Entropy Geometry in Vertex Models

Kari Eloranta

Department of Mathematics, University of Helsinki

Randomness, Integrability and Universality
Galileo Galilei Institute, Florence, June 1, 2022

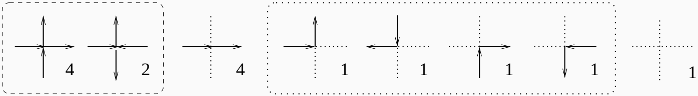
Vertex Models

Vertex Models

Definition (Zero Flux Vertex Model on the Square Lattice)

Assign arrows or unoriented edges between a vertex and its nearest neighbors in $\mathbf{Z}^2$ *respecting the* **rule***: there is the same number of incoming and outgoing arrows. The* **model** *consists of all configurations in which the rule is satisfied at every lattice point.*

No spin variables involved. Three well known examples:



Dotted line segments mean unoriented edges, numbers multiplicities.

Six-vertex (Ice) rule: The frame on the left.
15-vertex rule: The entire set.
19-vertex rule: Add the missing corner rules for isotropy.

Vertex Models

Note that while e.g. the **eight-vertex model** shares a similar generating mechanism, sources and sinks separate it into a different class of models.

Six-vertex rule has been intensely studied and serves here as a reference model. For preceding work on the 15/19-vertex models, see e.g. Izergin & Korepin, Fateev & Zamolodchikov (-81), Batchelor (with twisted boundary condition, -91), Inami & Odake & Zhang (-96), Pant and Wu (knot invariant, -97), more recently e.g. Galleas, Garbali, Hagendorf. Work mostly in unbounded or toral domains.

We will consider the models on the simplest bounded domain, a square, with special boundary condition(s) that will make comparison to Ice Model/Alternating Sign Matrix and still earlier dimer results possible.

The main questions

In Ice one has a fully packed loop soup and all entropy arises from a single action, the **reversal of unidirectional loops**. Short loops are the main carriers of entropy and with them one can localize the entropy generation.

15/19-vertex models relax the loop packing allowing a **diluted loop soup**. Entropy generation mechanism is more general because of unoriented edges. One has multiple generating actions each with its own distribution within the domain. How does this more involved **entropy geometry** look like?

Implications in terms of limit shapes?

Static bits

Definition

Height function *is a mapping from dual lattice* $\mathbf{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ *to integers. If one crosses a lattice arrow pointing left, it increases by one, if to the right, decreases by one and if no arrow encountered, height stays constant. Given a vertex configuration on a simply connected domain it is unique up to an additive constant. The graph of the height function can be wieved either as a stepped surface or as a Lipschitz-surface over the configuration (***tilt*** *extrema* ± 1*).*

Definition

Suppose that the configurations c *and* c' *on a domain have the same boundary configuration. Let* C *and* C' *be the solids under their height surfaces. Let* $c \succeq c'$ *if* $C \supseteq C'$*. Then* $\succeq$ *defines a* **partial order** *on the configurations.*

Definition

$(L, \vee, \wedge)$ is a **distributive lattice** *if for all $x,\ y$*
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and with min/max $(\wedge/\vee)$ interchanged.
Distributive lattices have always maximal and minimal elements.

Proposition

Our configuration sets on a square with the given partial order form a distributive lattice.

This facilitates e.g. irreducibility and coalescing arguments to work on the configuration spaces.

Dynamical Models

Dynamical Models

For the vertex models here any unidirectional loop or unidirectional infinite path can be reversed to generate a new configuration. In 19-vertex model additionally any such loop/path can be converted undirected or vice versa.

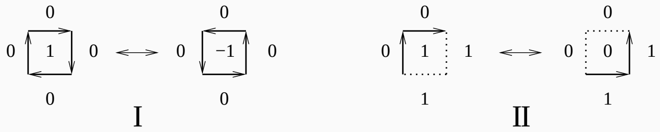
To implement an efficient algorithm one should do the allowed perturbations using the smallest legal **actions/local moves/flips**.

Dynamical Models

For the vertex models here any unidirectional loop or unidirectional infinite path can be reversed to generate a new configuration. In 19-vertex model additionally any such loop/path can be converted undirected or vice versa.

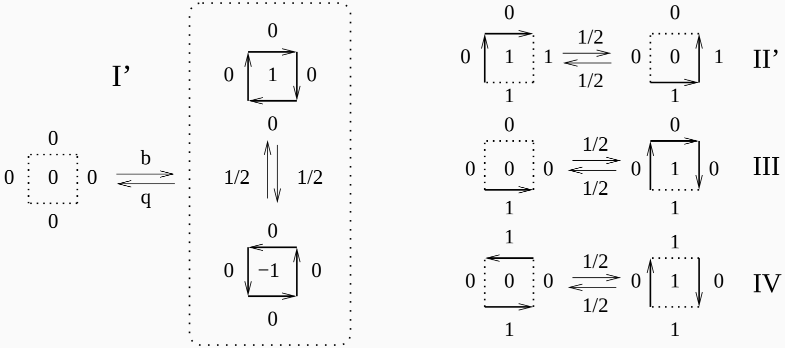
To implement an efficient algorithm one should do the allowed perturbations using the smallest legal **actions/local moves/flips**.

The left action I suffices to generate all six-vertex configurations.



I and II together with rotation by π generates for 15-vertex rule. Heights noted around and inside squares.

The minimal set of local moves for the 19-vertex rule (up to rotation and reflection):



Transition probabilities next to arrows. All reflections and rotations are included.

The three action sets cannot generate illegal configurations from legal ones i.e. these are necessary in each case. Conversely by careful study of the height surfaces one can show that these suffice.

One can utilize a **deposition/sublimation** model. Depending on the case one uses one or two types of volumes: unit cubes or upright $1 \times 1 \times 2$ -pieces. With these one can fill local minima, cut down local maxima and reach the maximal/minimal element for the given boundary condition. Irreducibility of the configuration sets under the action sets follow.

The three action sets cannot generate illegal configurations from legal ones i.e. these are necessary in each case. Conversely by careful study of the height surfaces one can show that these suffice.

One can utilize a **deposition/sublimation** model. Depending on the case one uses one or two types of volumes: unit cubes or upright $1 \times 1 \times 2$ -pieces. With these one can fill local minima, cut down local maxima and reach the maximal/minimal element for the given boundary condition. Irreducibility of the configuration sets under the action sets follow.

Theorem (Irreducibility and ergodicity)

Given a 19-vertex configuration on a bounded domain, any other legal configuration with the same boundary condition can be generated from the former using a finite sequence of elementary actions I'-IV. A strict subset of actions will not suffice. When $0 < b, q < 1$ the Markov Chain on the graph of legal configurations is ergodic.

The conclusions hold for the subsets of actions of six-vertex and 15-vertex rules. Unique stationary measures exist for all.

Computational bits

The actual computation of 6/15/19-vertex model is most naturally done on a $N \times N$ array of symbols each coding the arrows around a lattice unit square (so $2^4 = 16$ or $3^4 = 81$ symbols in the alphabet). This array is a diamond enclosing the domain square.

The boundary condition is imposed on the diamond, specifically on its $4 \times 2N$ lattice edges. When done right it will force a desired boundary condition in the inscribed $N \times N$ square in all the models.

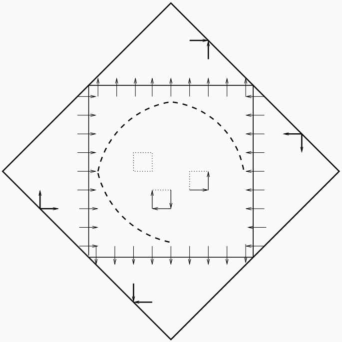
The Probabilistic Cellular Automaton (PCA) which implements the MC acts on the diamond array split as a checkerboard. The black and white subsets are updated alternatively. On each color the updating is done according to the MC, sitewise independently, followed by the update of the other color subset for consistency.

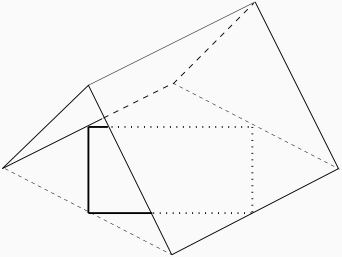
Some loss: uniform weights.
Some gain: MC is **monotone chain** and CFTP is available.

Boundaries

DWBC

For the sake of simplicity and comparisons we restrict to square domain. The Izergin-Korepin **Domain Wall Boundary Condition** or its relaxation are used:





DWBC is on the inscribed square. The diamond is due to computation alone. On the right the **ridge roof** height. It forces DWBC inside.

Proposition

In a 15-vertex configuration an unoriented interior edge implies unoriented boundary edges. Hence with DWBC the 15-vertex rule reduces to the six-vertex rule.

To choose genuinely non-six-vertex 15-vertex boundary conditions one observes:

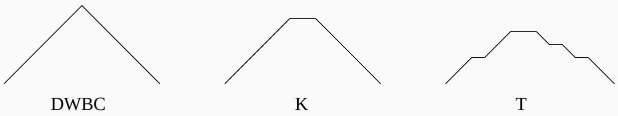
Proposition

For a given boundary condition the number of unoriented edges in the 15-vertex fill-in configuration is constant. An unoriented path cannot branch under 15-vertex rule.

Not true for 19-vertex configurations. We'll see DWBC can bound highly non-trivial and non-six-vertex-like configurations.

Non-DWBC for 15-vertex

Two generalizations of the ridge roof (left, imposes DWBC):

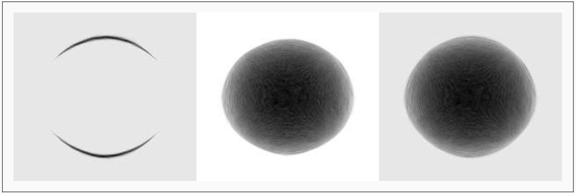


NW-SE cross cuts of the initial height over the diamond.

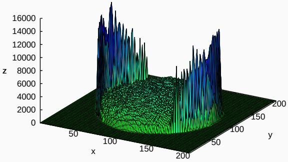
The horizontal flats correspond initially to unoriented SW-NE staircases. Their end points are fixed on the boundary. Under the iteration they behave like non-branching random bridges stretching across the diamond.

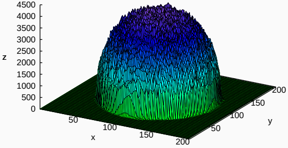
Samples

15-vertex equilibria, non-DWBC

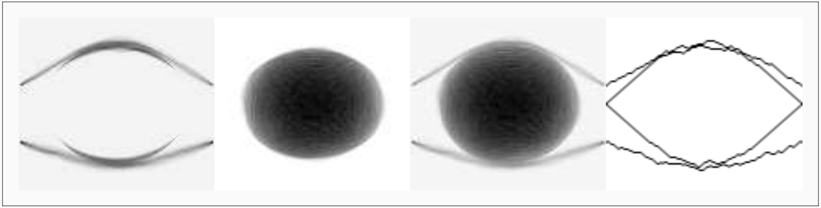


K-boundary, a pair of blank staircases at the top. Diamond tilted by $\pi/4$; ridge is horizontal. Densities of II, I and I+II. 206^2 diamond/square, equilibrium iterates 61-70.000. Below left action II weighted 10-fold.





Equilibrium from T-type boundary condition for 15-vertex model:

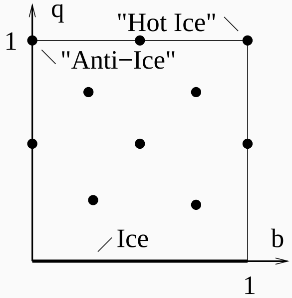


Cumulative distributions of II, I and I+II. On the right the locations of the unoriented edges at termination. Square 106^2, ridge horizontal, 10.000 iterates at the equilibrium.

19-vertex parametrization (and back to DWBC)

- Bottom (all b) is six-vertex since no arrow vacancy can come about. 1-enumeration point of ASM in the disordered phase ($\Delta = \frac{1}{2}$). Limit shape a bit off circle.
- "Anti-Ice": no trace of Ice-action since no unidirectional loops are reversed.
- $b + q$ is the unidirectional 1-loop creation-annihilation rate, "Ice-temperature".

• Sample points.



DWBC and density

DWBC (which is just “maximal tilt boundary segments without cul-de-sacs”) can be generalized from the square to more complicated domains yielding more exotic limit shapes to 6/19-vertex.

On the boundary the arrow density is one, but

Proposition (Interior arrow density)

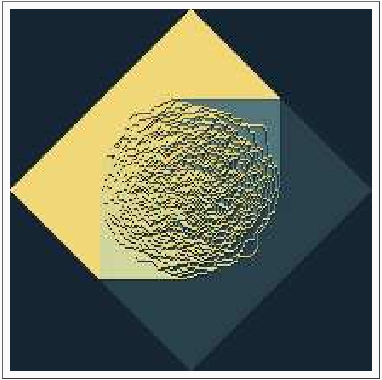
The arrow density of a 19-vertex configuration over a domain with DWBC is always at least $1/2$. *Bound is tight.*

We do not know of any boundary condition with density of boundary arrows strictly below one yielding non-trivial limit shapes.

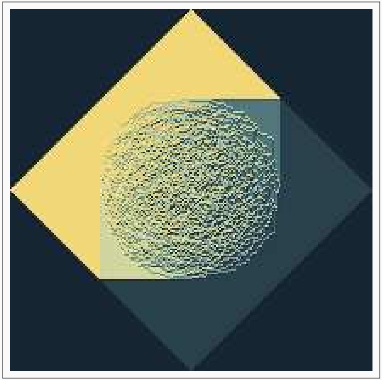
Ergo, all subsequent 19-vertex samples are with DWBC at equilibrium.

From Ice to Hot-Ice

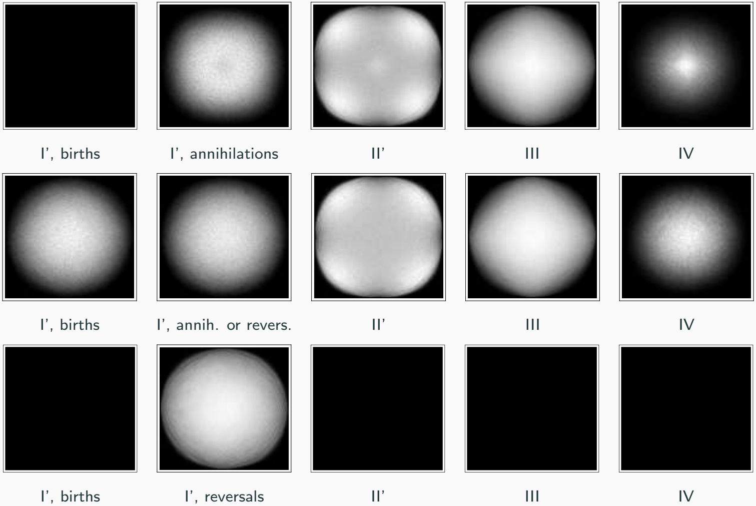
Configurations after 40.000 iterates with DWBC on the maximal square inside 106^2 diamond/square. 1-square arrow arrangements color coded.



Ice $((b,q) = (0,0))$

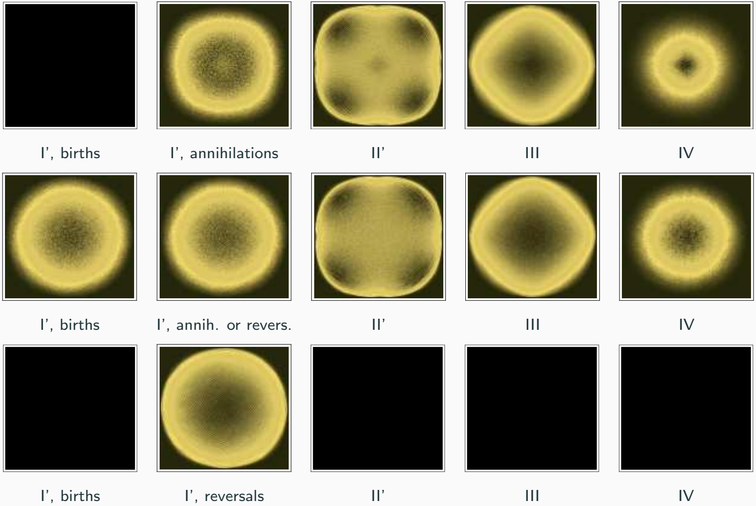


Hot-Ice $((1,1))$



Action densities at equilibrium. Captions indicate the (sub)actions.
Top row: Anti-Ice ($(b, q) = (0, 1)$). **Middle row**: diagonal $q = b > 0$.
Bottom row: 1-weight ASM i.e. Ice-case ($(b, 0)$, any b).
106^2 square. Lighter is more active. Individually scaled for contrast.

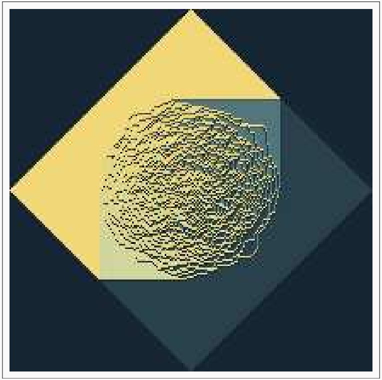
Same data, enhanced:



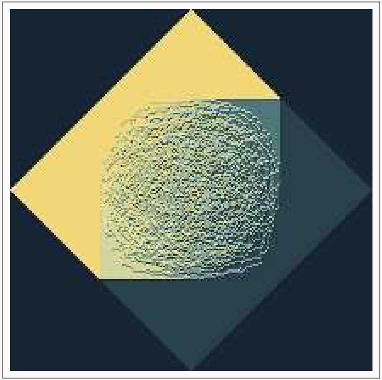
Rough order of the intensities: III>II'>I'(ann. & rev.)$\geq$I'(birth)$\approx$IV.

Ice versus Anti-Ice

Configurations after 40.000 iterates on 106^2 diamond, DWBC:



Ice



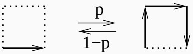
Anti-Ice $((0,1))$

Action III' skewed

Actions I'(reversal), II' and IV do not change the arrow density.

Action III changes the arrow density and it alters the geometry of oriented paths. Moreover it is the highest intensity action.

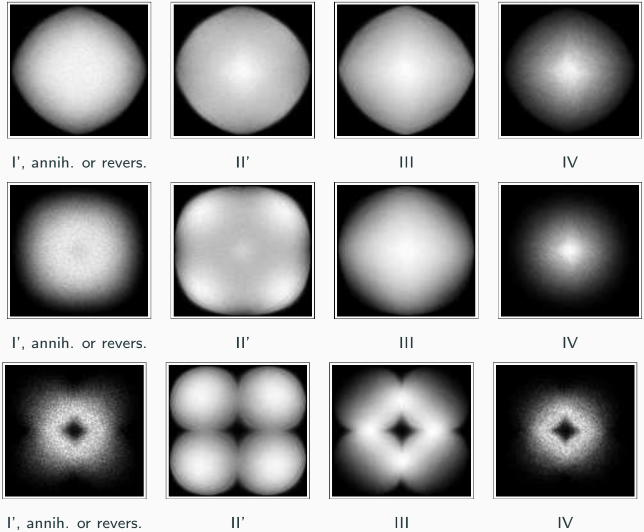
III'



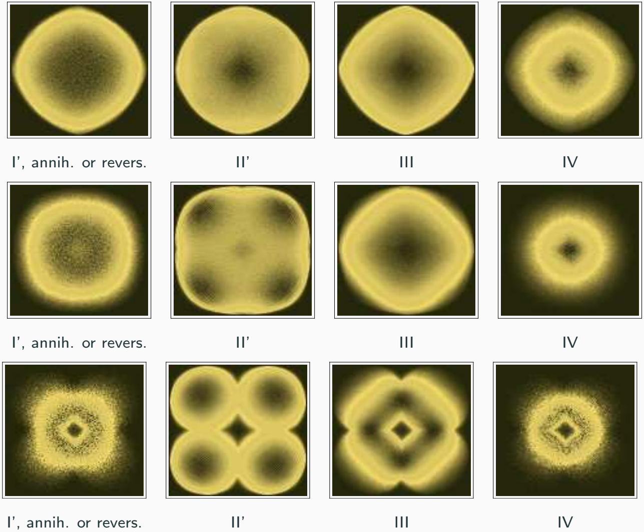
As $p \downarrow 0$, the oriented paths should straighten out and form thinner ensembles, hence further lowering the intensities of other actions.

As $p \uparrow 1$, the oriented paths get more convoluted, perhaps approaching lattice filling. This should even out the action distribution differences.

The effect should be most pronounced in the directions of lattice axes, but what are the other **global consequences?**

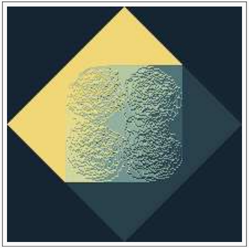


Skewed action III' at $(b, q) = (0, 1)$. **Middle row**: Anti-Ice ($p = \frac{1}{2}$). **Top row**: $p = \frac{7}{8}$ and **bottom row**: $p = \frac{1}{8}$. No I'-births in any: first blank column removed. 106^2 square.



Skewed action III' at $(b, q) = (0, 1)$, enhanced rendering. Middle row is Anti-Ice ($p = \frac{1}{2}$). Top row: $p = \frac{7}{8}$ and bottom row $p = \frac{1}{8}$.

Grazie!



syndetica.net/math