Randomness, Integrability and Universality

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Exact results for quenched randomness at criticality

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Based on:

G. Delfino, Phys. Rev. Lett. 118 (2017) 250601

G. Delfino and E. Tartaglia, Phys. Rev. E 96 (2017) 042137; J. Stat. Mech. (2017) 123303

G. Delfino and N. Lamsen, JHEP 04 (2018) 077; J. Stat. Mech. (2019) 024001; EPJB 92 (2019) 278

G. Delfino, Eur. Phys. J. B 94 (2021) 65 (colloquium)

Introduction

quenched disorder: some degrees of freedom take too long to reach thermal equilibrium and can be considered as random variables

disorder average is taken on the free energy $F(\{J\})$

$$\overline{F} = \sum_{\{J\}} P(\{J\}) F(\{J\})$$

with a probability distribution $P(\{J\})$

examples: impurities in magnets, spin glasses, ...

numerics/experiments: exists "random" criticality with new critical exponents

theoretical state of the art until recently (short range interactions):

- perturbative results in few cases for weak disorder, only numerics for strong disorder

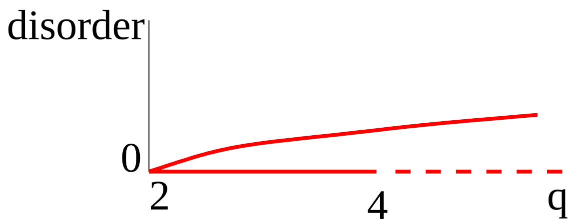
- surprising absence of exact results in 2D (pure systems solved in '80s)

2D bond disorder softens 1st order transitions [Aizenman, Wehr '89; Hui, Berker '89] in favor of 2nd order ones making more room for conformal invariance, but corresponding CFT's were never found

historically, the Potts model received a special attention

2D random bond Potts model:

$$H = -\sum_{\langle i,j\rangle} J_{ij}\delta_{s_i,s_j} \qquad s_i = 1, 2, \ldots, q$$



- permutational invariance S_q; exists continuation to q real

- transition of pure ferromagnet ($J_{ij} = J > 0$) 2nd order up to $q = 4$, 1st order after

- bond disorder yields 2nd order transition extending to $q = \infty$ [Aizenman, Wehr '89; Hui, Berker '89]

- perturbative random critical point for $q \to 2$ [Ludwig '90, Dotsenko, Picco, Pujol '95]

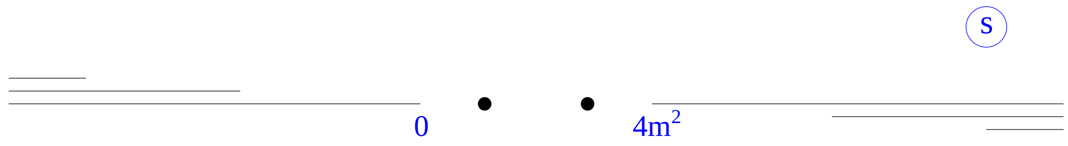
- numerical hints of q-independent exponents [Chen et al '92,'95; Domany, Wiseman '95; Kardar et al '95]. Superuniversality?

- q-dependent exponents (weakly for ν) [Cardy, Jacobsen '97 (numerical transfer matrix)] random critical points are exactly accessible in 2D [GD '17]

Criticality from scale invariant scattering [GD '13]

- Euclidean field theory in 2D $\longleftrightarrow$ relativistic quantum field theory in (1+1)D $\longrightarrow$ particles $\longrightarrow$ scattering theory

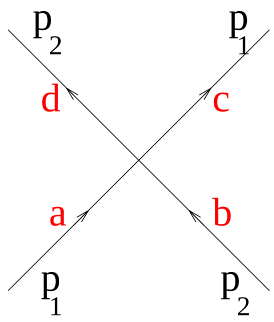
- analytic structure of scattering amplitudes:



generically, infinitely many branch points collapse on top of each other as $m \to 0$: intractable

- at criticality in (1+1)D ∞-dimensional conformal symmetry leaves only the elastic thresholds and avoids the catastrophe

$$S_{ab}^{cd} \quad =$$



center of mass energy only relativistic invariant, dimensionful $\Rightarrow$ **energy-independent amplitudes** by scale invariance

unitarity: $$\sum_{e,f} S_{ab}^{ef} \left[S_{ef}^{cd}\right]^* = \delta_{ac}\delta_{bd}$$

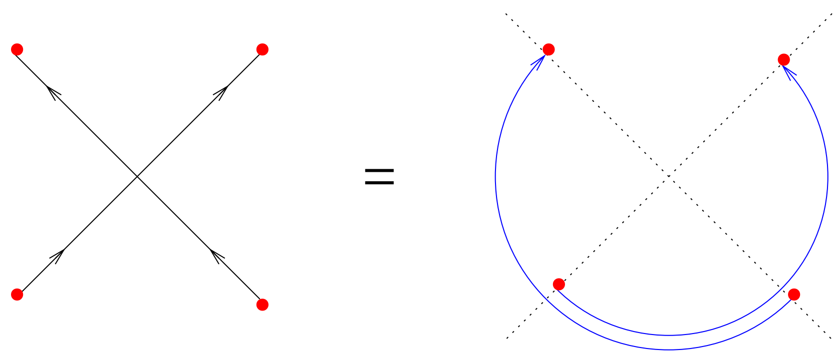
crossing: $$S_{ab}^{cd} = \left[S_{\bar{d}a}^{\bar{b}c}\right]^*$$

- the energy-independent amplitudes are related to statistics:

$\eta(x)$ with conformal dimensions $(\Delta_\eta, \bar{\Delta}_\eta)$ creates a particle if

$$\langle p|\eta(0)|0\rangle \propto (E+p)^{\Delta_\eta}(E-p)^{\bar{\Delta}_\eta} \neq 0$$

$\Rightarrow$ $\bar{\Delta}_\eta$ (Δ_η) vanishes for right (left) movers



$$S \qquad = \qquad e^{-2i\pi\Delta_\eta}$$

bosons/fermions for Δ_η =integer/half-integer; generalized statistics otherwhise

O(N) model [GD,Lamsen '18,'19]

$$H = -\sum_{\langle i,j \rangle} J_{ij}\,\mathbf{s}_i \cdot \mathbf{s}_j\,, \qquad \mathbf{s}_i = N\text{-component unit vector}$$

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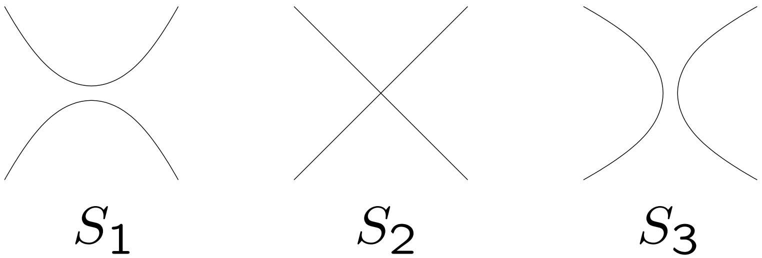
pure case: $J_{ij} = J$

particles: $a = 1, \ldots, N$

scattering amplitudes:



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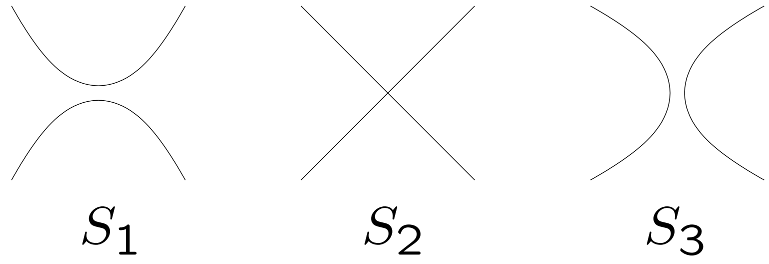


crossing symmetry: $S_1 = S_3^* \equiv \rho_1\, e^{i\phi}$ $\qquad S_2 = S_2^* \equiv \rho_2$

unitarity: $\rho_1^2 + \rho_2^2 = 1$

$\rho_1 \rho_2 \cos\phi = 0$

$N\rho_1^2 + 2\rho_1\rho_2 \cos\phi + 2\rho_1^2 \cos 2\phi = 0$

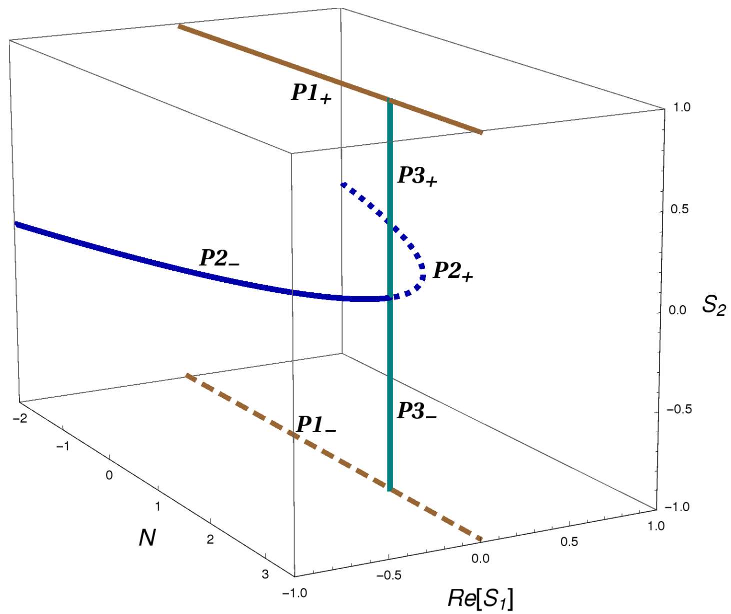
solutions are $O(N)$-invariant RG fixed points solutions:

Solution	N	ρ_1	ρ_2	$\cos\phi$
$P1_\pm$	$(-\infty,\infty)$	0	± 1	-
$P2_\pm$	$[-2,2]$	1	0	$\pm\frac{1}{2}\sqrt{2-N}$
$P3_\pm$	2	$[0,1]$	$\pm\sqrt{1-\rho_1^2}$	0

P1$_\pm$: free bosons/fermions

P2$_\pm$: dense/dilute self-avoiding loops

P3$_+$: BKT phase



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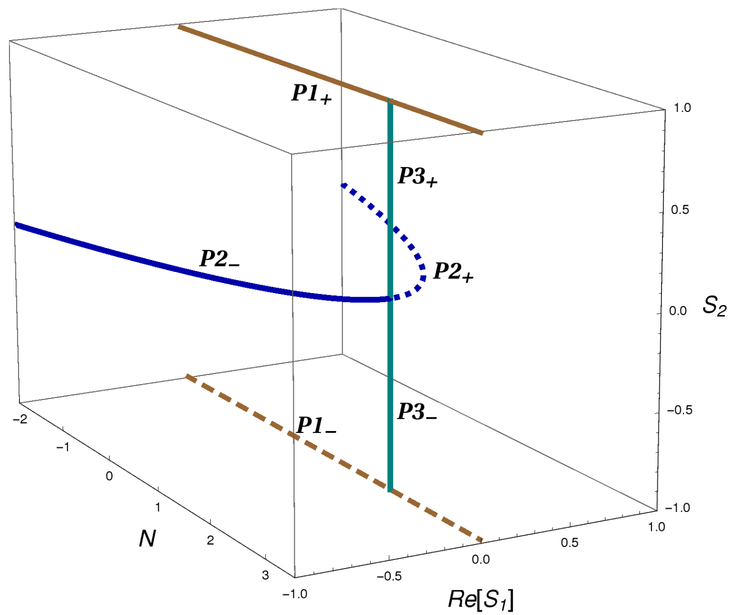

conformal data (match known results):

Solution	N	c	Δ_η	Δ_ε	Δ_s
$P1_-$	$(-\infty,\infty)$	$\frac{N}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{16}$
$P1_+$	$(-\infty,\infty)$	$N-1$	0	1	0
$P2_-$	$2\cos\frac{\pi}{p}$	$1-\frac{6}{p(p+1)}$	$\Delta_{2,1}$	$\Delta_{1,3}$	$\Delta_{\frac{1}{2},0}$
$P2_+$	$2\cos\frac{\pi}{p+1}$	$1-\frac{6}{p(p+1)}$	$\Delta_{1,2}$	$\Delta_{3,1}$	$\Delta_{0,\frac{1}{2}}$
$P3_\pm$	2	1	$\frac{1}{4b^2}$	b^2	$\frac{1}{16b^2}$

$$\Delta_{\mu,\nu} = \frac{[(p+1)\mu - p\nu]^2 - 1}{4p(p+1)}$$

$$NS_1 + S_2 + S_3 = e^{-2i\pi\Delta_\eta}$$

disordered case:

replica method: $$\overline{F} = -\overline{\ln Z} = -\lim_{n \to 0} \frac{\overline{Z^n} - 1}{n}$$

n replicas coupled by average over disorder

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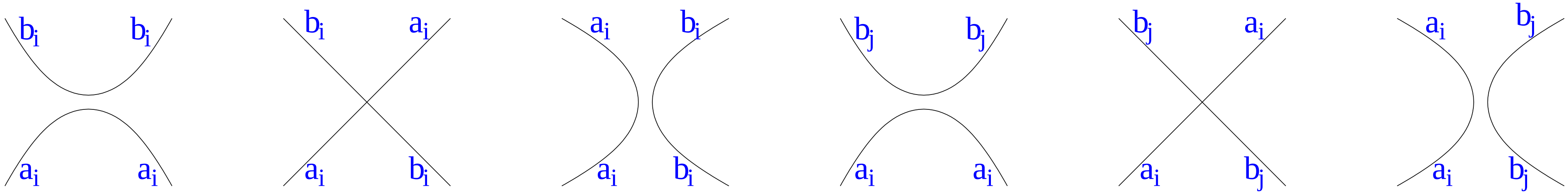
particles: a_i $\quad a = 1, \ldots, N \quad i = 1, \ldots, n$

amplitudes:



S_1 S_2 S_3 S_4 S_5 S_6

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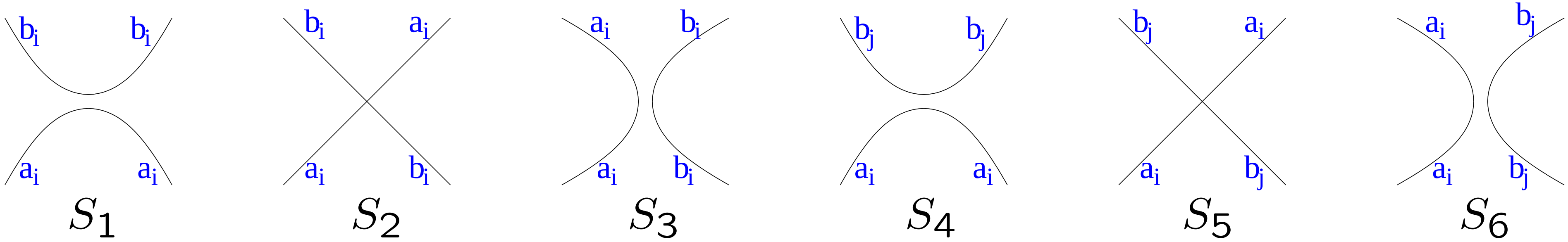


crossing symmetry: $S_1 = S_3^* \equiv \rho_1 \, e^{i\phi}$ $\qquad S_2 = S_2^* \equiv \rho_2$

$S_4 = S_6^* \equiv \rho_4 \, e^{i\theta}$ $\qquad S_5 = S_5^* \equiv \rho_5$

unitarity: $\rho_1^2 + \rho_2^2 = 1$ $\qquad \rho_1\rho_2 \cos\phi = 0$

$\rho_4^2 + \rho_5^2 = 1$ $\qquad \rho_4\rho_5 \cos\theta = 0$

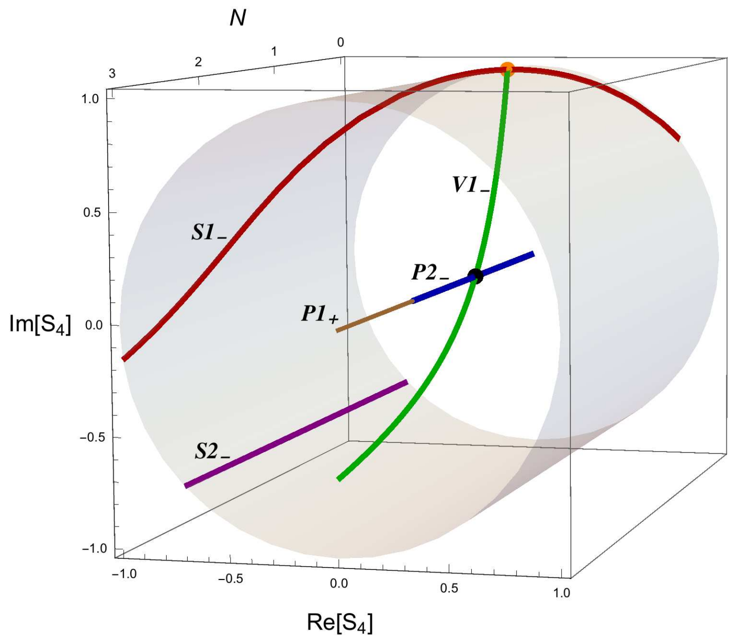
$$N\rho_1^2 + N(n-1)\rho_4^2 + 2\rho_1\rho_2\cos\phi + 2\rho_1^2\cos 2\phi = 0$$

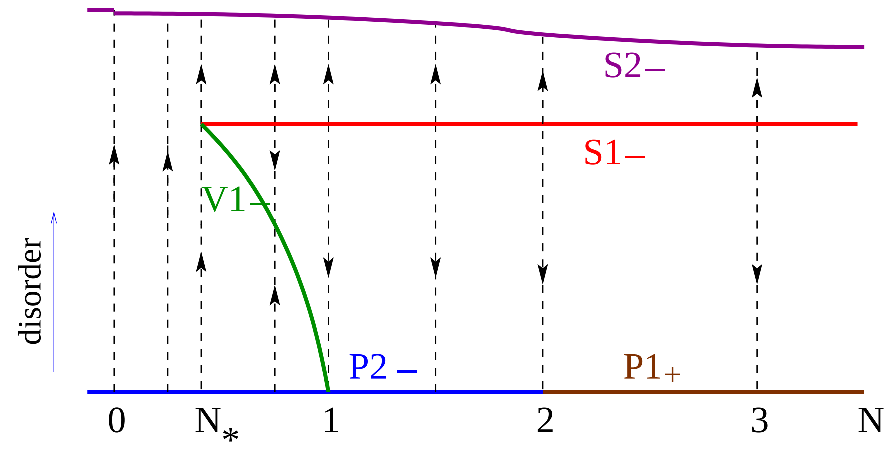
$$2N\rho_1\rho_4\cos(\phi-\theta) + N(n-2)\rho_4^2 + 2\rho_2\rho_4\cos\theta + 2\rho_1\rho_4\cos(\phi+\theta) = 0$$

solutions with $n = 0$ extending to positive N:

Solution	N	ρ_2	$\cos\phi$	ρ_4	$\cos\theta$
$P1_\pm$	$(-\infty, \infty)$	± 1	-	0	-
$P2_\pm$	$[-2, 2]$	0	$\pm\frac{1}{2}\sqrt{2-N}$	0	-
$P3_\pm$	2	$\pm\sqrt{1-\rho_1^2}$	0	0	-
$V1_\pm$	$[\sqrt{2}-1, \infty)$	0	$\pm\dfrac{1}{N+1}$	$\dfrac{N-1}{N+1}\sqrt{\dfrac{N+2}{N}}$	0
$S1_\pm$	$(-\infty, \infty)$	0	$\pm\frac{1}{\sqrt{2}}$	1	$\pm\frac{N^2+2N-1}{\sqrt{2}(N^2+1)}$
$S2_\pm$	$(-\infty, \infty)$	0	$\pm\frac{1}{\sqrt{2}}$	1	$\pm\frac{1}{\sqrt{2}}$

$\rho_4 = 0$ yields decoupled replicas (pure case) $\longrightarrow$ $\rho_4 \sim$ disorder strength





- V_1 perturbatively accessible for $N \to 1^-$ [Shimada '09]
- $N_* = \sqrt{2} - 1 = 0.414..$
(cfr numerical estimate $N_* \approx 0.5$ of [Shimada,Jacobsen,Kamiya '14])
- S1_-: Nishimori-like multicritical line
- S2_-: zero temperature line
- disordered polymers ($N = 0$) renormalize on S2_-

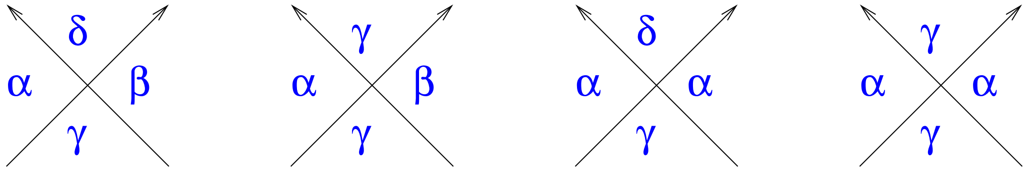
Potts model [GD '17; GD,Tartaglia '17; GD,Lamsen '19]

$$H = -\sum_{\langle i,j\rangle} J_{ij}\, s_i s_j\,, \qquad s_i = 1,2,\ldots,q$$

pure case: $J_{ij} = J$

particles: $\alpha|\beta$ $\qquad \alpha,\beta = 1,\ldots,q\,, \qquad \alpha\neq\beta$

amplitudes:



S_0 $\quad S_1$ $\quad S_2$ $\quad S_3$

crossing: $\quad S_0 = S_0^* \equiv \rho_0, \qquad S_1 = S_2^* \equiv \rho\, e^{i\varphi}, \qquad S_3 = S_3^* \equiv \rho_3$

unitarity:

$$\rho_3^2 + (q-2)\rho^2 = 1$$
$$2\rho\rho_3 \cos\varphi + (q-3)\rho^2 = 0$$
$$\rho^2 + (q-3)\rho_0^2 = 1$$
$$2\rho_0\rho \cos\varphi + (q-4)\rho_0^2 = 0$$

solutions:

Solution	Range	ρ_0	ρ	$2\cos\varphi$	ρ_3
I	$q=3$	$0,\ 2\cos\varphi$	1	$[-2,2]$	0
$\mathrm{II}_\pm$	$q\in[-1,3]$	0	1	$\pm\sqrt{3-q}$	$\pm\sqrt{3-q}$
$\mathrm{III}_\pm$	$q\in[0,4]$	±1	$\sqrt{4-q}$	$\pm\sqrt{4-q}$	$\pm(3-q)$
$\mathrm{IV}_\pm$	$q\in[\frac{1}{2}(7-\sqrt{17}),3]$	$\pm\sqrt{\frac{q-3}{q^2-5q+5}}$	$\sqrt{\frac{q-4}{q^2-5q+5}}$	$\pm\sqrt{(3-q)(4-q)}$	$\pm\sqrt{\frac{q-3}{q^2-5q+5}}$
$\mathrm{V}_\pm$	$q\in[4,\frac{1}{2}(7+\sqrt{17})]$	$\pm\sqrt{\frac{q-3}{q^2-5q+5}}$	$\sqrt{\frac{q-4}{q^2-5q+5}}$	$\mp\sqrt{(3-q)(4-q)}$	$\pm\sqrt{\frac{q-3}{q^2-5q+5}}$

- solution III ending at $q=4$: ferromagnet
- solutions up to $q_{max}=\frac{1}{2}(7+\sqrt{17})=5.56..$ (larger than usually expected value 4)
- room for 2nd order transition in q=5 antiferromagnet (numerical candidate in [Deng et al '11], revisited in [Salas '20], more candidates in [Huang et al '13])
- lattice realization of solution I: $q=3$ antiferromagnet on self-dual quadrangulations [Lv et al '18]

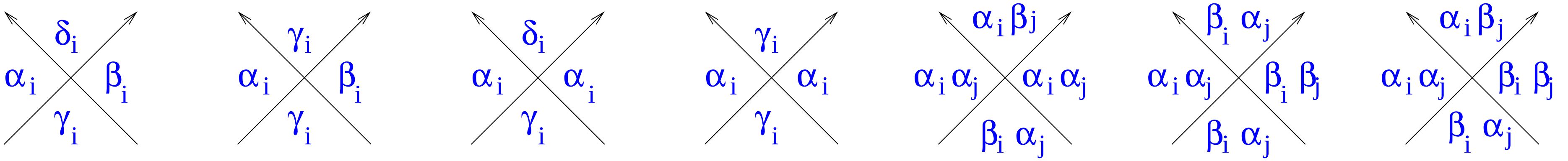
Solution	$\sqrt{q}$	Potts	c	Δ_ε	Δ_η	Δ_σ
$\mathrm{III}_-^{\sin\varphi<0}$	$2\cos\frac{\pi}{(p+1)}$	F critical	$1-\frac{6}{p(p+1)}$	$\Delta_{2,1}$	$\Delta_{1,3}$	$\Delta_{\frac{1}{2},0}$
$\mathrm{III}_-^{\sin\varphi>0}$	$2\cos\frac{\pi}{p}$	F tricritical	$1-\frac{6}{p(p+1)}$	$\Delta_{1,2}$	$\Delta_{3,1}$	$\Delta_{0,\frac{1}{2}}$
$\mathrm{III}_-^{\sin\varphi<0}$	$2\cos\frac{\pi}{(N+2)}$	AF square	$\frac{2(N-1)}{N+2}$	$\frac{N-1}{N}$	$\frac{2}{N+2}$	$\frac{N}{8(N+2)}$

$$S_3 + (q-2)S_2 = e^{-2i\pi\Delta_\eta}$$

disordered case:

particles: $\alpha_i|\beta_i$ $\qquad \alpha, \beta = 1, \ldots, q\,, \qquad \alpha \neq \beta\,, \qquad i = 1, \ldots, n$

amplitudes:



S_0 $\quad S_1$ $\quad S_2$ $\quad S_3$ $\quad S_4$ $\quad S_5$ $\quad S_6$

crossing:

$$S_0 = S_0^* \equiv \rho_0, \qquad S_1 = S_2^* \equiv \rho\, e^{i\varphi}, \qquad S_3 = S_3^* \equiv \rho_3, \qquad S_4 = S_5^* \equiv \rho_4\, e^{i\theta}, \qquad S_6 = S_6^* \equiv \rho_6$$

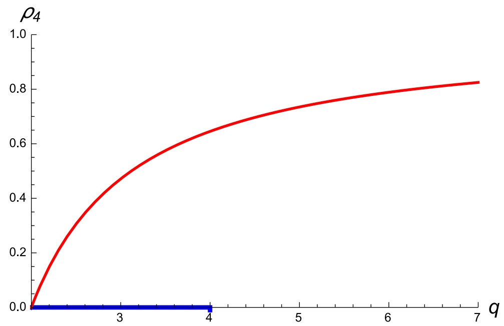
unitarity:

$$\rho_3^2 + (q-2)\rho^2 + (n-1)(q-1)\rho_4^2 = 1$$
$$2\rho\rho_3 \cos\varphi + (q-3)\rho^2 + (n-1)(q-1)\rho_4^2 = 0$$
$$2\rho_3\rho_4 \cos\theta + 2(q-2)\rho\rho_4 \cos(\varphi+\theta) + (n-2)(q-1)\rho_4^2 = 0$$
$$\rho^2 + (q-3)\rho_0^2 = 1$$
$$2\rho_0\rho \cos\varphi + (q-4)\rho_0^2 = 0$$
$$\rho_4^2 + \rho_6^2 = 1$$
$$\rho_4\rho_6 \cos\theta = 0$$

$n = 0$:

exists and is unique solution with disorder vanishing as $q \to 2$ and defined $\forall q \geq 2$:

$$\cos\theta = \rho_0 = 0, \quad \rho = 1, \quad \rho_3 = 2\cos\varphi = -\frac{2}{q}, \quad \rho_4 = \frac{q-2}{q}\sqrt{\frac{q+1}{q-1}}$$



- softening of 1st order transition by disorder exhibited exactly

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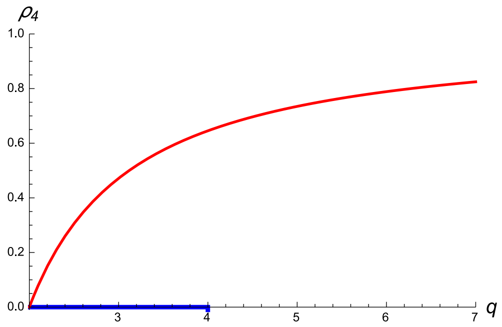
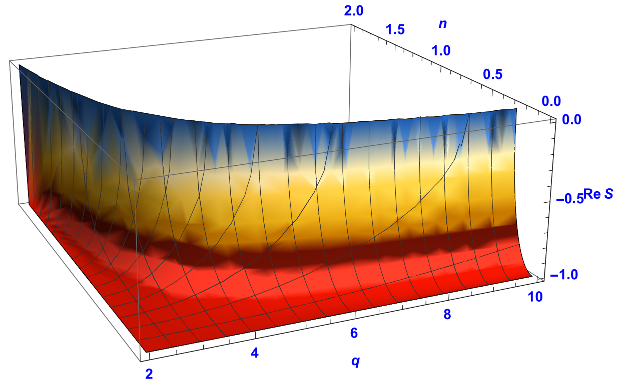


- softening of 1st order transition by disorder exhibited exactly

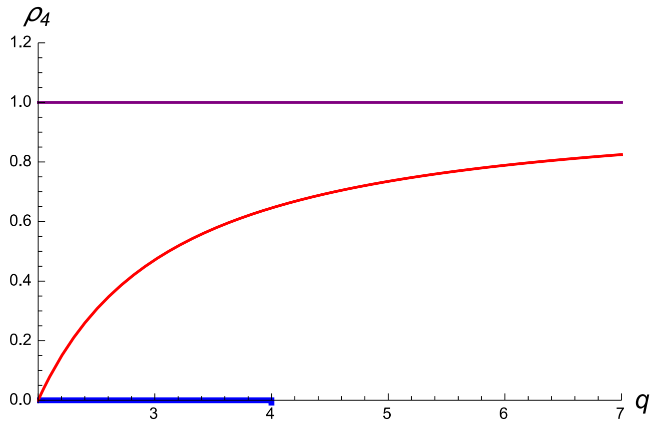
- color singlet amplitude becomes superuniversal at $n = 0$:

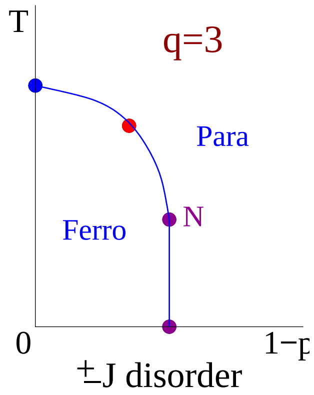
consistent with q-independence of ν and q-dependence of β

numerics never found appreciable deviations from $\nu = 1$ even for large q



- there are solutions strongly disordered ($\rho_4 = 1$) for any q
- Nishimori-like and T=0 critical points belong to this class





from simulations in [Huse et al '98], [Jacobsen, Picco, '02]

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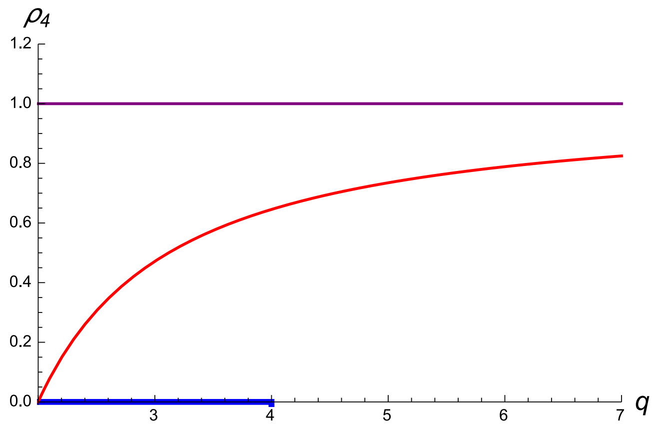

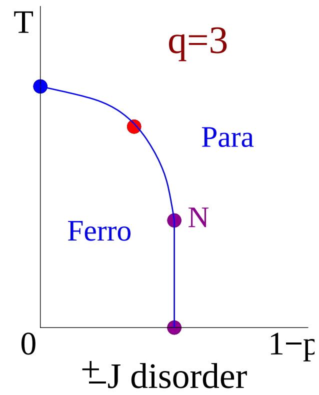

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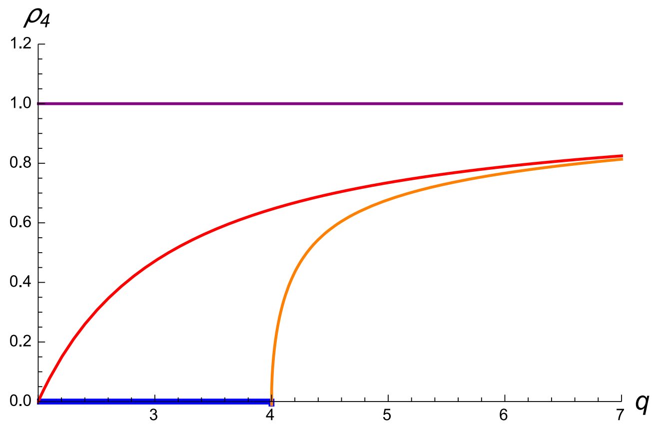
- there is solution able to provide origin of the RG flow for $q > 4$



Conclusion

- random criticality can be exactly accessed in 2D (for any disorder strength)

- symmetry-independent (superuniversal) sectors emerge as characteristic of random criticality

- characterization of corresponding CFTs beyond scattering approach is one of the challenges ahead