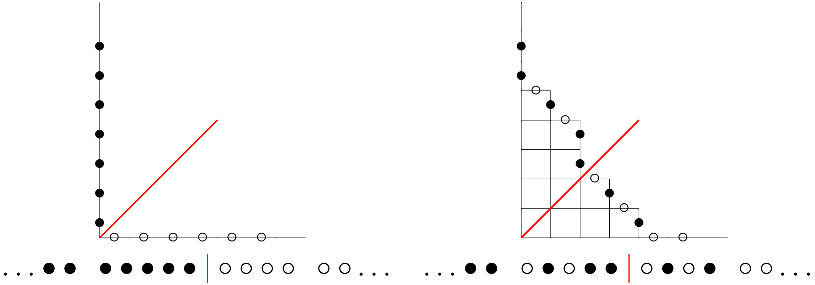


$$(2,2) \preceq' (3,3,1)$$

Maya diagrams

Maya diagram of a partition:

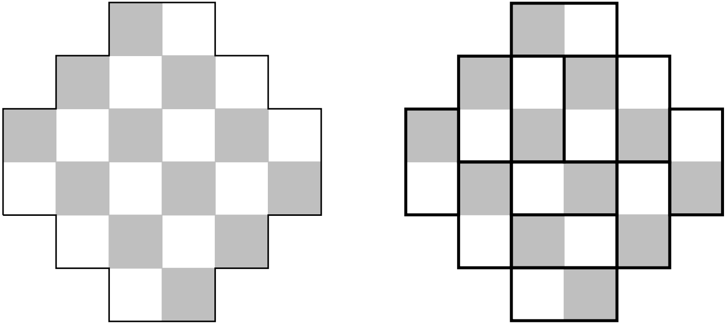
$\lambda = \emptyset$ $\qquad$ $\lambda = (4, 3, 2, 2, 1)$



In red, we have indicated the center of the diagram.

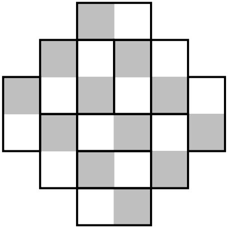
Domino tilings of the Aztec diamond

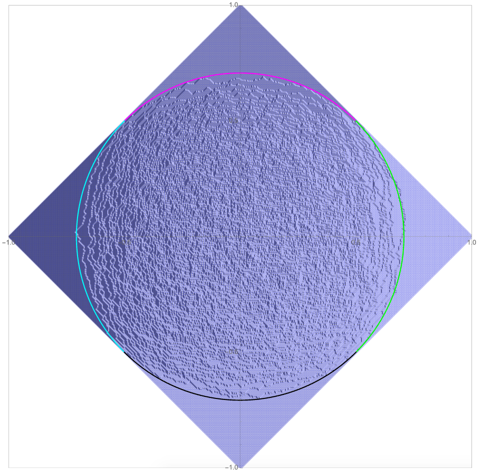
Domino tilings of the Aztec diamond were first introduced by Elkies, Kuperberg, Larsen, and Propp in 1992.



The Aztec diamond of rank $m = 3$ and one possible domino tiling.

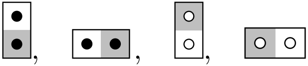
Domino tilings of the Aztec Diamond

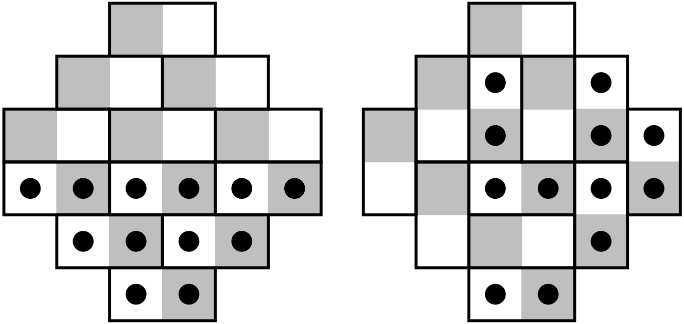




Domino tilings and sequences of partitions

Assign 'particles' and 'holes' to our dominos according to the rules





Domino tilings and sequences of partitions

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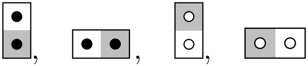
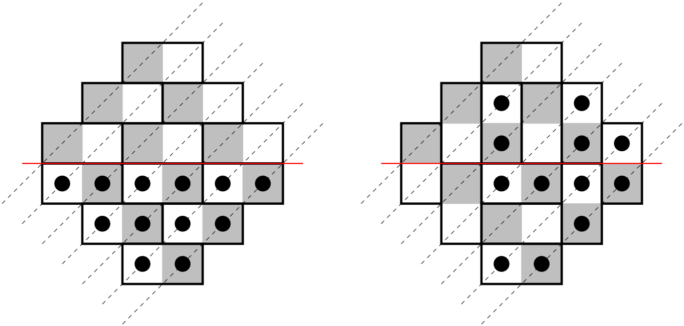




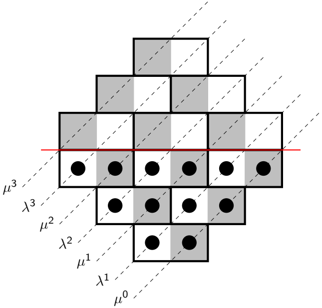
We think of each diagonal slice as having a string of particles corresponding to the Maya diagram of a partition.

Domino tilings and sequences of partitions

Specifying a domino tiling is equivalent to specifying a sequence

$$\emptyset = \mu^0 \preceq' \lambda^1 \succeq \mu^1 \preceq' \ldots \preceq' \lambda^m \succeq \mu^m = \emptyset$$

of $2m+1$ interlacing partitions. For example:



We have the sequence

$$\emptyset \preceq' \emptyset \succeq \emptyset \preceq' \emptyset \succeq \emptyset \preceq' \emptyset \succeq \emptyset$$

Domino tilings and sequences of partitions

Specifying a domino tiling is equivalent to specifying a sequence

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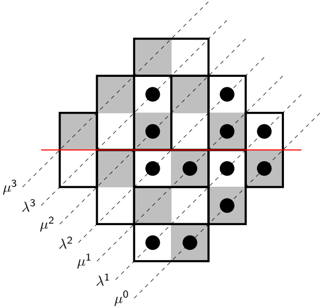


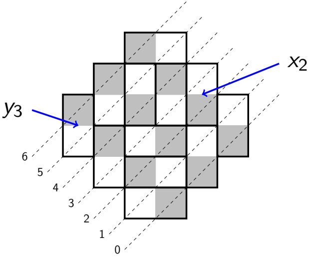
We have the sequence

$$\emptyset \preceq' (1,1) \succeq (1,1) \preceq' (2,1) \succeq (1) \preceq' (2) \succeq \emptyset$$

Domino tilings of the Aztec diamond

Let's add weights to the dominos according to the rules

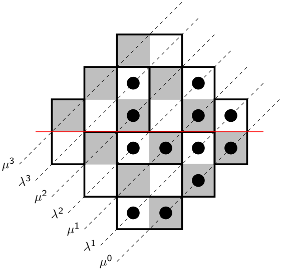
- A domino of the form  whose top square is on slice $2i-1$ gets a weight of x_i.
- A domino of the form  whose bottom square is on slice $2i-1$ gets a weight of y_i.
- All other dominos get a weight of 1.
- We want to compute the partition function $Z_{AD}(X_m, Y_m)$.



weight: $x_1^2 x_2 x_3 y_2^2 y_3^2$

Domino tilings and sequences of partitions

In terms of partitions, the degree of x_i in the weight is given by $|\lambda^i/\mu^{i-1}|$, and for y_i it is given by $|\lambda^i/\mu^i|$.



The sequence

$$\emptyset \preceq' (1,1) \succeq (1,1) \preceq' (2,1) \succeq (1) \preceq' (2) \succeq \emptyset$$

has weight

$$x_1^{|(1,1)/(0,0)|} y_1^{|(1,1)/(1,1)|} x_2^{|(2,1)/(1,1)|} y_2^{|(2,1)/(1)|} x_3^{|(2)/(1)|} y_3^{|(2)/(0)|} = x_1^2 x_2 x_3 y_2^2 y_3^2$$

Domino tilings and Schur polynomials

Recall the Schur polynomials of shape λ/μ is given by

$$s_{\lambda/\mu}(x_1,\dots,x_n) = \sum_{T\in \mathrm{SSYT}(\lambda/\mu)} x^T.$$

When we only have a single variable this simplifies to

$$s_{\lambda/\mu}(x_1) = \begin{cases} x_1^{|\lambda/\mu|}, & \text{if } \mu \preceq \lambda \\ 0, & \text{o.w.} \end{cases}$$

So given

$$\emptyset = \mu^0 \preceq' \lambda^1 \succeq \mu^1 \preceq' \dots \preceq' \lambda^m \succeq \mu^m = \emptyset$$

we can write it's weight as

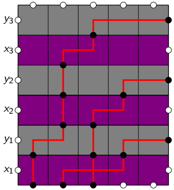
$$s_{(\lambda^1/\mu^0)'}(x_1)s_{\lambda^1/\mu^1}(y_1)\dots s_{(\lambda^{m-1}/\mu^m)'}(x_m)s_{\lambda^m/\mu^m}(y_m)$$

Tilings of the Aztec diamond are an example of a Schur process.

A vertex model formulation

We'll give one more formulation in terms of lattice paths.

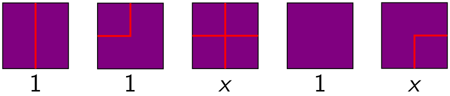
1. An example of the kind of path configurations we will consider:

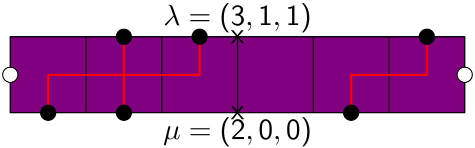


2. We will identify the configurations of the paths along a horizontal slice with the Maya diagram of a partition.
3. The different color faces represent different choices of weights, which will impose different interlacing requirements on these partitions.
4. Yang-Baxter integrability will allow us to compute the partition function.

A vertex model formulation

We need two different five vertex models. First consider the 5-vertex model:





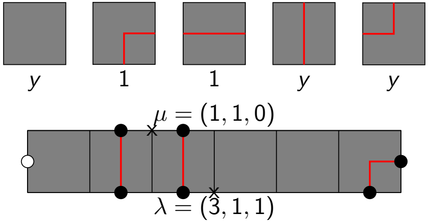
$$x^{|\lambda/\mu|}, \ \mu \preceq' \lambda$$

Note:

The boundary conditions are indexed by co-interlacing partitions, and they uniquely determine the paths.

The weight agrees with the weight we assigned to co-interlacing partitions previously.

Now consider a different 5-vertex model:



$$y^{(\#\text{ of paths}-1)}y^{|\lambda/\mu|},\ \lambda \succeq \mu$$

Note:

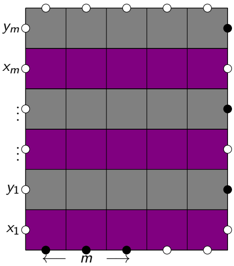
The boundary conditions are indexed by interlacing partitions, and they uniquely determine the paths.

The weight agrees with the weight we assigned to interlacing partitions previously (up to an overall factor).

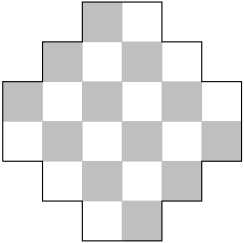
A vertex model formulation

We have a bijection between

paths on



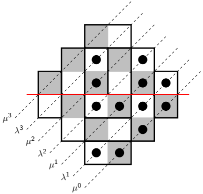
and tilings of



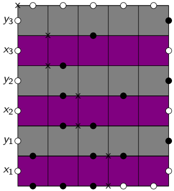
given by looking at the sequence of interlacing partitions.

A vertex model formulation

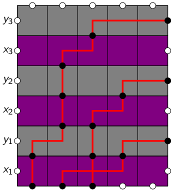
$$\emptyset \preceq' (1,1) \succeq (1,1) \preceq' (2,1) \succeq (1) \preceq' (2) \succeq \emptyset$$

Tiling:  with weight $x_1^2 x_2 x_3 y_2^2 y_3$

On the other hand, the partitions at each row



determine the paths

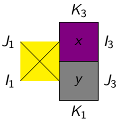


and weight: $(y_1^2 y_2) x_1^2 x_2 x_3 y_2^2 y_3$. The weights agree (up to an overall factor).

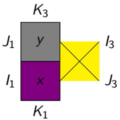
A vertex model formulation: the Yang-Baxter Equation

These vertex models satisfy the Yang-Baxter equation (YBE). We have

$\displaystyle\sum_{\text{interior paths}} w$ (

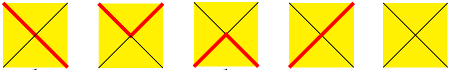


) $=$ $\displaystyle\sum_{\text{interior paths}} w$ (



)

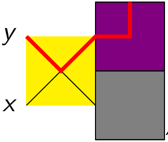
for any choice of boundary condition $I_1, J_1, K_1, I_3, J_3, K_3$. Here

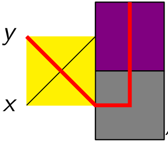


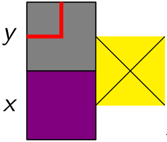
$\frac{1}{1+xy}$ $\quad \frac{xy}{1+xy}$ $\quad \frac{1}{1+xy}$ $\quad \frac{xy}{1+xy}$ $\quad 1$

A vertex model formulation: the Yang-Baxter Equation

For example:

$w\Bigg($ 

$\Bigg) + w\Bigg($ 

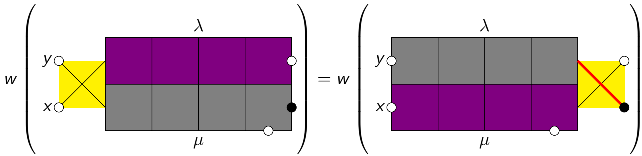
$\Bigg) = w\Bigg($ 

$\Bigg)$

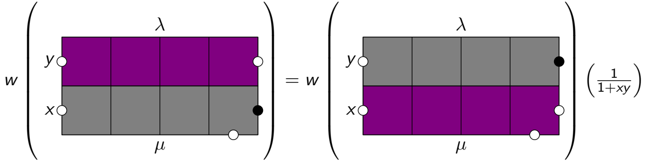
$$\left(\frac{xy}{1+xy}\right) y + \left(\frac{1}{1+xy}\right) y = y$$

A vertex model formulation: the Yang-Baxter Equation

We can use this to swap rows! By repeatedly using the YBE we have



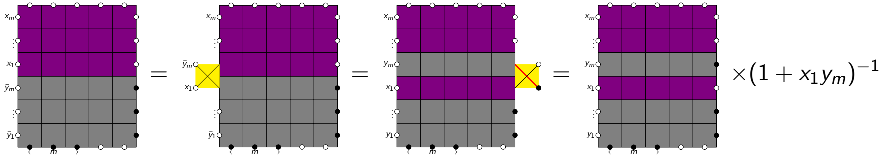
Then removing the yellow faces (but keeping the weight) gives



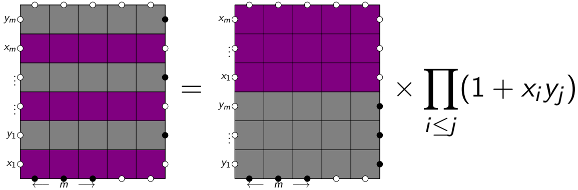
It turns out this makes the partition function of the vertex model simple to compute.

A vertex model formulation: the partition function

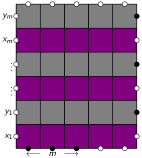
Using the YBE we have



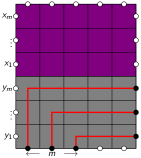
Now repeated applications of the YBE gives



A vertex model formulation: the partition function



$=$



$$\times \prod_{i\leq j}(1+x_iy_j) = \underbrace{(y_1^{m-1}y_2^{m-2}\dots y_m^0)}_{\text{overall factor}} \prod_{i\leq j}(1+x_iy_j)$$

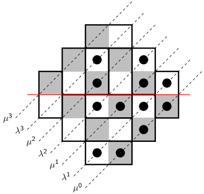
It follows that the partition function for the Aztec diamond $Z_{AD}(X_m, Y_m)$ is given by

$$Z_{AD}(X_m, Y_m) = \prod_{i\leq j}(1+x_iy_j).$$

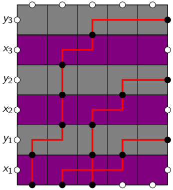
With uniform weights we recover: $\#$ of tilings $= 2^{\binom{m+1}{2}}$.

Summary so far:

1. We showed there is a bijection between tilings of the Aztec diamond and sequences of interlacing partitions.
2. We further showed there was a bijection between the sequences of partitions and a certain vertex model.
3. In the vertex model formalism we could use the YBE to compute the partition function.



$$\leftrightarrow \quad \emptyset \preceq' (1,1) \succeq (1,1) \preceq' (2,1) \succeq (1) \preceq' (2) \succeq \emptyset \quad \leftrightarrow$$



Part 2: Interacting k-tilings of the Aztec diamond

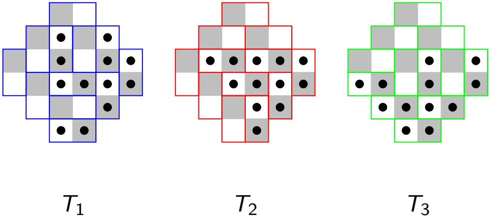
k-tilings

tiling $\to$ k-tiling

sequence of interlacing partitions $\to$ sequence of k-tuples of interlacing partitions

5-vertex model $\to$ k superimposed copies of the 5-vertex model

$$\begin{aligned}
\boldsymbol{\mu}^0 &= (\emptyset, \emptyset, \emptyset) \\
\boldsymbol{\lambda}^1 &= ((1,1),(1,1,1),(1)) \\
\boldsymbol{\mu}^1 &= ((1,1),(1,1),\emptyset) \\
\boldsymbol{\lambda}^2 &= ((2,1),(1,1),(1)) \\
\boldsymbol{\mu}^2 &= ((1),(1),\emptyset) \\
\boldsymbol{\lambda}^3 &= ((2),(1),\emptyset) \\
\boldsymbol{\mu}^3 &= (\emptyset, \emptyset, \emptyset)
\end{aligned}$$



We'll refer to the tilings a being different colors. We order the colors.

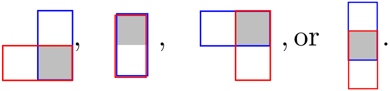
Weights of the k-tiling

Assign weights to the dominos according to the rules

- A domino of the form  whose top square is on slice $2i-1$ gets a weight of x_i.
- A domino of the form  whose bottom square is on slice $2i-1$ gets a weight of y_i.
- All other dominos get a weight of 1.

for each color.

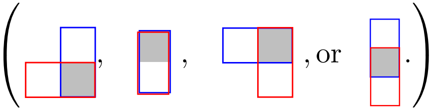
For every pair of colors $a<b$, each 'interaction' gives a power of t where we define 'interaction' according to the rule



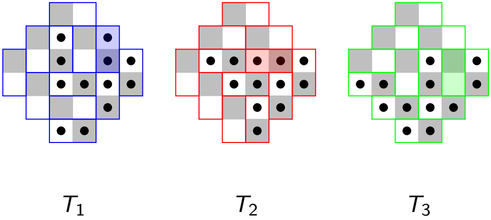
where here blue $<$ red.

k-tilings

For example,



$\boldsymbol{\lambda}^0 = (\emptyset, \emptyset, \emptyset)$
$\boldsymbol{\lambda}^1 = ((1,1),(1,1,1),(1))$
$\boldsymbol{\lambda}^2 = ((1,1),(1,1),\emptyset)$
$\boldsymbol{\lambda}^3 = ((2,1),(1,1),(1))$
$\boldsymbol{\lambda}^4 = ((1),(1),\emptyset)$
$\boldsymbol{\lambda}^5 = ((2),(1),\emptyset)$
$\boldsymbol{\lambda}^6 = (\emptyset, \emptyset, \emptyset)$



which has weight $x_1^2 x_2 y_2^2 x_3 y_3^2 x_1^3 y_1 y_2 y_3 x_1 y_1 x_2 y_2 \underbrace{t^4}_{b-r} \underbrace{t^3}_{b-g} \underbrace{t^4}_{r-g}$.

Aside: LLT polynomials

Recall that for a single tiling the weight was given by

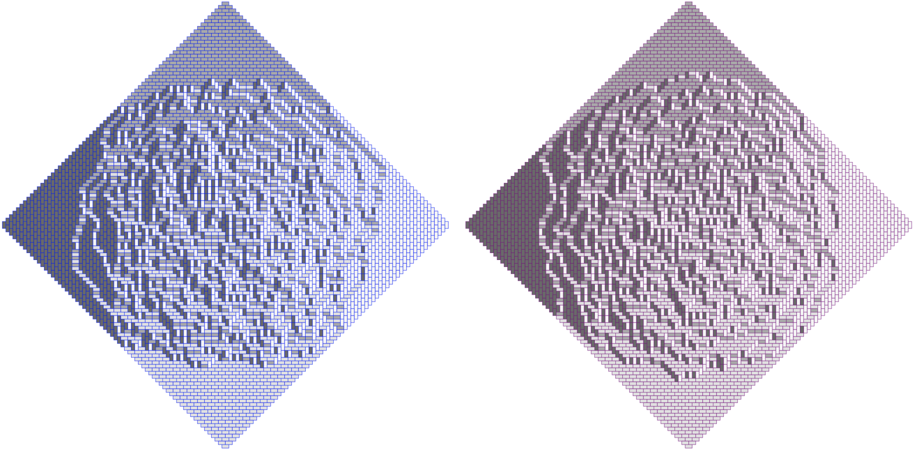
$$s_{(\lambda^1/\mu^0)'}(x_1)s_{\lambda^1/\mu^1}(y_1)\dots s_{(\lambda^{m-1}/\mu^m)'}(x_m)s_{\lambda^m/\mu^m}(y_m).$$

For the k-tiling the weight can be written as

$$t^{\#}\mathcal{L}_{(\boldsymbol{\lambda}^1/\boldsymbol{\mu}^0)'}(x_1;t^{-1})\mathcal{L}_{\boldsymbol{\lambda}^1/\boldsymbol{\mu}^1}(y_1;t)\dots\mathcal{L}_{(\boldsymbol{\lambda}^{m-1}/\boldsymbol{\mu}^m)'}(x_m;t^{-1})\mathcal{L}_{\boldsymbol{\lambda}^m/\boldsymbol{\mu}^m}(y_m;t)$$

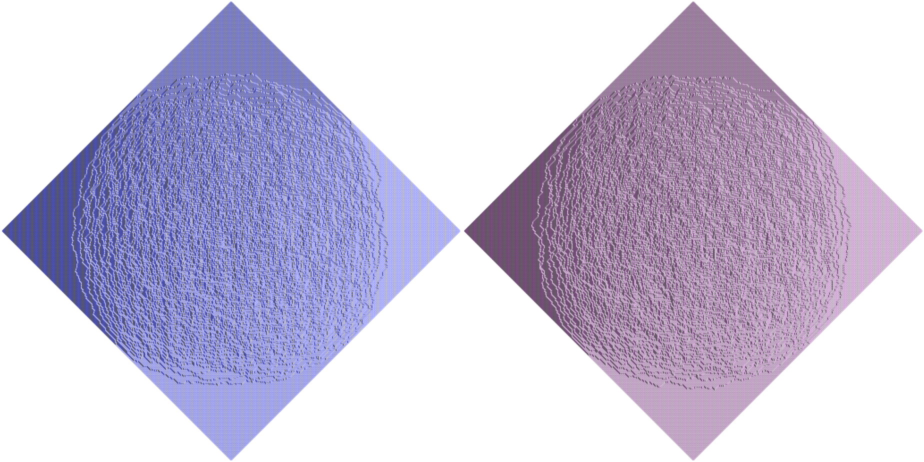
where $\mathcal{L}_{\boldsymbol{\lambda}}(x_1;t)$ is the coinversion LLT polynomial.

Simulations



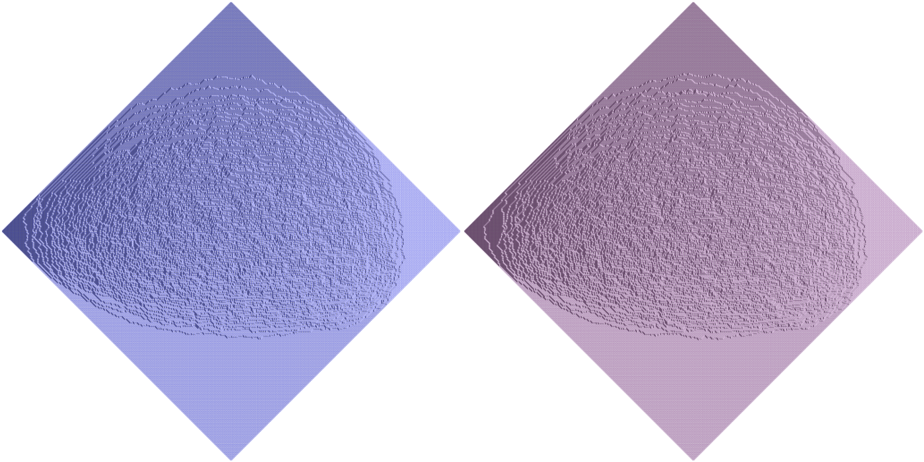
Simulation of a 2-tiling of the rank-64 Aztec diamond at $t = 1$.

Simulations



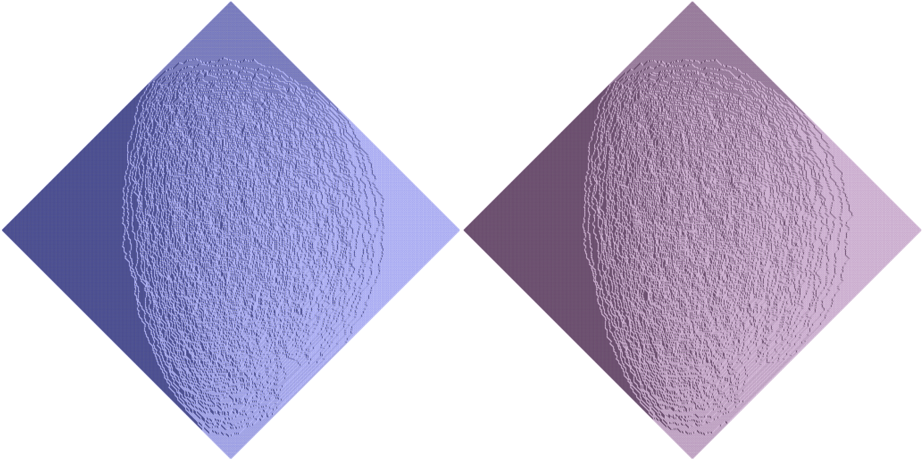
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 1$.

Simulations



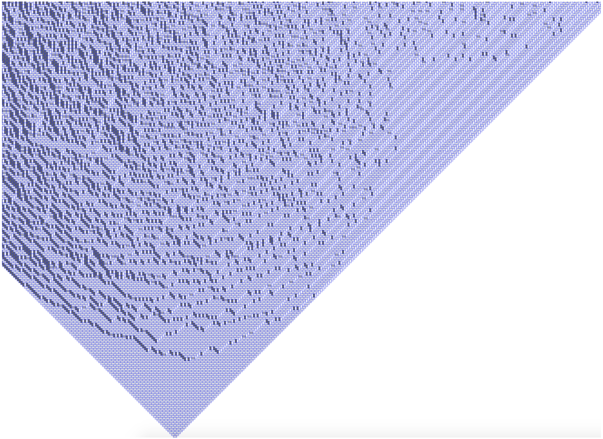
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 0.2$.

Simulations



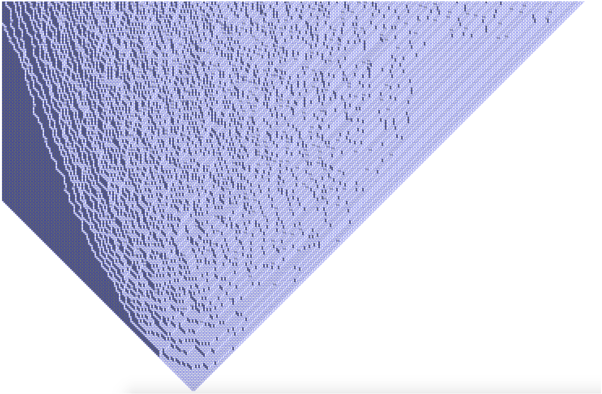
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 5$.

Simulations



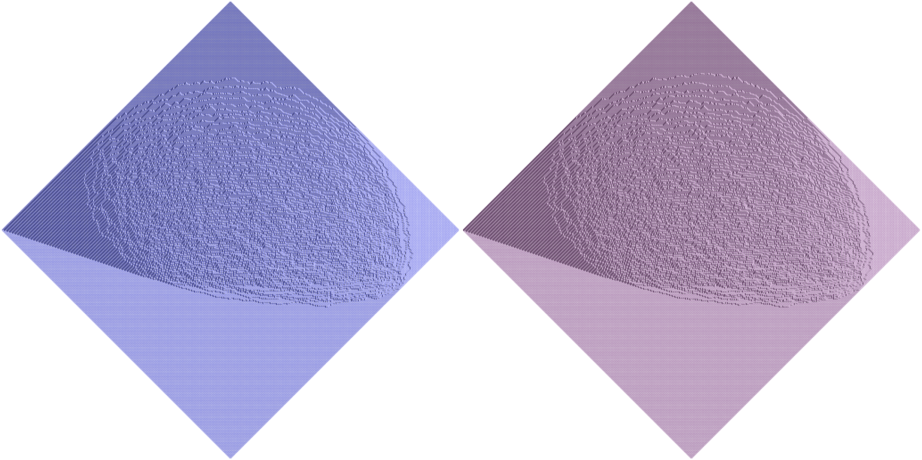
Close-up of southern corner of blue in a 2-tiling of the rank-512 Aztec diamond at $t = 5$.

Simulations



Close-up of southern corner of blue in a 3-tiling of the rank-512 Aztec diamond at $t = 5$.

Simulations



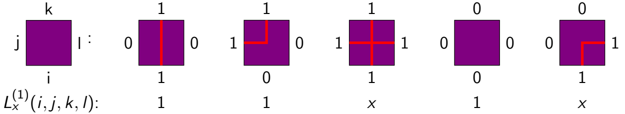
Simulation of a 2-tiling of the rank-256 Aztec diamond at $t \approx 0$.

Vertex model formulation

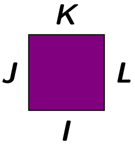
Now we'd like to take k copies of our vertex model, superimpose them, and add an interaction weight that agrees with our domino weights.
Constraint: Still need the YBE to hold.
Our vertex weights are a degeneration of those of Aggarwal, Borodin, and Wheeler where the weights come from an R-matrix of $U_q(\hat{\mathfrak{sl}}(2|k))$. We inherit the YBE from them.

The purple weights

We need to generalize our purple weights. Recall the weights for one color are



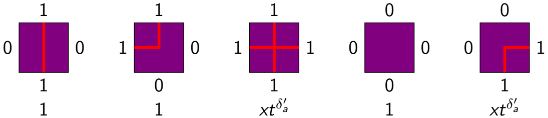
and the weights for k colors are



$$, \quad L_x^{(k)}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L}) = \prod_{i=1}^{k} L^{(1)}_{xt^{\delta'_a}}(I_i, J_i, K_i, L_i)$$

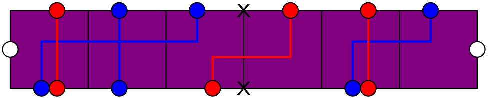
where $\delta'_a = \#$ colors greater than a that are vertical.

The purple weights



$\delta'_a = \#$ colors greater than a that are vertical,

$$\boldsymbol{\lambda} = ((3,1,1),(2,2,0))$$



$x^3 x t^2$

$$\boldsymbol{\mu} = ((2,0,0),(2,1,0))$$

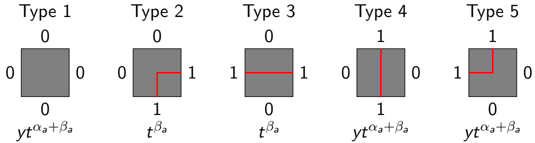
Here blue is a smaller color than red. In general,

$$x^{|\boldsymbol{\lambda}/\boldsymbol{\mu}|} t^{\sum_a \delta'_a}, \ \boldsymbol{\mu} \preceq' \boldsymbol{\lambda}$$

where $\boldsymbol{\lambda} = \sum_a |\lambda|$ and $\boldsymbol{\mu} \preceq' \boldsymbol{\lambda}$ means $\mu^{(a)} \preceq' \lambda^{(a)}$ for each $a = 1, \ldots, k$.

The gray weights

We need to generalize the gray faces. The weight at a face is the product of the weight for each color and the weight for color a is

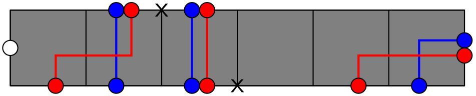


where

$$\alpha_a = \#\text{ colors greater than } a \text{ of Type 1,}$$
$$\beta_a = \#\text{ colors greater than } a \text{ of Type 4 or 5.}$$

The gray weights

$\boldsymbol{\lambda} = ((1,1),(1,1))$



$\boldsymbol{\mu} = ((3,1,1),(2,1,0))$

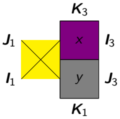
Here blue is a smaller color than red. In general,

$$y^{k(\#\text{ of paths}-1)} t^{\binom{k}{2}(\#\text{ of paths}-1)} y^{|\boldsymbol{\mu}/\boldsymbol{\lambda}|} t^{\sum_a \alpha_a}, \quad \boldsymbol{\mu} \succeq \boldsymbol{\lambda},$$

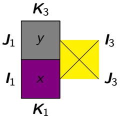
where $\boldsymbol{\lambda} = \sum_a |\lambda|$ and $\boldsymbol{\mu} \succeq \boldsymbol{\lambda}$ means $\mu^{(a)} \succeq \lambda^{(a)}$ for each $a = 1, \ldots, k$.

With these choices of vertex weight, the YBE still holds.

$$\sum_{\text{interior paths}} w \left(\vphantom{\int} \right.$$

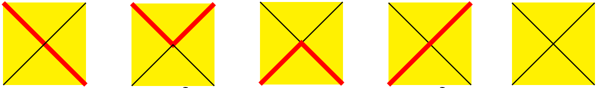


$$\left. \vphantom{\int} \right) = \sum_{\text{interior paths}} w \left(\vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right)$$

for any choice of boundary condition $\boldsymbol{I}_1, \boldsymbol{J}_1, \boldsymbol{K}_1, \boldsymbol{I}_3, \boldsymbol{J}_3, \boldsymbol{K}_3$. Here the contribution from color a is given by

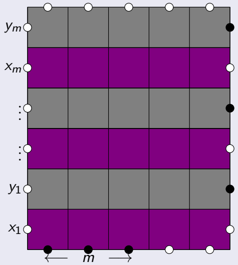


$$\frac{1}{1+xyt^{\delta_a}} \qquad \frac{xyt^{\delta_a}}{1+xyt^{\delta_a}} \qquad \frac{1}{1+xyt^{\delta_a}} \qquad \frac{xyt^{\delta_a}}{1+xyt^{\delta_a}} \qquad 1$$

where $\delta_a = \#$ colors larger than a that are present. (The total weight at the face is product over all the colors.)

Theorem (Corteel, Gitlin, and K. (2022))

The partition function of the lattice



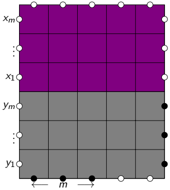
with k colors is equal to $(y_1^{m-1} y_2^{m-2} \dots y_m^{m-m})^k t^{\binom{m}{2}\binom{k}{2}}$ *times the partition function of the k-tiling of the Aztec diamond of rank m. In particular, we have*

$$Z_{AD}^{(k)}(X_m; Y_m; t) = \prod_{\ell=0}^{k-1} \prod_{i \le j} \left(1 + x_i y_j t^{\ell}\right).$$

Proof of the theorem

The proof works just as in the one color case.

1. There is a bijection between k-tilings and k-color path configurations going through interlacing sequences of tuples of partitions.
2. From the vertex model formulation, the partition function is easy to compute by using the YBE to rearrange the rows of



Some special values of t

$$Z_{AD}^{(k)}(X_m; Y_m; t) = \prod_{\ell=0}^{k-1} \prod_{i \leq j} \left(1 + x_i y_j t^{\ell}\right).$$

Clearly, when $t = 1$ we have

$$Z_{AD}^{(k)}(X_m; Y_m; 1) = (Z_{AD}(X_m, Y_m))^k$$

that is, we have k independent tilings.

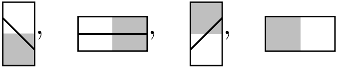
At $t = 0$ we have

$$\begin{aligned} Z_{AD}^{(k)}(X_m; Y_m; 0) &= \prod_{\ell=0}^{k-1} \prod_{i \leq j} \left(1 + x_i y_j t^{\ell}\right)|_{t=0} \\ &= \prod_{i \leq j} (1 + x_i y_j) = Z_{AD}(X_m, Y_m) \end{aligned}$$

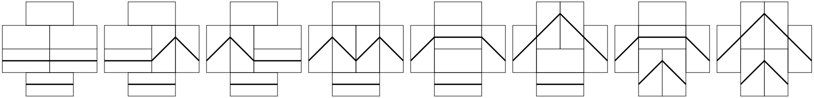
We'll give a bijective proof of this. It turns out this is easiest to see in terms of Schröder paths.

Schröder paths

For each color, we can assign paths to the dominos according to the following rules:



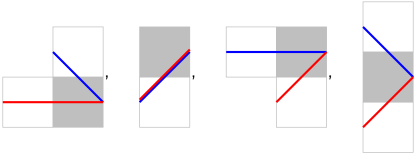
For rank-2 we have



Note for each tiling the paths will be nonintersecting.

Schröder paths

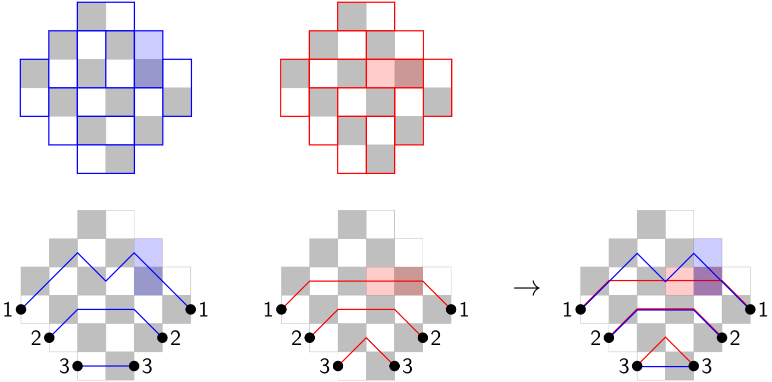
Translating our weights over to the Schröder path picture we have that the power of t is the number of interactions of the form



where blue is a smaller color than red.

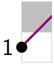
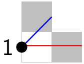
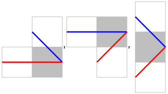
Schröder paths

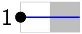
For example:

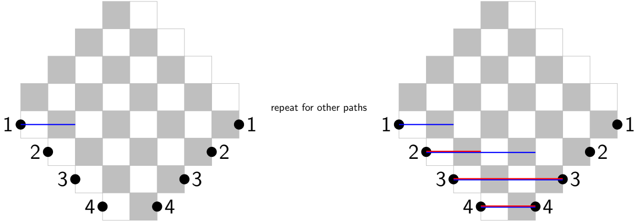


$t = 0$ and Schröder paths

Taking $t = 0$ imposes strict restriction on the Schröder paths.
Consider the starting point of the top most paths of each color:

1. If  then we get a power of t.
2. If  then eventually  and we get a power of t.

So when $t = 0$ we are forced to have .



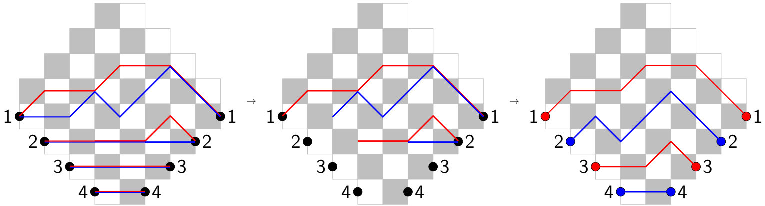
$t = 0$ and Schröder paths

Similar arguments show blue path i will be weakly below red path i, and strictly above red path $i + 1$.

Proposition

There is a bijection between 2-tilings of the Aztec diamond at $t = 0$ and 1-tilings of the Aztec diamond by removing the forced paths and sliding.

For example,



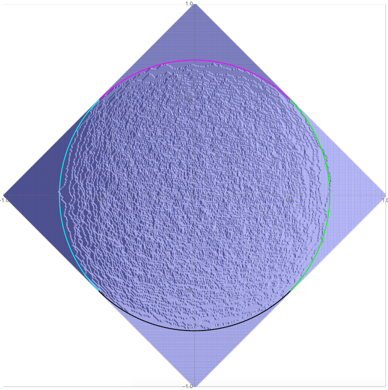
The Arctic circle

It is well-known that tilings of the Aztec diamond exhibit the limit shape phenomenon. As we take the rank to infinity, there are 'frozen' regions in each corner separated from a 'disordered' region in the center by a deterministic curve known as the Arctic curve.

Theorem (Jockusch, Propp, Schor, '98)

The Arctic curve of the Aztec diamond is given by

$$x^2 + y^2 = \frac{1}{2}.$$



The Arctic circle

Reversing the bijection, we can map a single tiling of the Aztec diamond to a k-tiling at $t = 0$.

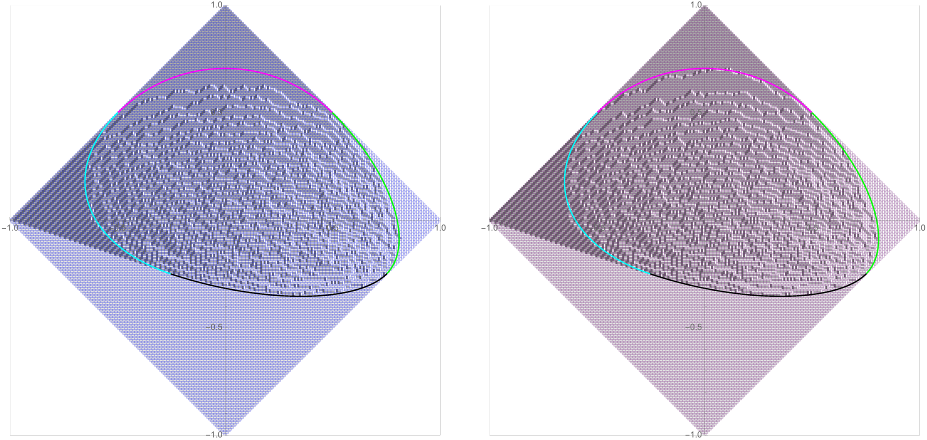
Theorem (Corteel,Gitlin,K.)

The arctic curve for 2-tilings of the Aztec diamond when $t = 0$ is given by

$$\begin{cases} x^2 + y^2 = \frac{1}{2}, & x \in [-1/2, 1/2],\ y > 1/2 \\ (x+y)^2 + (2y)^2 = \frac{1}{2}, & x \in [-1/4, 3/4],\ y < -1/4 \\ \left(\frac{3x+y-1}{2}\right)^2 + \left(\frac{3y+x-1}{2}\right)^2 = \frac{1}{2}, & y \in [-1/4, 1/2],\ x > -\frac{1}{3}y + \frac{2}{3} \\ \left(\frac{3x+y-1}{4}\right)^2 + \left(\frac{5y-x-1}{4}\right)^2 = \frac{1}{2}, & y \in [-1/4, 1/2],\ x < -\frac{1}{3}y - \frac{1}{3} \end{cases}$$

for both colors.

The Arctic circle



Simulation of a 2-tiling of the rank-128 Aztec diamond at $t = 0$ with the computed Arctic curve overlaid.

$t \to \infty$

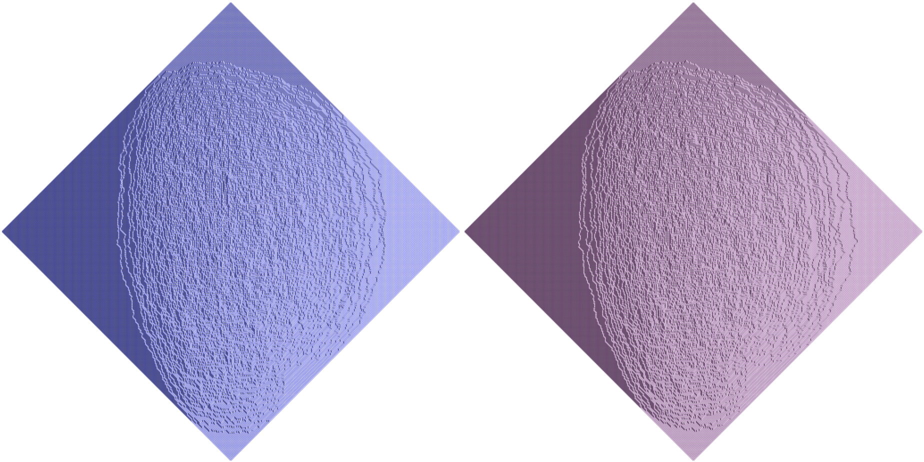
Lemma

Let $\boldsymbol{T}$ be a k-tiling of the Aztec diamond of rank m with j interactions. Let $\phi(\boldsymbol{T})$ be the involution that flips the tilings across the line $y = x$. Then $\phi(\boldsymbol{T})$ is a k-tiling with $\binom{k}{2}\binom{m+1}{2} - j$ interactions.

Proposition

There is a bijection between tilings of the Aztec diamond at $t = 0$ and tilings of the Aztec diamond in the limit $t \to \infty$ given by reflecting across the line $y = x$. In particular, the Arctic curve in the limit $t \to \infty$ is the same as that for $t = 0$ up to the reflection.

Other values of t?



Simulation of a 2-tiling of the rank-256 Aztec diamond at $t = 5$.

Summary

1. We can use the machinery from the LLT vertex model to construct a model of k coupled tilings of the Aztec diamond and compute its partition function.
2. In certain limits of the interaction strength $(t = 0, 1, \infty)$ we have bijections relating the k-tilings to the usual 1-tilings.
3. We know that the model has a symmetry with respect to mapping $t \mapsto \frac{1}{t}$.
4. One can generalize the domino shuffle algorithm to the interacting tilings.
5. It would very interesting to understand the asymptotic behavior for values t outside of the special cases.

End!

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Thank You!