Quantum Spectral Curve: from Anomalous Dimensions to Structure Constants

Arthur Klemenchuk Sueiro

École Normale Supérieure (LPENS)

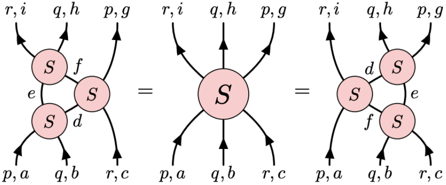
Work in progress with B. Basso and A. Georgoudis

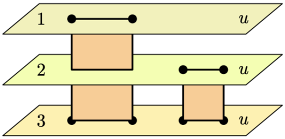
Randomness, Integrability and Universality - Galileo Galilei Institute

May 31, 2022

Introduction

- Integrability: powerful tool to solve 1+1d models
  - Spin chains, vertex models, sigma models, PDEs
  - Mathematical structures: Bethe ansatz, quantum groups, algebraic curve...
- 4d Quantum Field Theories: ubiquitous in physics
  - Complicated: perturbation theory
  - Exact results ?
  - Planar $\mathcal{N} = 4$ Super Yang-Mills
- New integrable structures: Quantum Spectral Curve and Hexagons





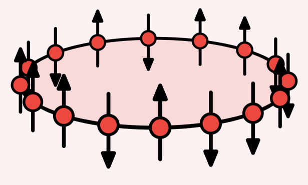
Overview

1. Revisiting the Heisenberg spin chain
2. Planar $\mathcal{N} = 4$ Super Yang-Mills and QSC
3. Hexagons and structure constants

Heisenberg $XXX_{1/2}$ spin chain

- L sites with spin $1/2 \Rightarrow su(2)$ symmetry

$$\hat{H} = 2g^2 \sum_{j=1}^{L} (\mathbb{I} - \mathbb{P}_{j,j+1})$$



- Diagonalized via Bethe Ansatz: start with vacuum $|0\rangle = |\uparrow \ldots \uparrow\rangle$
  - Build spin-down excitations: magnons
  - Eigenstates: N-magnon states $\{u_1, \ldots, u_N\}$

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j \neq k}^{N} \frac{u_k - u_j + i}{u_k - u_j - i}$$

$$E = \sum_{j=1}^{N} \frac{2g^2}{u_j^2 + 1/4}$$

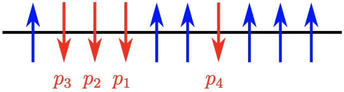
- Analogue of Schrödinger equation ?

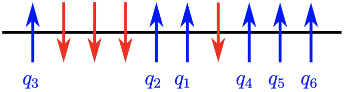
Baxter equation

- State encoded in Baxter polynomial: $\boxed{Q_1(u) = \prod_{j=1}^{N}(u - u_j)}$
  - Asymptotics $\Rightarrow$ charges: $Q_1(u) \sim u^N$
- BAE equivalent to polinomiality of Baxter equation:

$$\boxed{T(u)Q_1(u) = \left(u + \frac{i}{2}\right)^L Q_1^{[-2]} + \left(u - \frac{i}{2}\right)^L Q_1^{[+2]}}$$

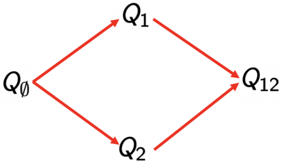
  - Notation: $Q_1^{[+n]} = Q_1\left(u + n\frac{i}{2}\right)$
- 2 independent solutions due to vacuum choice: Q_1 and Q_2





- Wronskian relation (like ODEs)

$$\begin{vmatrix} Q_1^- & Q_1^+ \\ Q_2^- & Q_2^+ \end{vmatrix} = u^L = Q_\emptyset Q_{12}$$

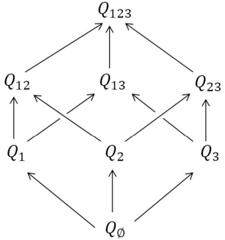


- New Q functions $Q_\emptyset = u^L$ and $Q_{12} = 1 \Rightarrow$ Hasse diagram (universal)
- Converse: given $Q_\emptyset$ and polinomiality of Q functions (model dependent) $\Rightarrow$ BAE !

- Works for any $gl(N|M)$ integrable spin chain, eg $gl(3)$

$$\frac{Q_\emptyset^+}{Q_\emptyset^-} = - \left. \frac{Q_a^{++}}{Q_a^{--}} \frac{Q_{ab}^-}{Q_{ab}^+} \right|_{u=u_{a,n}}$$

$$1 = - \left. \frac{Q_{ab}^{++}}{Q_{ab}^{--}} \frac{Q_a^-}{Q_a^+} \right|_{u=u_{ab,n}}$$



In summary

Symmetry (Hasse diagram) + Analyticity $\Rightarrow$ Bethe equations

Planar $\mathcal{N} = 4$ Super Yang-Mills

- Simplest gauge theory in 4d
- Field content $\boxed{A_\mu, \varphi^I, \psi^a, \bar{\psi}_a} \to$ Adjoint rep. of $SU(N)$
- Single trace operators:

$$\boxed{\mathcal{O}(x) = \mathrm{Tr}[D^{j_1}(\phi^I(x))^{j_2} \ldots (\psi^a(x))^{j_n} \ldots]}$$

- Planar limit $\Rightarrow$ integrable

$$\boxed{g = \frac{\sqrt{\lambda}}{4\pi}, \quad \lambda = g_{YM}^2 N, \quad g_{YM} \to 0, \quad N \to \infty}$$

- Conformal invariance: only need (Δ, C_{123})

Conformal data

- Scaling dimension: $\Delta = \Delta_0 + \gamma(g)$
  - 2 point correlator:

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle = \frac{1}{|x-y|^{2\Delta_1}}\delta_{\Delta_1,\Delta_2}$$

- Structure constant: C_{123}
  - 3 point correlator:

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \frac{C_{123}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|x-z|^{\Delta_1+\Delta_3-\Delta_2}|y-z|^{\Delta_2+\Delta_3-\Delta_1}}$$

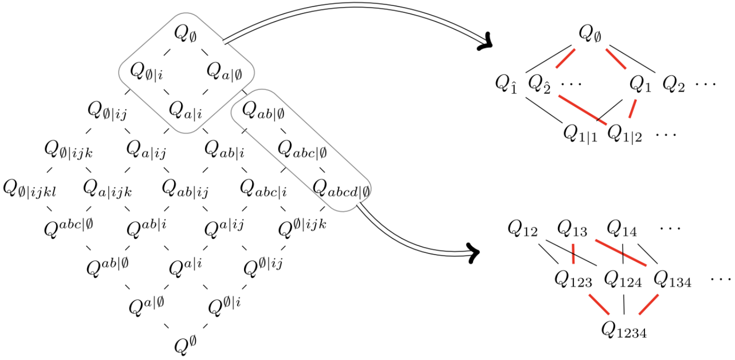
Global symmetry group + integrability

- CFT + SUSY $\Rightarrow$ $psu(2,2|4)$: similar to $gl(4|4)$
  - Bosonic part $SU(2,2) \times SU(4)$ + fermionic part
  - Cartan charges: $(\Delta; S_1, S_2; J_1, J_2, J_3)$
- Spectrum of Anomalous dimensions
  - Operators mapped to spin chain states [Minahan, Zarembo]

Vacuum: $\mathrm{Tr}[Z^L]$ Excited states: $\mathrm{Tr}[Z^{L-M}X^M], \mathrm{Tr}[Z^{L-M}D^M]$...

  - $\gamma(g)$ mapped to energy E
  - Can write BAE [Beisert, Staudacher,...]: problems after 3 loops
- Finite size corrections ? Quantum Spectral Curve [Gromov, Kazakov,Leurent, Volin]

- Goal: find full non-perturbative spectrum using $psu(2,2|4)$ symmetry
- Inspiration from spin chain: build $gl(4|4)$ Q-system
  - $2^{4+4} = 256$ Q functions: $\mathcal{Q}_{..ab..|..ij..}$
  - Unimodularity: $\mathcal{Q}_{\emptyset|\emptyset} = \mathcal{Q}_{1234|1234} = 1$
- Hasse diagram



QSC - Algebra

- QQ relations

$$\mathcal{Q}_{A|I}\mathcal{Q}_{Aab|I} = \mathcal{Q}^+_{Aa|I}\mathcal{Q}^-_{Ab|I} - \mathcal{Q}^-_{Aa|I}\mathcal{Q}^+_{Ab|I}$$
$$\mathcal{Q}_{A|I}\mathcal{Q}_{A|Iij} = \mathcal{Q}^+_{A|Ii}\mathcal{Q}^-_{A|Ij} - \mathcal{Q}^-_{A|Ii}\mathcal{Q}^+_{A|Ij}$$
$$\mathcal{Q}_{Aa|I}\mathcal{Q}_{A|Ii} = \mathcal{Q}^+_{Aa|Ii}\mathcal{Q}^-_{A|I} - \mathcal{Q}^+_{A|I}\mathcal{Q}^-_{Aa|Ii}$$

- Only need $4+4$ elementary Q-functions

$$P_a = \mathcal{Q}_{a|\emptyset}, \quad Q_i = \mathcal{Q}_{\emptyset|i}, \quad P_a Q_i = \mathcal{Q}^+_{a|i} - \mathcal{Q}^-_{a|i}$$

- Hodge duals: P^a and Q^i

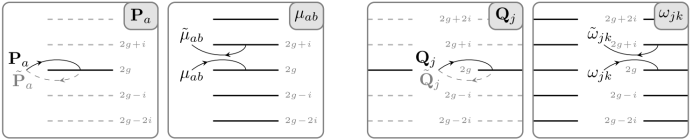
$$\mathcal{Q}^{A|I} \equiv (-1)^{|A'||I|}\epsilon^{A'A}\epsilon^{I'I}\mathcal{Q}_{A'|I'}$$

QSC - Analytical structure

- Strong coupling: classical Type IIB Superstrings in $AdS_5 \times S^5$
  - Integrable σ-model $\Rightarrow$ finite-gap solution $\Rightarrow$ algebraic curve
  - Quasi-momenta $\left\{ e^{ip_1}, e^{ip_2}, e^{ip_3}, e^{ip_4} \mid e^{ip_{\hat{1}}}, e^{ip_{\hat{2}}}, e^{ip_{\hat{3}}}, e^{ip_{\hat{4}}} \right\}$
  - Square root branch cuts [Gromov, Vieira]
- Analytical structure:

- Monodromy properties

$$\tilde{P}_a = \mu_{ab} P^b, \qquad \tilde{Q}_i = \omega_{ij} Q^j$$



QSC - Asymptotics

- Large u asymptotics $\Rightarrow$ charges of state !

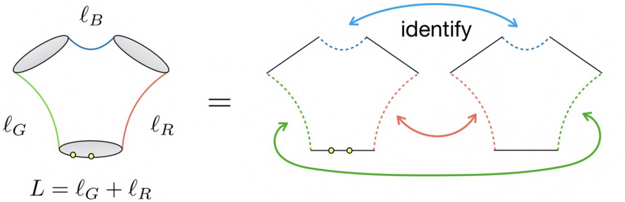
$$\mathbf{P}_a \sim A_a u^{-\tilde{M}_a}, \quad \mathbf{P}^a \sim A^a u^{\tilde{M}_a - 1}, \quad \mathbf{Q}_i \sim B_i u^{\hat{M}_i - 1}, \quad \mathbf{Q}^i \sim B^i u^{-\hat{M}_i}$$

$$\tilde{M}_a = \left\{ \frac{J_1 + J_2 - J_3 + 2}{2}, \frac{J_1 - J_2 + J_3}{2}, \dots \right\}$$
$$\hat{M}_i = \left\{ \frac{\Delta - S_1 - S_2 + 2}{2}, \frac{\Delta + S_1 + S_2}{2}, \dots \right\}$$

- Given operator charges, can obtain Δ
- Any local, some non-local (cusped Wilson-lines) operators, BFKL, Hagedorn temperature...
- 11 loops $\gamma(g)$ for Konishi $\mathrm{Tr}[D^2 Z^2]$

Hexagons and structure constants

- Special configurations: 2 BPS, 1 non protected $sl(2)$

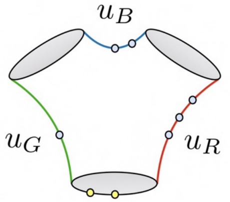


- Hexagons: building blocks of correlators
  - Integrable bootstrap [Basso, Komatsu, Vieira]
- Large operators + large bridge ⇒ asymptotic factorization

$$C_{123} = \frac{1}{\mathcal{N}_G} \sum_{\alpha \cup \bar{\alpha} = \{1,\dots,M\}} (-1)^{|\bar{\alpha}|} \prod_{j \in \bar{\alpha}} e^{i p_j \ell_R} \prod_{\substack{j<k \\ j \in \bar{\alpha}, k \in \alpha}} S(u_i, u_j)\, \mathcal{H}(\alpha) \mathcal{H}(\bar{\alpha})$$

Wrapping corrections

- Small operators: finite size corrections from mirror magnons



$$\mathcal{N} \sum_{\text{flavor}} \int du \frac{W_R(u_R) W_G(u_G) W_B(u_B)}{\Delta((u_R, u_G))}$$

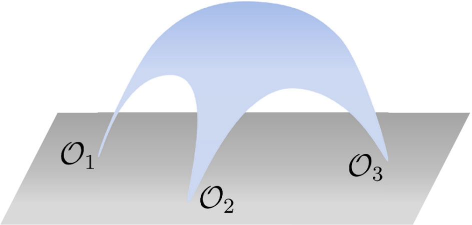
- Divergences $\Rightarrow$ renormalization of weights $\Rightarrow$ Q functions
- Conjecture for bottom bridge

$$W_B = w_a = \sum_{i=1}^{4} Q^{i[+a]} Q_i^{[-a]}$$

- Weak coupling: reproduces 5-loop Konishi OPE coefficient [Georgoudis, Gonçalves, Pereira]

Strong coupling

- String theory: worldsheet area
- Classical string computation done by Komatsu
- Q-functions at strong coupling



$$Q^i(u) = \mathcal{P}^{\emptyset|i} e^{\left(-g \int_0^{u/g} p_{\hat{i}}(z)dz\right)}, \quad Q_i(u) = \mathcal{P}_{\emptyset|i} e^{\left(+g \int_0^{u/g} p_{\hat{i}}(z)dz\right)}$$

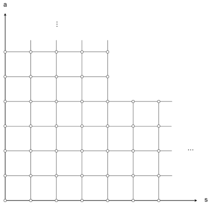
- Bottom weight reduce to $gl(4|4)$ character $\Rightarrow$ agree with string theory

$$w_a = (-1)^a \mathbf{t}_{a,1} = \sum_{j=1}^{4} \frac{\prod_{s=1}^{4}(y_j - x_s)}{\prod_{i\neq j}(y_j - y_i)} y_j^{a-1}, \quad (x_a|y_i) = (e^{-ip_{\tilde{a}}}|e^{-ip_{\hat{i}}})$$

Checks

- These characters satisfy Classical Hirota equation on L-hook

$$\mathbf{t}_{a,s}^2 = \mathbf{t}_{a+1,s}\mathbf{t}_{a-1,s} + \mathbf{t}_{a,s+1}\mathbf{t}_{a,s-1}$$



- Adjacent channels at strong coupling: weights reduces to $su(2,2|4)$ characters: T-hook
- Asymptotic limit: weight related to $su(2|2)$ Transfer matrices $\Rightarrow$ Quantum Hirota equation

Conclusion

- QSC: powerful mathematical tool to study multiple observables
  - Insights from classical and quantum integrability
  - Origins of integrability ?
- UV completion of hexagons ?
- Extend to other configurations
- Generalise to other theories ? ABJM ?

Thanks for your attention!