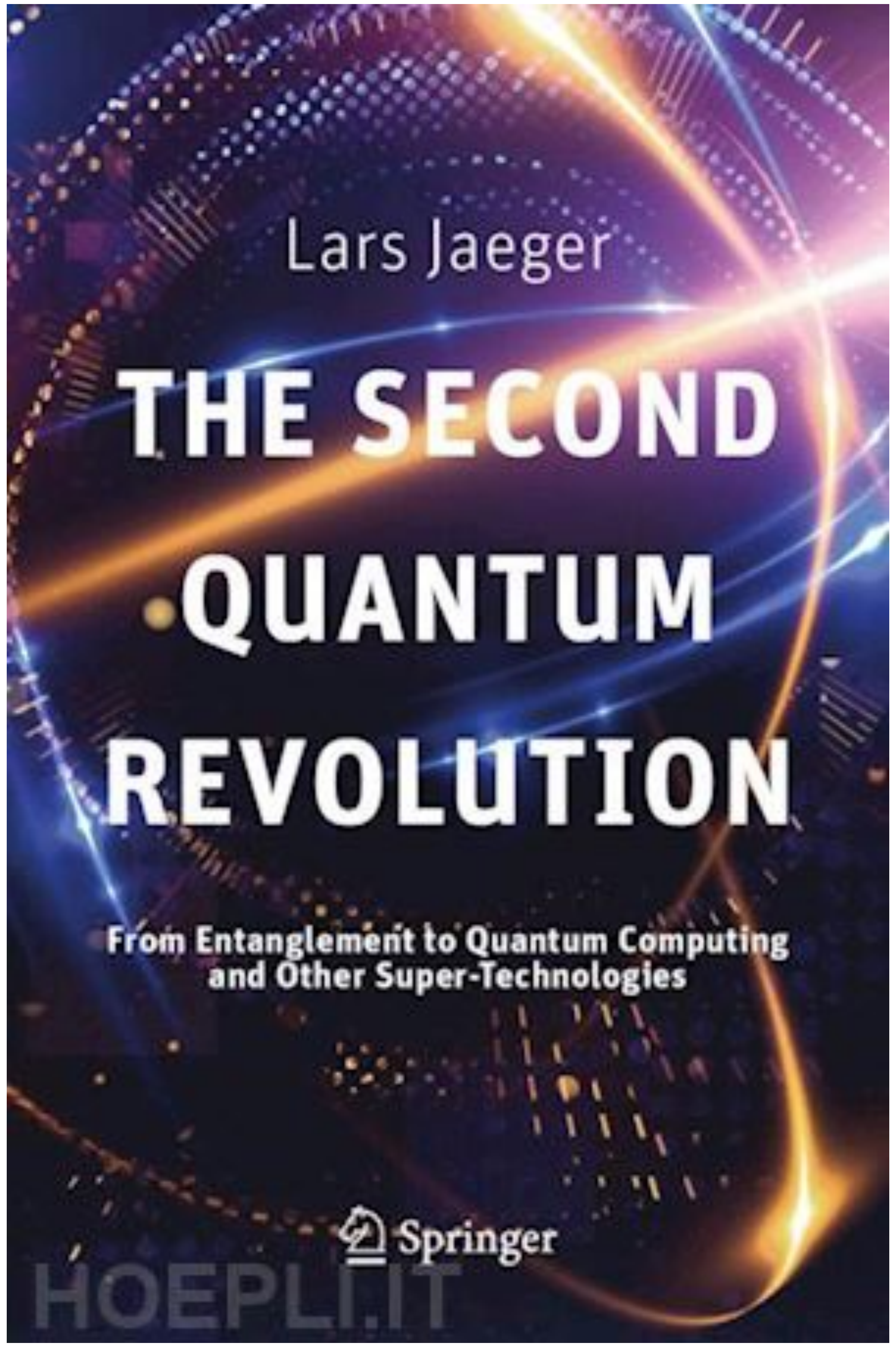
Why entanglement matters

Pasquale Calabrese

SISSA-Trieste

GGI Tea break, June 2022

The second quantum revolution



Noisy intermediate-scale quantum era

From Wikipedia, the free encyclopedia

In the **noisy intermediate-scale quantum (NISQ) era**[1] the leading quantum processors contain about 50 to a few hundred qubits, but are not advanced enough to reach fault-tolerance nor large enough to profit sustainably from quantum supremacy.[2][3] The term was coined by John Preskill in 2018.[4][1] It is used to describe the current state of the art in the fabrication of quantum processors.[5]

However, this talk is not about quantum technologies

What is entanglement

$$\frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\right)$$

It depends to whom you ask and when

- It is a spooky action at distance (1930-70)
- It is a resource for quantum tech (1980-90)
- It is a tool to study and characterise (new) phases of matter (2000)
- It is the key to understand fundamental laws of nature (2010)

From few to many body entanglement

Entanglement between two (few) particles is studied theoretically and experimentally from decades

$$\frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\right)$$

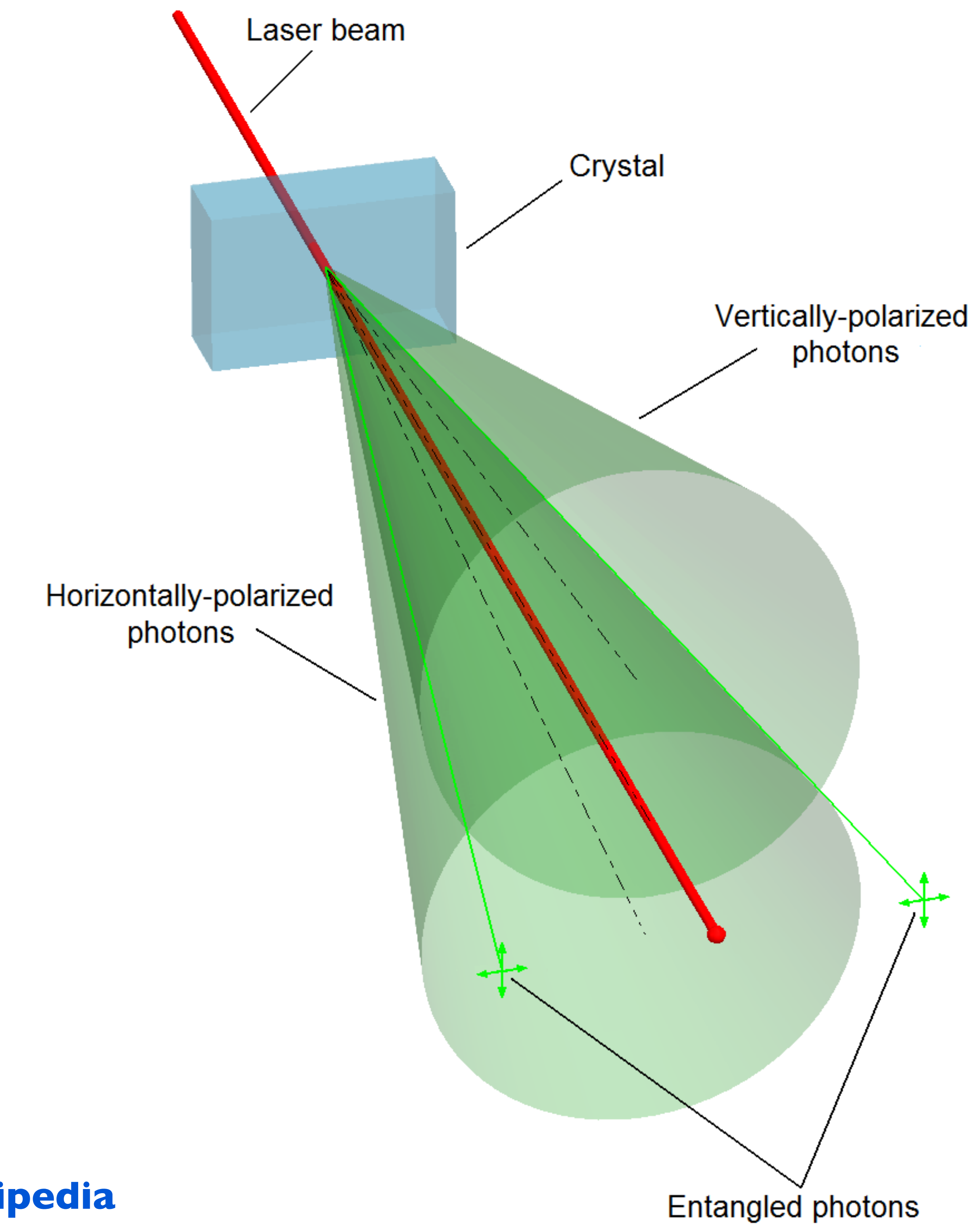


Figure from Wikipedia

Many-body entanglement became a mature subject only in the last 20 (theory) or 10 (experiment) years

$$|\psi\rangle_{AB} = \sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B\,.$$

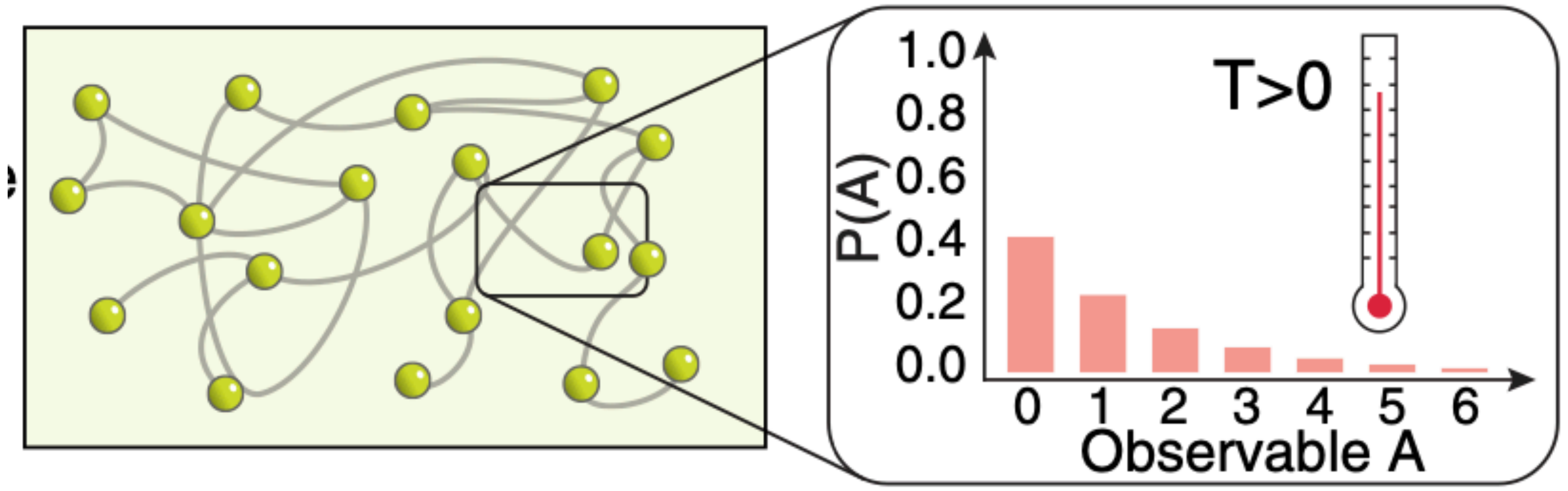
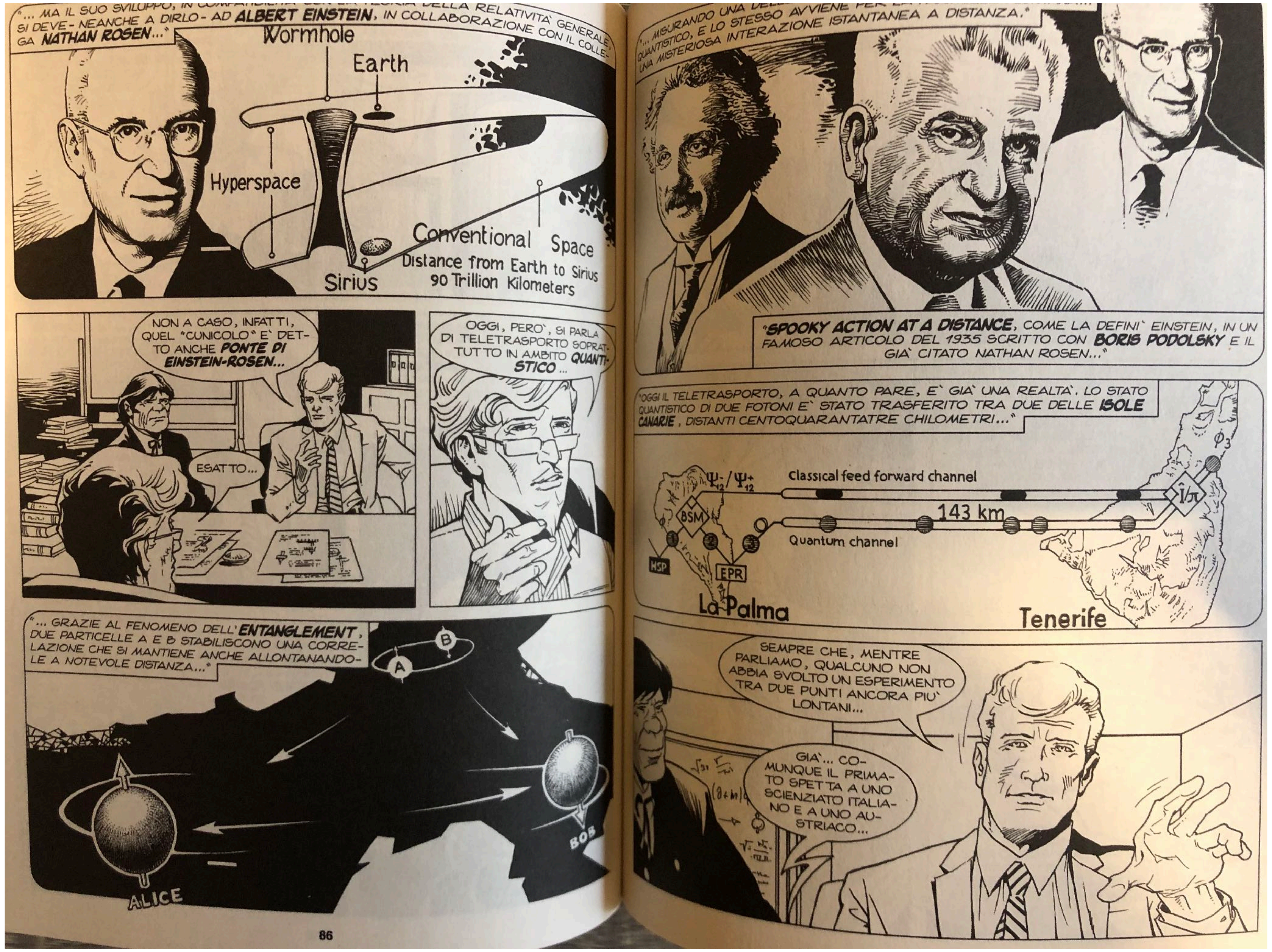


Figure from Kaufman et al, Science 353, 794 (2016)

Many body entanglement in popular culture



Martin Mystere (# 368, April 2020) talks about entanglement and ER=EPR conjecture by Maldacena and Susskind

ER = EPR

From Wikipedia, the free encyclopedia

ER = EPR is a conjecture in physics stating that two entangled particles (a so-called Einstein–Podolsky–Rosen or EPR pair) are connected by a wormhole (or Einstein–Rosen bridge)[1][2] and may be a basis for unifying general relativity and quantum mechanics into a theory of everything.[1]

Cool horizons for entangled black holes

J. Maldacena, L. Susskind

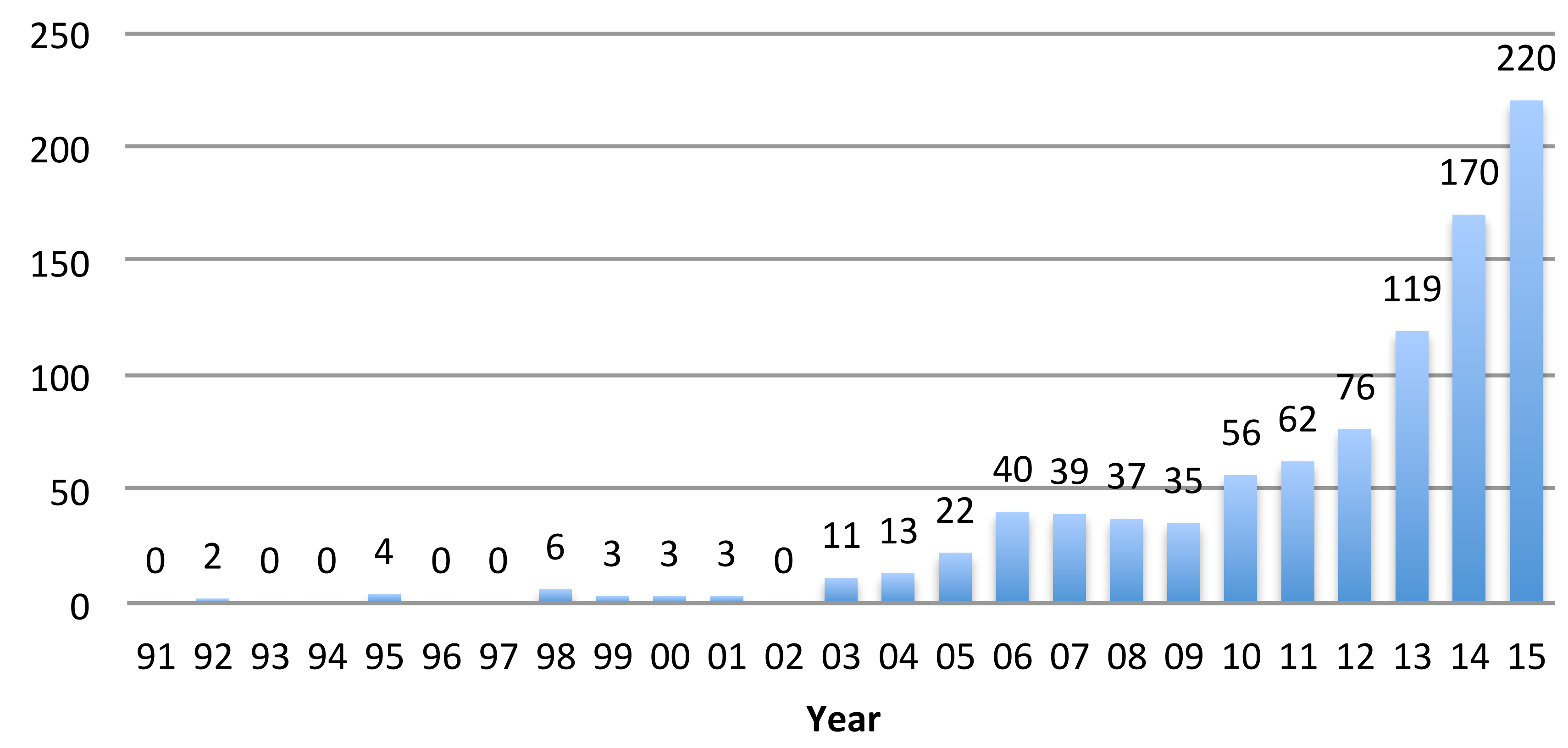
First published: 01 August 2013 | https://doi.org/10.1002/prop.201300

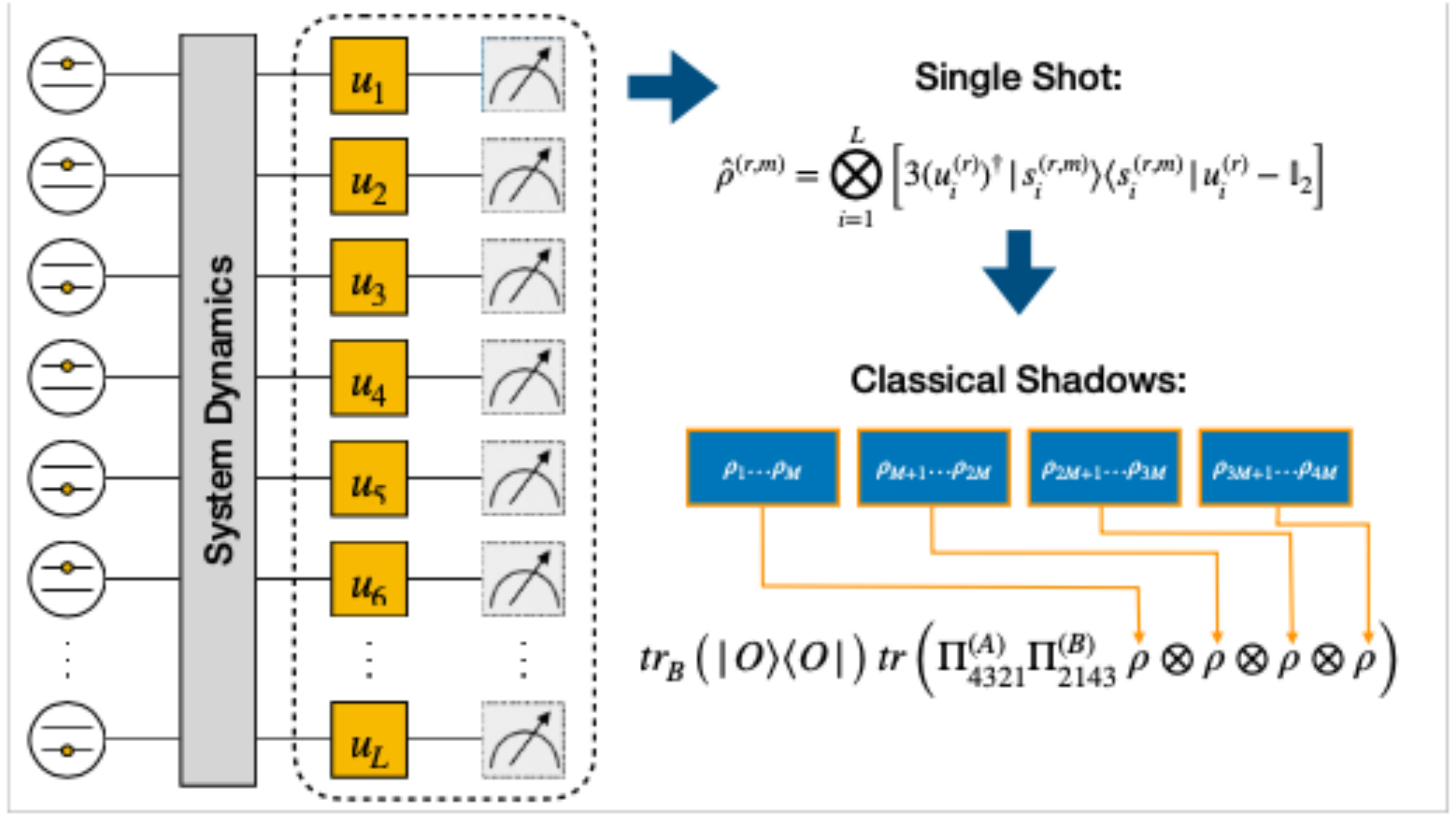
Goal of the talk

I’ll show (too?) many apparently unrelated results in disparate fields of physics that have as common denominator many-body entanglement.
(I worked only on few of them, so be patient with things I do not master)

The goal is to convene the message that many-body entanglement brought new and fresh ideas for a deeper understanding of nature

hep-th arXiv preprints with entanglement in the title

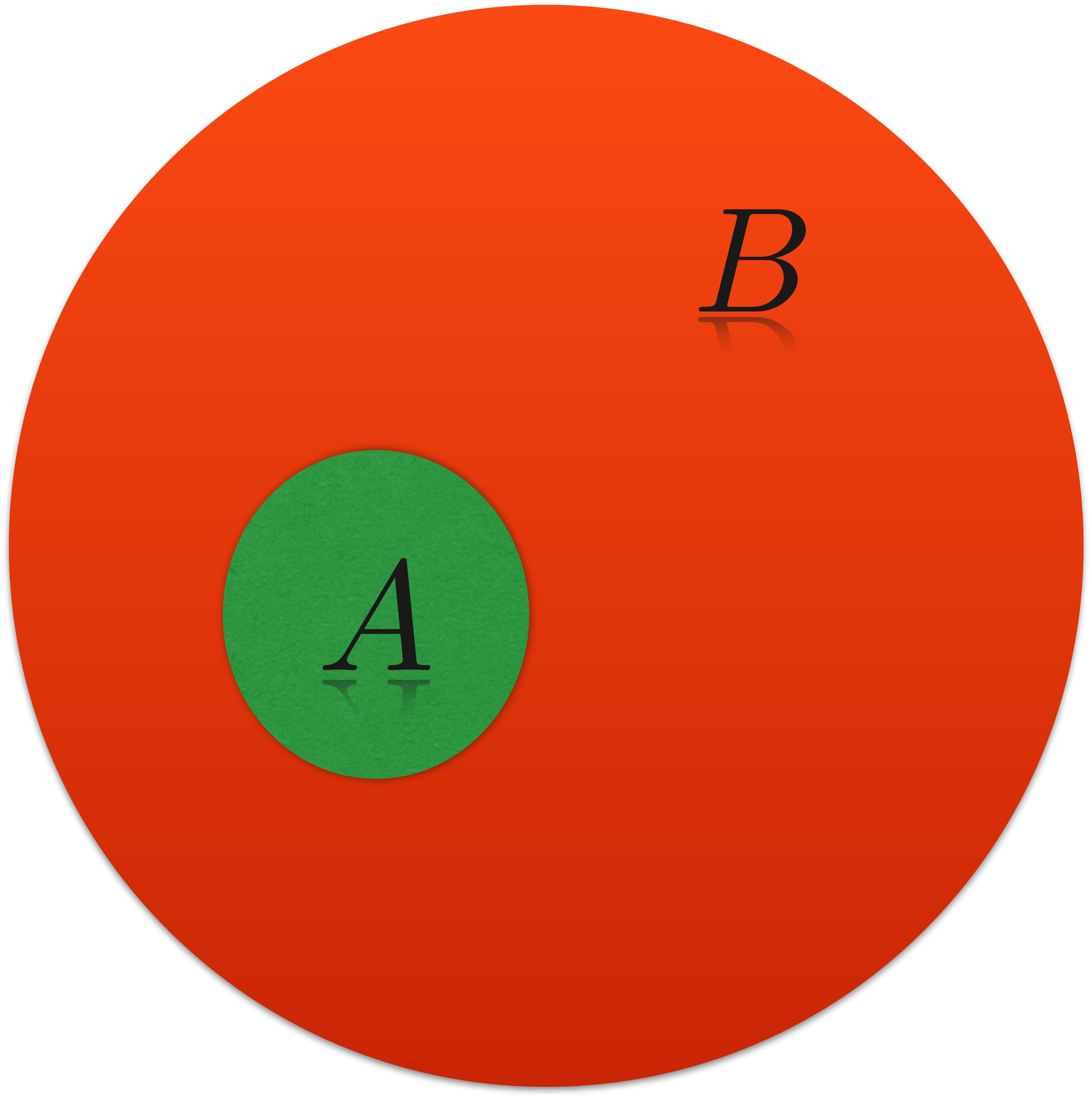




How to quantify entanglement: the entanglement entropy

Consider a system in a pure quantum state $|\psi\rangle, \quad \rho = |\psi\rangle\langle\psi|$

$$\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B \qquad |\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$$



The reduced density matrix of A is $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$

The entanglement entropy

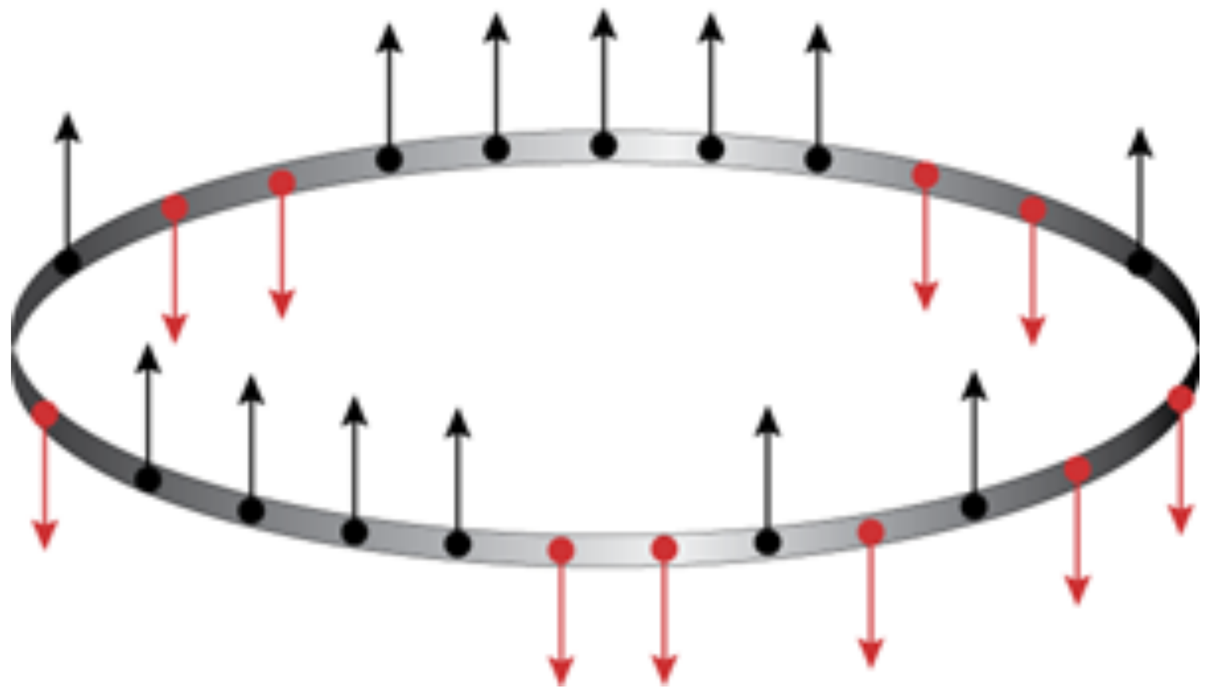
$$\boxed{S_A = -\,\mathrm{Tr}\left(\rho_A \log \rho_A\right)}$$

measures the bipartite entanglement between A & B

Note: In the remainder of the talk A & B refer to a spatial bipartition

How to access the (many-body) entanglement entropy

- **Possibility 1**: with a quantum computer (analog or digital)



$$|\Psi\rangle = \sum_{s_i=\pm} A_{s_1 s_2 \dots s_N} |s_1, s_2, \dots s_N\rangle$$

2^N coefficients: too many for a classical PC

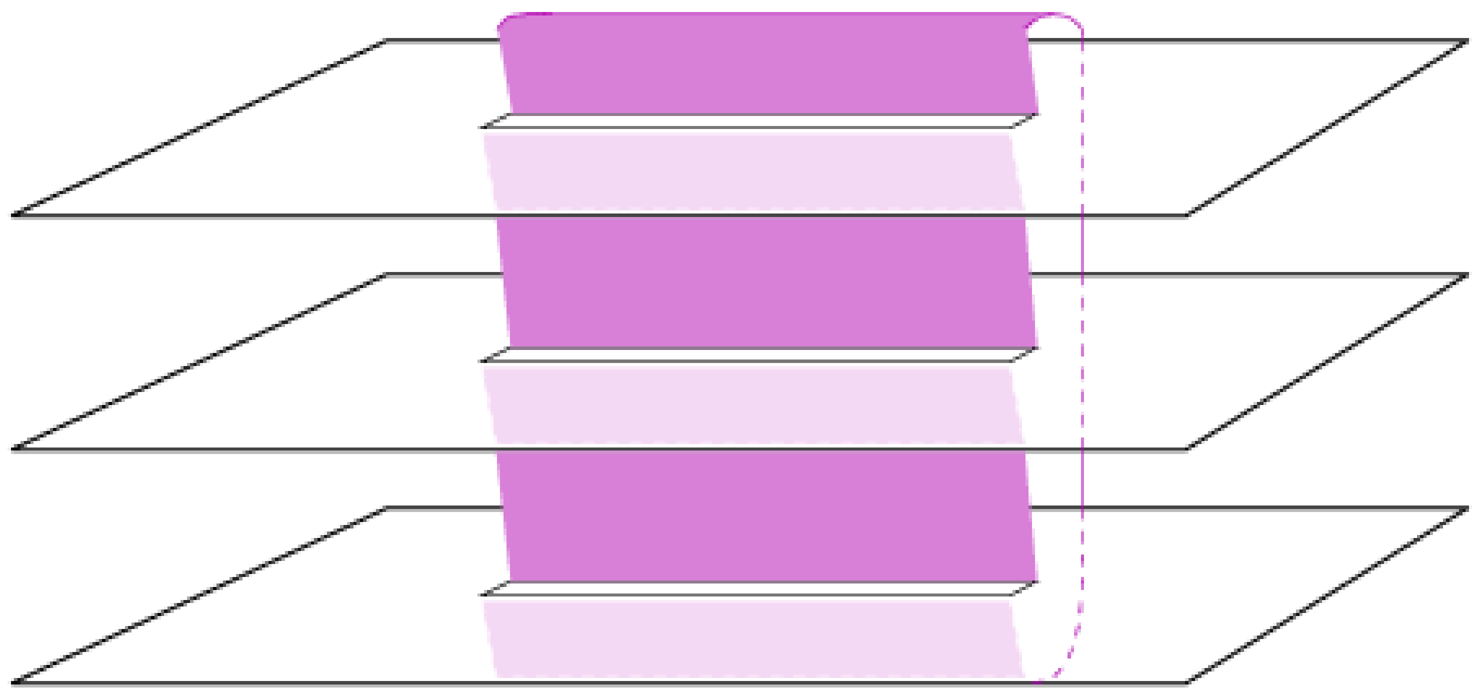
- **Possibility 2**: developing appropriate theoretical or numerical frameworks

Replica approach to the entanglement entropy

PC, J Cardy 2004

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho_A^n$$

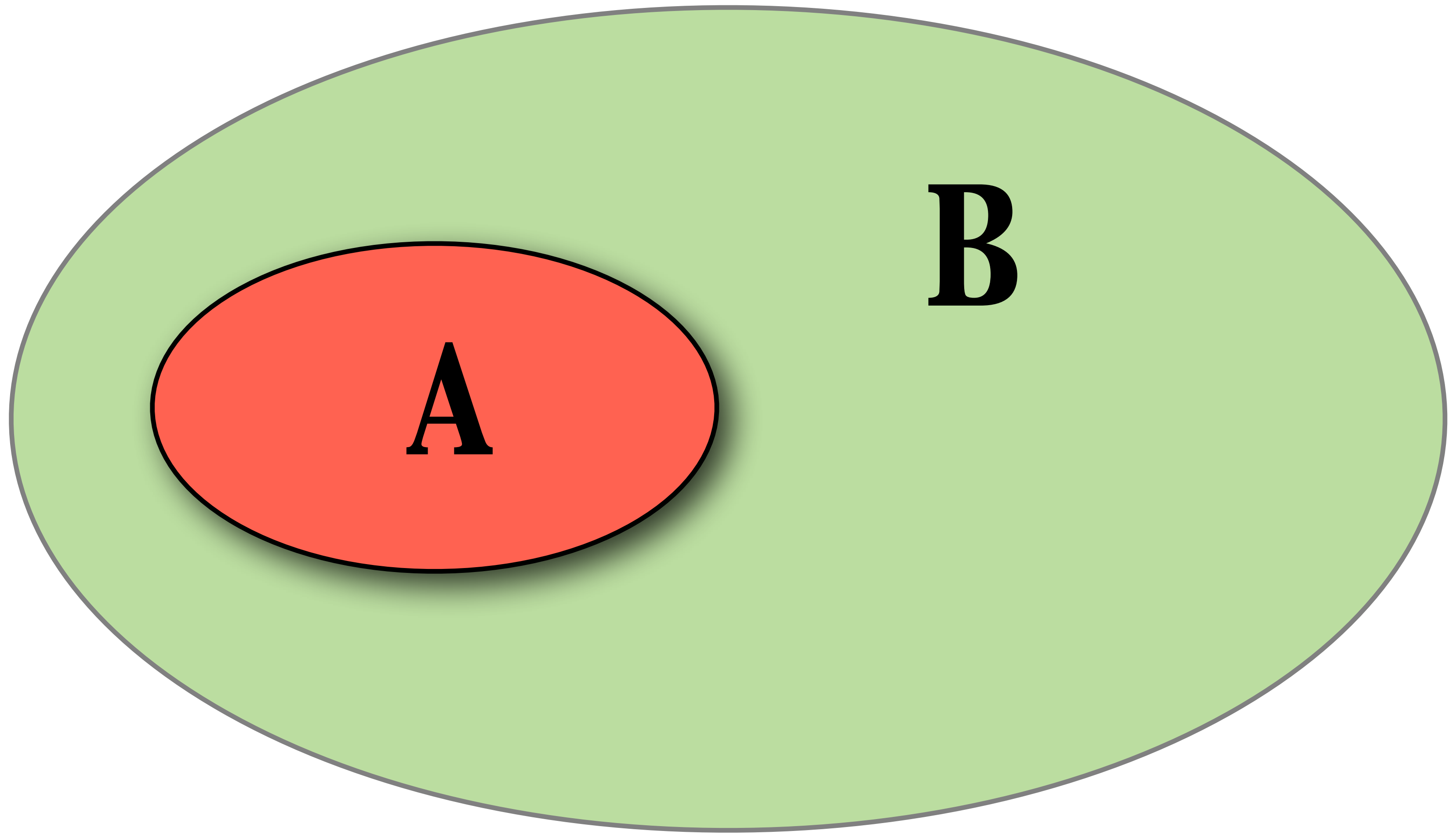
For n integer, the ground state $\mathrm{Tr}\,\rho_A^n$ is obtained by sewing cyclically n cut planes resulting in a partition function on a n-sheeted Riemann surface

$\mathrm{Tr}\rho_A^n =$ 

In (1+1)D, $\mathrm{Tr}\,\rho_A^n$ is equivalent to the 2-point function of some **twist fields**

$$\mathrm{Tr}\rho_A^n = \langle \mathcal{T}_n(u)\, \bar{\mathcal{T}}_n(v) \rangle$$

Entanglement in the ground state of physical systems



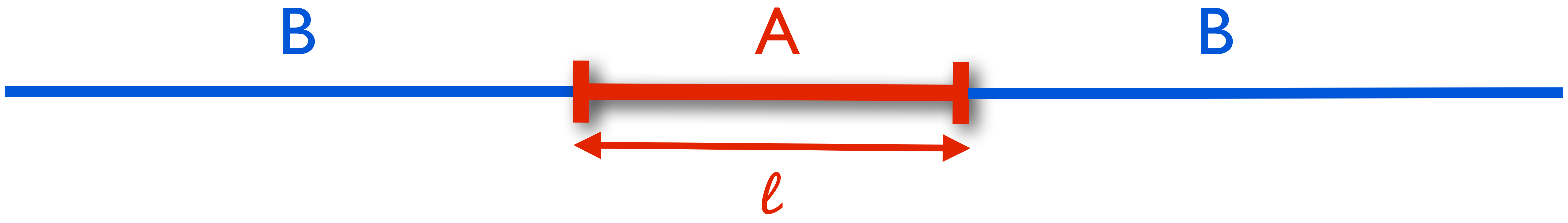
If $|\psi\rangle$ is the ground state of a local gapped Hamiltonian:

Area Law

$S_A \propto$ Area separating A and B

Srednicki ’93
+many more

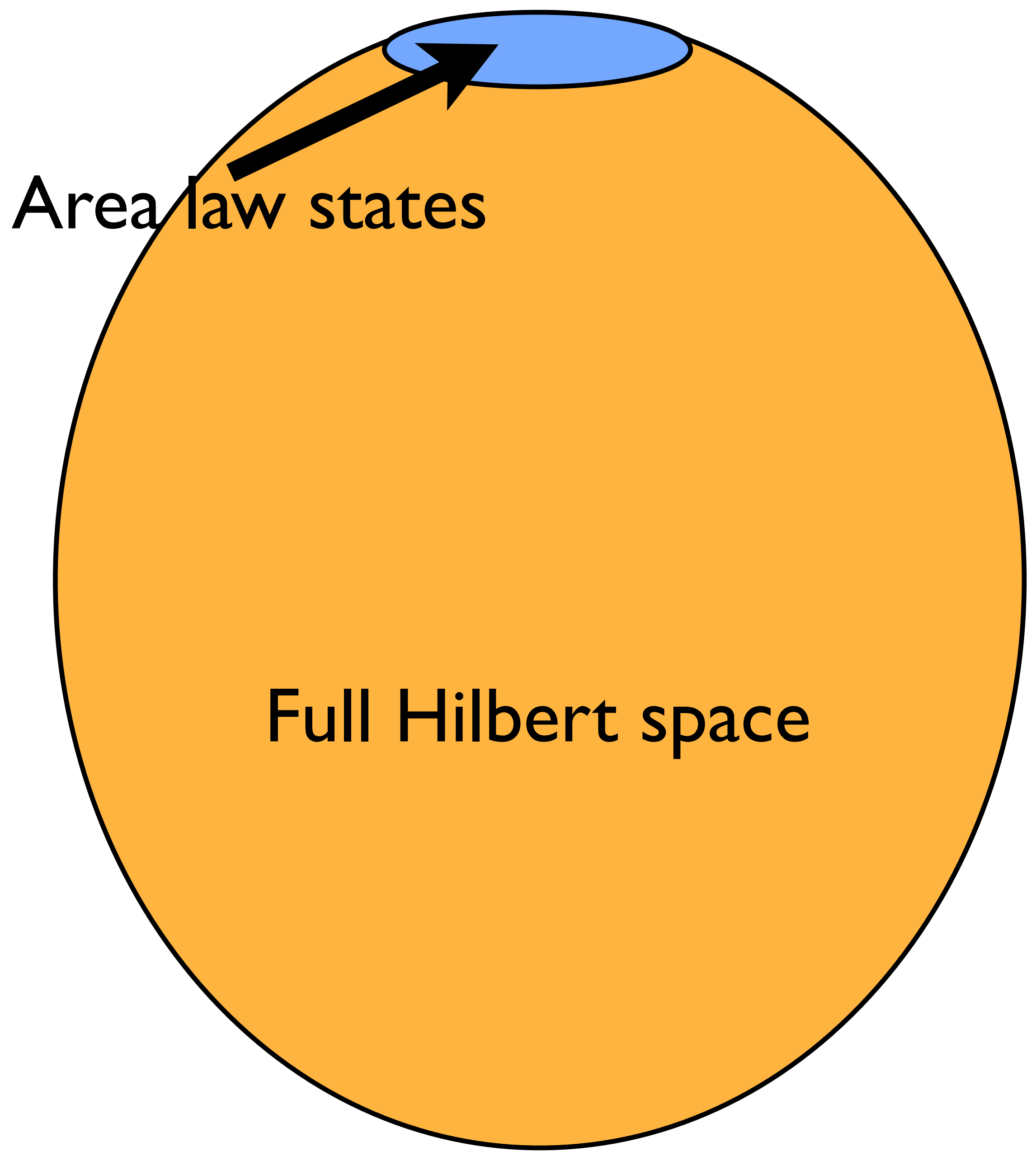
In a 1+1 D CFT Holzhey, Larsen, Wilczek ’94



$$\boxed{S_A = \frac{c}{3}\ln \ell}$$

This is the most effective way to determine the central charge

One application



Only a tiny fraction of states satisfy the area law (or small violations)

→ If we can limit the search for the ground state to this small subset, the complexity of the problem would be exponentially reduced

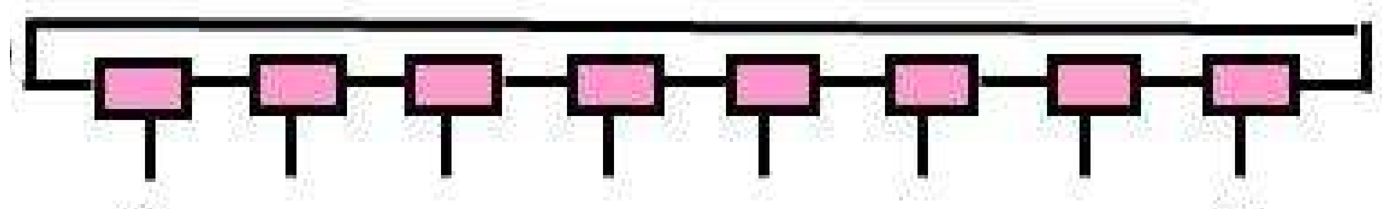
→ One meaning of S_A:

$\exp(S_A)$ gives the minimal amount of classical information required to specify $|\Psi\rangle$

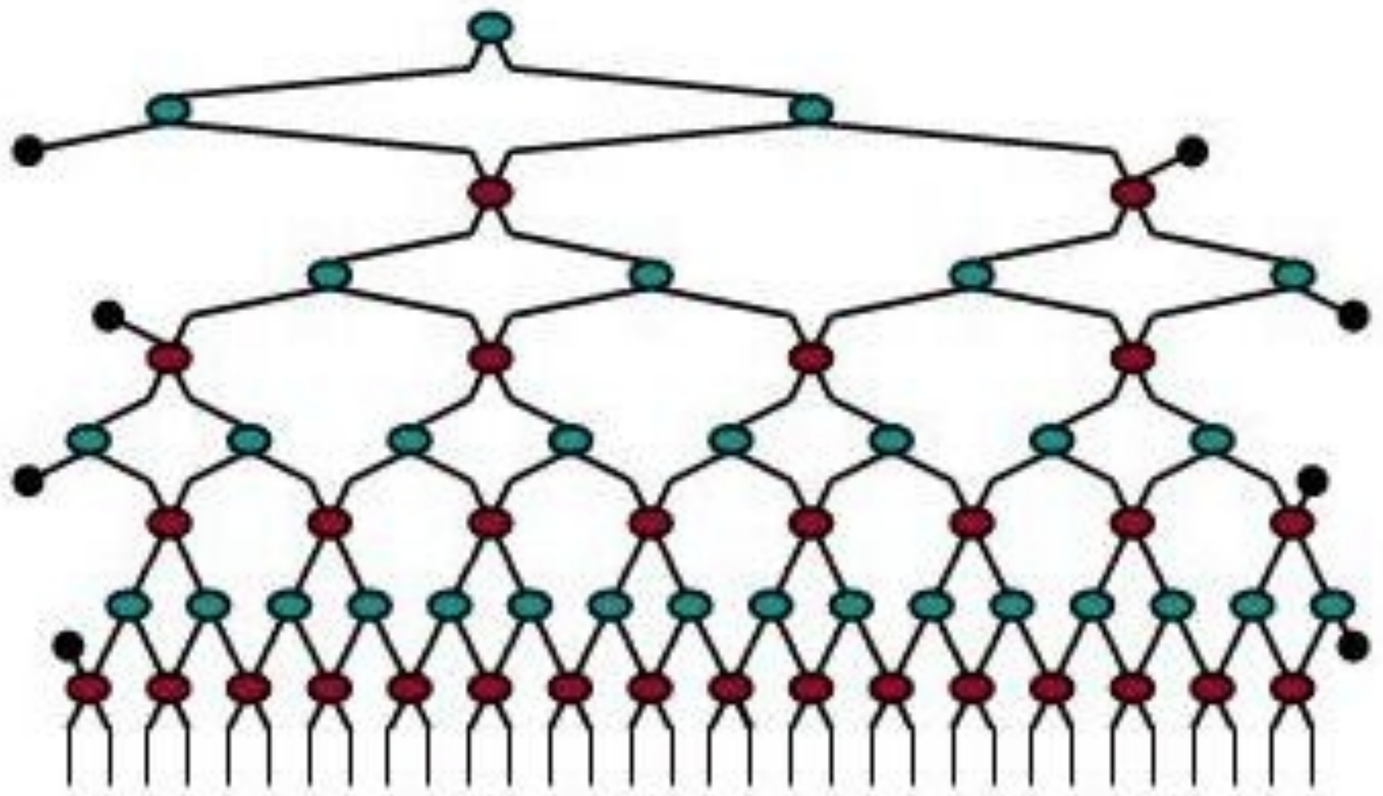
Tensor network states

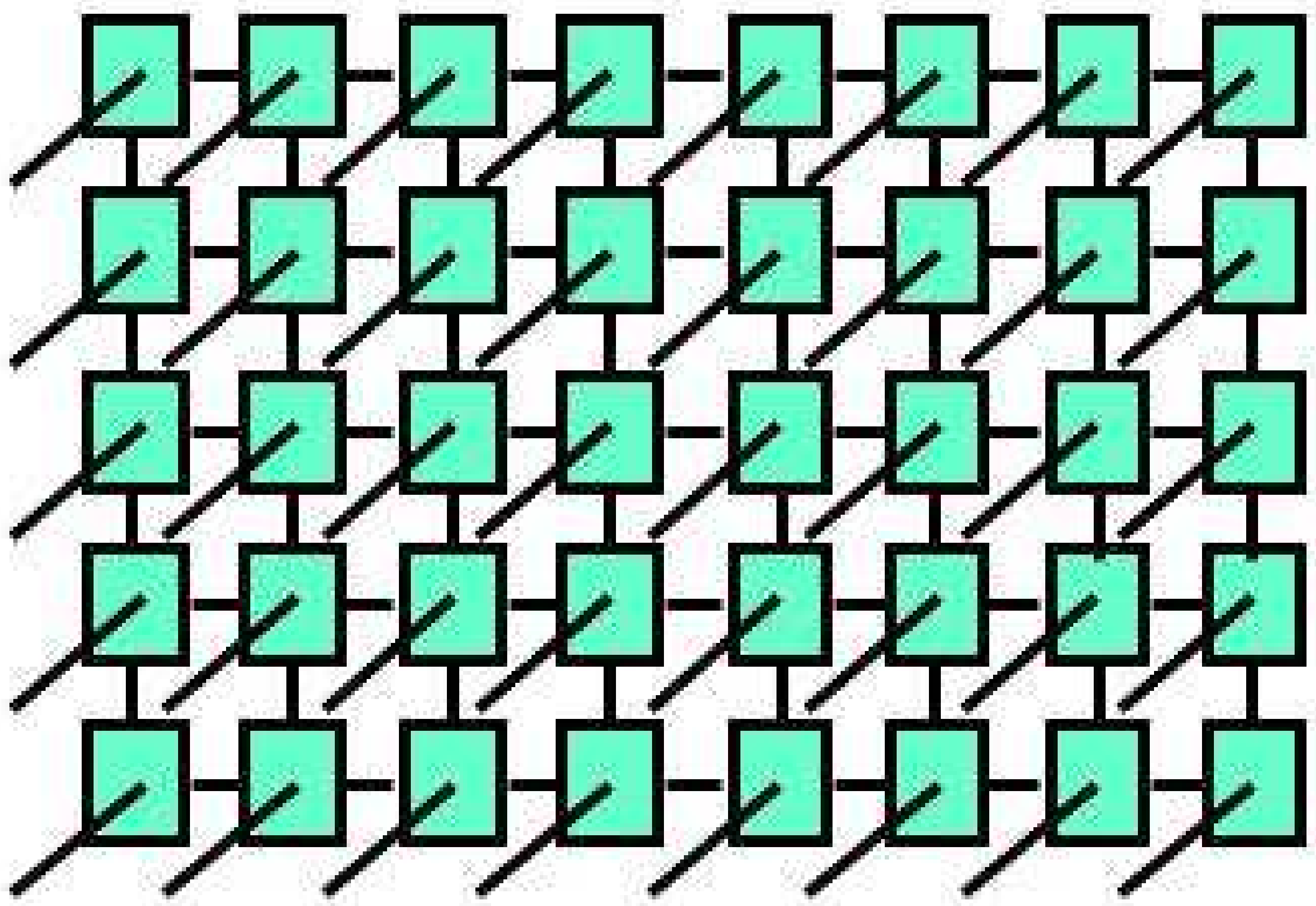
A powerful set of numerical methods based on entanglement content of quantum states

MPS



MERA





PEPS

"Alphabet soup of proposals"

Subir Sachdev

From CFT to cold atoms

ARTICLE

doi:10.1038/nature15750

Nature 528, 77 (2015)

Measuring entanglement entropy in a quantum many-body system

Rajibul Islam[1], Ruichao Ma[1], Philipp M. Preiss[1], M. Eric Tai[1], Alexander Lukin[1], Matthew Rispoli[1] & Markus Greiner[1]

Entanglement is one of the most intriguing features of quantum mechanics. It describes non-local correlations between quantum objects, and is at the heart of quantum information sciences. Entanglement is now being studied in diverse fields ranging from condensed matter to quantum gravity. However, measuring entanglement remains a challenge. This is especially so in systems of interacting delocalized particles, for which a direct experimental measurement of spatial entanglement has been elusive. Here, we measure entanglement in such a system of itinerant particles using quantum interference of many-body twins. Making use of our single-site-resolved control of ultracold bosonic atoms in optical lattices, we prepare two identical copies of a many-body state and interfere them. This enables us to directly measure quantum purity, Rényi entanglement entropy, and mutual information. These experiments pave the way for using entanglement to characterize quantum phases and dynamics of strongly correlated many-body systems.

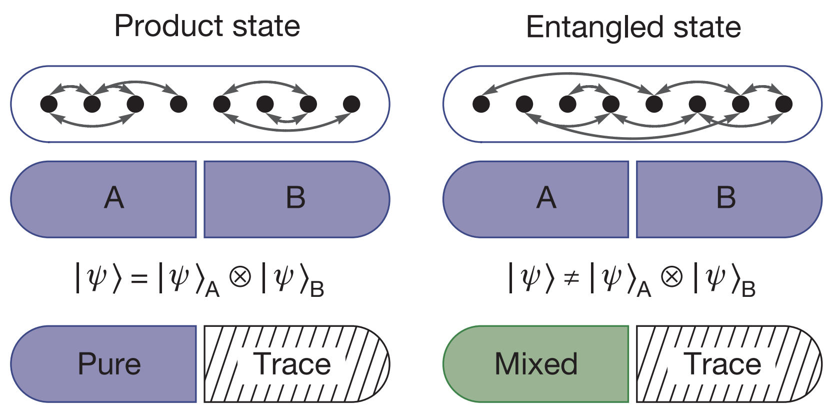


Figure 1 | Bipartite entanglement and partial measurements. A generic pure quantum many-body state has quantum correlations (shown as arrows) between different parts. If the system is divided into two subsystems A and B, the subsystems will be bipartite entangled with each other when there are quantum correlations between them (right column). Only when there is no bipartite entanglement present, the partitioned system $|\psi_{AB}\rangle$ can be described as a product of subsystem states $|\psi_A\rangle$ and $|\psi_B\rangle$ (left column). A path for measuring the bipartite entanglement emerges from the concept of partial measurements: ignoring all information about subsystem B (indicated as 'Trace') will put subsystem A into a statistical mixture, to a degree given by the amount of bipartite entanglement present. Finding ways of measuring the many-body quantum state purity of the system and comparing that of its subsystems would then enable measurements of entanglement. For an entangled state, the subsystems will have less purity than the full system.

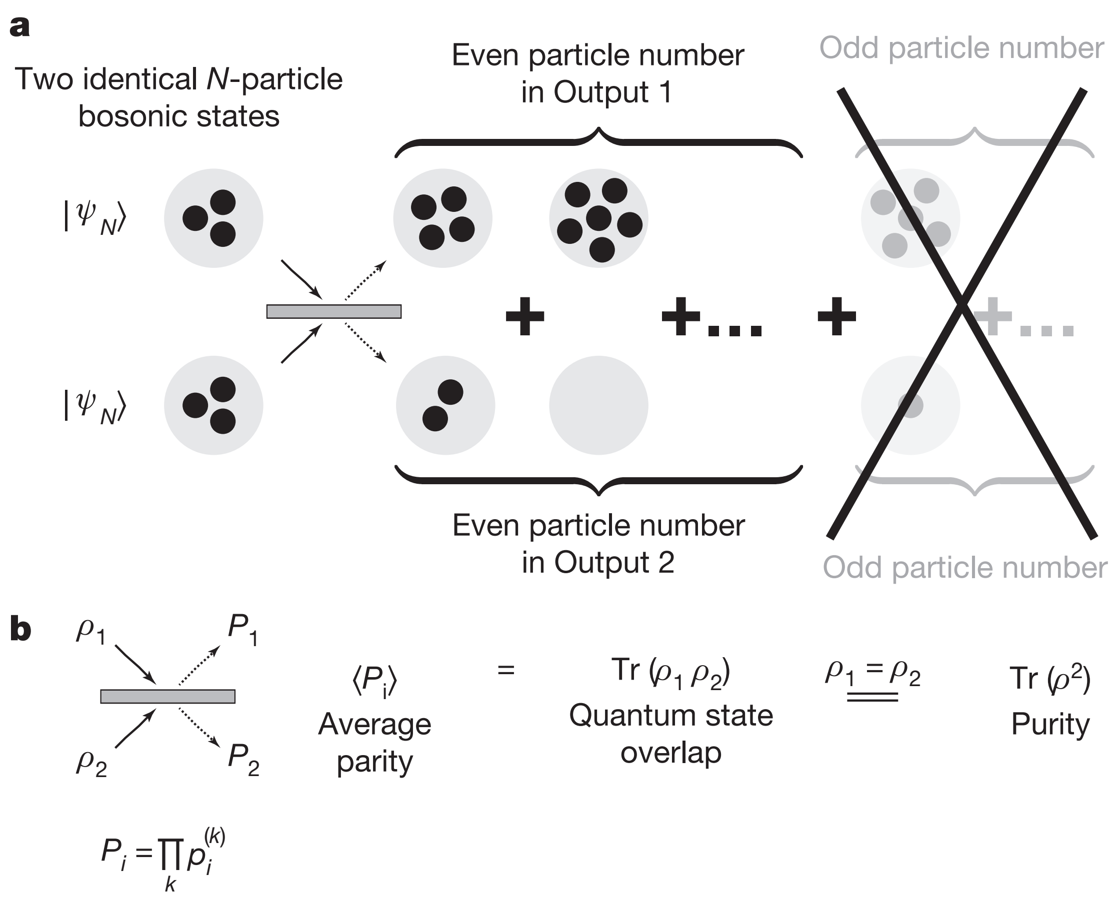
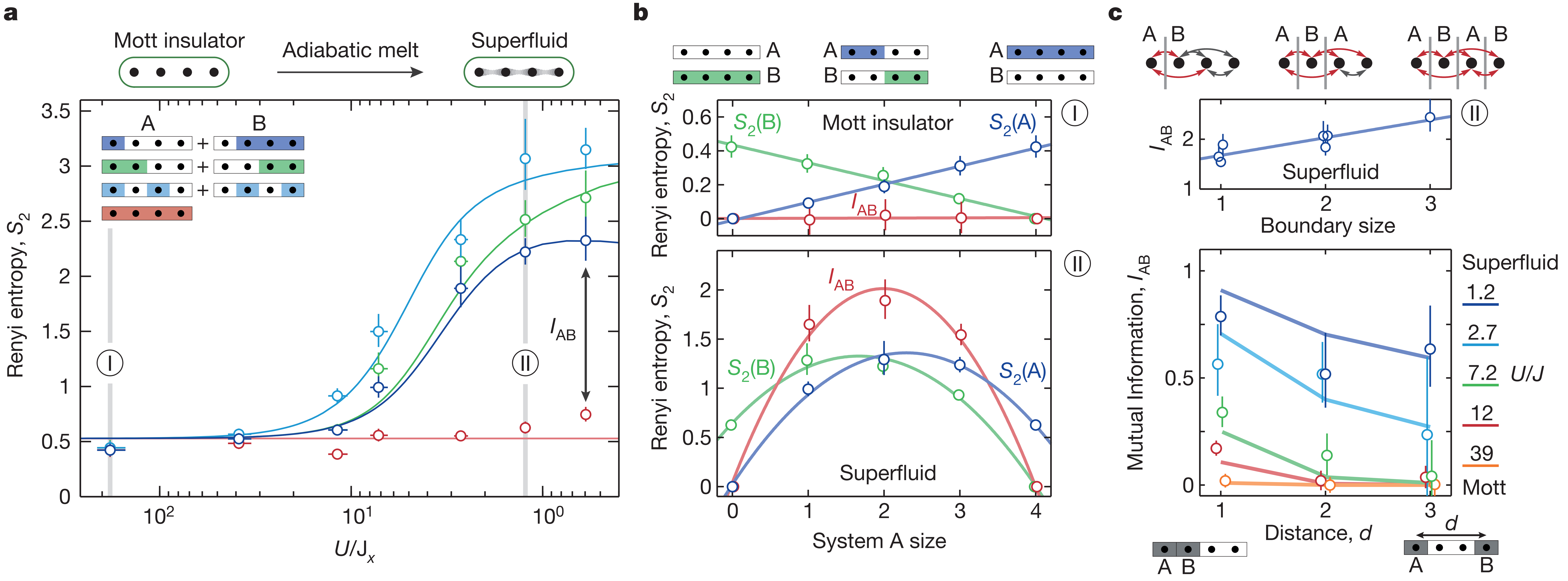


Figure 2 | Measurement of quantum purity with many-body bosonic interference of quantum twins. a, When two N-particle bosonic systems that are in identical pure quantum states are interfered on a 50%–50% beam splitter, they always produce output states with an even number

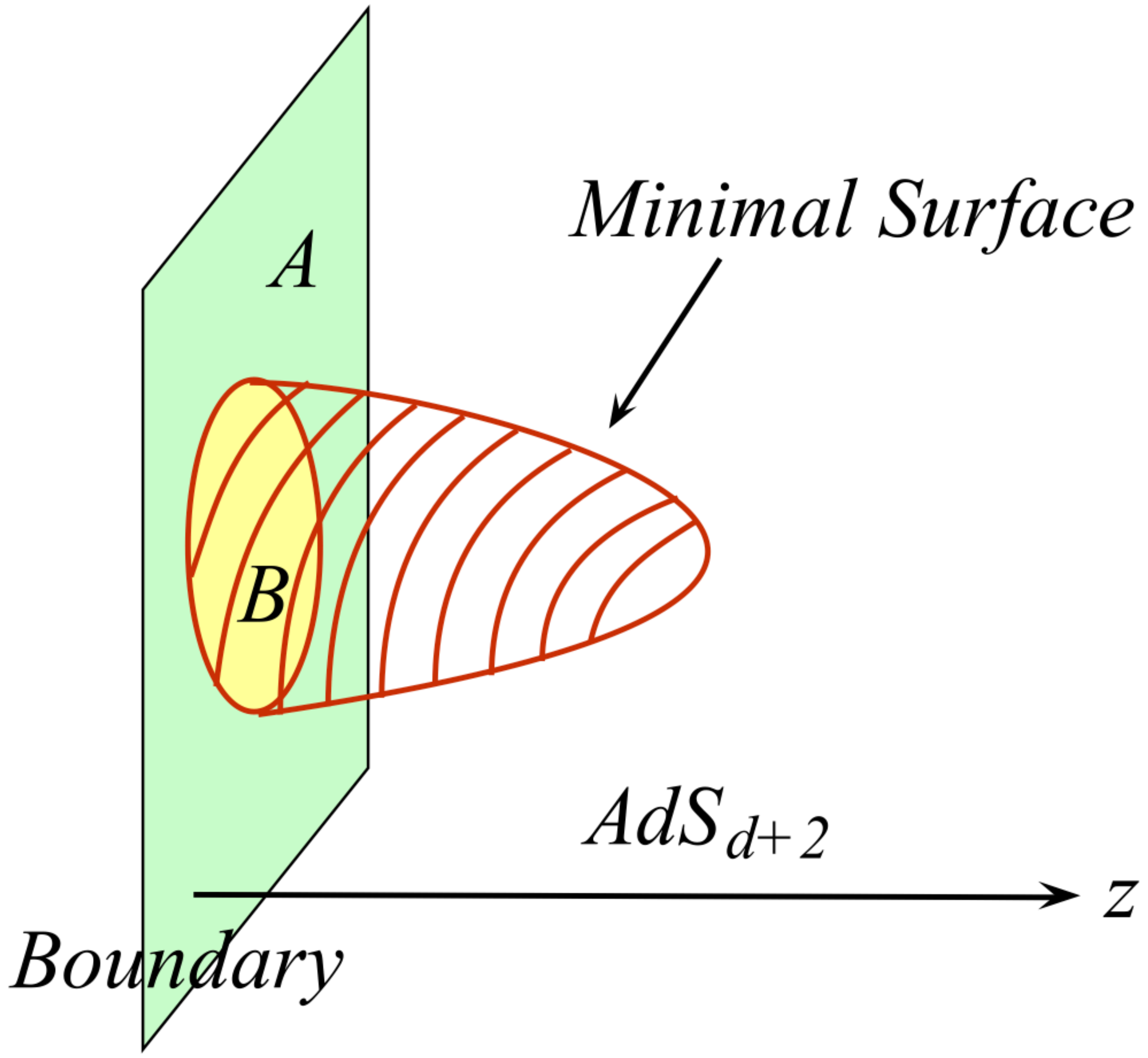
Renyi entanglement entropies $S_n = \dfrac{1}{1-n}\mathrm{Tr}\,\rho_A^n$

From CFT to cold atoms



Holographic entanglement entropy

In the framework of the AdS/CFT correspondence, Ryu and Takayanagi (2006) proposed the nowadays extremely famous formula



$$S_A = \frac{\text{Area of } \gamma_A}{4G_N^{(d+2)}},$$

where γ_A is the d-dimensional static minimal surface in AdS_{d+2} whose boundary is given by ∂A

General Relativity and Gravitation (2011)

Mark Van Raamsdonk

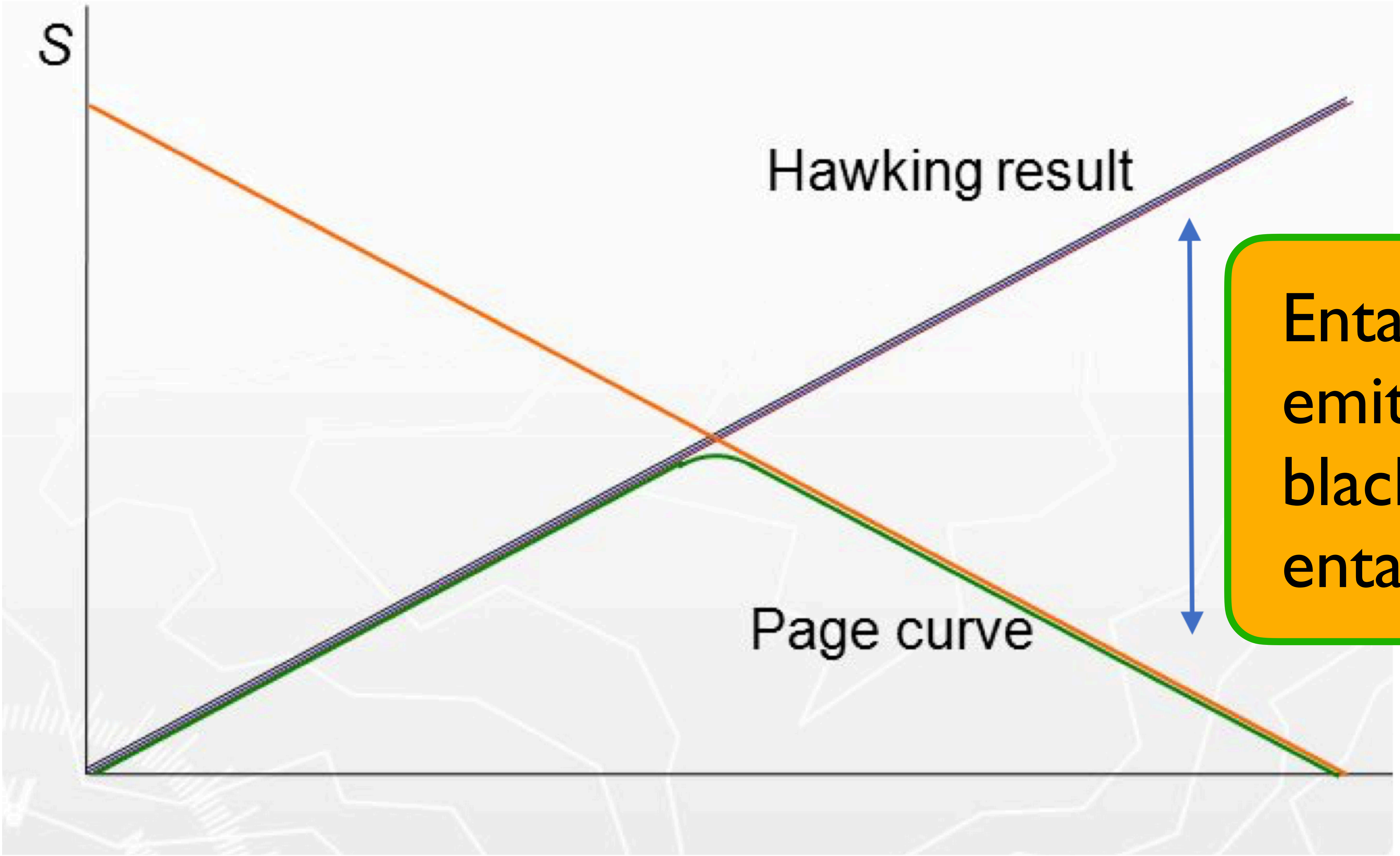
Building up spacetime with quantum entanglement

Abstract In this essay, we argue that the emergence of classically connected spacetimes is intimately related to the quantum entanglement of degrees of freedom in a non-perturbative description of quantum gravity. Disentangling the degrees of freedom associated with two regions of spacetime results in these regions pulling apart and pinching off from each other in a way that can be quantified by standard measures of entanglement.

Keywords AdS/CFT, Emergent spacetime, Quantum entanglement

Black hole information paradox and Page curve

As a block hole evaporates, it emits radiation which is entangled with the back hole interior. A semiclassical calculation by Hawking predicts to a liner increase of the entropy, leading to the information paradox when the black hole fully evaporated



Entangled particles are emitted as radiation by the black hole, causing the entanglement reduction

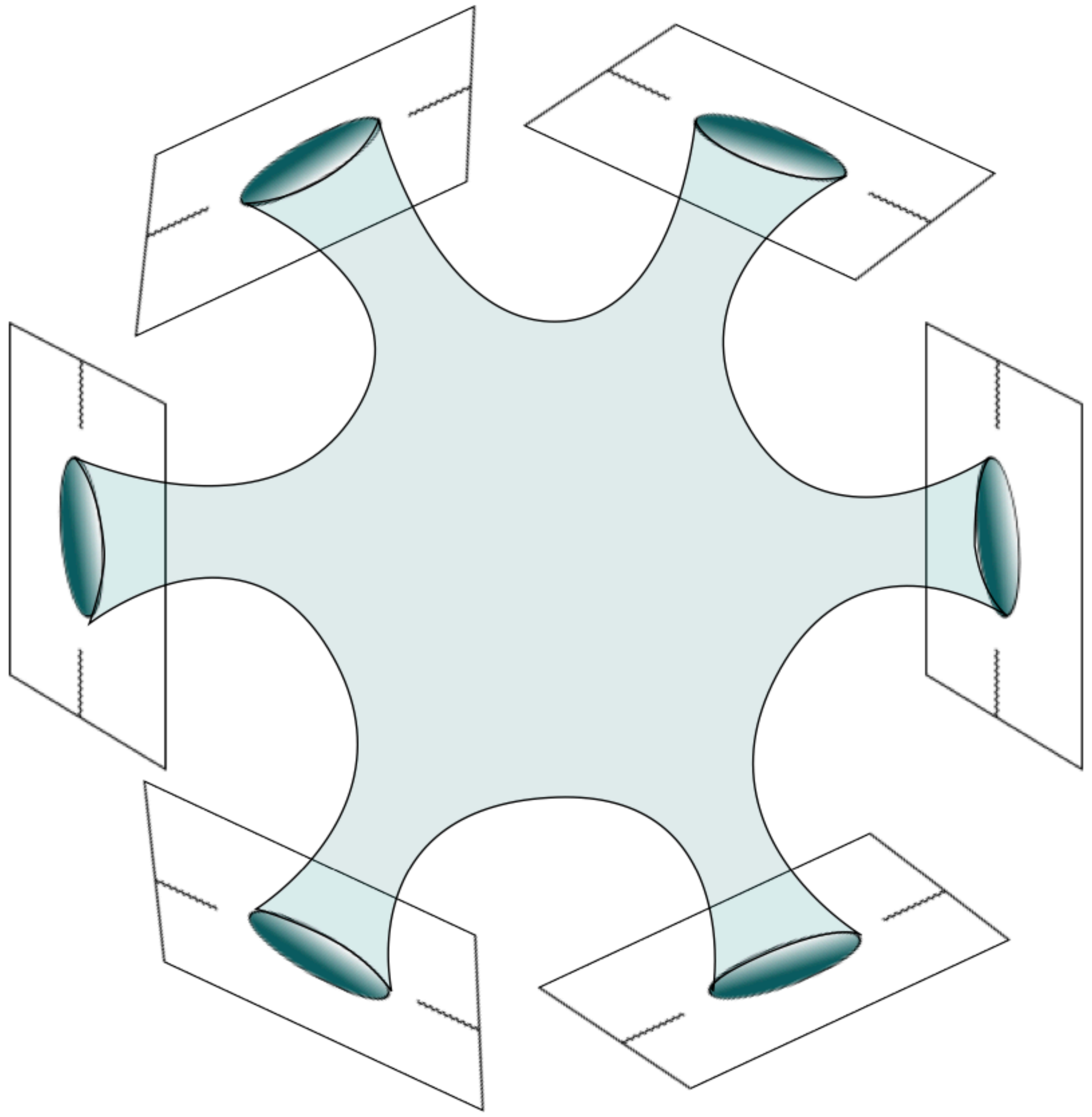
Black hole information paradox and replica wormholes

JHEP Published for SISSA by Springer

Replica wormholes and the entropy of Hawking radiation

Ahmed Almheiri,[a] **Thomas Hartman,**[b] **Juan Maldacena,**
and Amirhossein Tajdini[b]

ABSTRACT: The information paradox can be realized in anti
a Minkowski region. In this setting, we show that the large
Neumann entropy as calculated by Hawking and the require
including new saddles in the gravitational path integral. The
method as complexified wormholes connecting different cop
replica number $n \to 1$, the presence of these wormholes le
computation of the fine-grained gravitational entropy. We di
explicitly in two-dimensional Jackiw-Teitelboim gravity coup



JHEP Published for SISSA by Springer

Replica wormholes and the black hole interior

Shenker, Douglas Stanford and Zhenbin Yang

own how to obtain the Page curve of an evaporating
utations of entanglement entropy. We show how these
ng the replica trick, from geometries with a spacetime
eplicas. In a simple model, we study the Page transition
netries with different topologies. We compute related
complicated models, including JT gravity coupled to
odel. Separately, we give a direct gravitational argument
ction using an explicit formula known as the Petz map;
an important role. We discuss an interpretation of the
me ensemble average implicit in the gravity description.

Topological entanglement entropy

A (non-local) order parameter for topological phases

Kitaev & Preskill 2006; Levin & Wen, 2006

Topological entropy in physics

From Wikipedia, the free encyclopedia

For the mathematical concept in ergodic theory, see topological entropy.

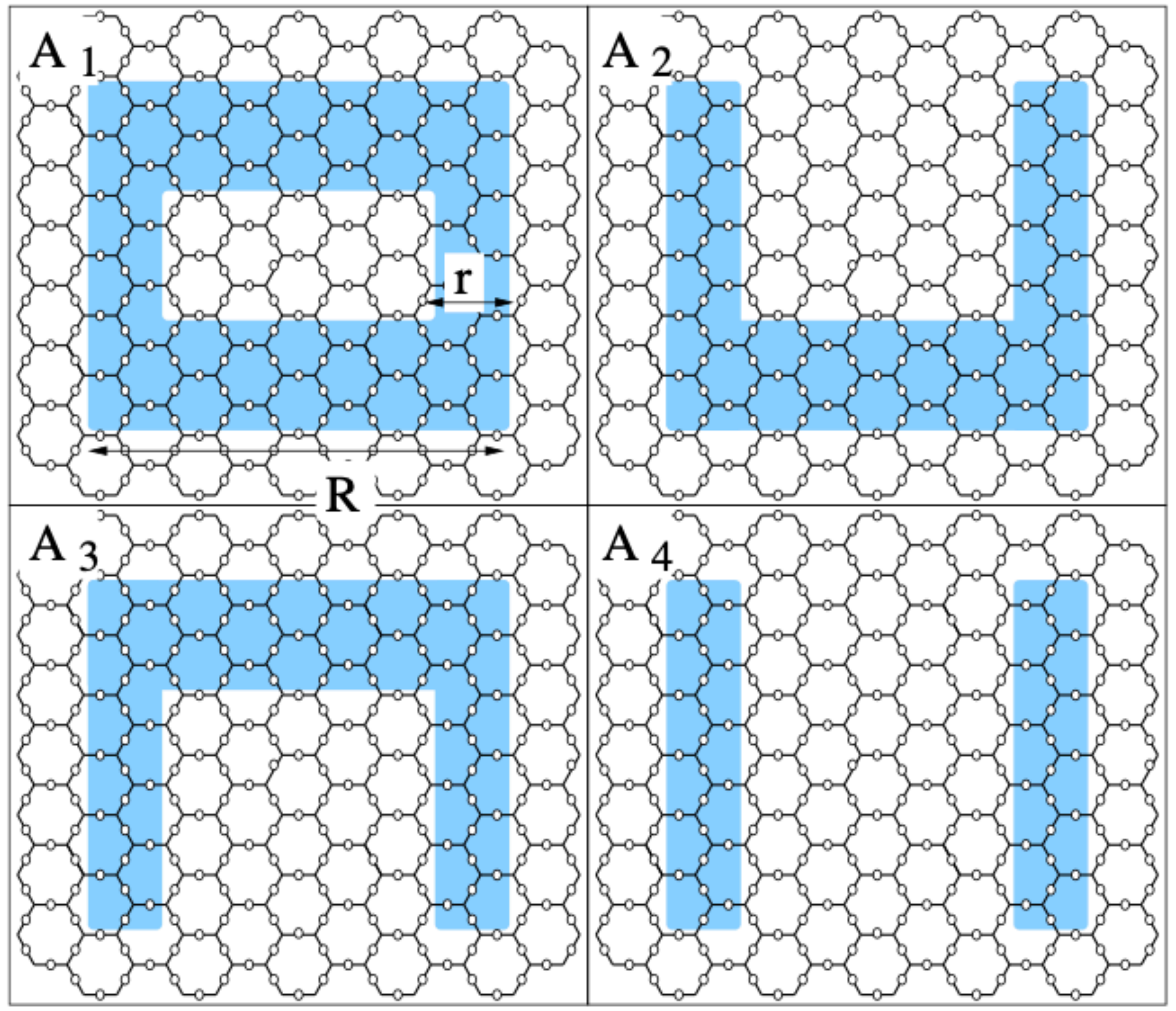
The **topological entanglement entropy**[1][2][3] or *topological entropy*, usually denoted by γ, is a number characterizing
order.

A non-zero topological entanglement entropy reflects the presence of long range quantum entanglements in a many-boc
entanglement entropy links topological order with pattern of long range quantum entanglements.

Given a topologically ordered state, the topological entropy can be extracted from the asymptotic behavior of the Von Ne
entanglement between a spatial block and the rest of the system. The entanglement entropy of a simply connected regic
dimensional topologically ordered state, has the following form for large L:

$$S_L \longrightarrow \alpha L - \gamma + \mathcal{O}(L^{-\nu})\,, \qquad \nu > 0$$

where $-\gamma$ is the topological entanglement entropy.



The topological entanglement entropy is equal to the logarithm of the total quantum dimension of the quasiparticle excitations of the state.

For example, the simplest fractional quantum Hall states, the Laughlin states at filling fraction $1/m$, have $\gamma = ½\log(m)$. The Z_2 fractionalized states, such as topologically ordered states of Z_2 spin-liquid, quantum dimer models on non-bipartite lattices, and Kitaev's toric code state, are characterized $\gamma = \log(2)$.

Topological order and entanglement spectrum

PRL **101,** 010504 (2008) PHYSICAL REVIEW LETTERS week ending 4 JULY 2008

Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States

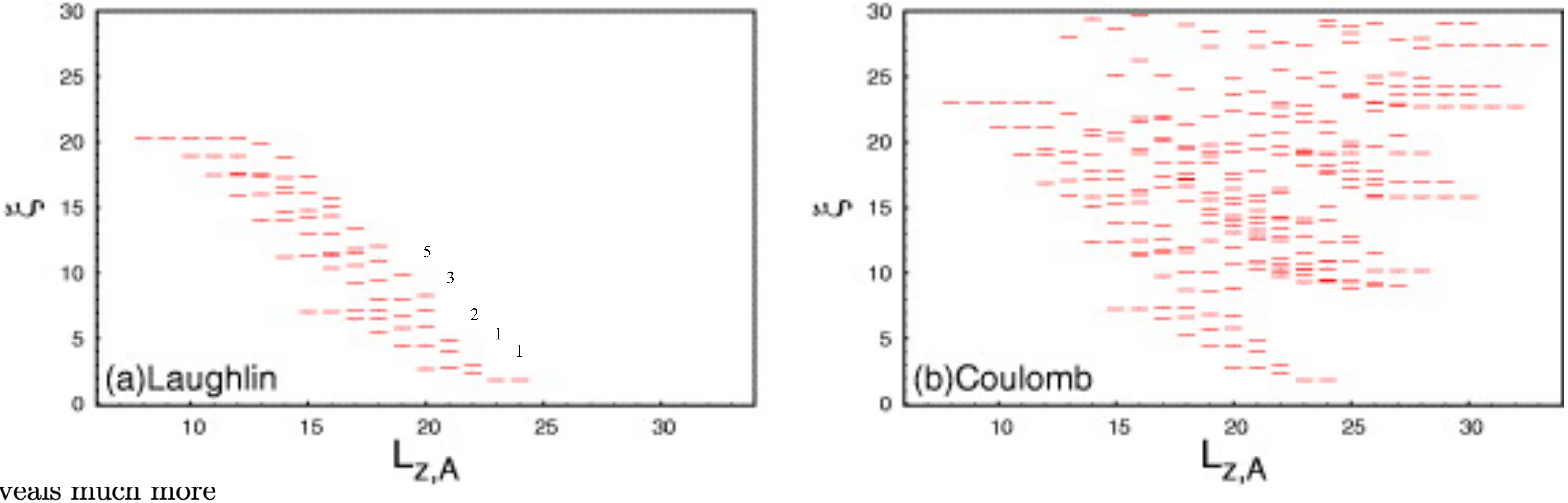
Hui Li and F. D. M. Haldane

Physics Department, Princeton University, Princeton, New Jersey 08544, USA

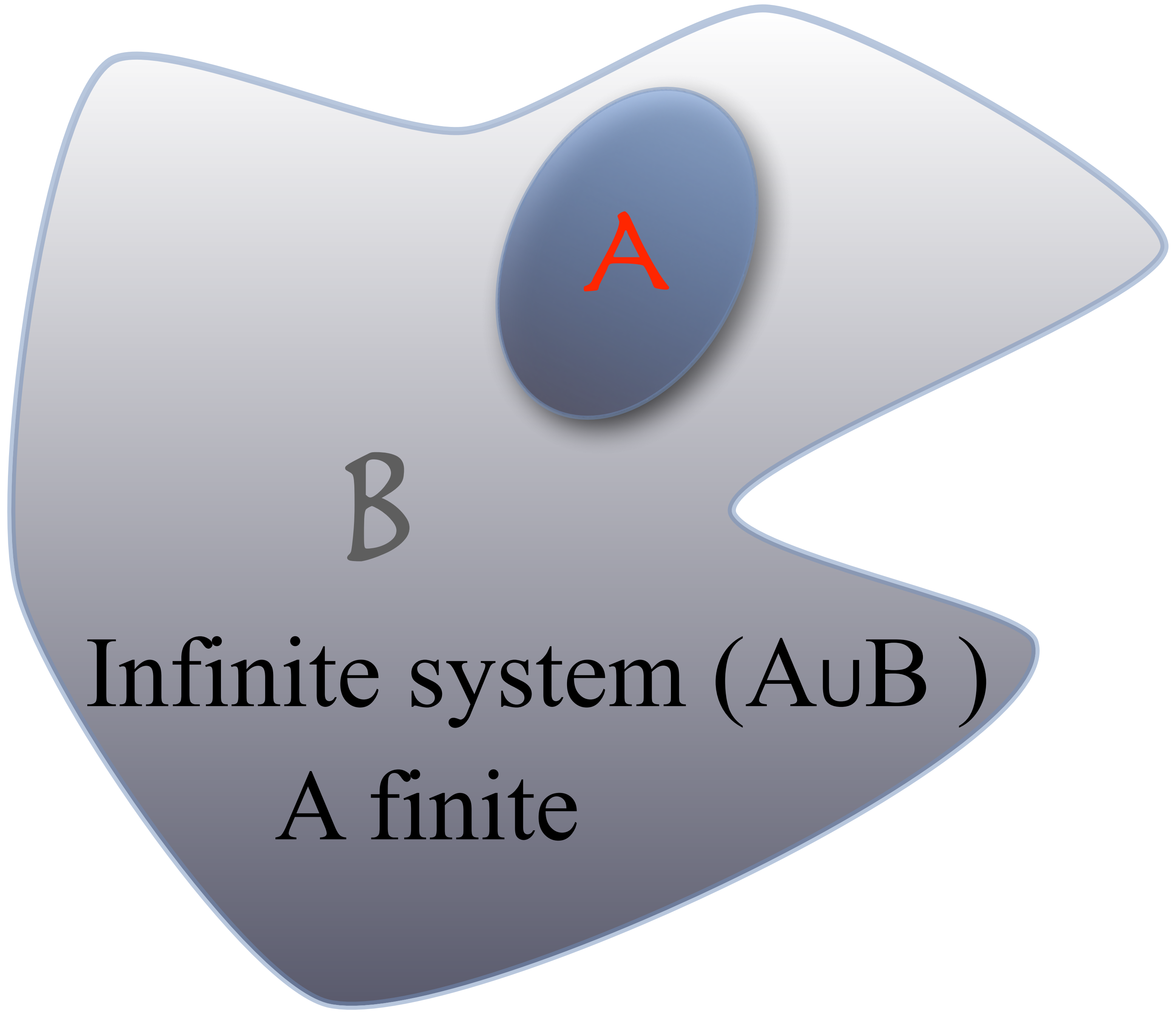
(Received 2 May 2008; published 3 July 2008)

We study the "entanglement spectrum" (a presentation of the Schmidt decomposition analogous to a set of "energy levels") of a many-body state, and compare the Moore-Read model wave function for the $\nu = 5/2$ fractional quantum Hall state with a generic 5/2 state obtained by finite-size diagonalization of the second-Landau-level-projected Coulomb interactions. Their spectra share a common "gapless" structure, related to conformal field theory. In the model state, these are the *only* levels, while in the "generic" case, they are separated from th
appears to remain finite in the thermody
spectrum can be used as a "fingerprint" to

value spectrum. The density matrix ma
the form $\hat{\rho} = \exp(-\hat{H})$, so that the entan
is equivalent to the thermodynamic entr
described by a hermitian "Hamiltonian"
ature" $T = 1$; in the case of a weak en
"excited states" eigenvalues of $\hat{H}$ are sep
ground state eigenvalue by a large "ener
comes infinite in the limit of a simple pr
vanishing entanglement entropy. In this
out that the spectrum of the "Hamilto
we call the *"entanglement spectrum"*, reveals much more complete information than the entanglement entropy, a single number.



Entanglement & thermodynamics in non-equilibrium systems



- prepare a many-body quantum system in a pure state $|\Psi_0\rangle$ that is not an eigenstate of the Hamiltonian

$$|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$$

- Reduced density matrix: $\rho_A(t) = \mathrm{Tr}_B\, \rho(t)$
- $S_A(t) = -\mathrm{Tr}[\rho_A(t) \ln \rho_A(t)]$
- Stationary state: if exists the limit

$$\lim_{t\to\infty} \rho_A(t) = \rho_A(\infty)$$

Thermalization:

Consider the Gibbs ensemble for the entire system $\rho_T = e^{-\beta H}/Z$

The system thermalizes if $\boxed{\rho_{A,T} = \rho_A(\infty)}$

Entanglement vs Thermodynamics

The equivalence of reduced density matrices

$$\rho_{\mathrm{A,TD}} = \rho_{\mathrm{A}}(\infty)$$

Implies that the subsystem's entropies are the same: $\mathrm{S}_{\mathrm{A,TD}} = S_{\mathrm{A}}(\infty)$

The TD entropy S_{TD}=-Tr $\rho_{\mathrm{TD}} \ln \rho_{\mathrm{TD}}$ is extensive

$$\boxed{\frac{\mathrm{S}_{\mathrm{TD}}}{\mathrm{V}} \simeq \frac{\mathrm{S}_{\mathrm{A,TD}}}{\mathrm{V}_{\mathrm{A}}} = \frac{S_{\mathrm{A}}(\infty)}{\mathrm{V}_{\mathrm{A}}}}$$

For large time the entanglement entropy becomes thermodynamic entropy

The **entropy** of the stationary state is just the **entanglement** accumulated during time

Quantum thermalization through entanglement in an isolated many-body system

Adam M. Kaufman, M. Eric Tai, Alexander Lukin, Matthew Rispoli, Robert Schittko, Philipp M. Preiss, Markus Greiner*

Science 353, 794 (2016)

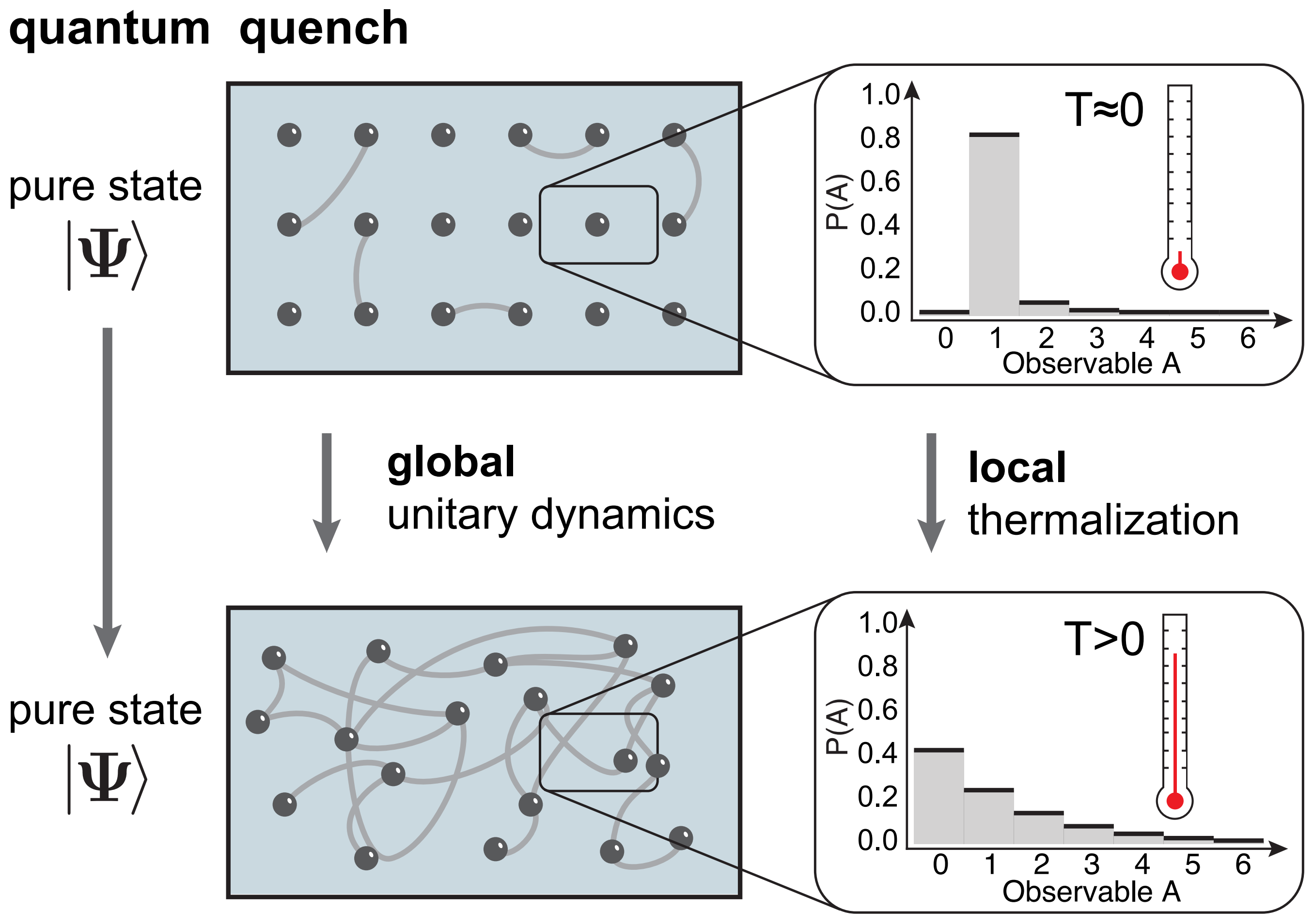
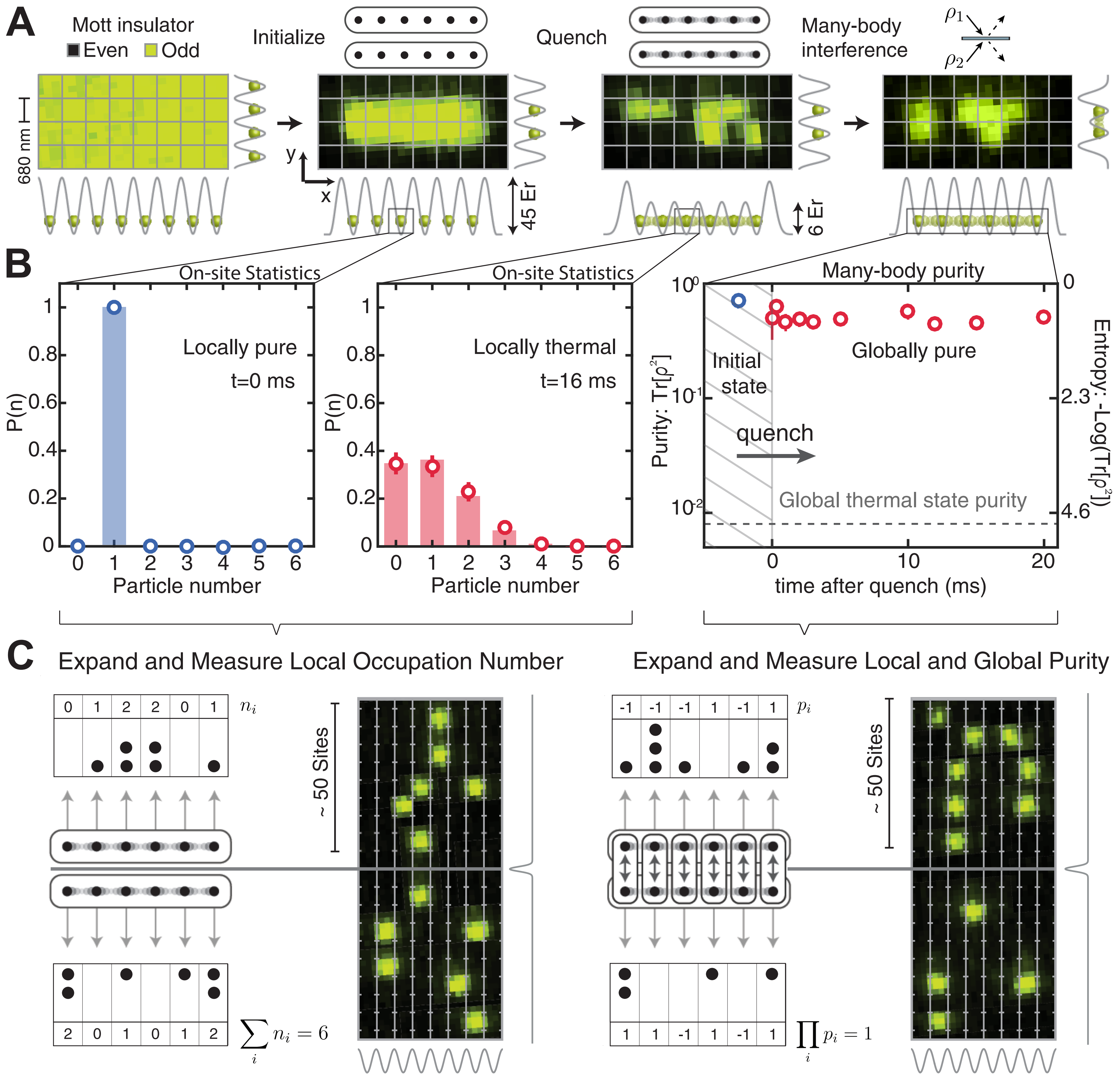
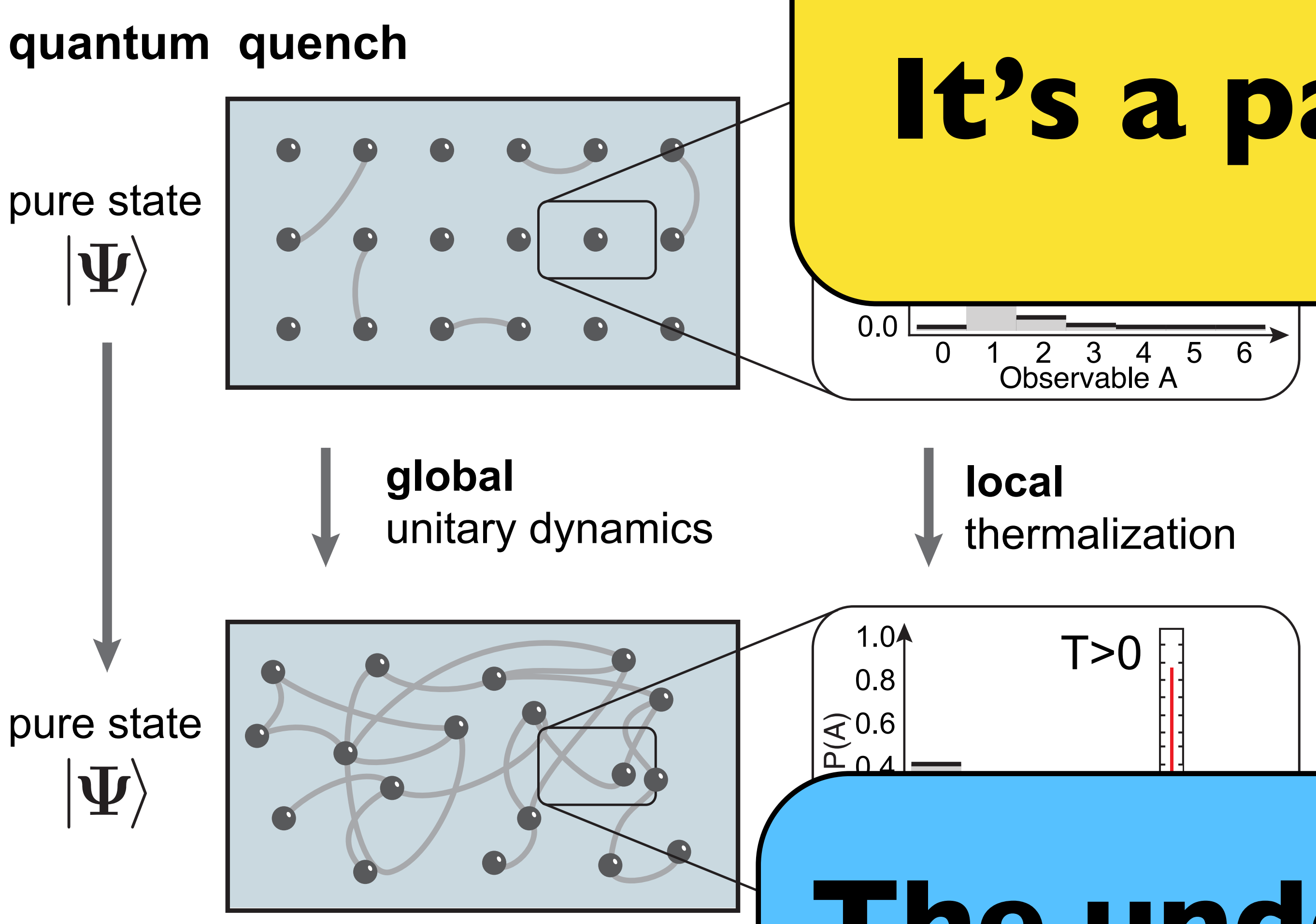


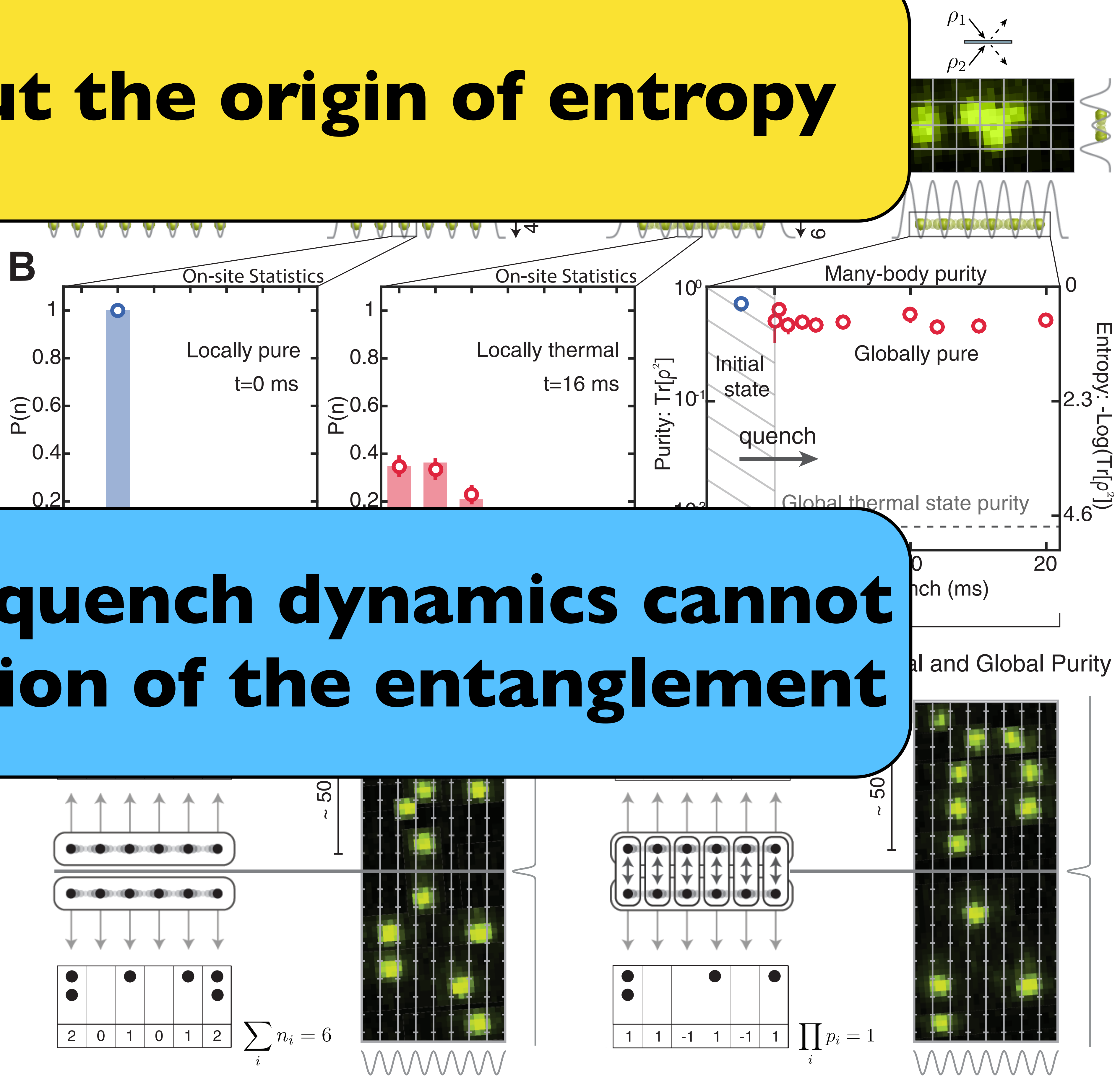
FIG. 1. **Schematic of thermalization dynamics in closed systems**. An isolated quantum system at zero temperature can be described by a single pure wavefunction $|\Psi\rangle$. Subsystems of the full quantum state appear pure, as long as the entanglement (indicated by grey lines) between subsystems is negligible. If suddenly perturbed, the full system evolves unitarily, developing significant entanglement between all parts of the system. While the full system remains in a pure, zero-entropy state, the entropy of entanglement causes the subsystems to equilibrate, and local, thermal mixed states appear to emerge within a globally pure quantum state.



Quantum thermalization through entanglement in an isolated many-body system

Adam M. Kaufman, M. Eric Tai, Alexander Lukin, Matthew Rispoli, Robert Schittko, Philipp M. Preiss, Markus Greiner*





It's a paradigm shift about the origin of entropy

The understanding of the quench dynamics cannot prescind the characterisation of the entanglement

FIG. 1. **Schematic of therm… closed systems**. An isolated qua… perature can be described by a sing… Subsystems of the full quantum state appear pure, as long as the entanglement (indicated by grey lines) between subsystems is negligible. If suddenly perturbed, the full system evolves unitarily, developing significant entanglement between all parts of the system. While the full system remains in a pure, zero-entropy state, the entropy of entanglement causes the subsystems to equilibrate, and local, thermal mixed states appear to emerge within a globally pure quantum state.

Kaufmann et al 2016

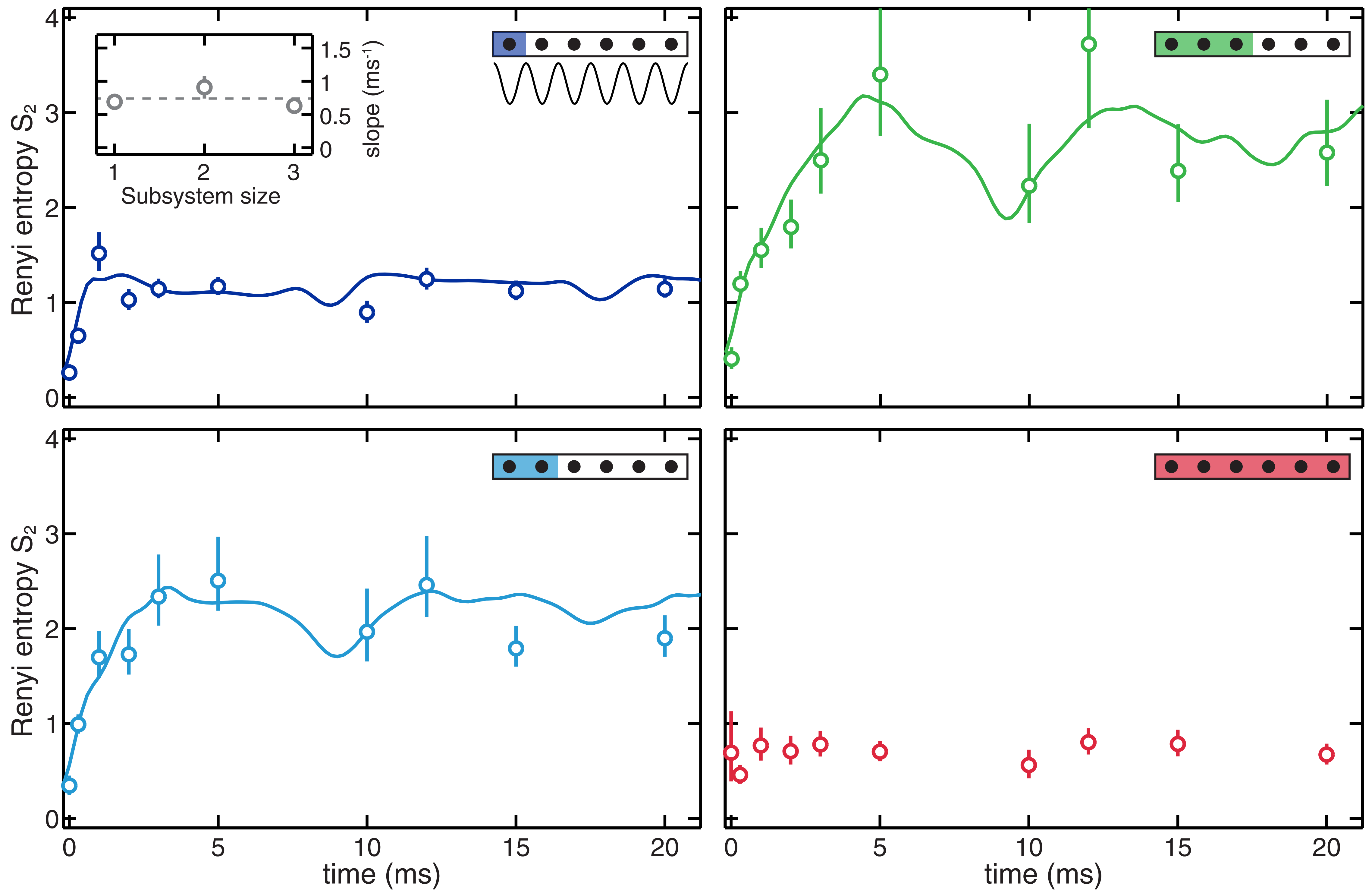
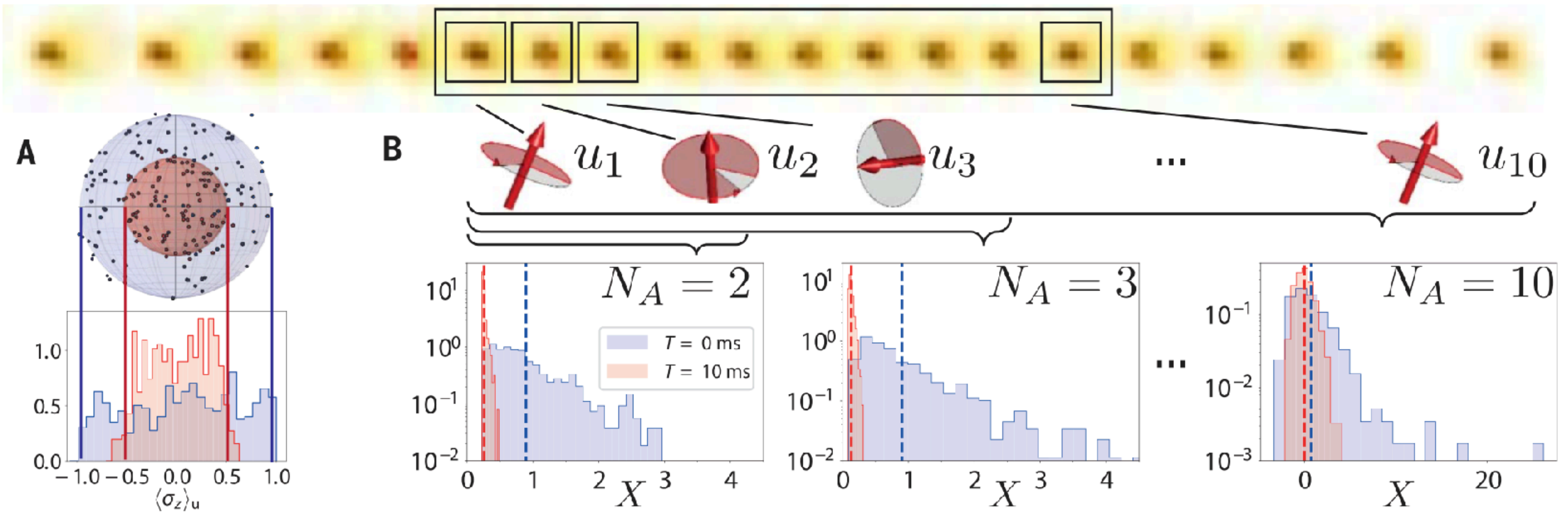


FIG. 3. **Dynamics of entanglement entropy.** Starting from a low-entanglement ground state, a global quantum quench leads to the development of large-scale entanglement between all subsystems. We quench a six-site system from the Mott insulating product state ($J/U \ll 1$) with one atom per site to the weakly interacting regime of $J/U = 0.64$ and measure the dynamics of the entanglement entropy. As it equilibrates, the system acquires local entropy while the full system entropy remains constant and at a value given by measurement imperfections. The dynamics agree with exact numerical simulations with no free parameters (solid lines). Error bars are the standard error of the mean (S.E.M.). For the largest entropies encountered in the three-site system, the large number of populated microstates leads to a significant statistical uncertainty in the entropy, which is reflected in the upper error bar extending to large entropies or being unbounded. Inset: slope of the early time dynamics, extracted with a piecewise linear fit (see Supplementary Material). The dashed line is the mean of these measurements.

Another experiment (ion traps)



QUANTUM ENTANGLEMENT

Probing Rényi entanglement entropy via randomized measurements

Tiff Brydges[1,2]*, Andreas Elben[1,2]*, Petar Jurcevic[1,2], Benoît Vermersch[1,2], Christine Maier[1,2], Ben P. Lanyon[1,2], Peter Zoller[1,2], Rainer Blatt[1,2], Christian F. Roos[1,2]†

Entanglement is a key feature of many-body quantum systems. Measuring the entropy of different partitions of a quantum system provides a way to probe its entanglement structure. Here, we present and experimentally demonstrate a protocol for measuring the second-order Rényi entropy based on statistical correlations between randomized measurements. Our experiments, carried out with a trapped-ion quantum simulator with partition sizes of up to 10 qubits, prove the overall coherent character of the system dynamics and reveal the growth of entanglement between its parts, in both the absence and presence of disorder. Our protocol represents a universal tool for probing and characterizing engineered quantum systems in the laboratory, which is applicable to arbitrary quantum states of up to several tens of qubits.

Brydges *et al.*, *Science* **364**, 260–263 (2019)   19 April 2019

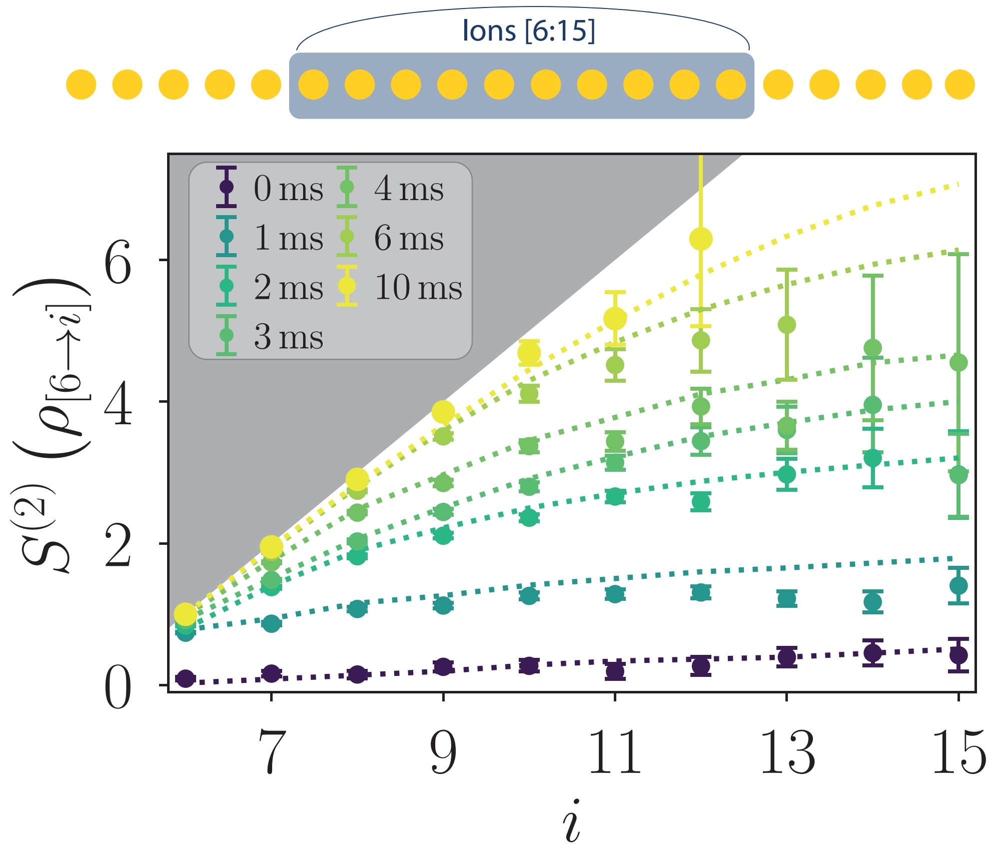


Figure 3: **Second-order Rényi entropy of 1- to 10-qubit partitions of a 20-qubit system.** The intial low-entropy Néel state evolves under H_{XY} ($J_0 = 370\ s^{-1}$, $\alpha = 1.01$) within 10 ms into a state with high-entropy partitions, corresponding to nearly fully mixed subsystems. For the data taken at 6 ms (10 ms) time evolution, the two (three) data points corresponding to highly mixed states are not shown due to their large statistical error bars. For details regarding numerical simulations (dotted curves) and error bars, see (*26*).

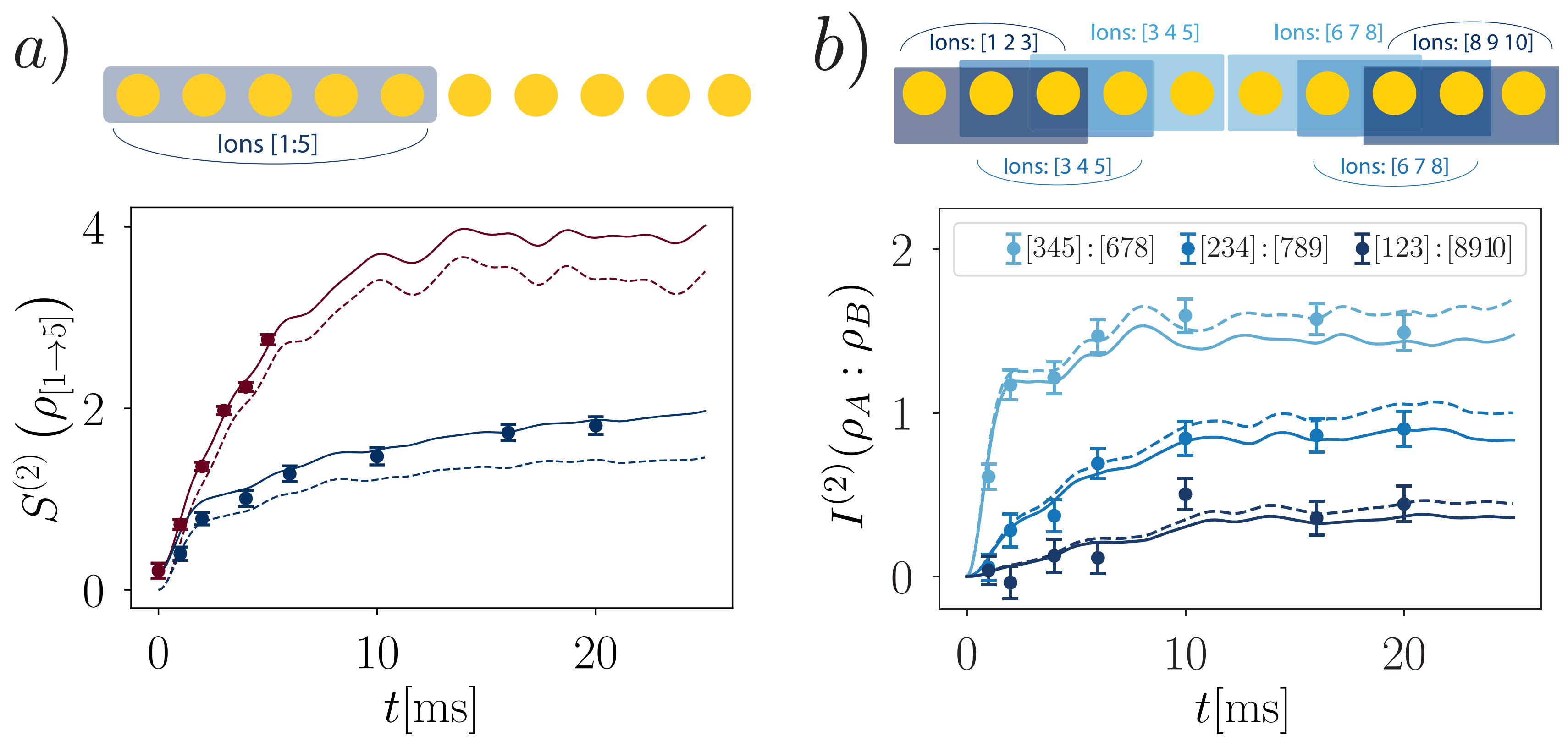
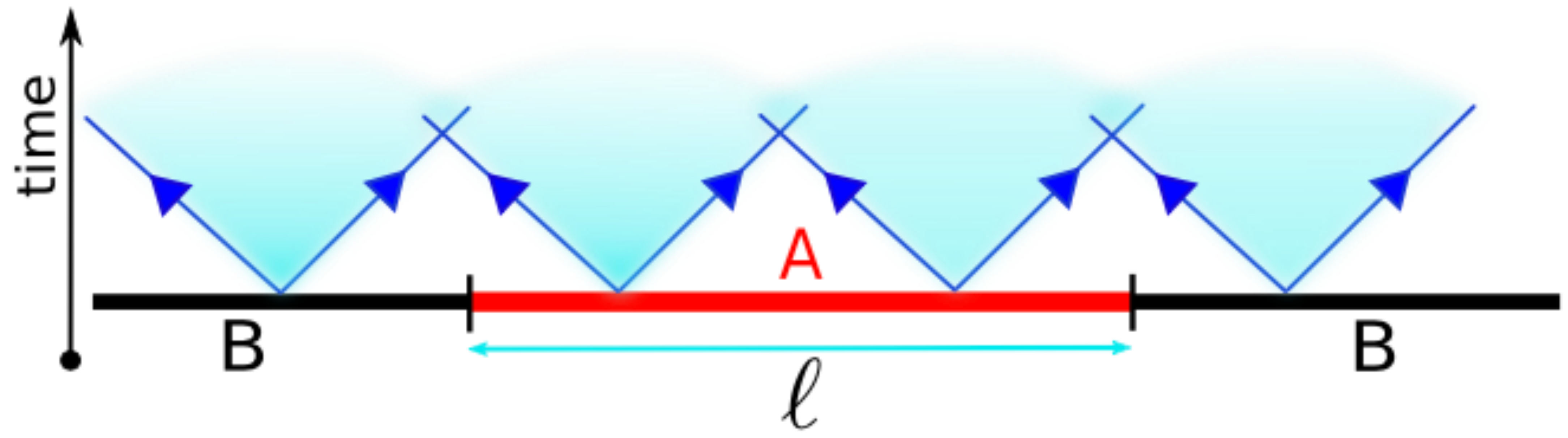


Figure 4: **Spread of quantum correlations under H_{XY} ($J_0 = 420\,\mathrm{s}^{-1}$, $\alpha = 1.24$) with and without disorder.** (a) Half-chain entropy growth versus time without disorder (red data points) and with disorder (blue data points). Numerical simulations based on unitary dynamics (dotted curves) including known sources of decoherence (full lines) are in agreement with the measured second-order Rényi entropies. (b) Quantum mutual information of selected subsystems versus time. The decrease of $I^{(2)}$ with distance between

Quasiparticle vs membrane picture

Integrable systems: stable quasiparticles

PC & J Cardy 2005
V Alba & PC 2017



Chaotic systems

Nahum et al 2017
Zhou & Nahum 2020

$$S(x,t) = \min_{\text{path}} \int s_{\text{eq}} \mathcal{E}(v) dt + S(y,0)$$

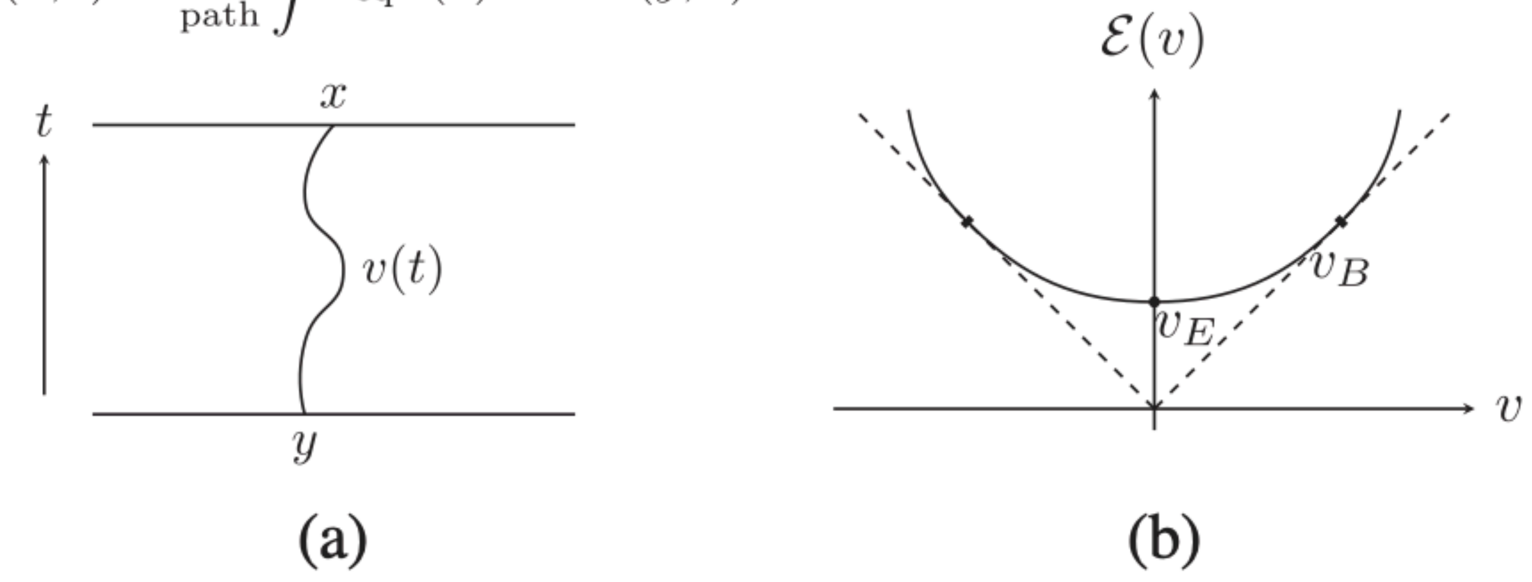
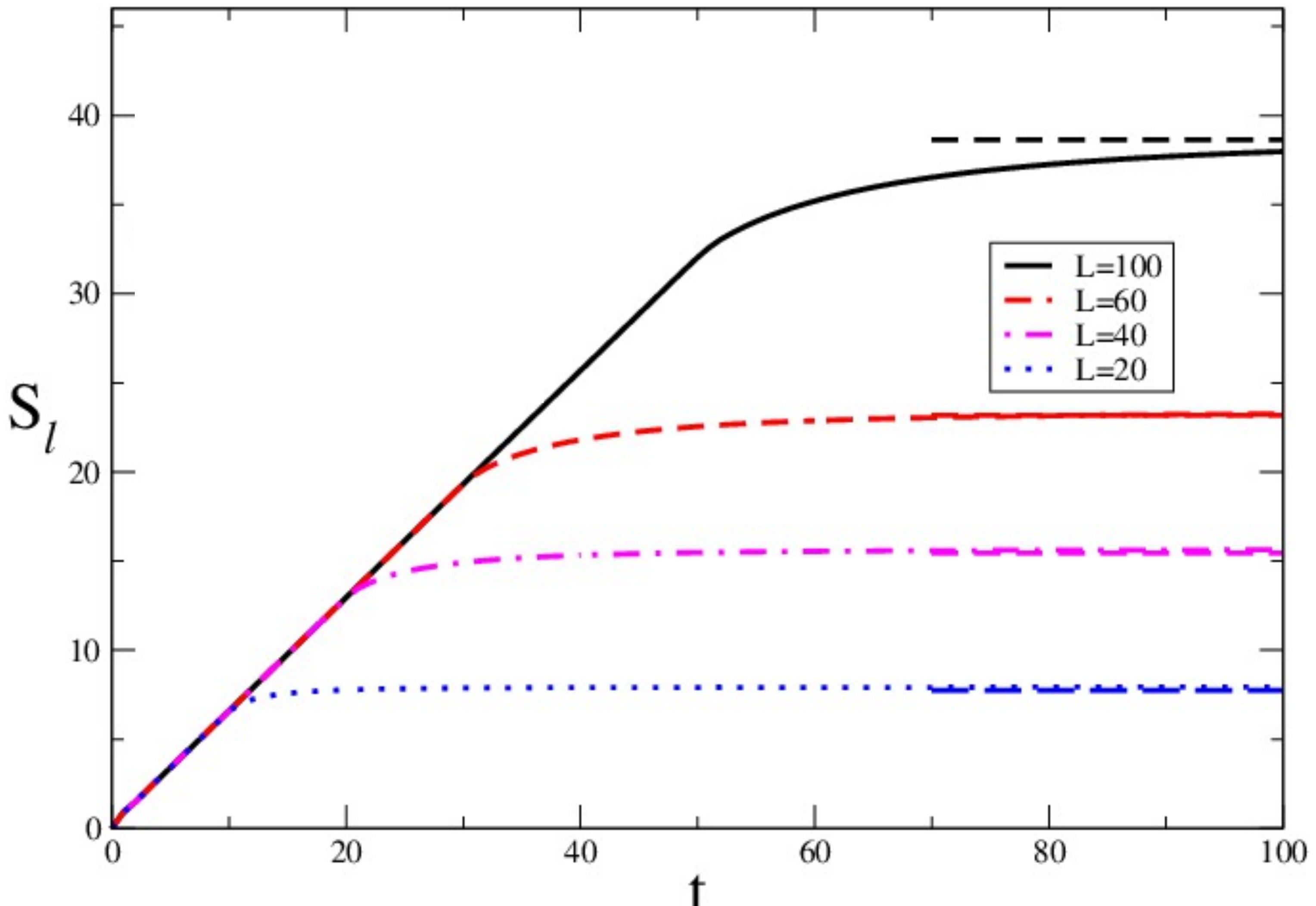


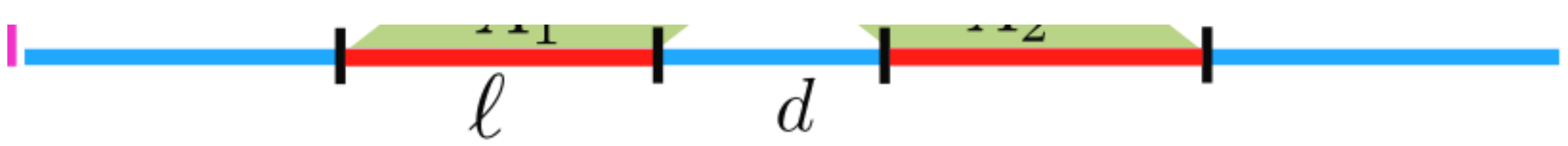
FIG. 1. The membrane picture of entanglement and other dynamical quantities. (a) Example of a membrane for evaluating Rényi entropy growth in a 1D quench. Minimizing the membrane tension gives the entanglement. The tension is a function of local

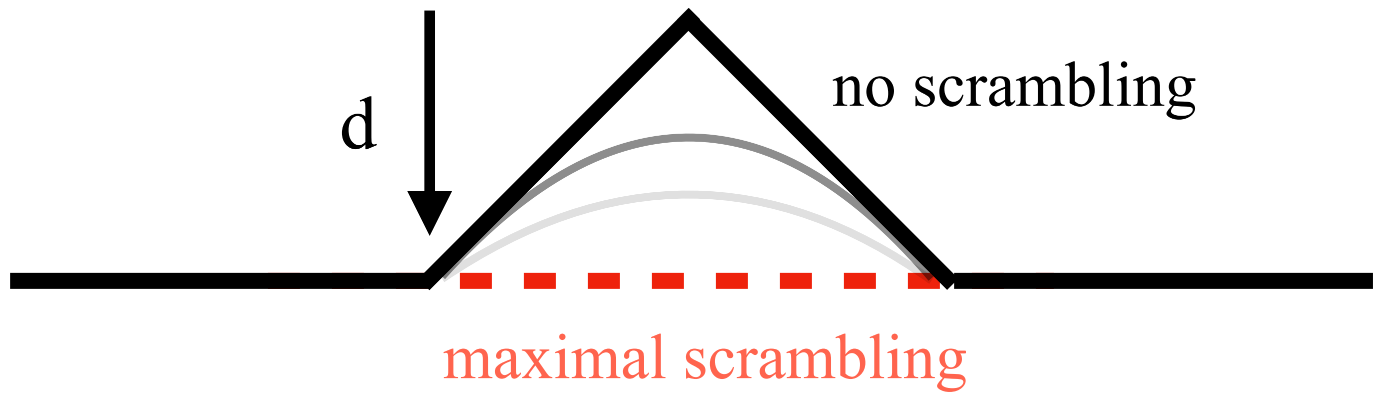
Both predict linear increase followed by saturation



Differences are observed with more complicated geometries:

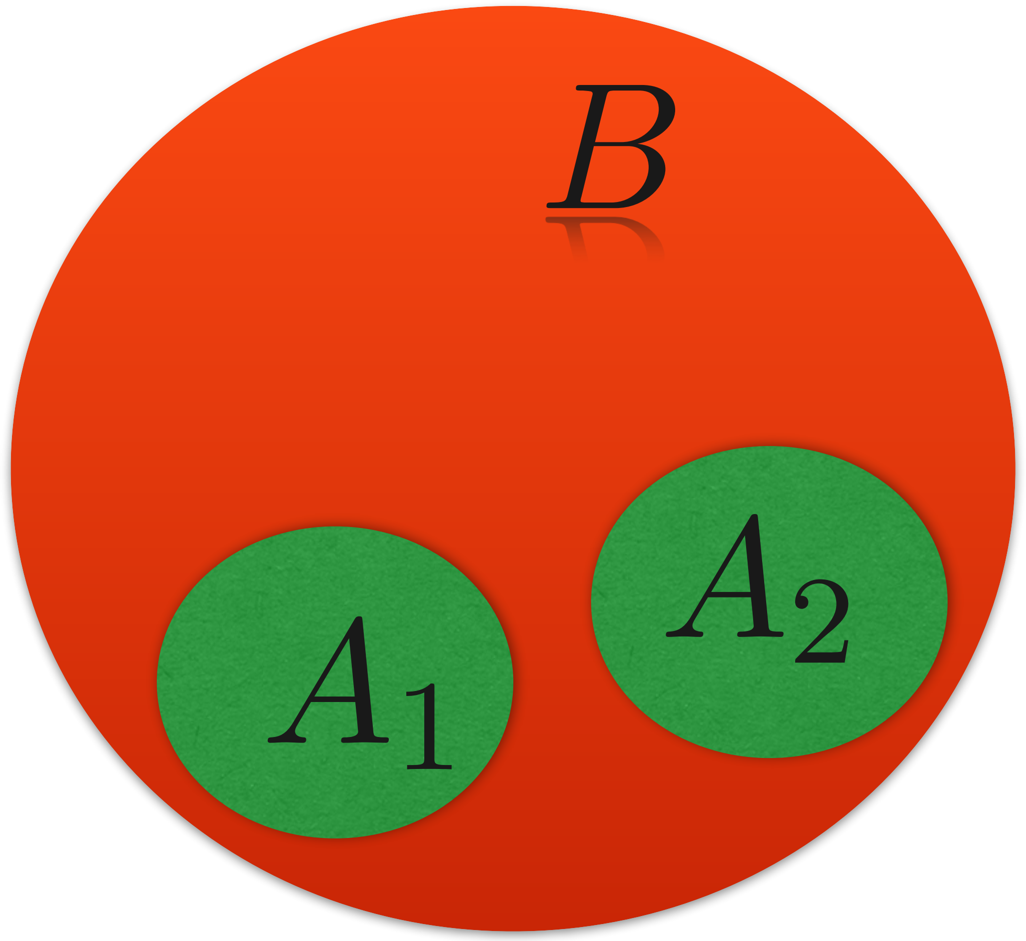
Asplund, Bernamonti, Galli, Hartman ‘15





Mixed state entanglement: Partial transpose and negativity

Q: what is the entanglement in a mixed state?



$|e_k^1\rangle$ and $|e_l^2\rangle$ bases of A_1 and A_2

$$(|e_i^1, e_j^2\rangle \langle e_k^1, e_l^2|)^{T_1} \equiv |e_k^1, e_j^2\rangle \langle e_i^1, e_l^2|$$

$$\rho_A = \sum_{ijkl} \langle e_i^1, e_j^2 | \rho_A | e_k^1, e_l^2 \rangle | e_i^1, e_j^2 \rangle \langle e_k^1, e_l^2 |$$

$$\Downarrow$$

$$\rho_A^{T_1} = \sum_{ijkl} \langle e_k^1, e_j^2 | \rho_A | e_i^1, e_l^2 \rangle | e_k^1, e_j^2 \rangle \langle e_i^1, e_l^2 |$$

PPT criterion:

If $\rho_A^{T_1}$ has negative eigenvalues ρ_A is entangled

Peres, 1996

The Negativity $= \mathcal{N} = \dfrac{\mathrm{Tr}|\rho_A^{T_1}| - 1}{2}$ measures how much the eigenvalues of $\rho_A^{T_1}$ are negative (it is also an entanglement monotone)

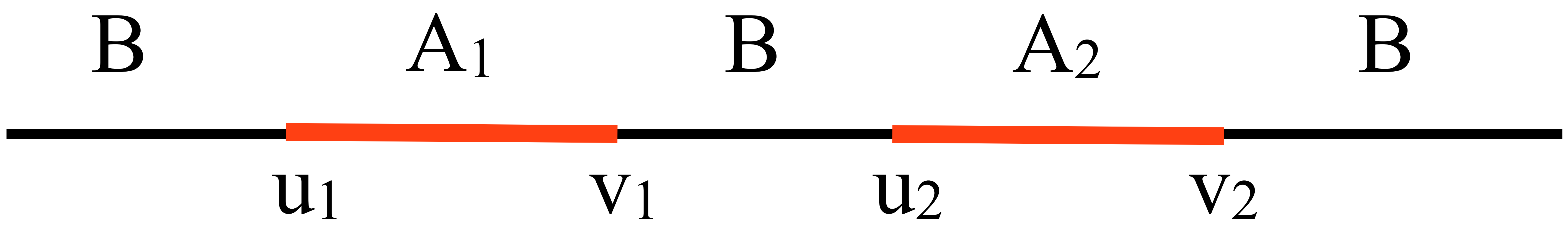
Vidal Werner 2002

Replica trick: $\mathrm{Tr}|\rho_A^{T_1}| = \lim_{n \to 1/2} \mathrm{Tr}(\rho_A^{T_1})^{2n}$

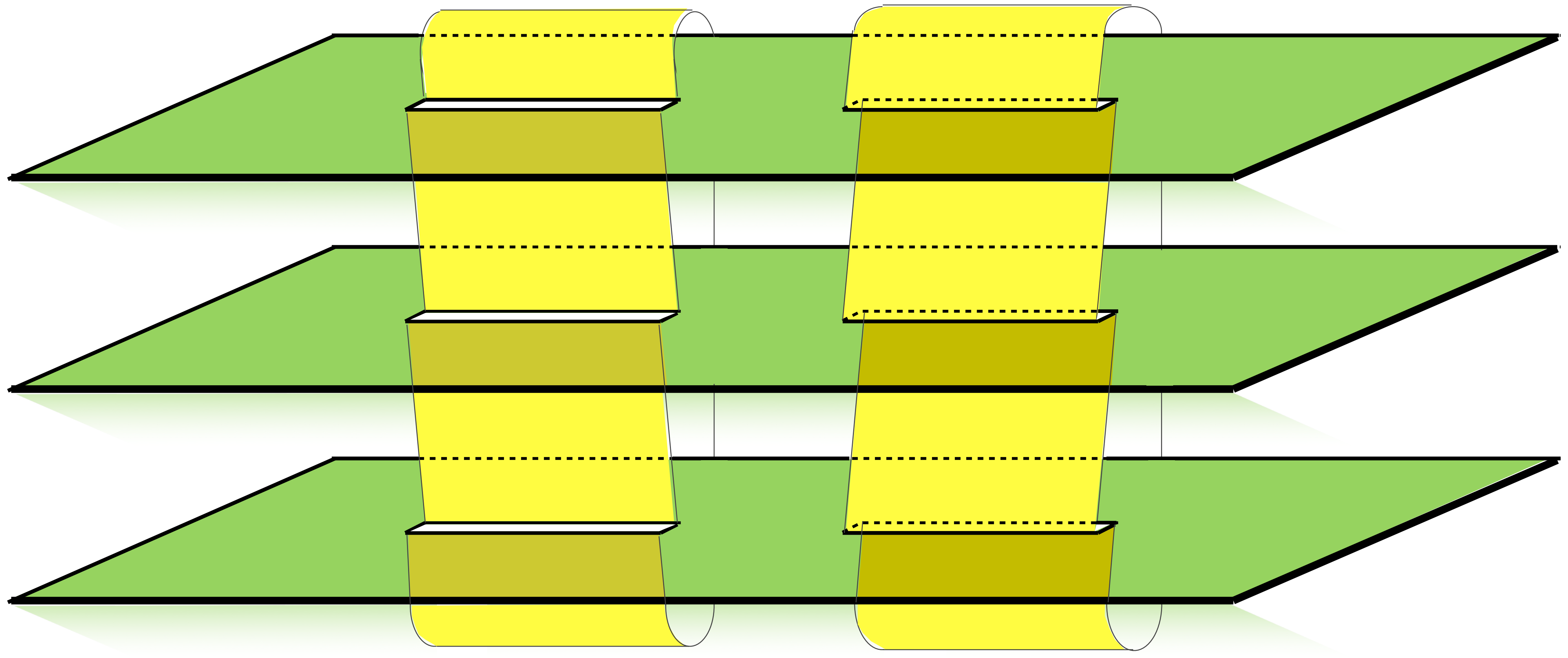
P. Calabrese, J. Cardy, E. Tonni 2012

Negativity and QFT

Tripartition:



$$\mathrm{Tr}(\rho_A^{T_2})^n =$$



$$= \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\bar{\mathcal{T}}_n(u_2)\mathcal{T}_n(v_2)\rangle$$

The partial transposition exchanges two twist operators

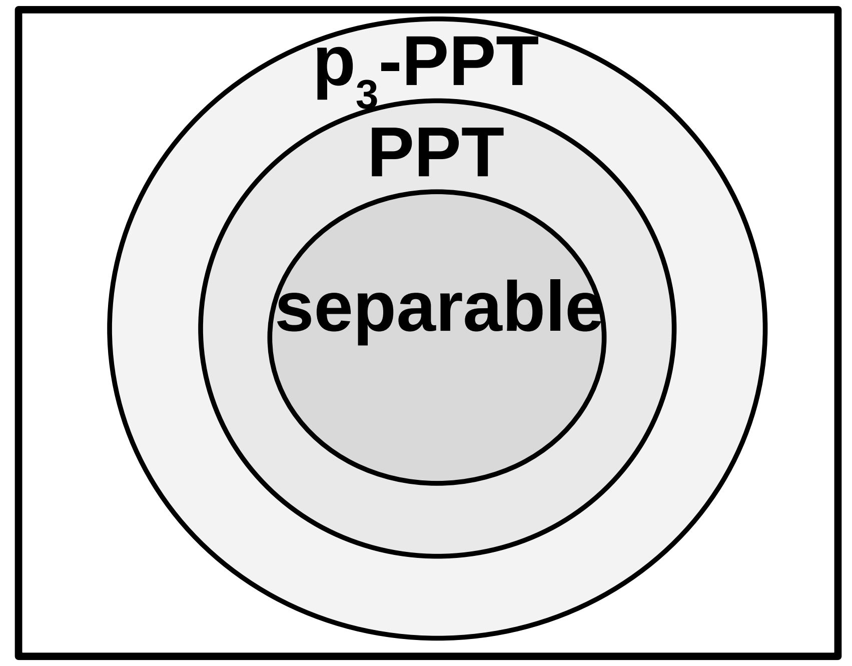
It led to different Riemann surfaces and new twist fields that found applications from topological matter to black holes

“Negativity” in experiments

The negativity is difficult to measure experimentally, but the moments of the partial transpose p_n can

E. Cornfeld, M. Goldstein, and E. Sela, PRA 98, 032302 (2018)
J. Gray, L. Banchi, A. Bayat, and S. Bose, Phys. Rev. Lett. 121, 150503 (2018)
A. Elben, R. Kueng, H.-Y. Huang, R. van Bijnen, C. Kokail, M. Dalmonte, P. Calabrese, B. Kraus, J. Preskill, P. Zoller, and B. Vermersch, PRL 125, 200501 (2020)

a)



p_n are obtained by performing local random measurements and post-processing using the classical shadows framework

p3-PPT condition: if $p_3 < p_2^2$, then PPT is violated and there is entanglement

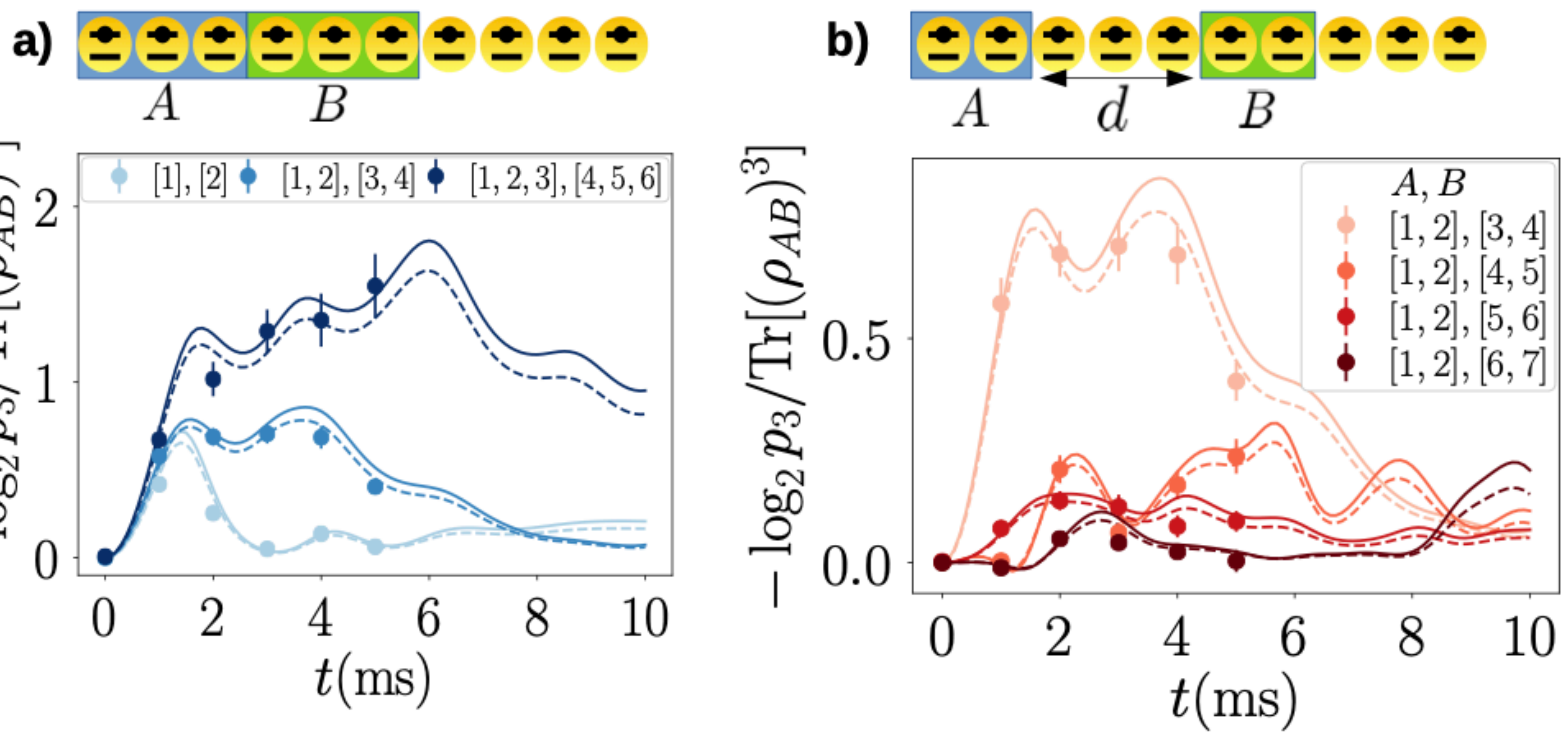
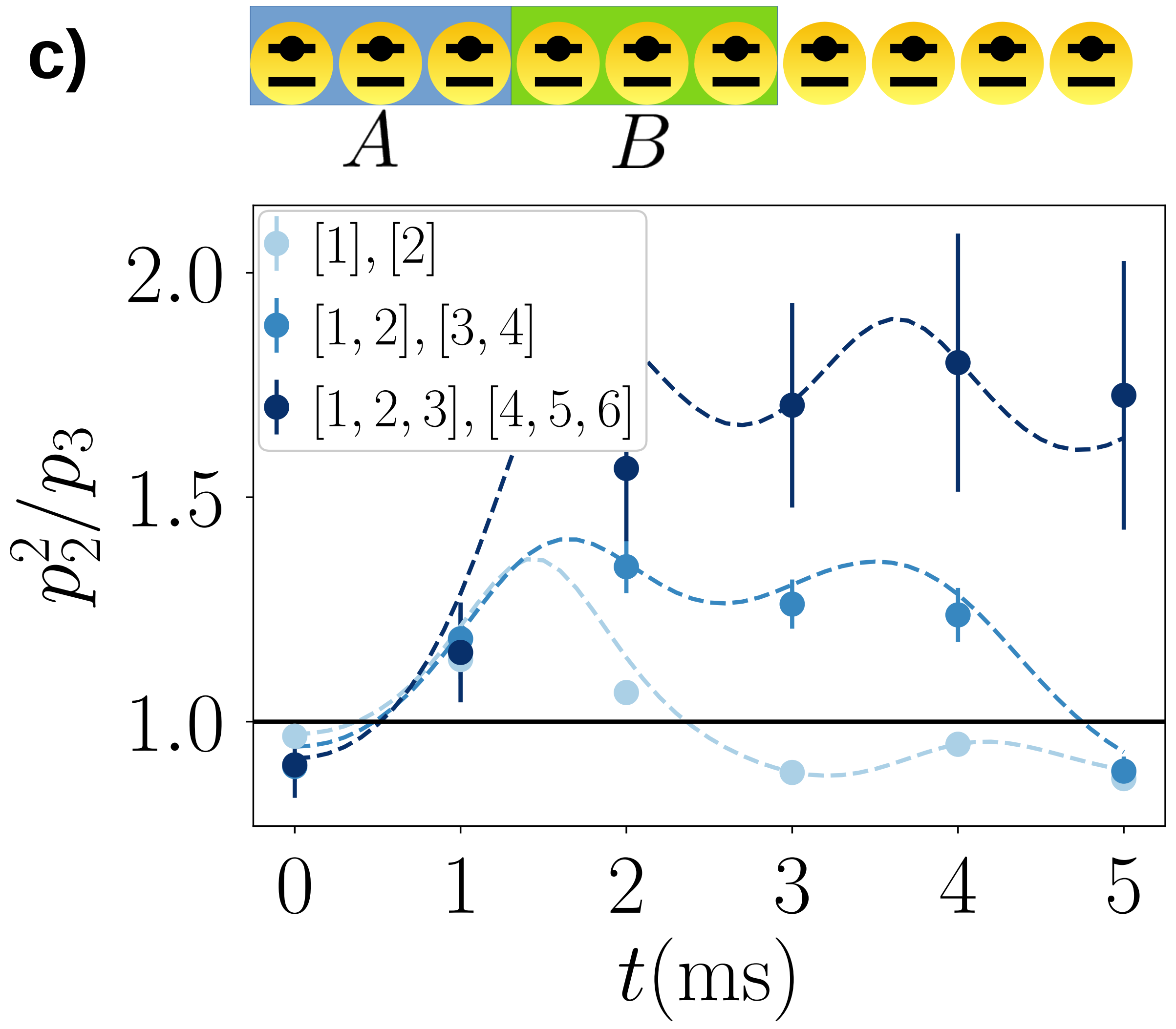


FIG. 4. *Evolution of the ratio R_3 from experimental data [10].*

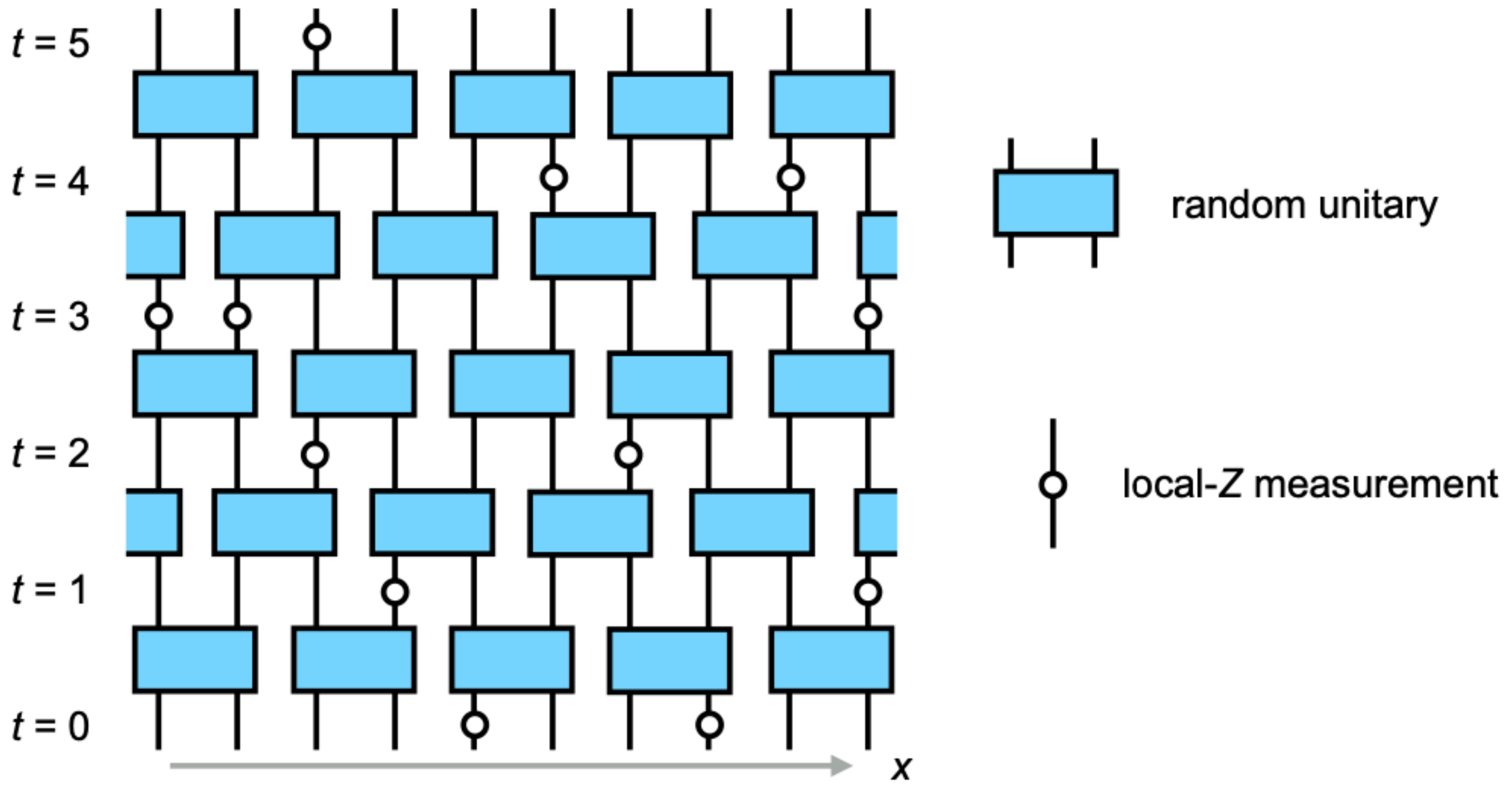
c)

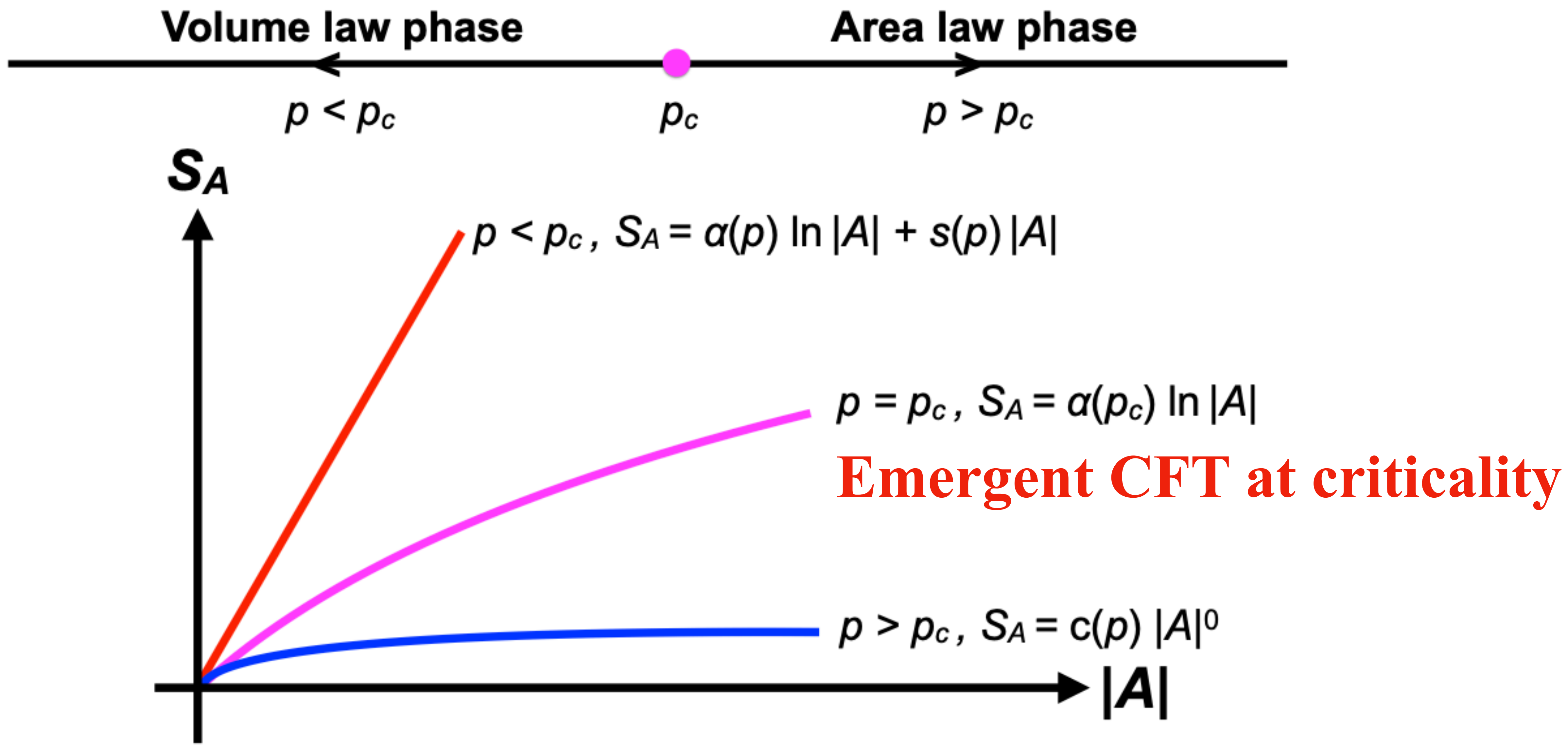


Quench from Néel state in a long range XX

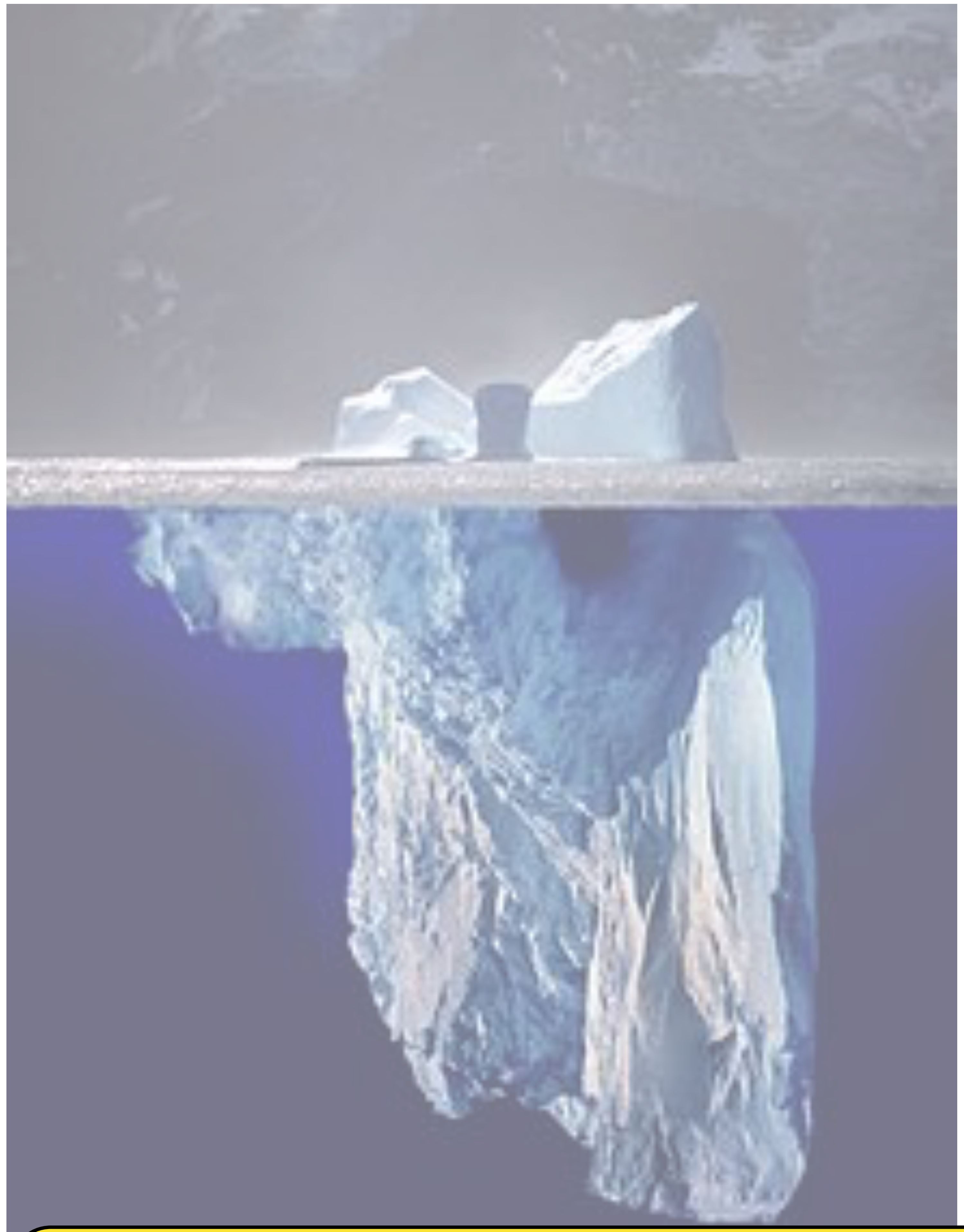
Measurement induced transitions

Li, Chen, Fisher 2019
Skinner, Ruhman, Nahum 2019
Chan et al, 2019





Without looking at entanglement, observing and identifying this transition would been very hard, likely impossible



Many fundamental results have not bee… …tioned here:

- Disordered systems and many… …lization
- Entanglement Hamiltonian… …aic QFT
- Gauge theories
- Symmetries and en… …t
- Operator entan…
- QFT results… …n, QNEC,…
- Quantu… …K, quantum circuits

…me message

Whenever you look at enta… …deep enough, you get much more than by other means

Thank you for your attention!