Quantum cellular automata: quantum field theories and their simulation

SQMS/GGI Summer School on Quantum Simulation of Field Theories

Galileo Galilei Institute - Firenze

Paolo Perinotti - July 28th

UNIVERSITÀ DI PAVIA

Istituto Nazionale di Fisica Nucleare

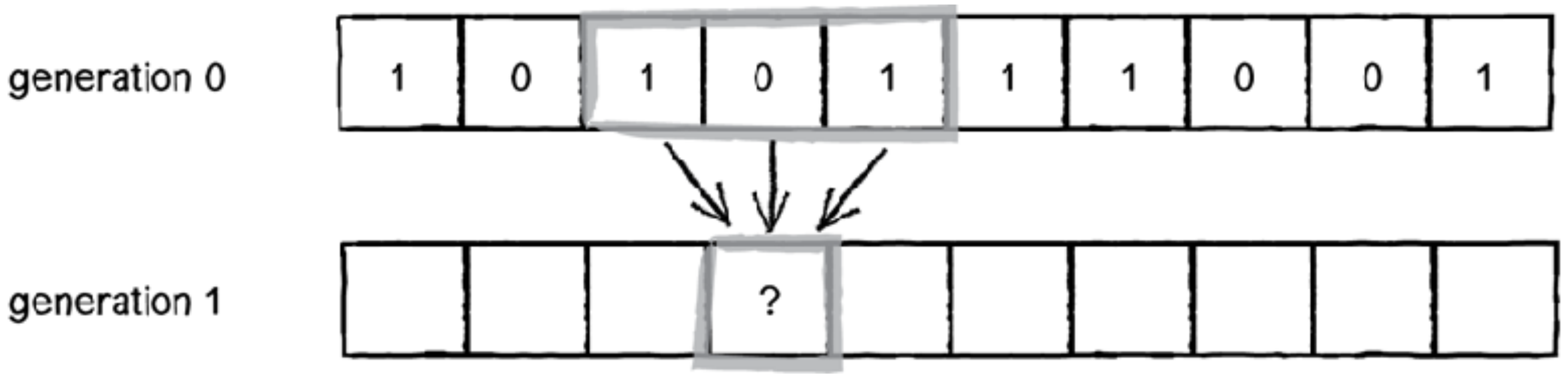
quantum information theory group

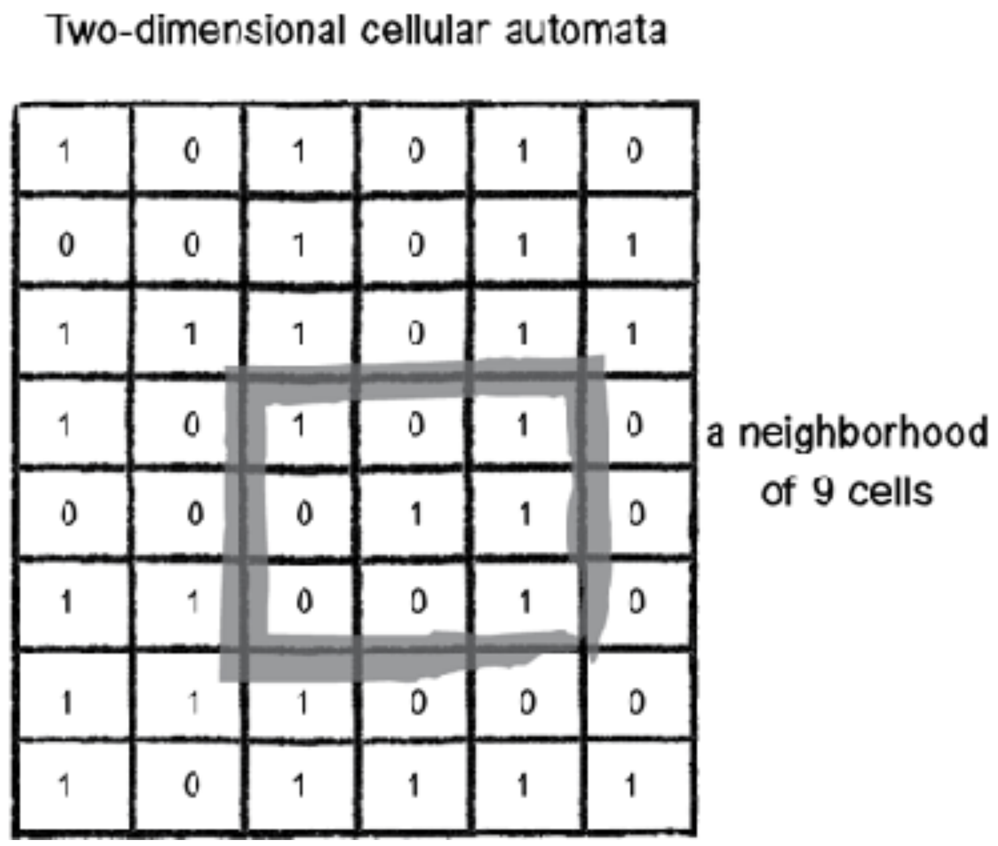
Summary

- Quantum cellular automata
- Fermionic cellular automata
- Abelian case - Fourier analysis and plane waves
- Maxwell's equation and boson statistics
- Path-sum solution
- Interactions and non-linearity
- Examples of interacting models
- Perturbative techniques

Cellular Automata

J. Von Neumann and A. W. Burks, “Theory of self-reproducing automata” 1966





E.g. Conway’s game of life

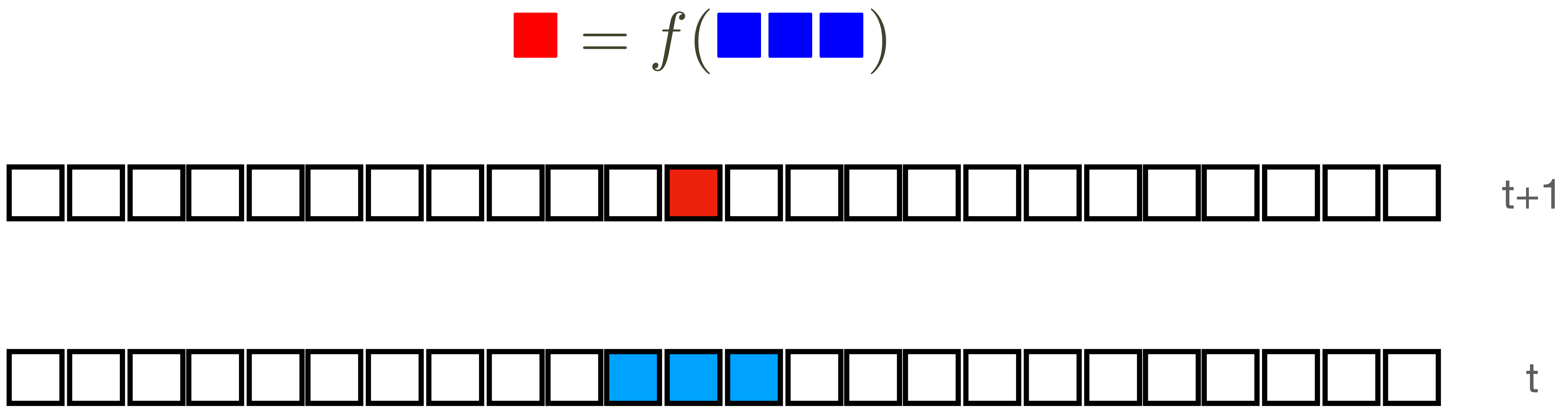
Definition (Cellular Automata). *A* cellular automaton *(CA) is a 4-tuple* $(\mathrm{L}, \Sigma, \mathcal{N}, f)$ *consisting of (1) a d-dimensional lattice of cells* L *indexed* $i \in \mathbb{Z}^d$*, (2) a finite set of states* Σ*, (3) a finite neighborhood scheme* $\mathcal{N} \subset \mathbb{Z}^d$*, and (4) a local transition function* $f : \Sigma^{\mathcal{N}} \to \Sigma$.

K. Wiesner, Quantum Cellular Automata. In: Meyers R. (eds.) Encyclopedia of Complexity and Systems Science.

Quantum cellular automata

Problems

- What is a local update rule?

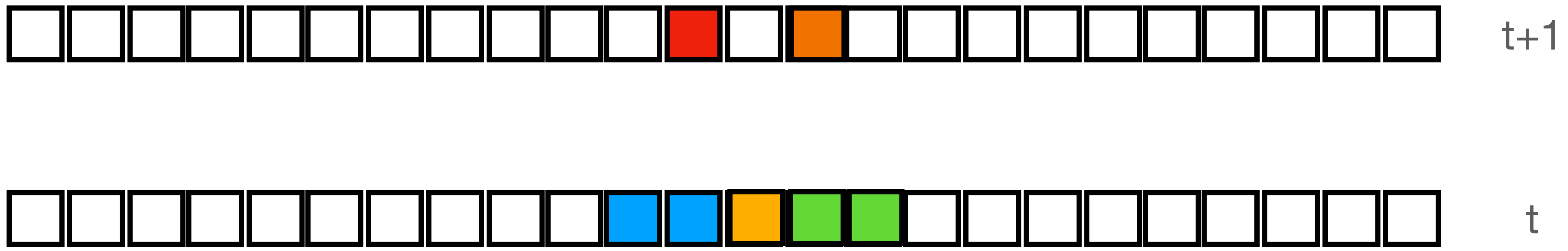


B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Quantum cellular automata

Problems

- What is a local update rule?

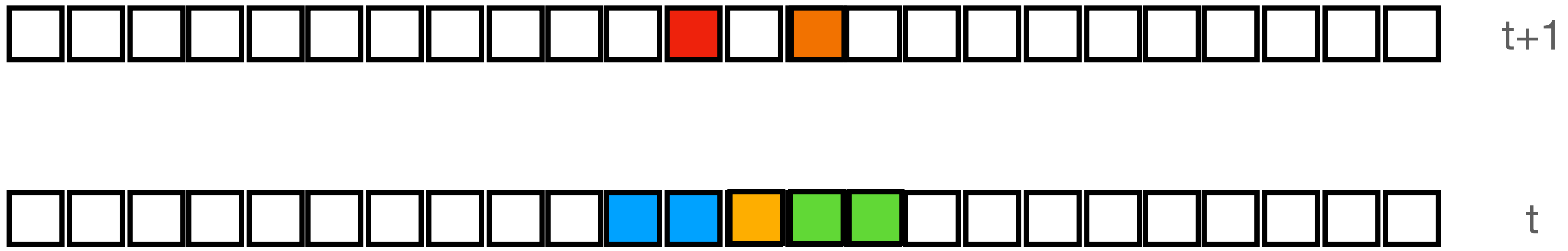


B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Quantum cellular automata

Problems

- What is a local update rule?
- No-cloning theorem
- Non commutativity of local operations

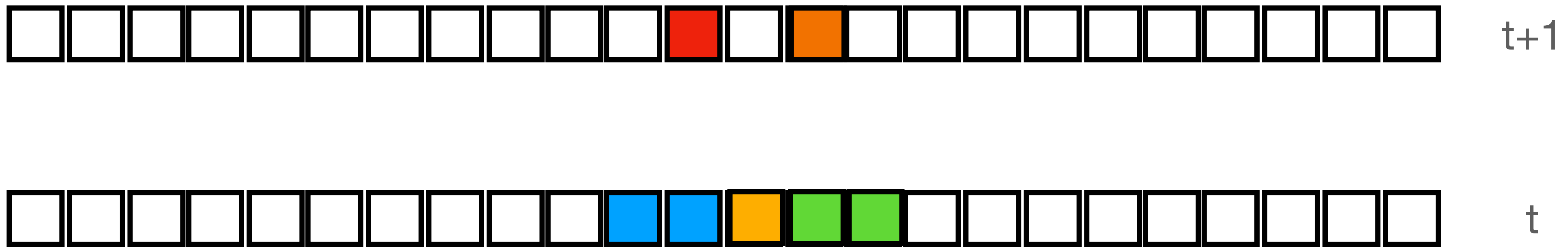


B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Quantum cellular automata

Problems

- What is a local update rule?
- No-cloning theorem
- Non commutativity of local operations
- Heart of the problem: definition through action on states



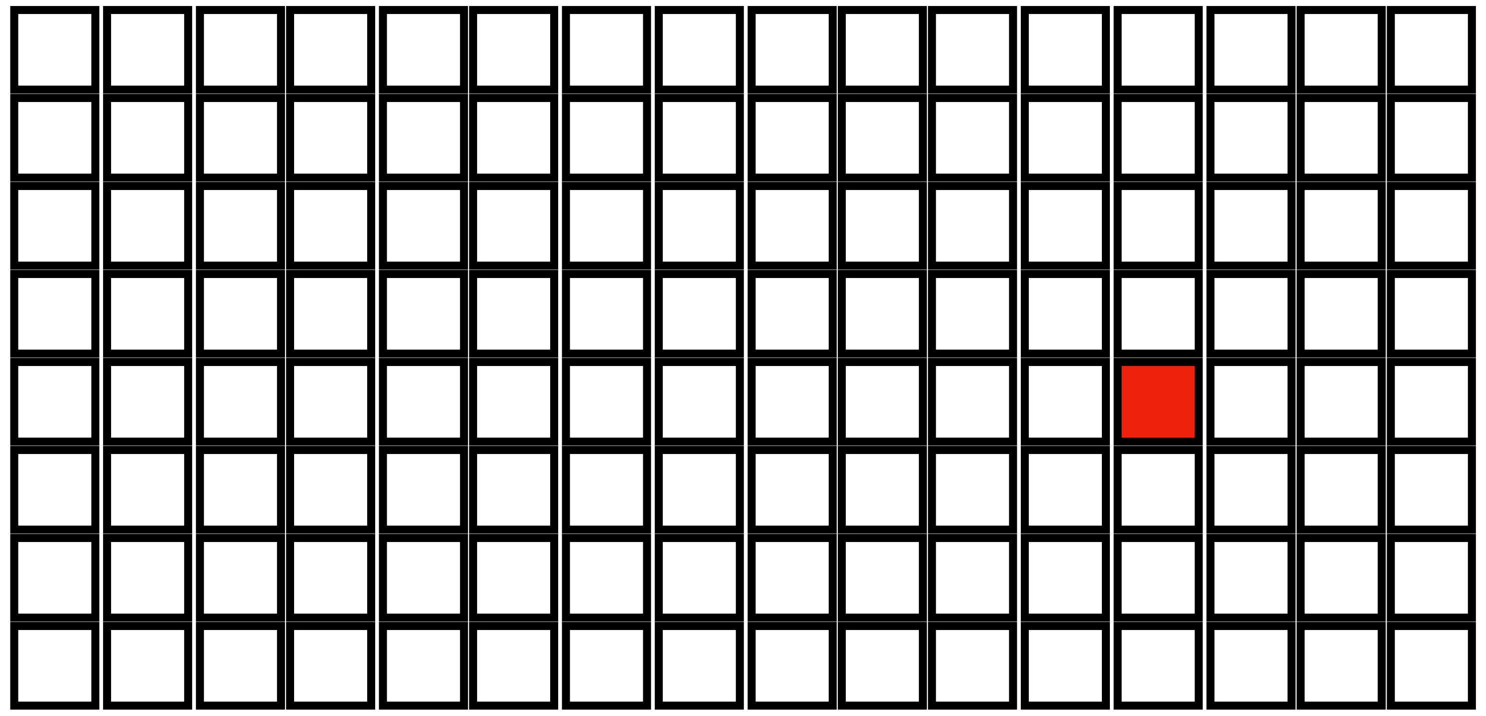
B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Quantum cellular automata

Finite case

 $= \mathcal{H}_x$

C.A.: $U : \bigotimes_x \mathcal{H}_x \to \bigotimes_x \mathcal{H}_x$

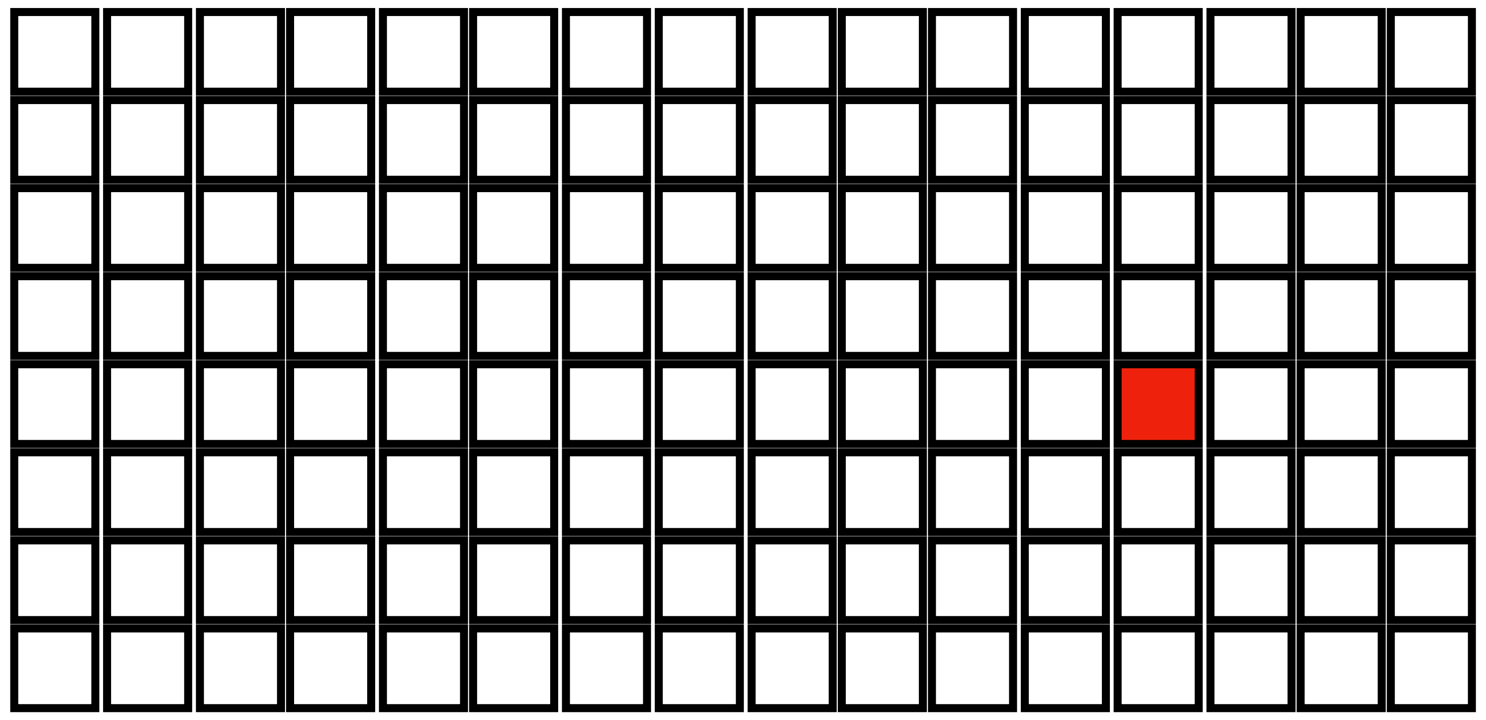


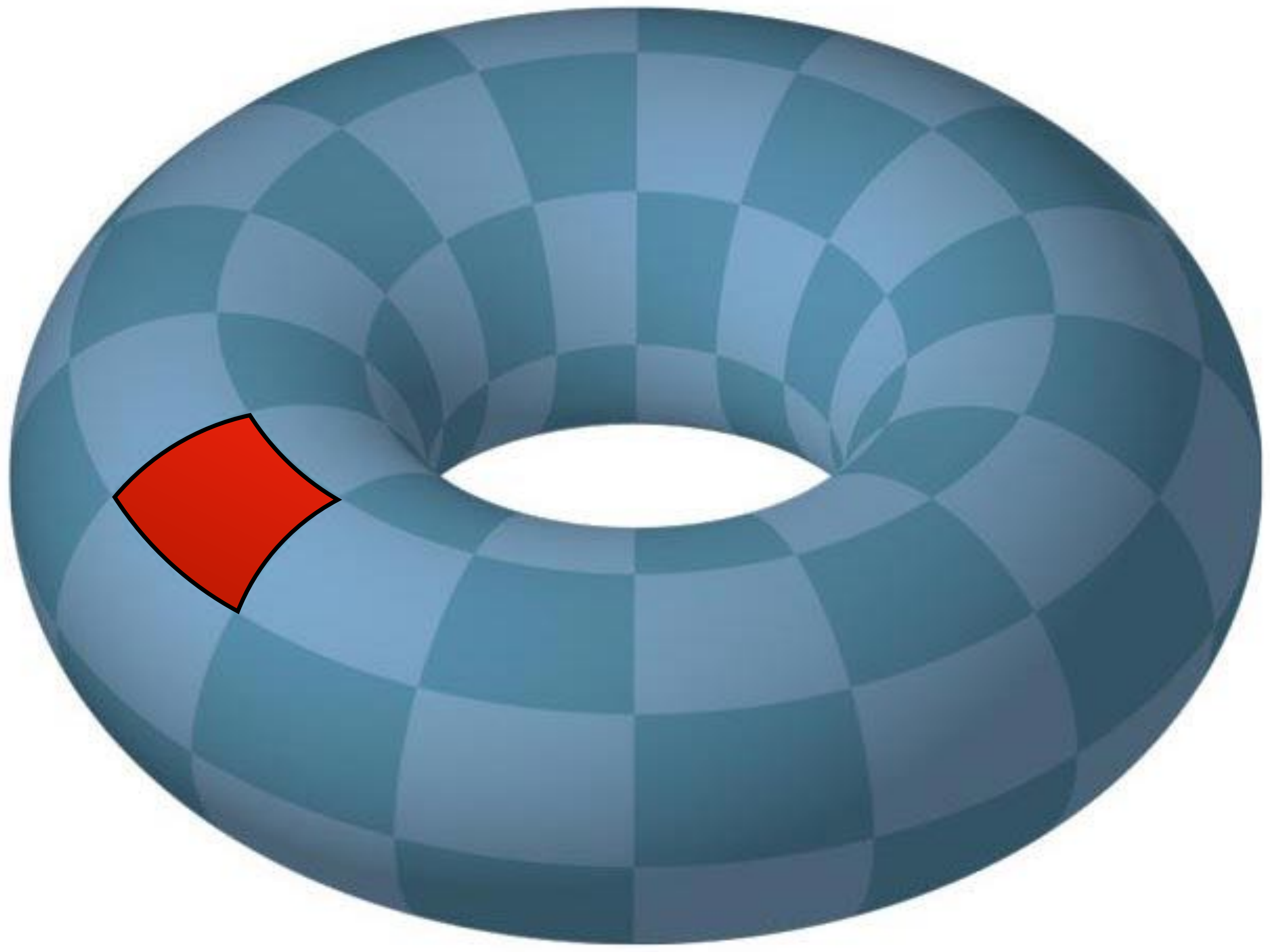
Quantum cellular automata

Finite case

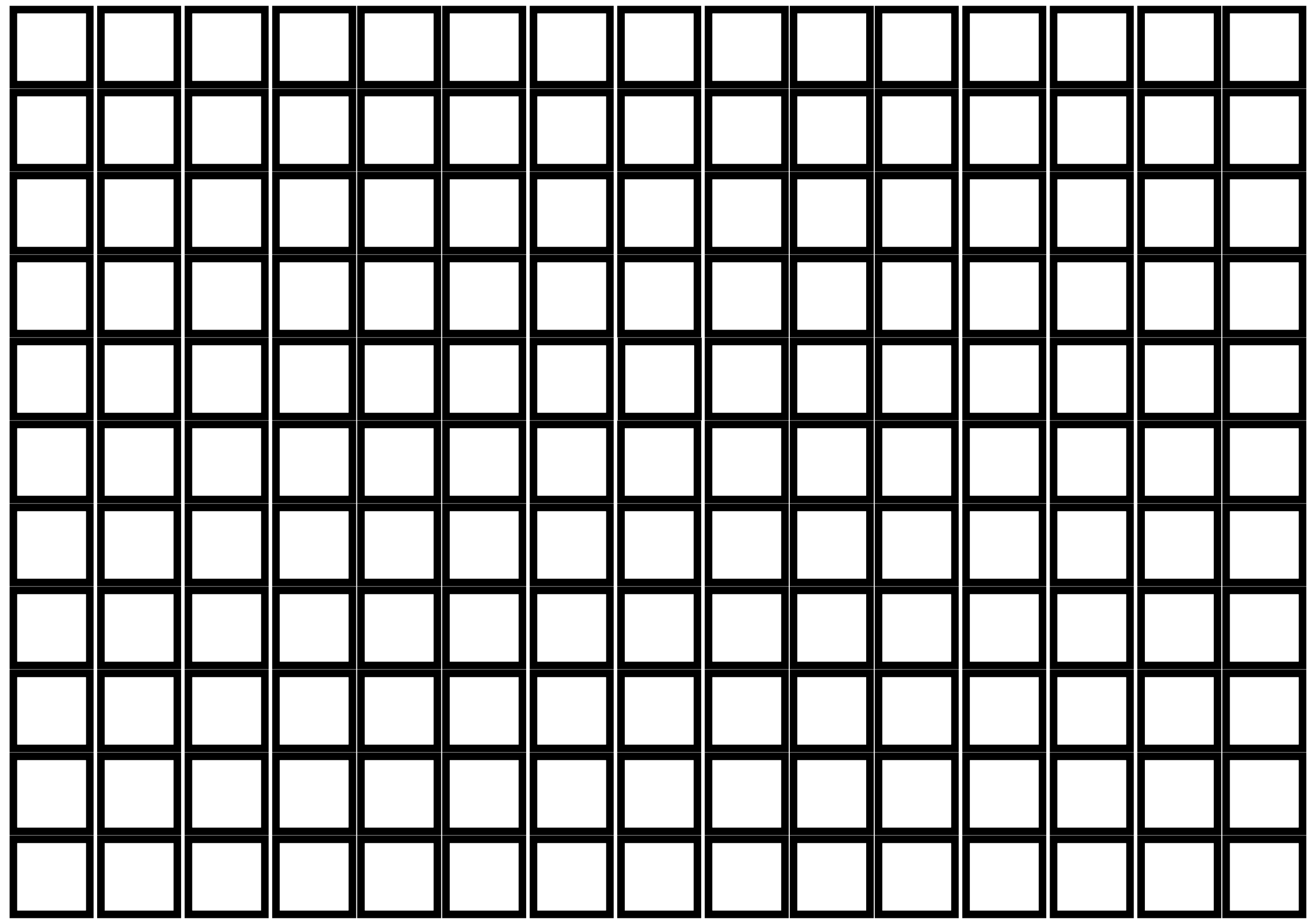
■ $= \mathcal{H}_x$

C.A.: $U : \bigotimes_x \mathcal{H}_x \to \bigotimes_x \mathcal{H}_x$



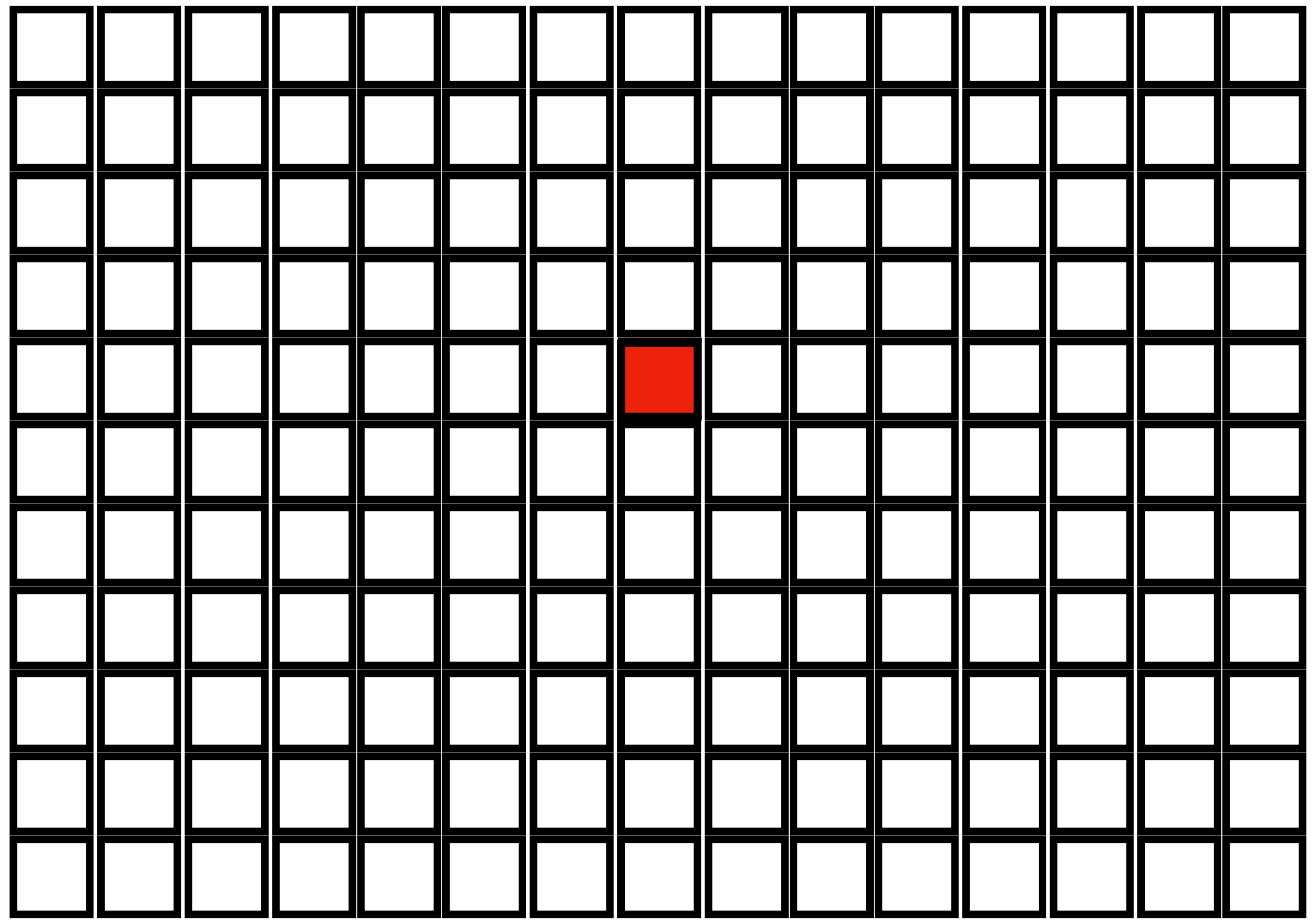


Local systems and observables



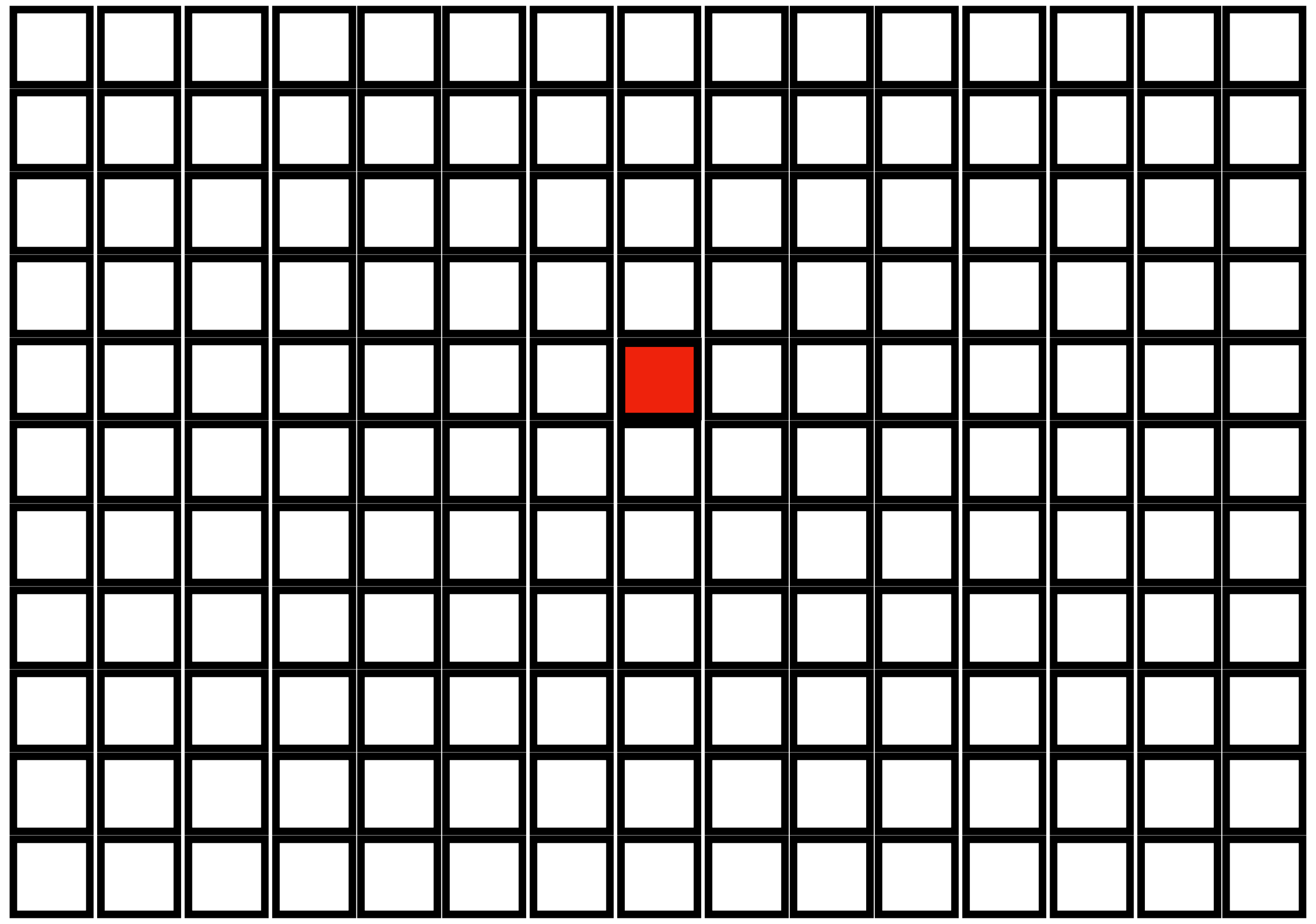
Local systems and observables

$$p(A_x = a \mid \rho) = \mathrm{Tr}[P(a)_x \rho] = \mathrm{Tr}[\{P(a) \otimes I_{\bar{x}}\}\rho]$$



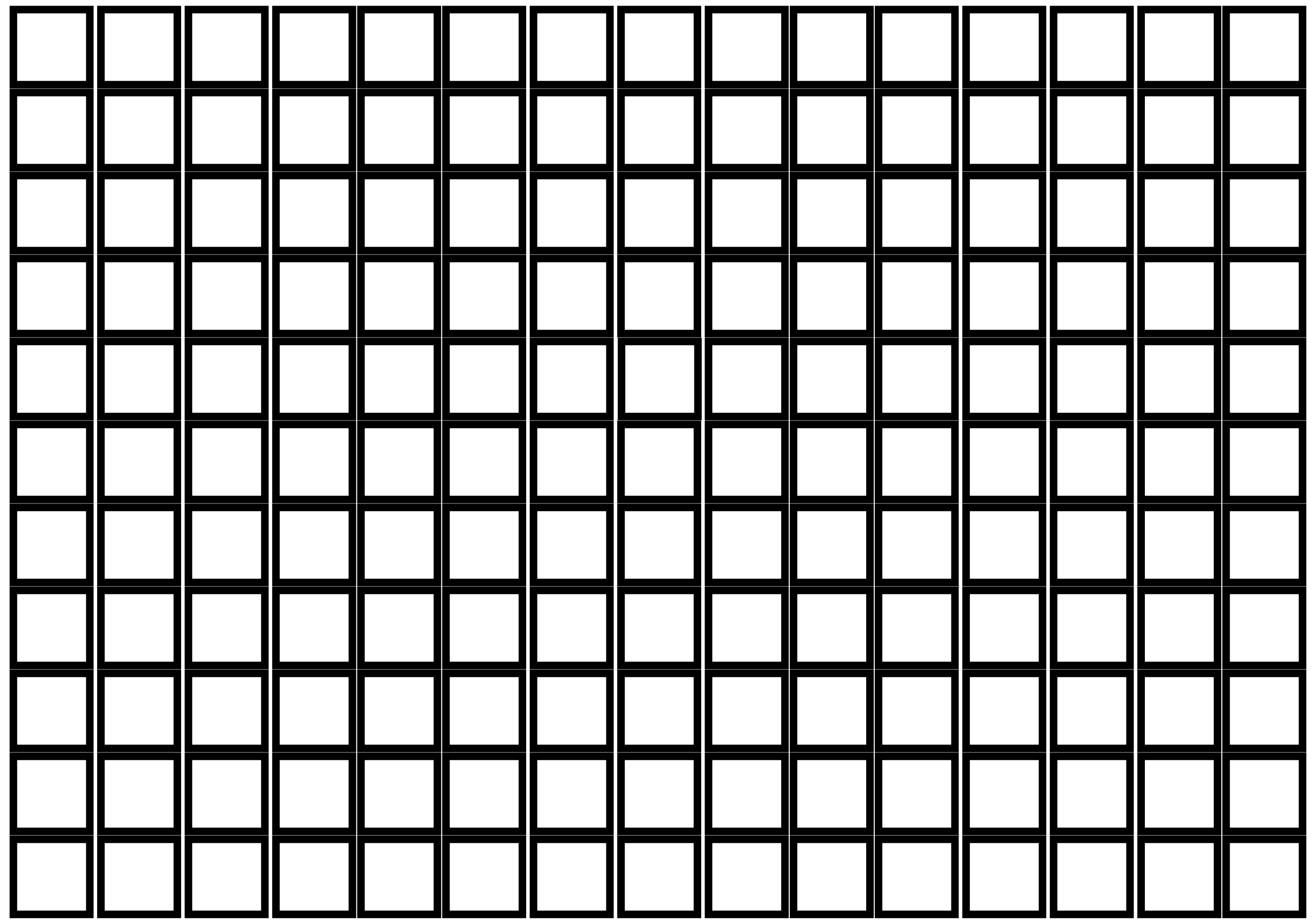
Local systems and observables

$$p(A_x = a \mid \rho) = \mathrm{Tr}[P(a)_x \rho] = \mathrm{Tr}[\{P(a) \otimes I_{\bar{x}}\}\rho]$$



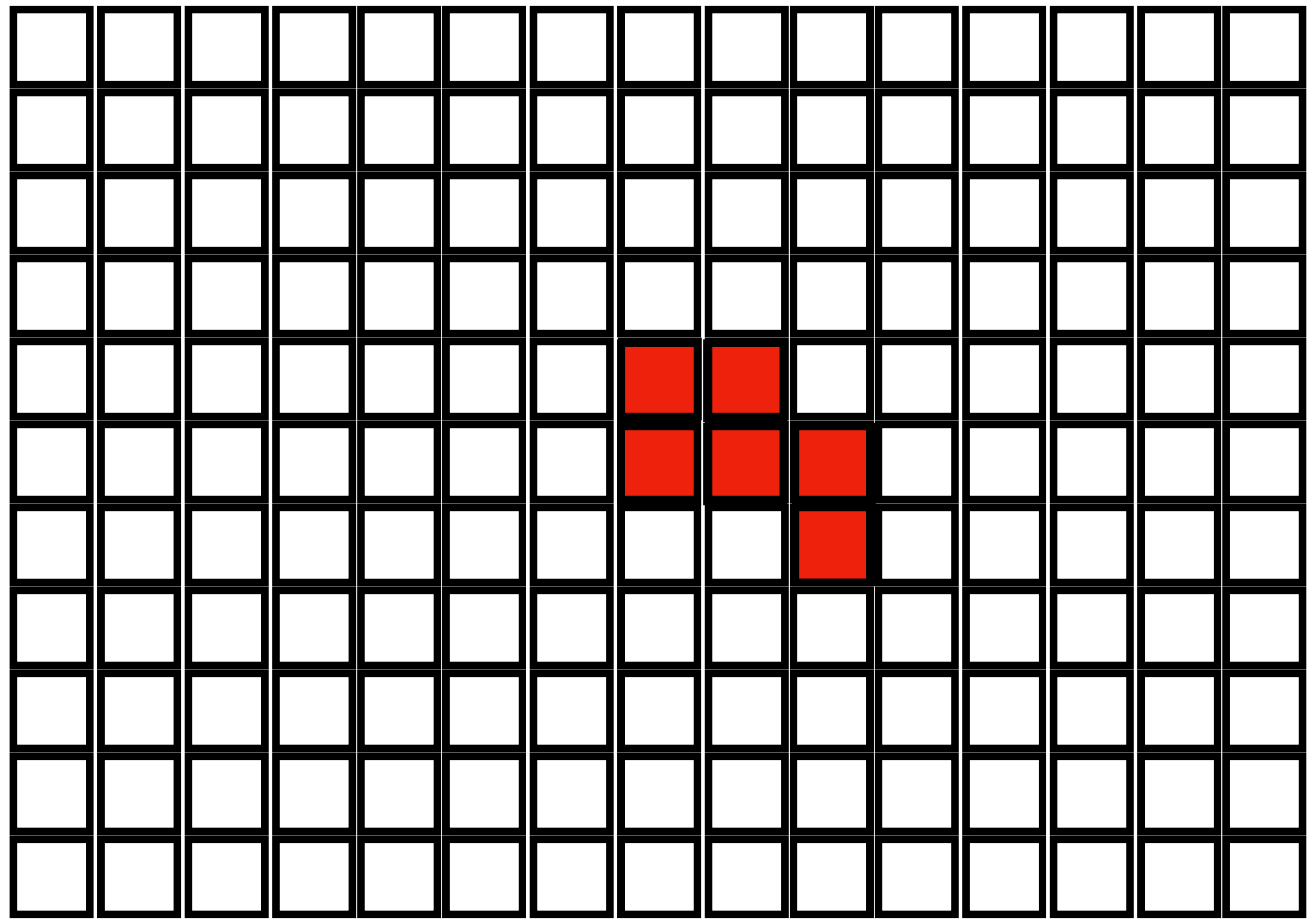
 $= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$

Local systems and observables



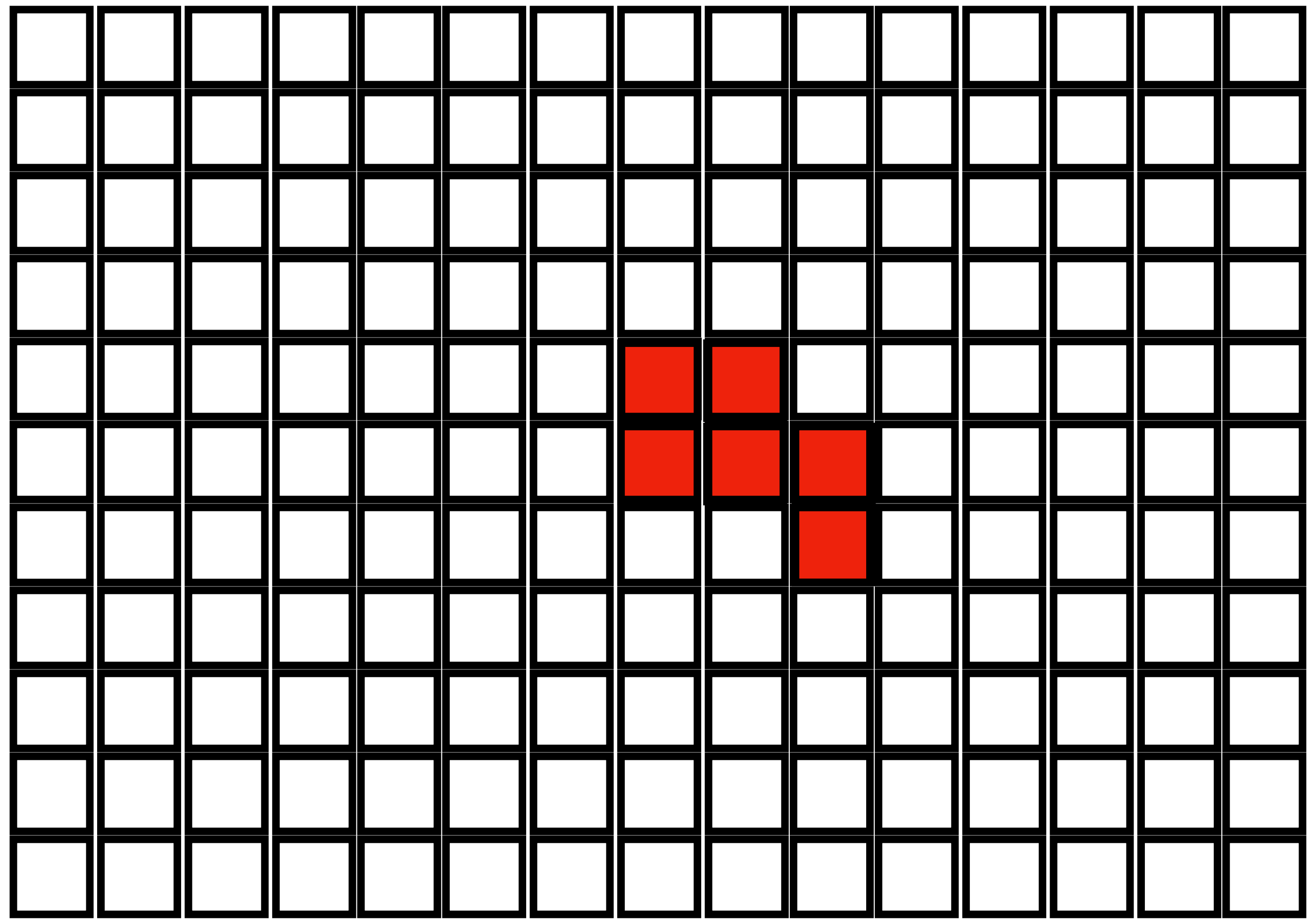
Local systems and observables

$$p(A_R = a \mid \rho) = \mathrm{Tr}[P(a)_R \rho] = \mathrm{Tr}[\{P(a) \otimes I_{\bar{R}}\}\rho]$$



Local systems and observables

$$p(A_R = a \mid \rho) = \mathrm{Tr}[P(a)_R \rho] = \mathrm{Tr}[\{P(a) \otimes I_{\bar{R}}\}\rho]$$



$$A_R \in \mathsf{Span}\left(\bigotimes_{x \in R} \mathsf{A}_x\right)$$

Heisenberg picture

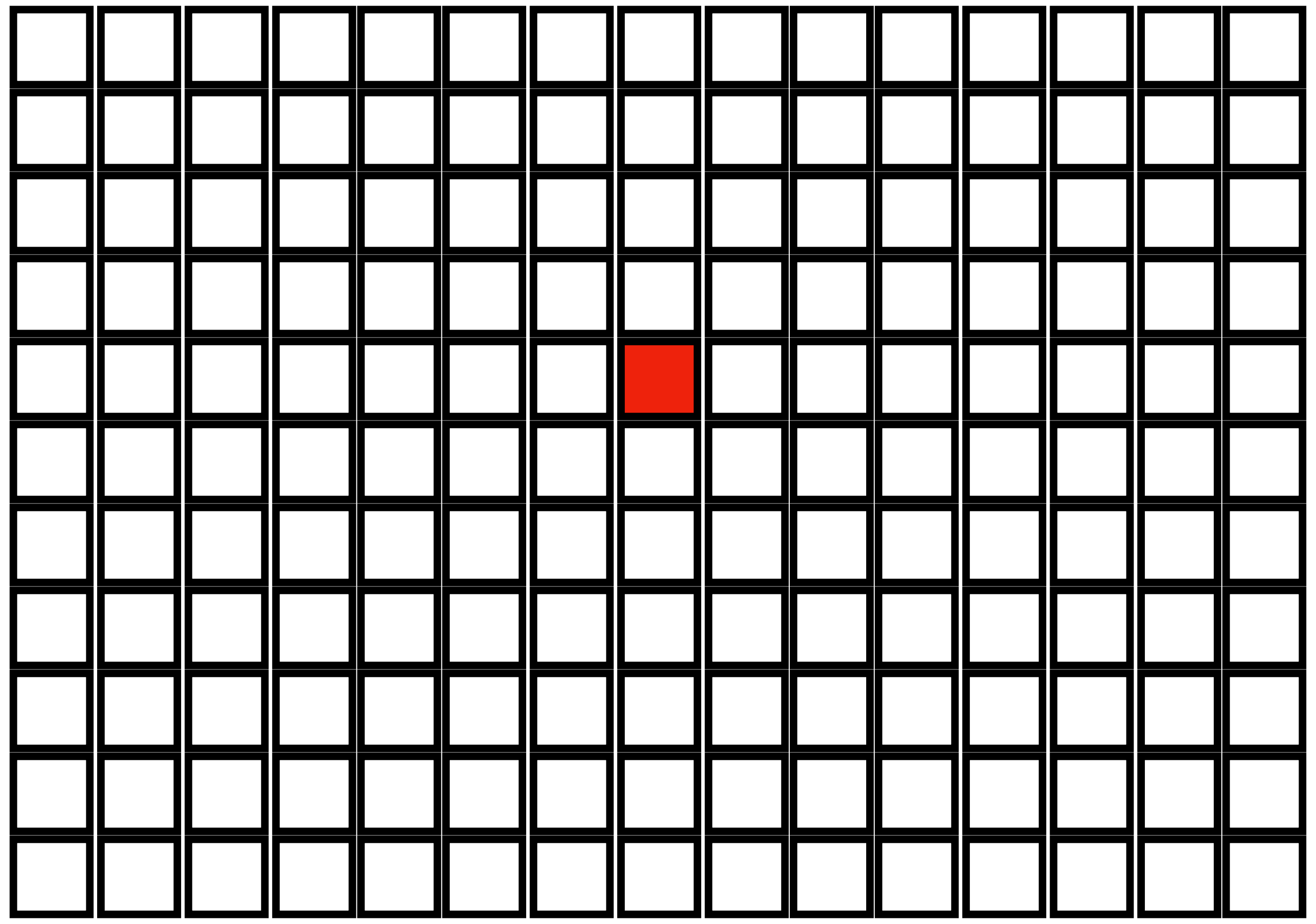
$$p(A_x = a \mid U\rho U^\dagger) = \mathrm{Tr}[P(a)_x U\rho U^\dagger] = \mathrm{Tr}[U^\dagger P(a)_x U\rho]$$

Locality

Neighbourhood of a cell

■ $= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$

C.A.: $U : \bigotimes_x \mathcal{H}_x \to \bigotimes_x \mathcal{H}_x$



Locality

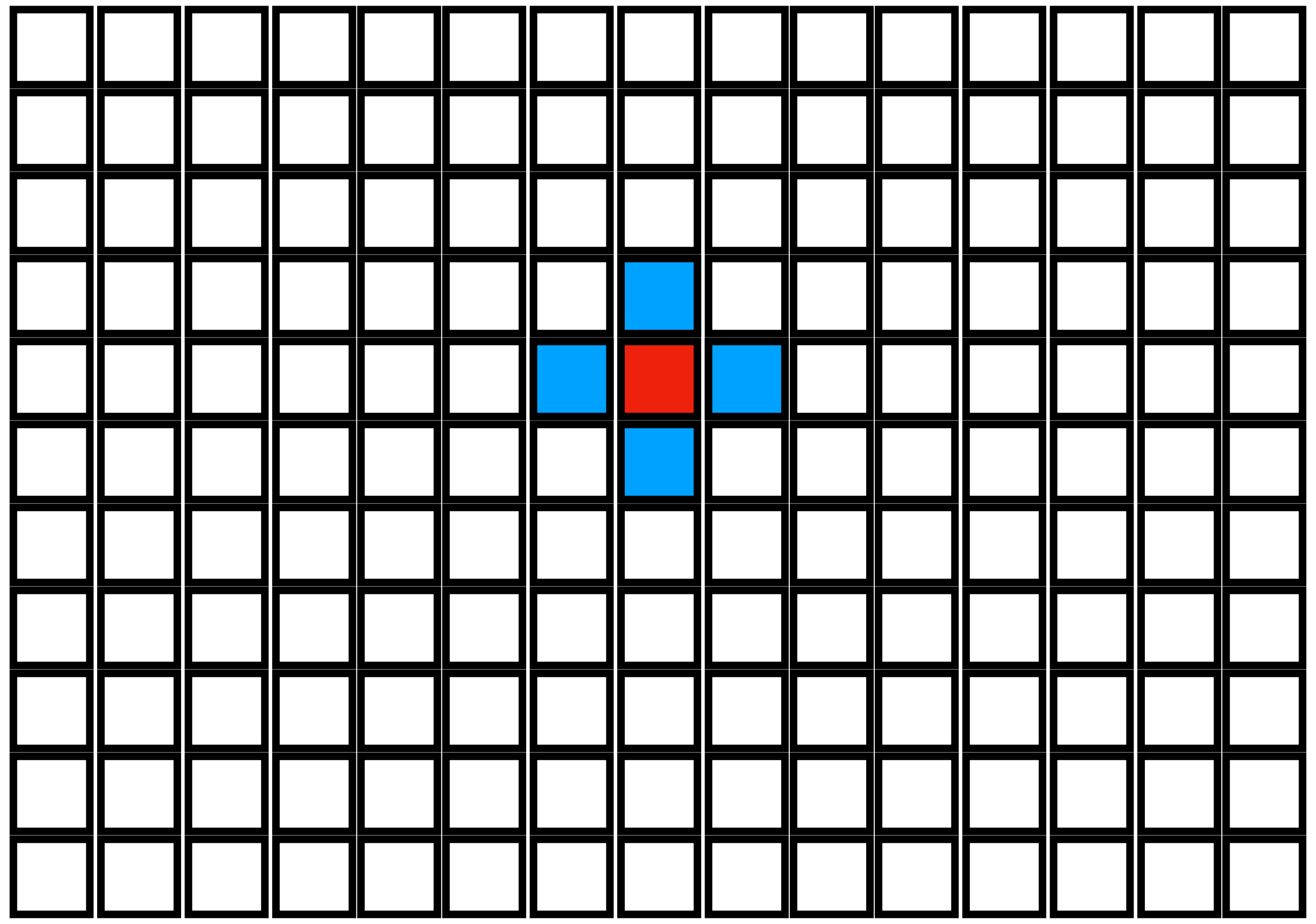
Neighbourhood of a cell

 $= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$

C.A.: $U : \bigotimes_x \mathcal{H}_x \to \bigotimes_x \mathcal{H}_x$

Backward neighbourhood of the cell x_0

$$U^{-1}\mathsf{A}_{x_0}U \subseteq \mathsf{A}_{\mathsf{N}^-(x_0)} \otimes I_{\bar{x}_0}$$



Locality

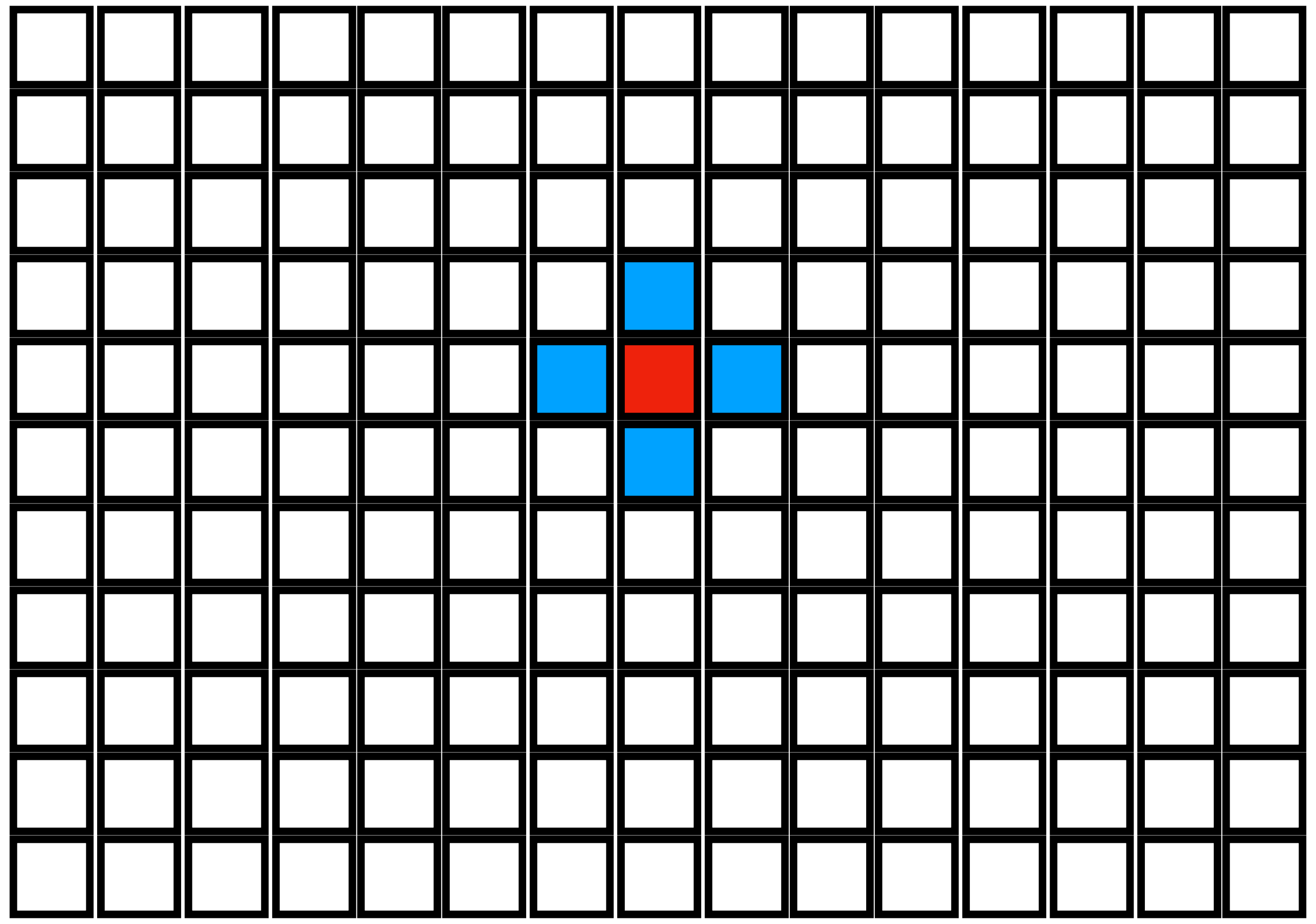
Neighbourhood of a cell

■ $= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$

C.A.: $U : \bigotimes_x \mathcal{H}_x \to \bigotimes_x \mathcal{H}_x$

Backward neighbourhood of the cell x_0

$$U^{-1}\mathsf{A}_{x_0}U \subseteq \mathsf{A}_{N^-(x_0)} \otimes I_{\bar{x}_0}$$



More precisely, $N^-(x_0)$ is the smallest set of cells such that $U^{-1}\mathsf{A}_{x_0}U \subseteq \mathsf{A}_{N^-(x_0)} \otimes I_{\bar{x}_0}$

Locality

Neighbourhood of a cell

 $= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$

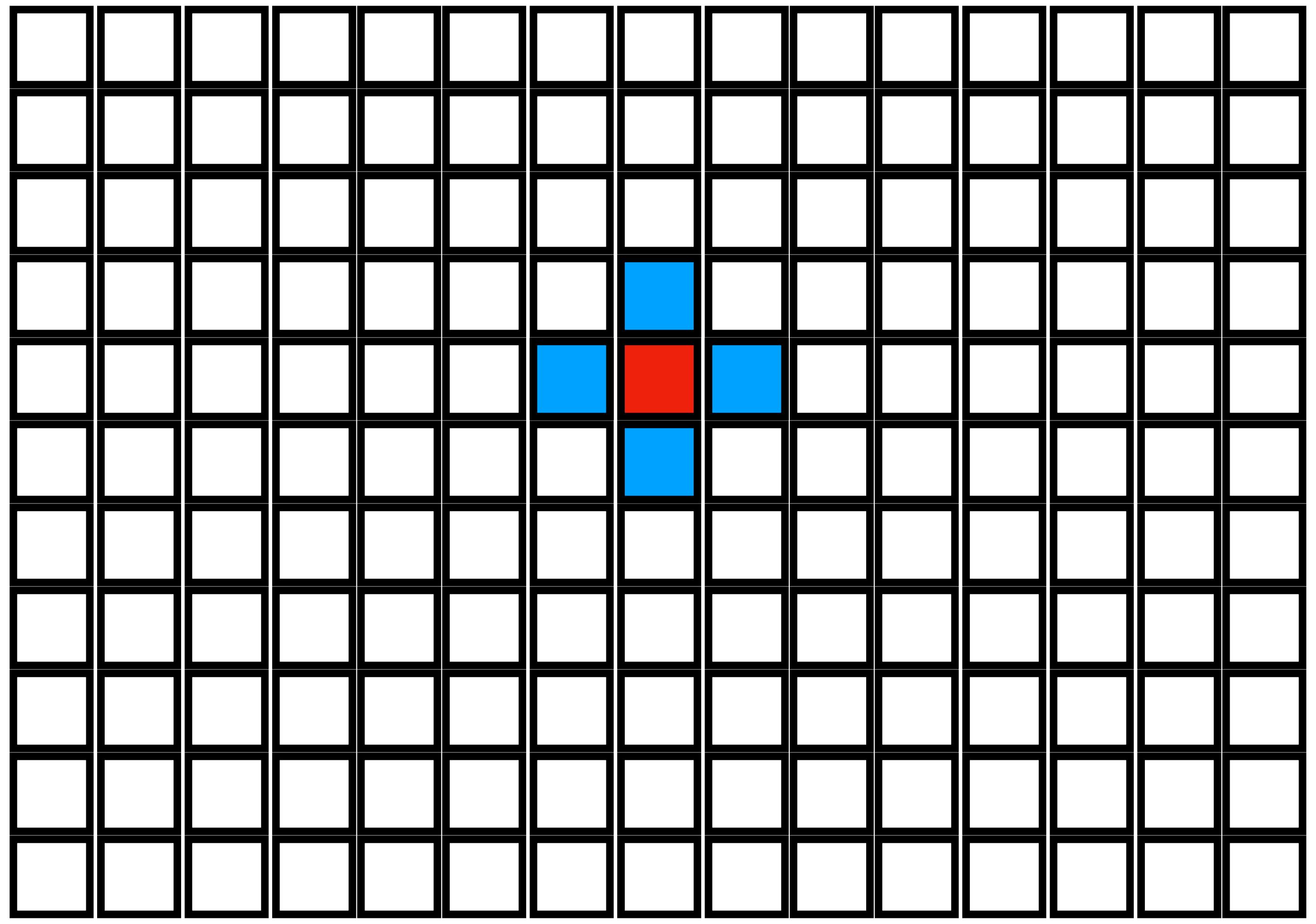
C.A.: $U : \bigotimes_x \mathcal{H}_x \to \bigotimes_x \mathcal{H}_x$

Backward neighbourhood of the cell x_0

$$U^{-1}\mathsf{A}_{x_0}U \subseteq \mathsf{A}_{N^-(x_0)} \otimes I_{\bar{x}_0}$$

Forward neighbourhood of the cell x_0

$$y \in N^+(x_0) \;\Leftrightarrow\; x_0 \in N^-(y)$$



More precisely, $N^-(x_0)$ is the smallest set of cells such that $U^{-1}\mathsf{A}_{x_0}U \subseteq \mathsf{A}_{N^-(x_0)} \otimes I_{\bar{x}_0}$

Locality

Neighbourhood of a cell

 $= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$

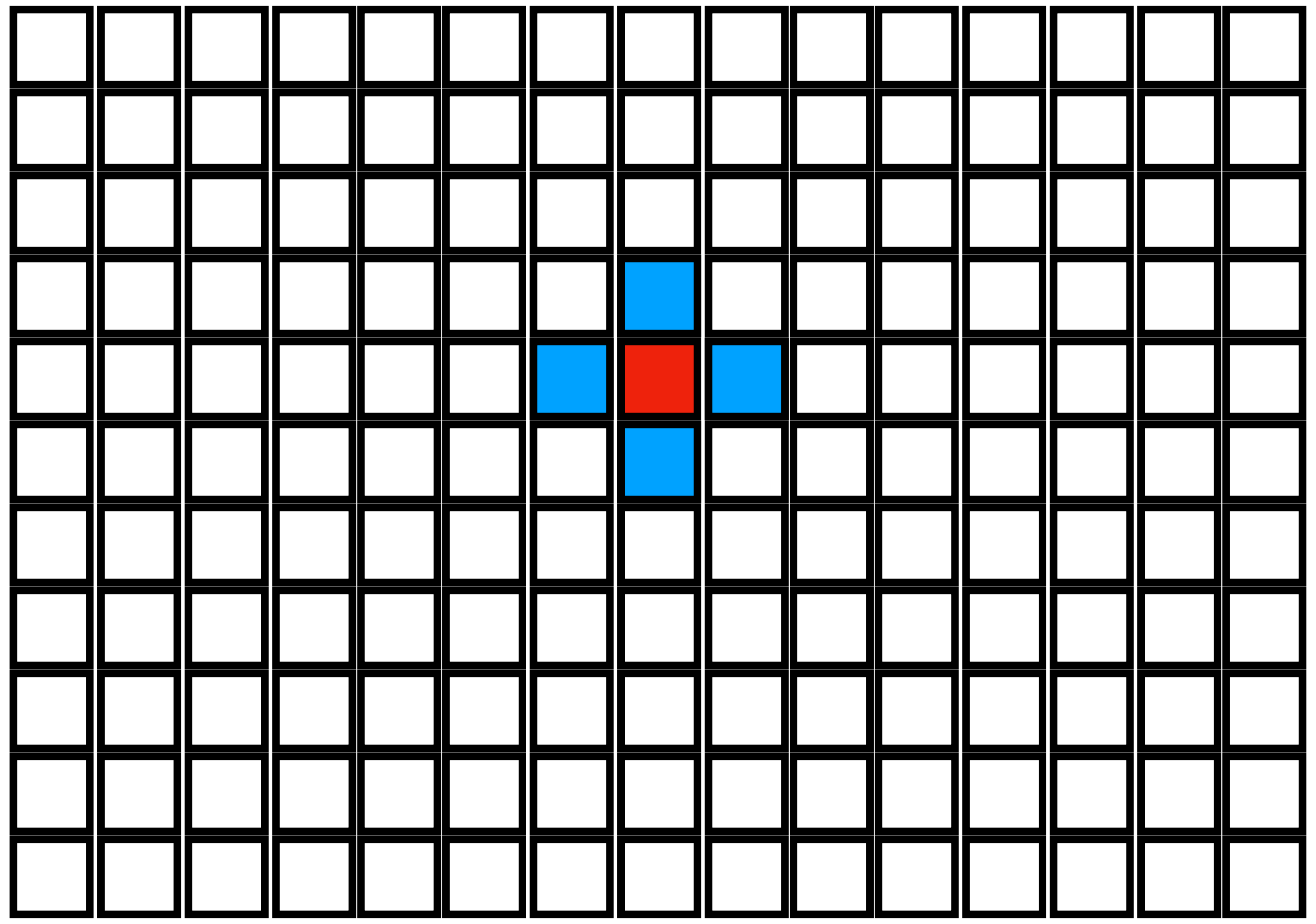
C.A.: $U : \bigotimes_x \mathcal{H}_x \to \bigotimes_x \mathcal{H}_x$

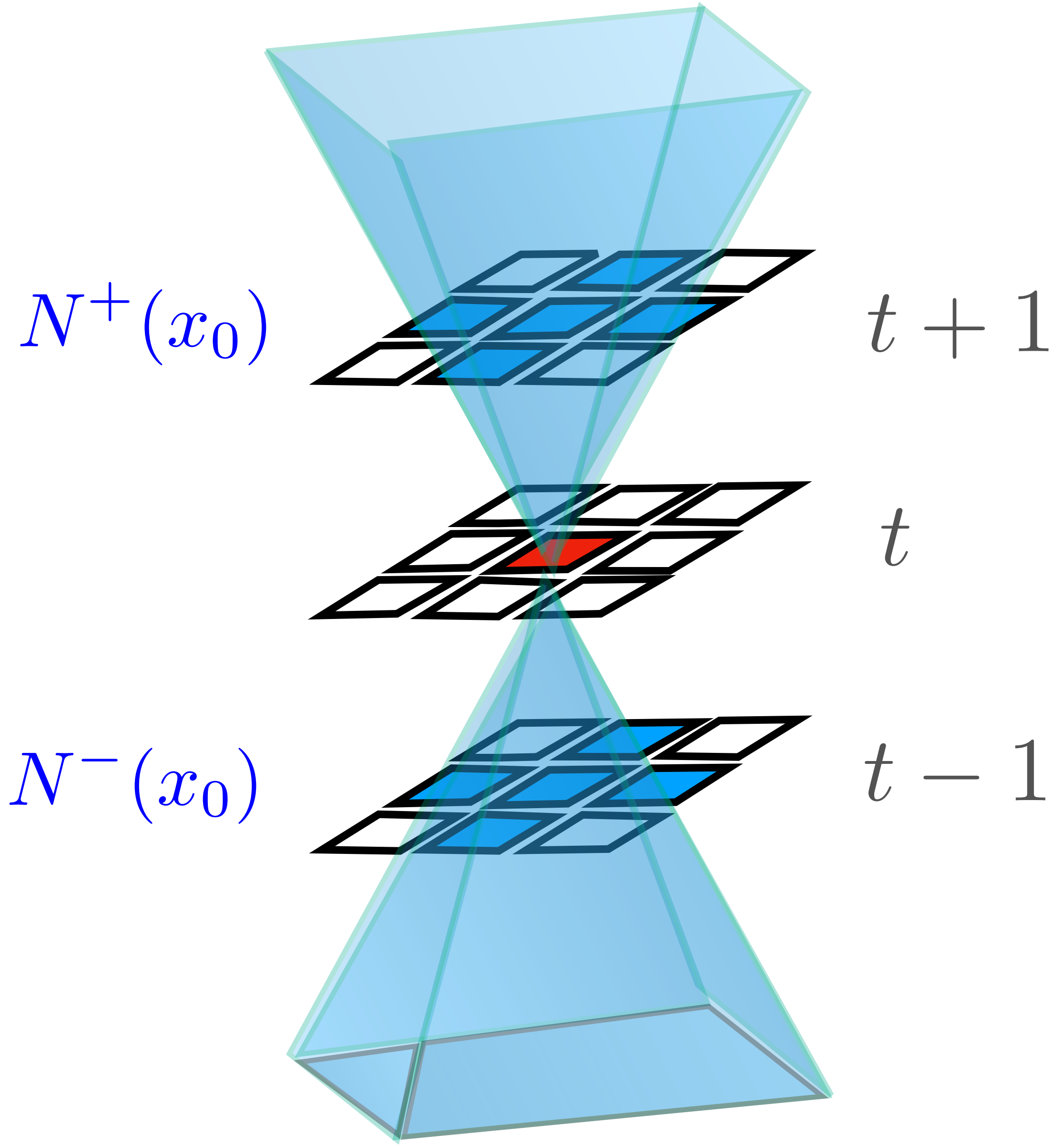
Backward neighbourhood of the cell x_0

$$U^{-1} \mathsf{A}_{x_0} U \subseteq \mathsf{A}_{N^-(x_0)} \otimes I_{\bar{x}_0}$$

Forward neighbourhood of the cell x_0

$$y \in N^+(x_0) \;\Leftrightarrow\; x_0 \in N^-(y)$$



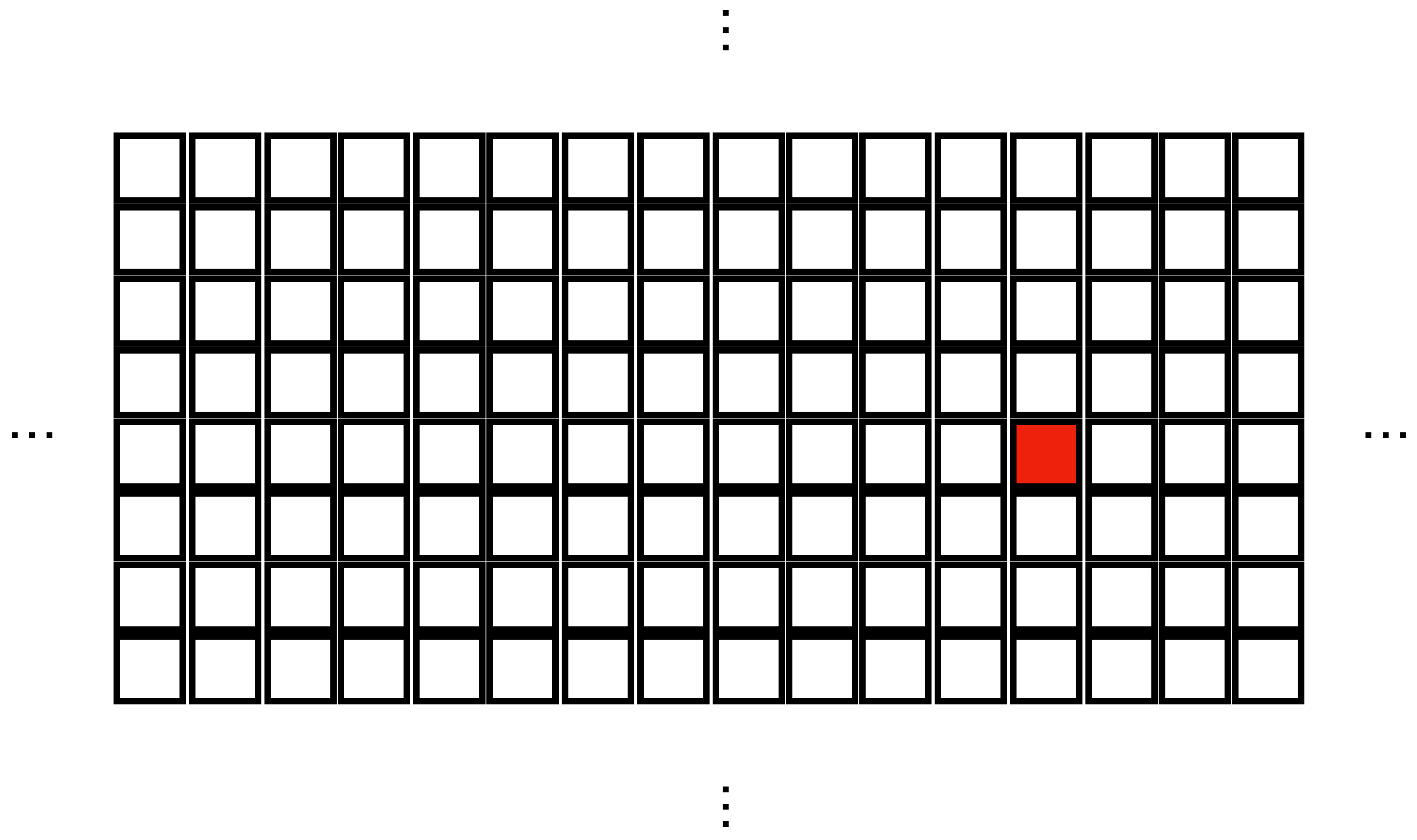


More precisely, $N^-(x_0)$ is the smallest set of cells such that $U^{-1} \mathsf{A}_{x_0} U \subseteq \mathsf{A}_{N^-(x_0)} \otimes I_{\bar{x}_0}$

Quantum cellular automata

Infinite case

 $= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$

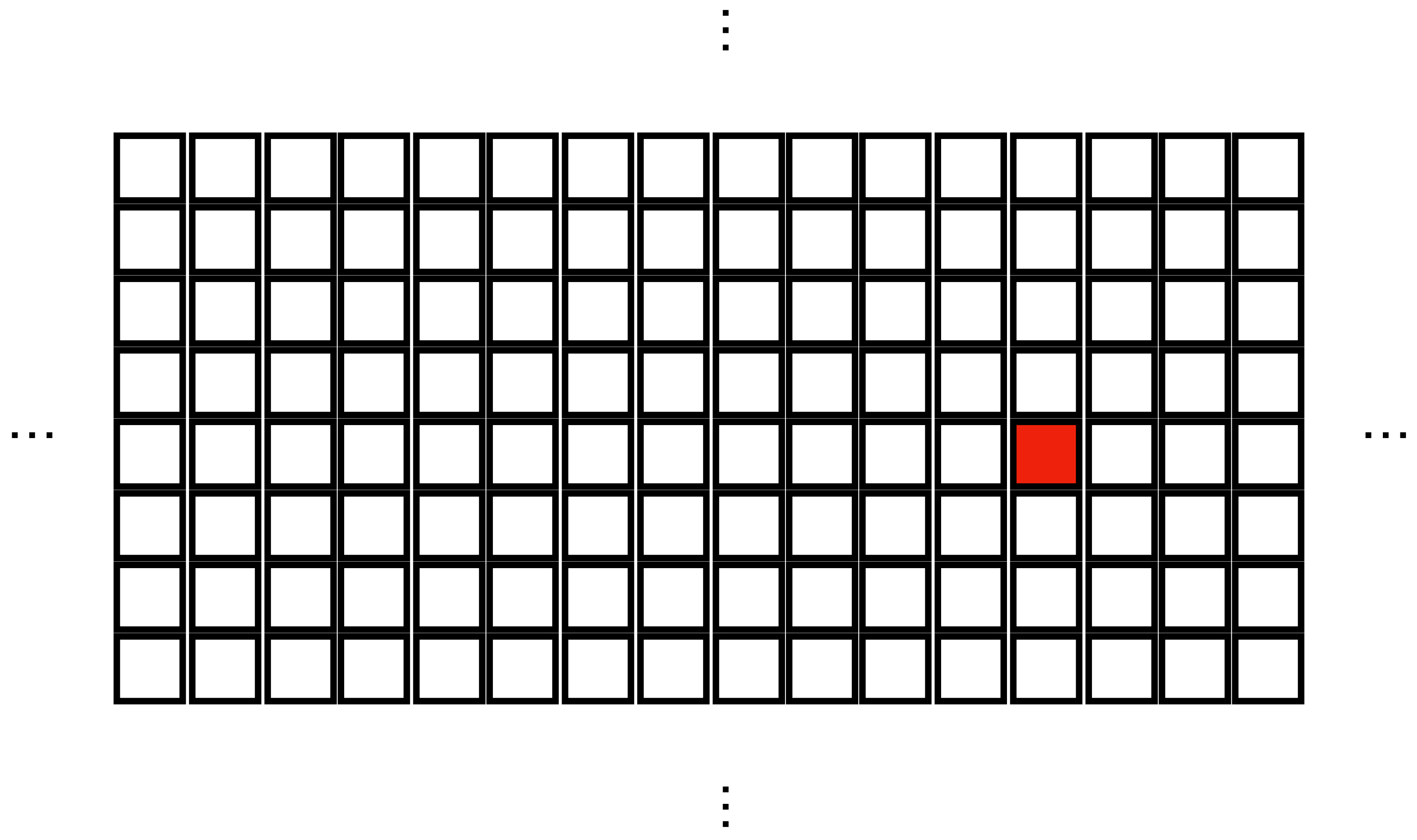


B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Quantum cellular automata

Infinite case

$= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$



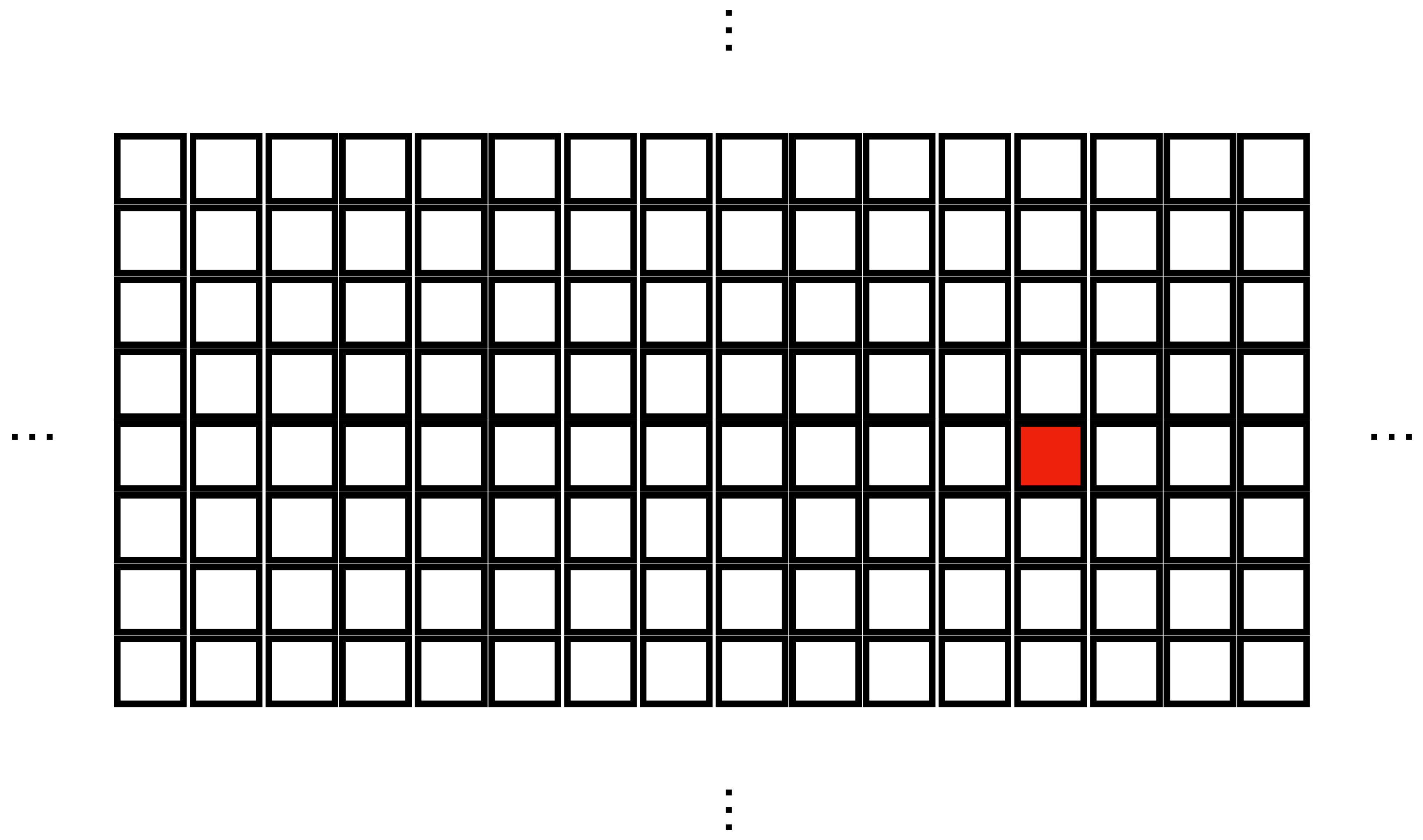
$$\mathsf{A} = \text{‘}\bigotimes_x\text{’}\mathsf{A}_x$$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Infinite system

Inductive limit

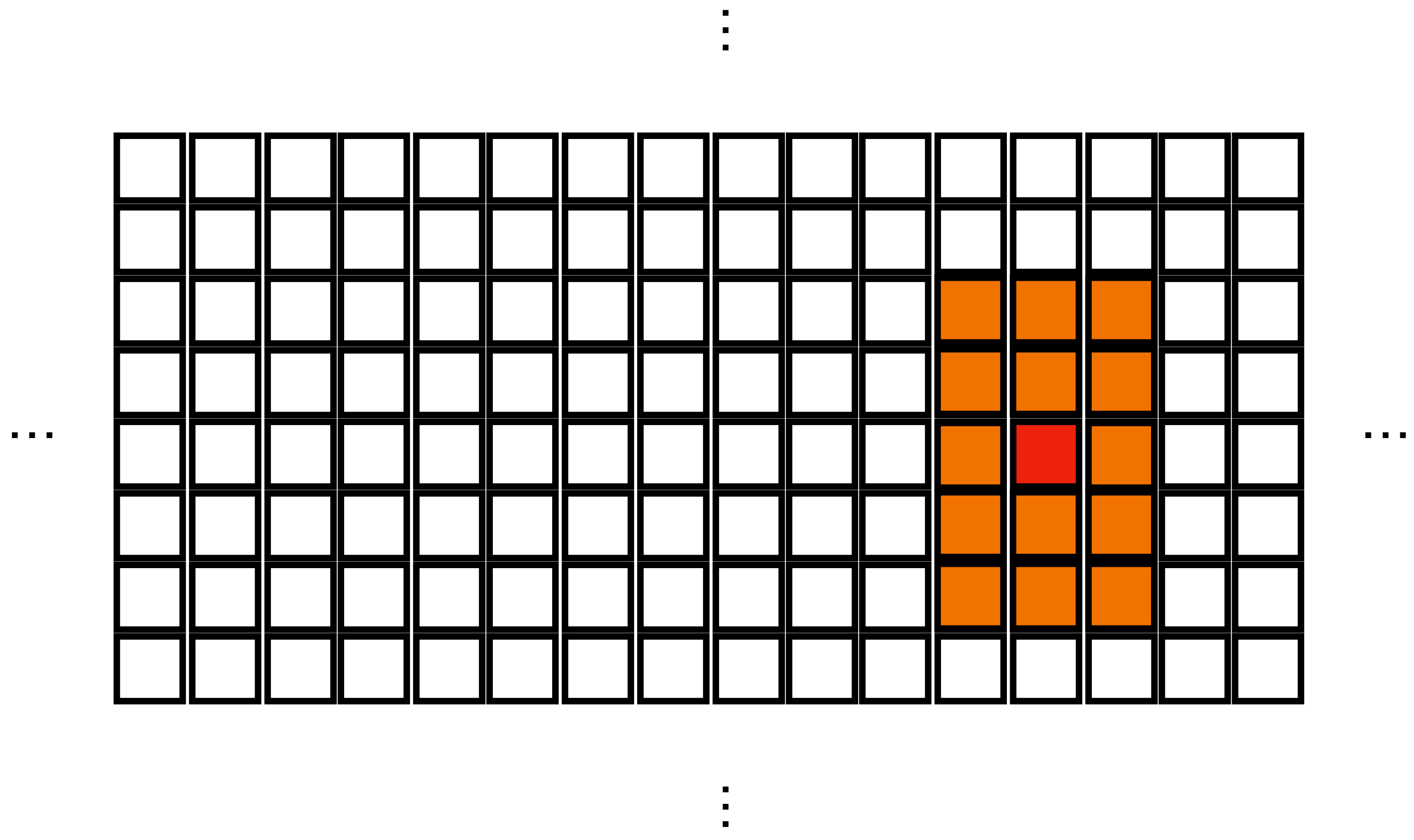
$= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$



Infinite system

Inductive limit

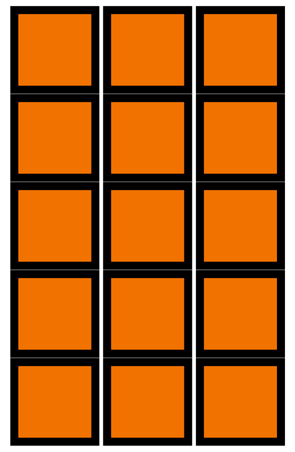
 $= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$



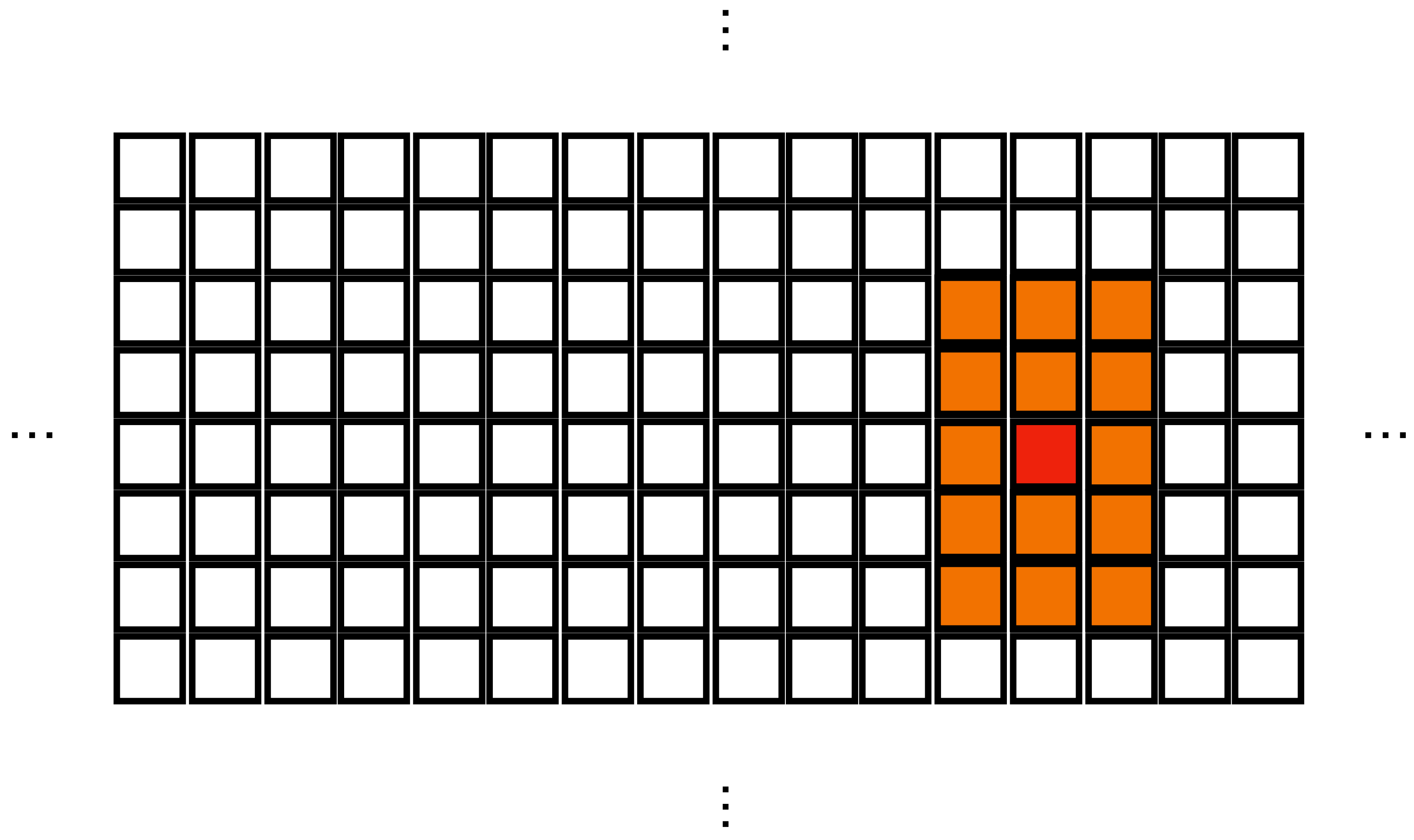
Infinite system

Inductive limit

$= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$



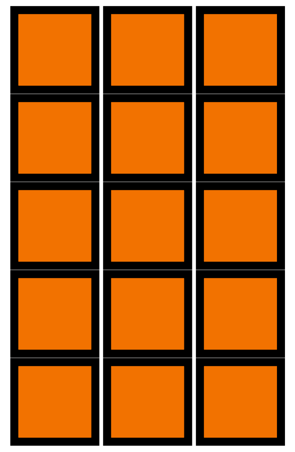
$$\mathsf{A}_R = \bigotimes_{x \in R} \mathsf{A}_x$$



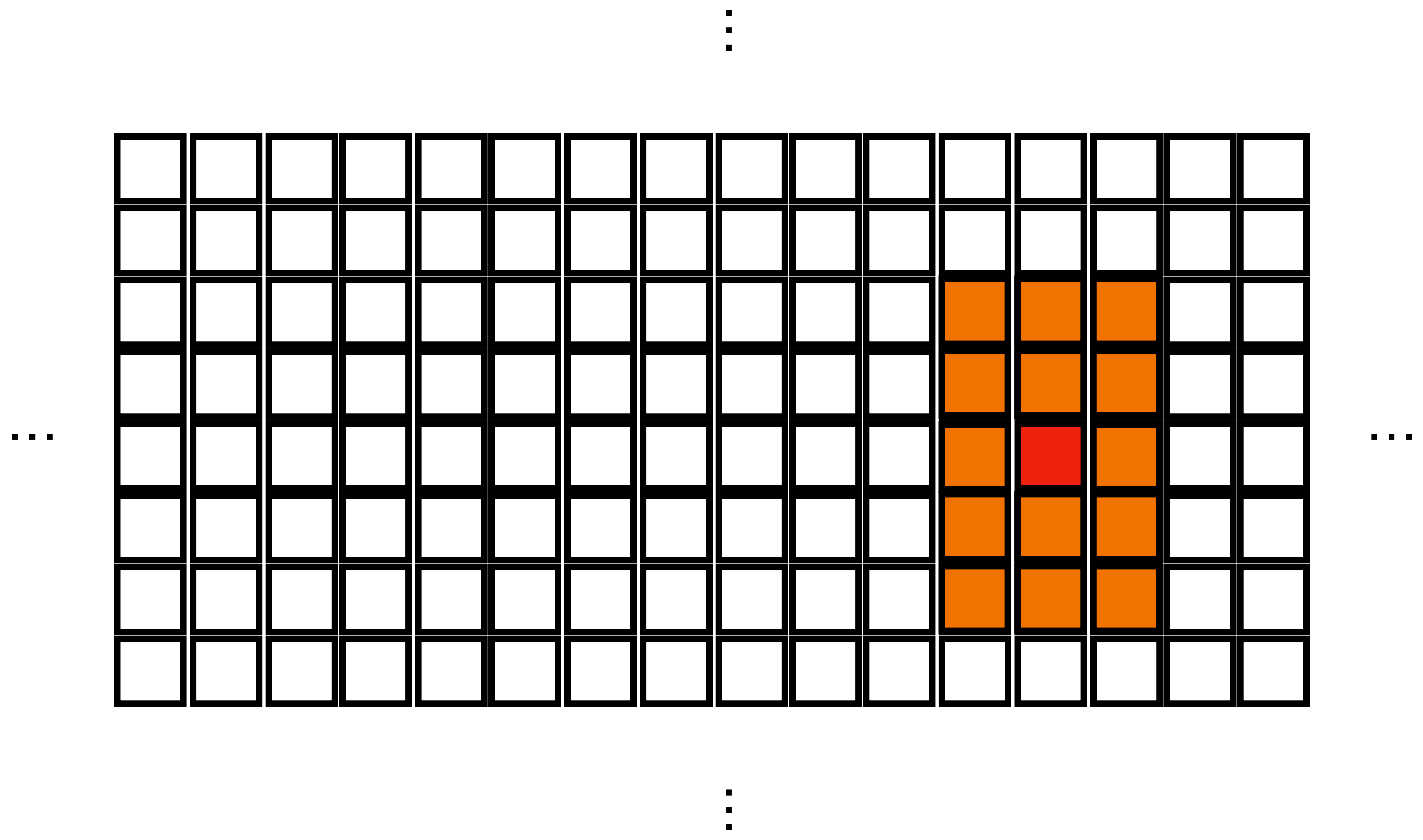
Infinite system

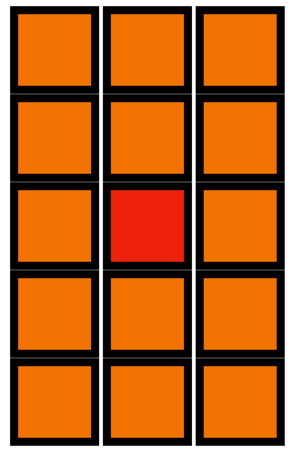
Inductive limit

 $= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$



$$\mathsf{A}_R = \bigotimes_{x \in R} \mathsf{A}_x$$



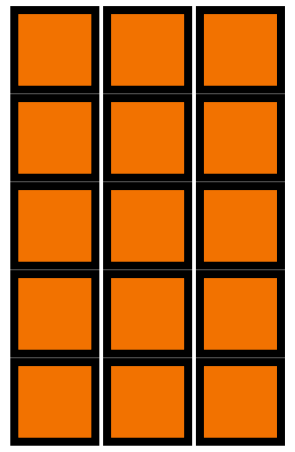


A_{x_0} ‘ $\subseteq$ ’ A_R

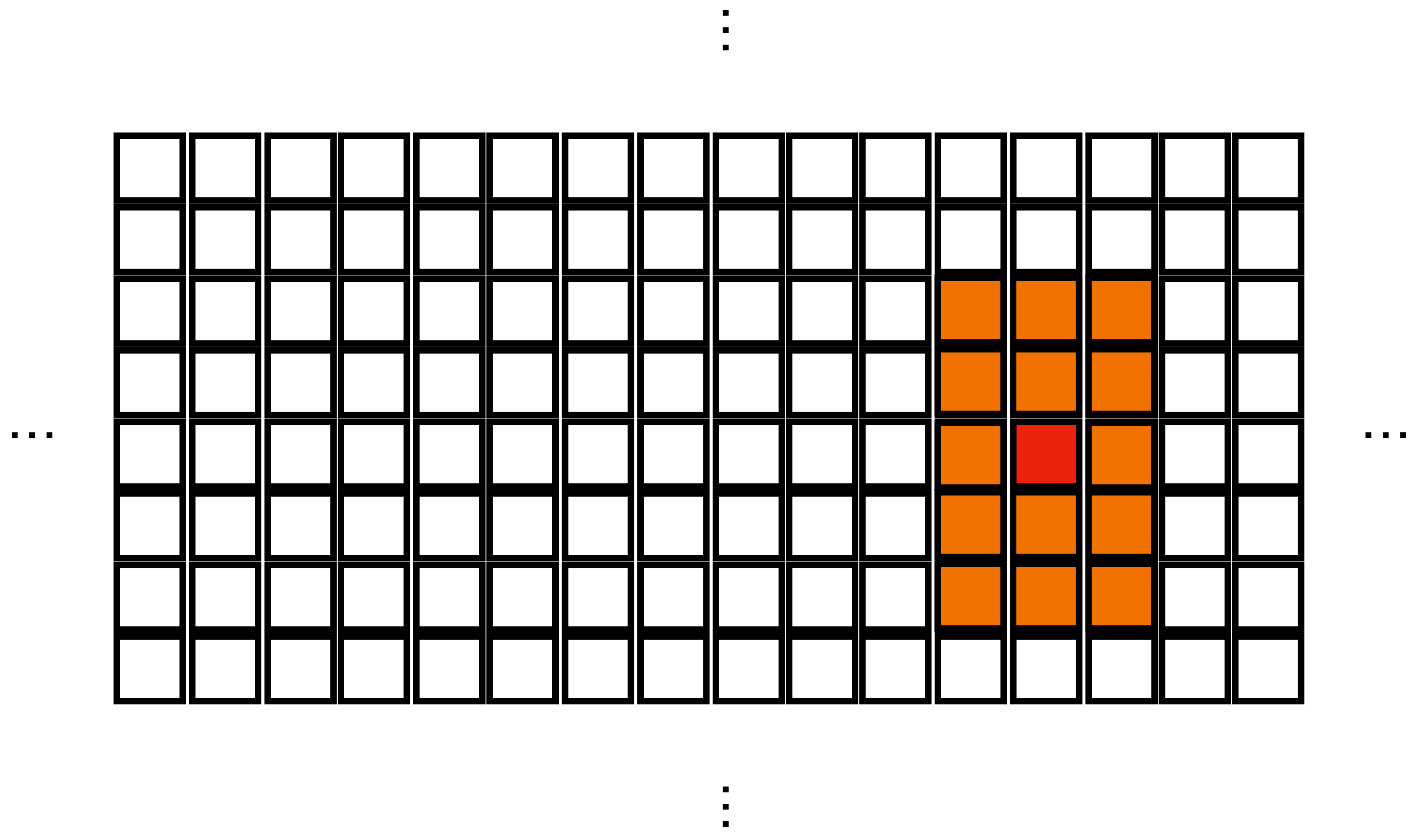
Infinite system

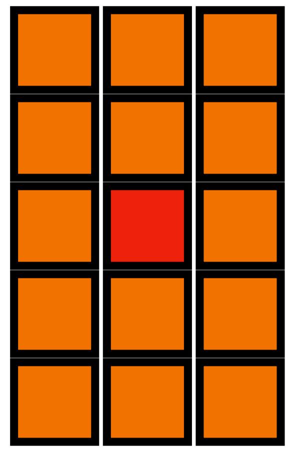
Inductive limit

 $= \mathcal{H}_x \leftrightarrow \mathsf{A}_x$



$$\mathsf{A}_R = \bigotimes_{x \in R} \mathsf{A}_x$$



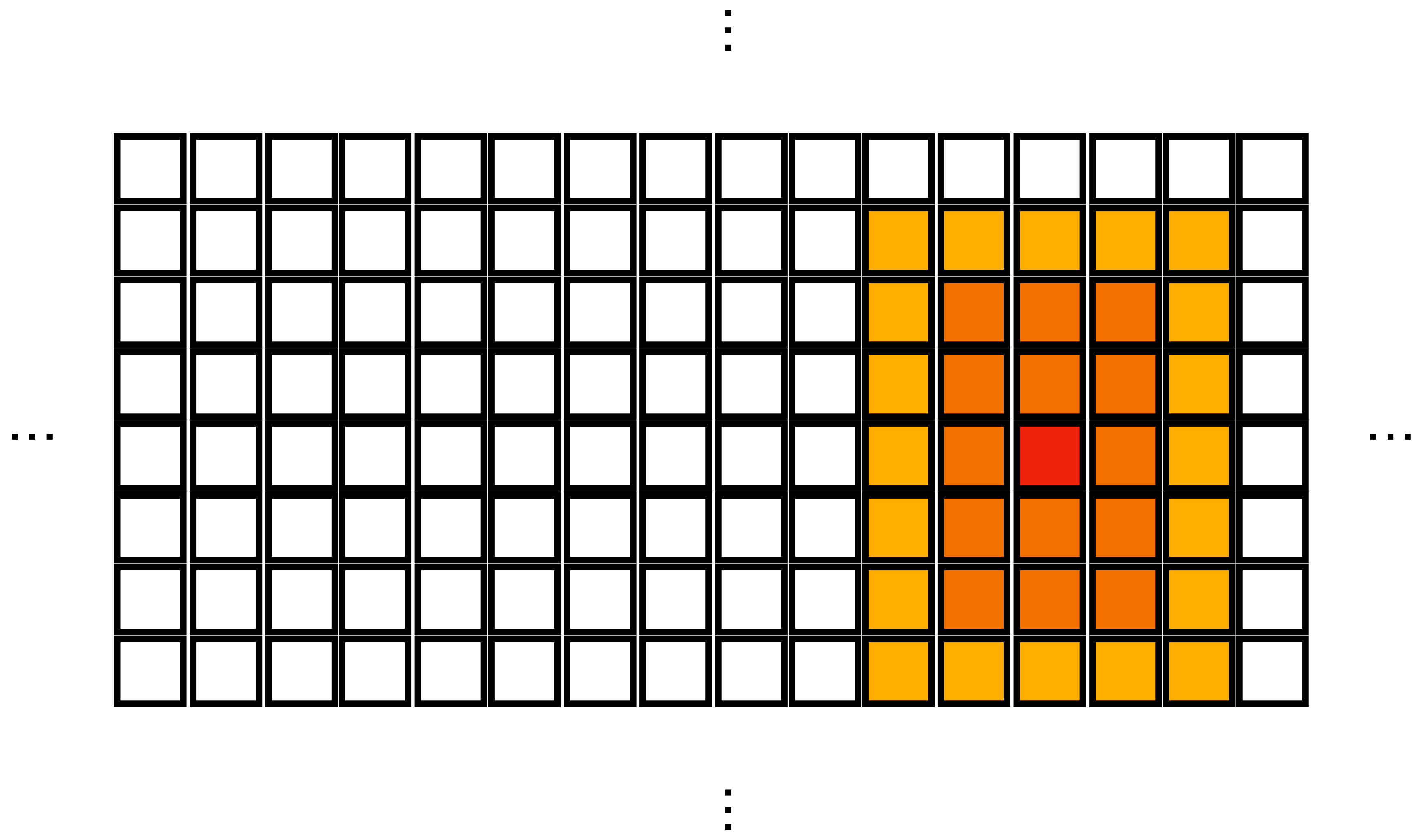


A_{x_0} ‘ $\subseteq$ ’A_R

$$f_{x_0,R} : \mathsf{A}_{x_0} \to \mathsf{A}_R$$
$$f_{x_0,R}(B) = B \otimes I_{R \setminus \{x_0\}}$$

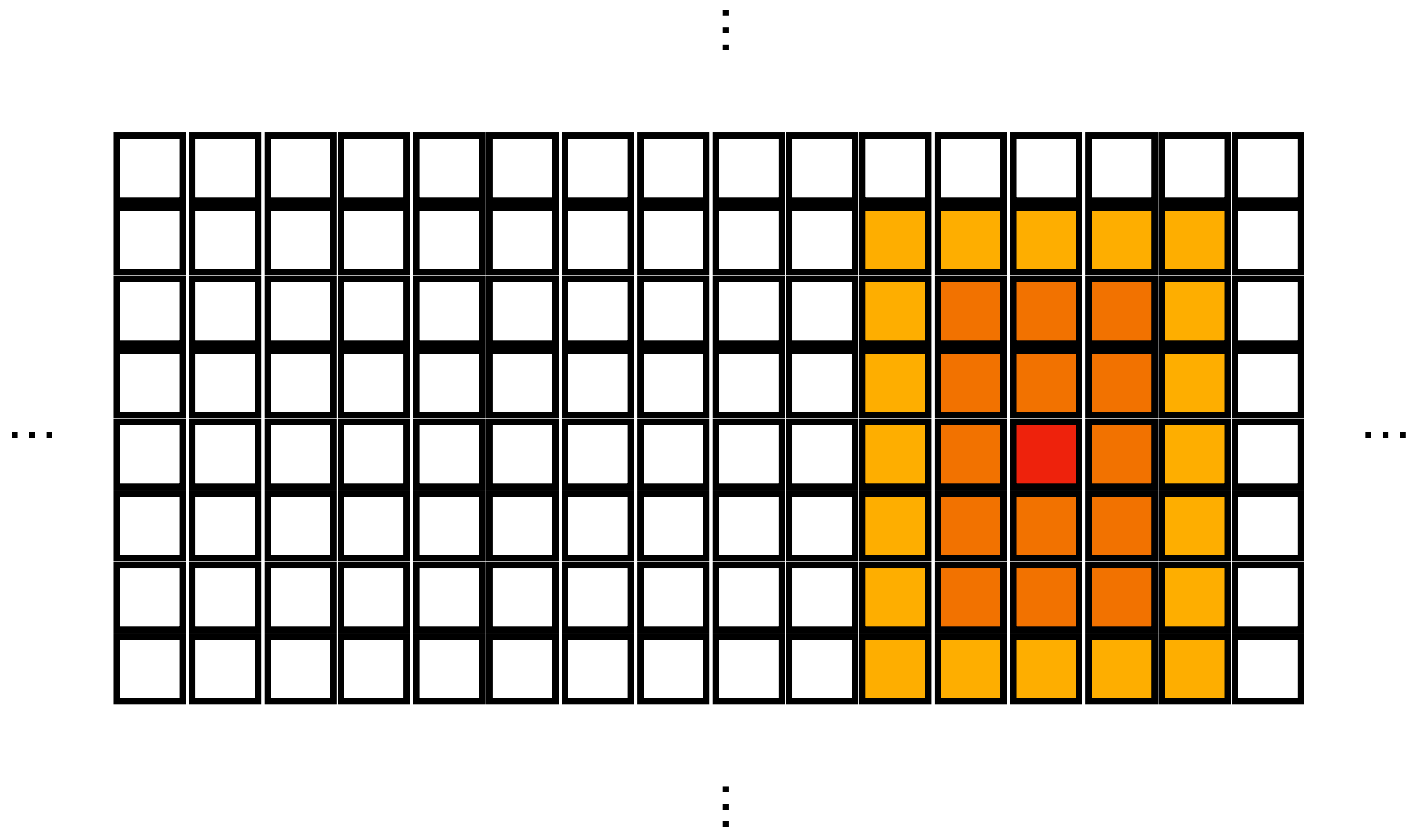
Infinite system

Inductive limit



Infinite system

Inductive limit



$R_0 \subseteq R_1 \subseteq R_2 \subseteq \dots$

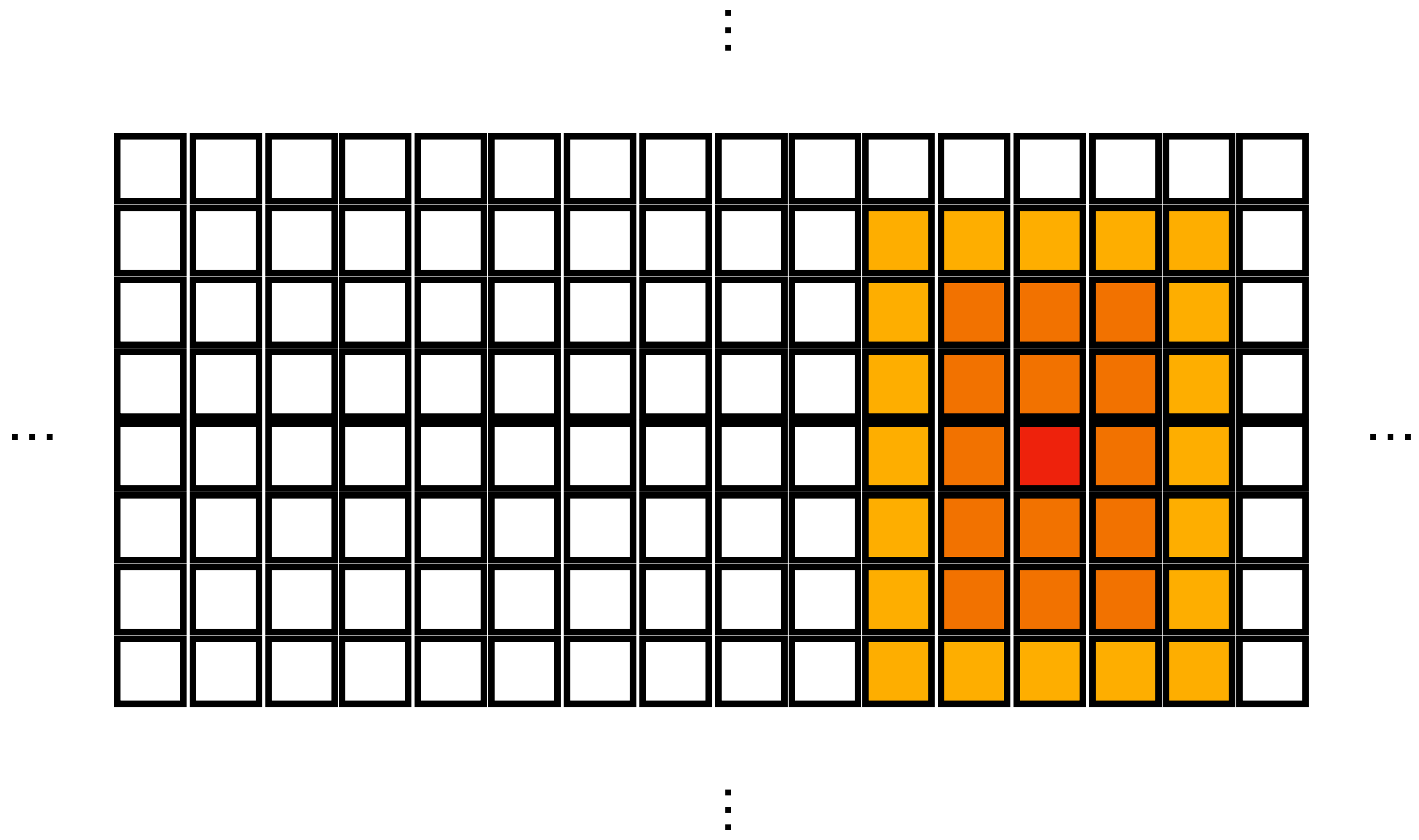
$\mathsf{A}_{R_0} \subseteq \mathsf{A}_{R_1} \subseteq \mathsf{A}_{R_2} \subseteq \dots$

$f_{ij} : \mathsf{A}_{R_i} \to \mathsf{A}_{R_j}$

$f_{ij}(B) = B \otimes I_{R_j \setminus R_i}$

Infinite system

Inductive limit



$R_0 \subseteq R_1 \subseteq R_2 \subseteq \dots$

$\mathsf{A}_{R_0} \subseteq \mathsf{A}_{R_1} \subseteq \mathsf{A}_{R_2} \subseteq \dots$

$f_{ij} : \mathsf{A}_{R_i} \to \mathsf{A}_{R_j}$

$f_{ij}(B) = B \otimes I_{R_j \setminus R_i}$

$B_j \sim B_i \;\text{ if }\; \exists k \geq i,j \;\text{ s.t. }\; f_{ik}(B_i) = f_{jk}(B_j)$

$$\mathsf{A}_L := \bigsqcup_i \mathsf{A}_{R_i} / \sim$$

Infinite system

Topological limit

- Inductive limit: all local operators on arbitrarily large but finite regions

Infinite system

Topological limit

- Inductive limit: all local operators on arbitrarily large but finite regions
- We equip A_L with the operational norm $\|B_i\|_{\text{op}} = \|B\|_{\infty,\mathcal{H}_{R_i}}$

Infinite system

Topological limit

- Inductive limit: all local operators on arbitrarily large but finite regions
- We equip A_L with the operational norm $\|B_i\|_{\text{op}} = \|B\|_{\infty,\mathcal{H}_{R_i}}$
- Let B_{i_n} be a Cauchy sequence in A_L

Infinite system

Topological limit

- Inductive limit: all local operators on arbitrarily large but finite regions
- We equip A_L with the operational norm $\|B_i\|_{\text{op}} = \|B\|_{\infty,\mathcal{H}_{R_i}}$
- Let B_{i_n} be a Cauchy sequence in A_L
  - $\|B_{i_n} - B_{i_m}\|_{\text{op}} \leq \varepsilon$ means that it is hard to discriminate B_{i_n} from B_{i_m}

Infinite system

Topological limit

- Inductive limit: all local operators on arbitrarily large but finite regions
- We equip A_L with the operational norm $\|B_i\|_{\text{op}} = \|B\|_{\infty,\mathcal{H}_{R_i}}$
- Let B_{i_n} be a Cauchy sequence in A_L
  - $\|B_{i_n} - B_{i_m}\|_{\text{op}} \leq \varepsilon$ means that it is hard to discriminate B_{i_n} from B_{i_m}
- $B_{i_n} \sim C_{i_n}$ if $\forall \varepsilon > 0 \exists n_0$ s.t. $\forall n \geq n_0$ $\|B_{i_n} - C_{i_n}\|_{\text{op}} \leq \varepsilon$

Infinite system

Topological limit

- Inductive limit: all local operators on arbitrarily large but finite regions
- We equip A_L with the operational norm $\|B_i\|_{\mathrm{op}} = \|B\|_{\infty,\mathcal{H}_{R_i}}$
- Let B_{i_n} be a Cauchy sequence in A_L
  - $\|B_{i_n} - B_{i_m}\|_{\mathrm{op}} \leq \varepsilon$ means that it is hard to discriminate B_{i_n} from B_{i_m}
- $B_{i_n} \sim C_{i_n}$ if $\forall \varepsilon > 0 \exists n_0 \;\; \text{s.t.} \;\; \forall n \geq n_0 \quad \|B_{i_n} - C_{i_n}\|_{\mathrm{op}} \leq \varepsilon$
- We can then define the algebra A of equivalence classes:
  it contains effects that can be approximated arbitrarily well by local effects

Infinite system

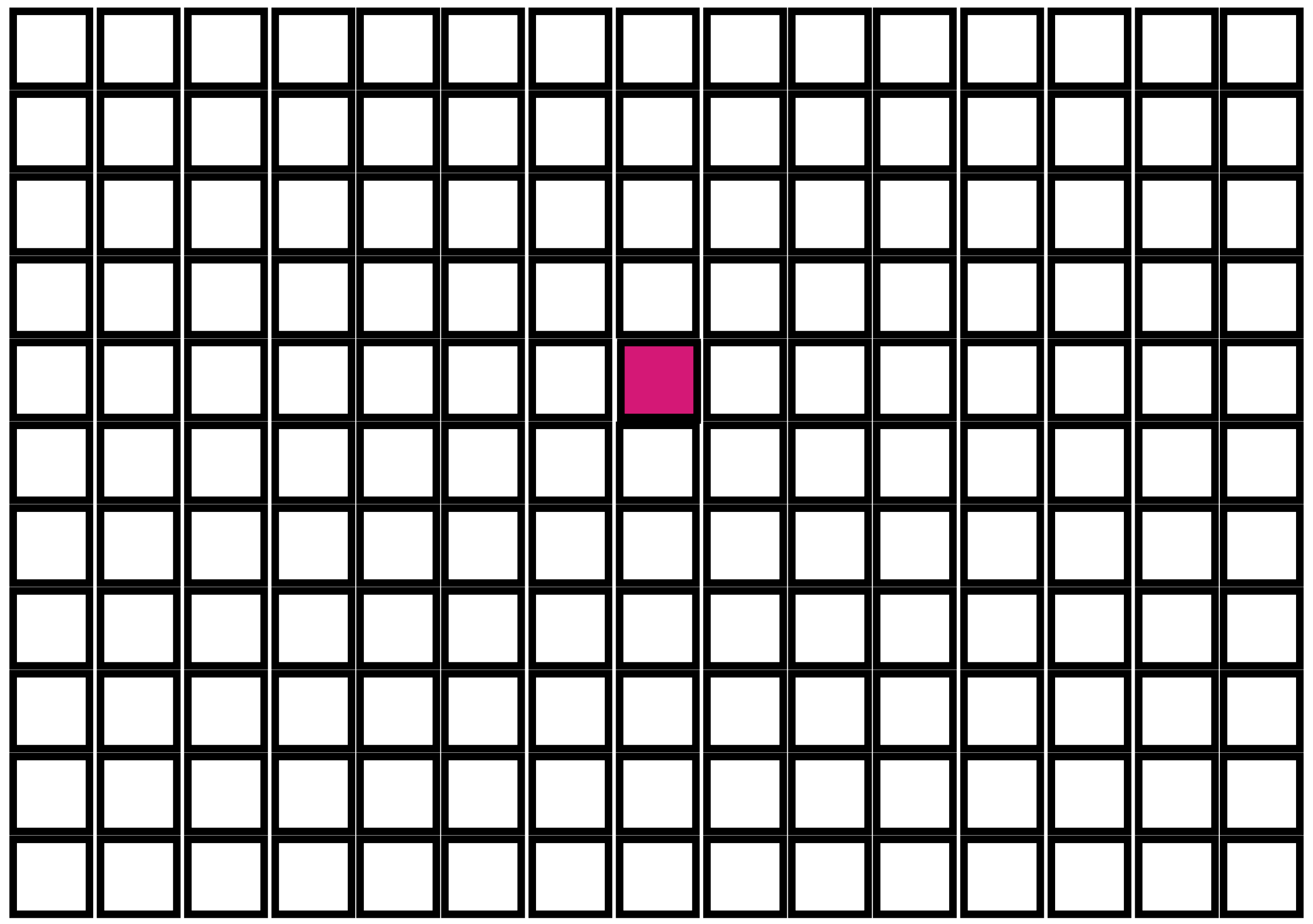
Topological limit

- Inductive limit: all local operators on arbitrarily large but finite regions
- We equip A_L with the operational norm $\|B_i\|_{\mathrm{op}} = \|B\|_{\infty,\mathcal{H}_{R_i}}$
- Let B_{i_n} be a Cauchy sequence in A_L
  - $\|B_{i_n} - B_{i_m}\|_{\mathrm{op}} \leq \varepsilon$ means that it is hard to discriminate B_{i_n} from B_{i_m}
- $B_{i_n} \sim C_{i_n}$ if $\forall \varepsilon > 0 \exists n_0 \ \text{s.t.} \ \forall n \geq n_0 \ \ \|B_{i_n} - C_{i_n}\|_{\mathrm{op}} \leq \varepsilon$
- We can then define the algebra A of equivalence classes:
  it contains effects that can be approximated arbitrarily well by local effects
- A is the quasi-local algebra

Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$

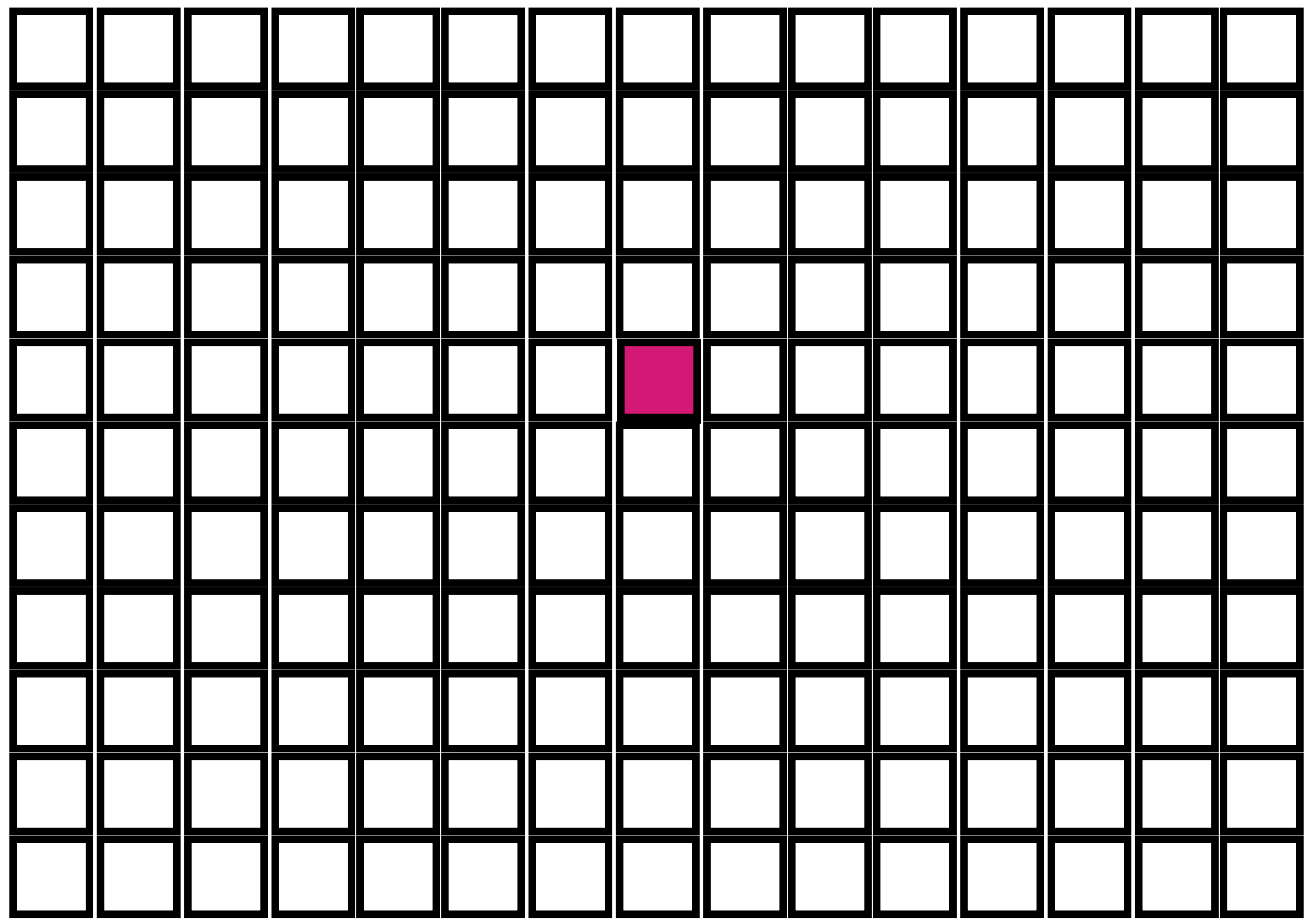


B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that



B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

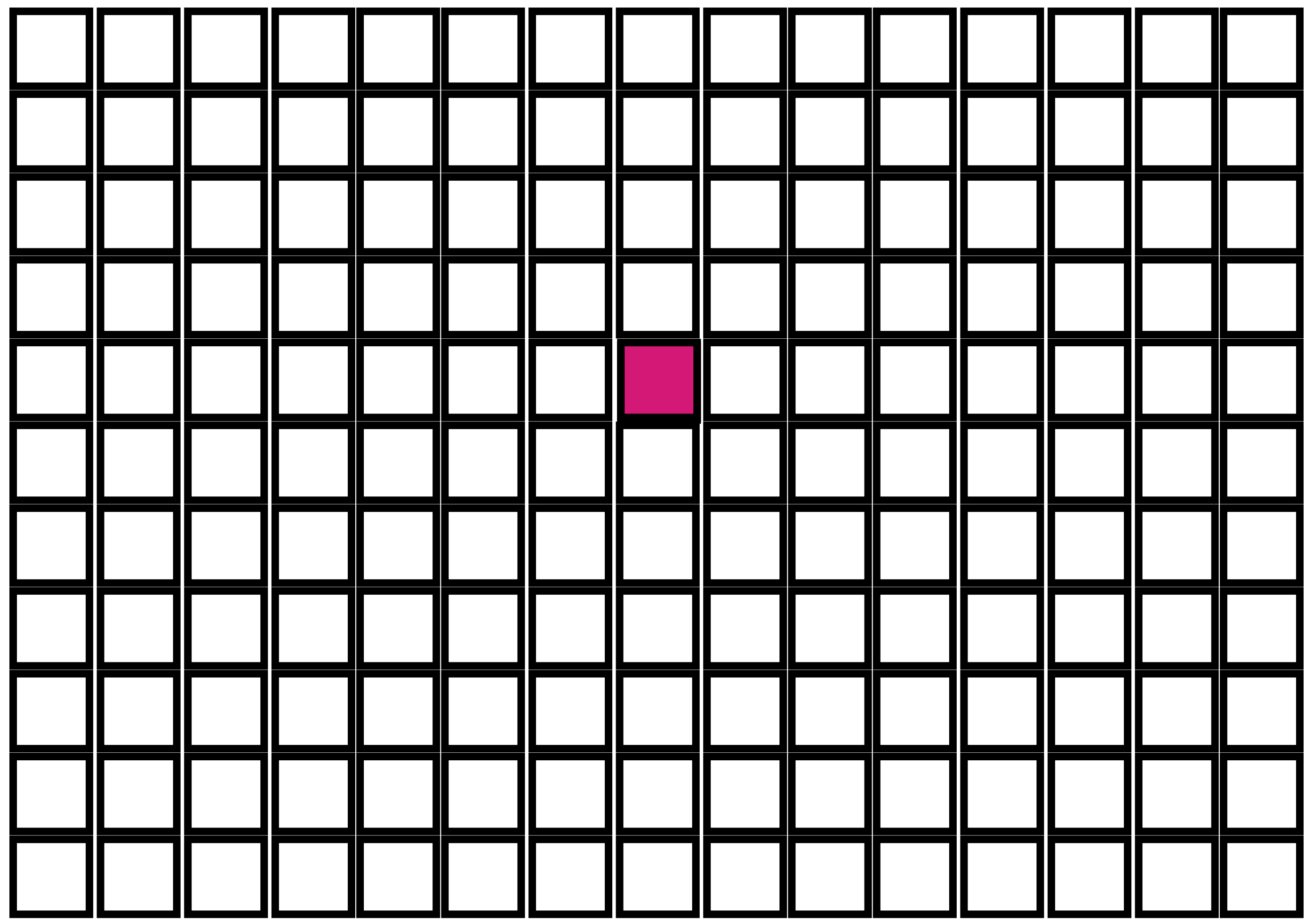
Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that
    - $\mathcal{V}$ satisfies locality

$$\forall x \ \exists \text{ a finite } N^+(x) \ \text{ s.t. } \ \mathcal{V}(\mathsf{A}_x) \subseteq \bigotimes_{y \in N^+(x)} \mathsf{A}_y$$

$$\boxed{\mathcal{V}(\mathsf{A}_L) \subseteq \mathsf{A}_L}$$



B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

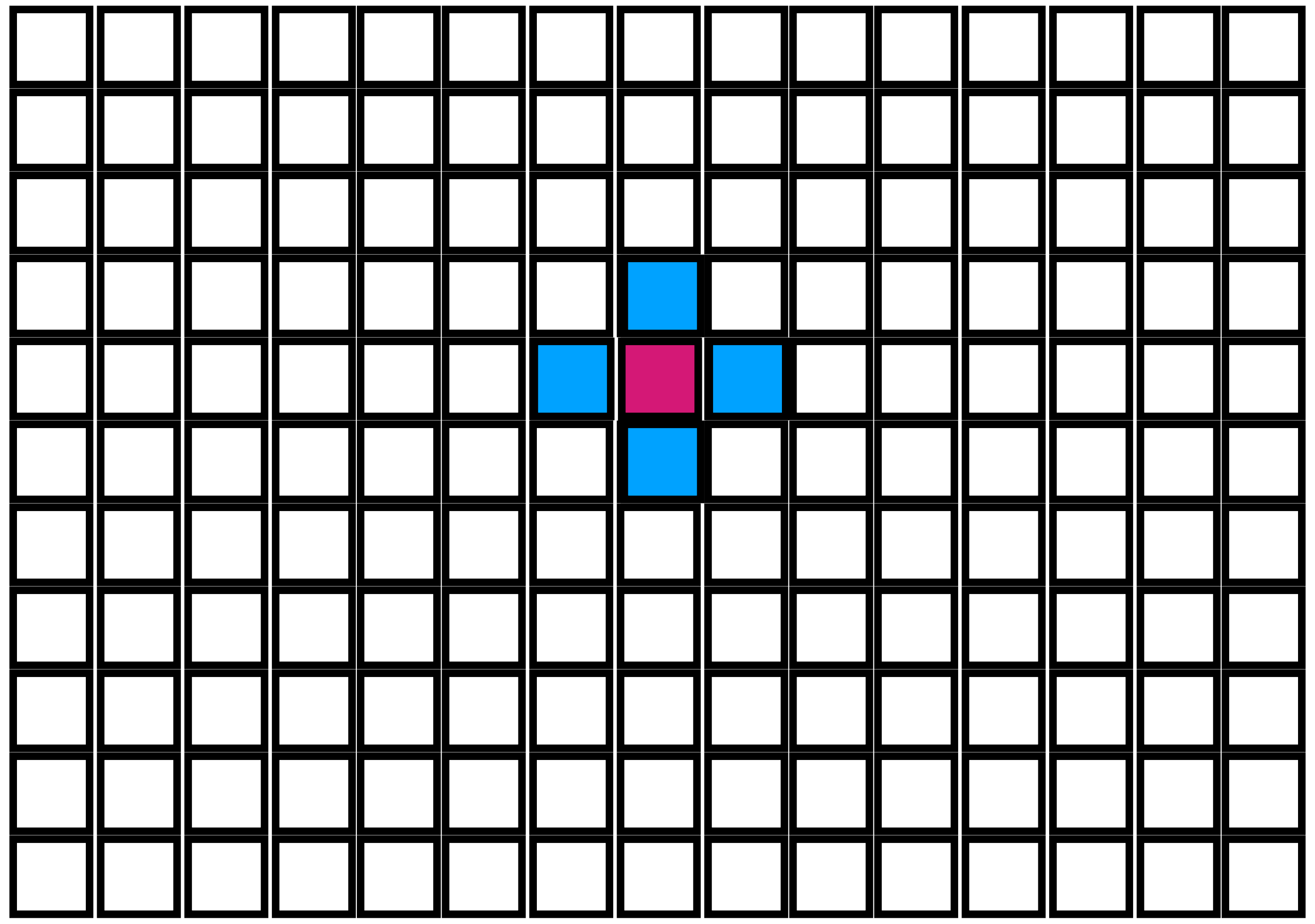
Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that
    - $\mathcal{V}$ satisfies locality

$$\forall x \ \exists \text{ a finite } N^+(x) \ \text{ s.t. } \ \mathcal{V}(\mathsf{A}_x) \subseteq \bigotimes_{y \in N^+(x)} \mathsf{A}_y$$

$$\mathcal{V}(\mathsf{A}_L) \subseteq \mathsf{A}_L$$



B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

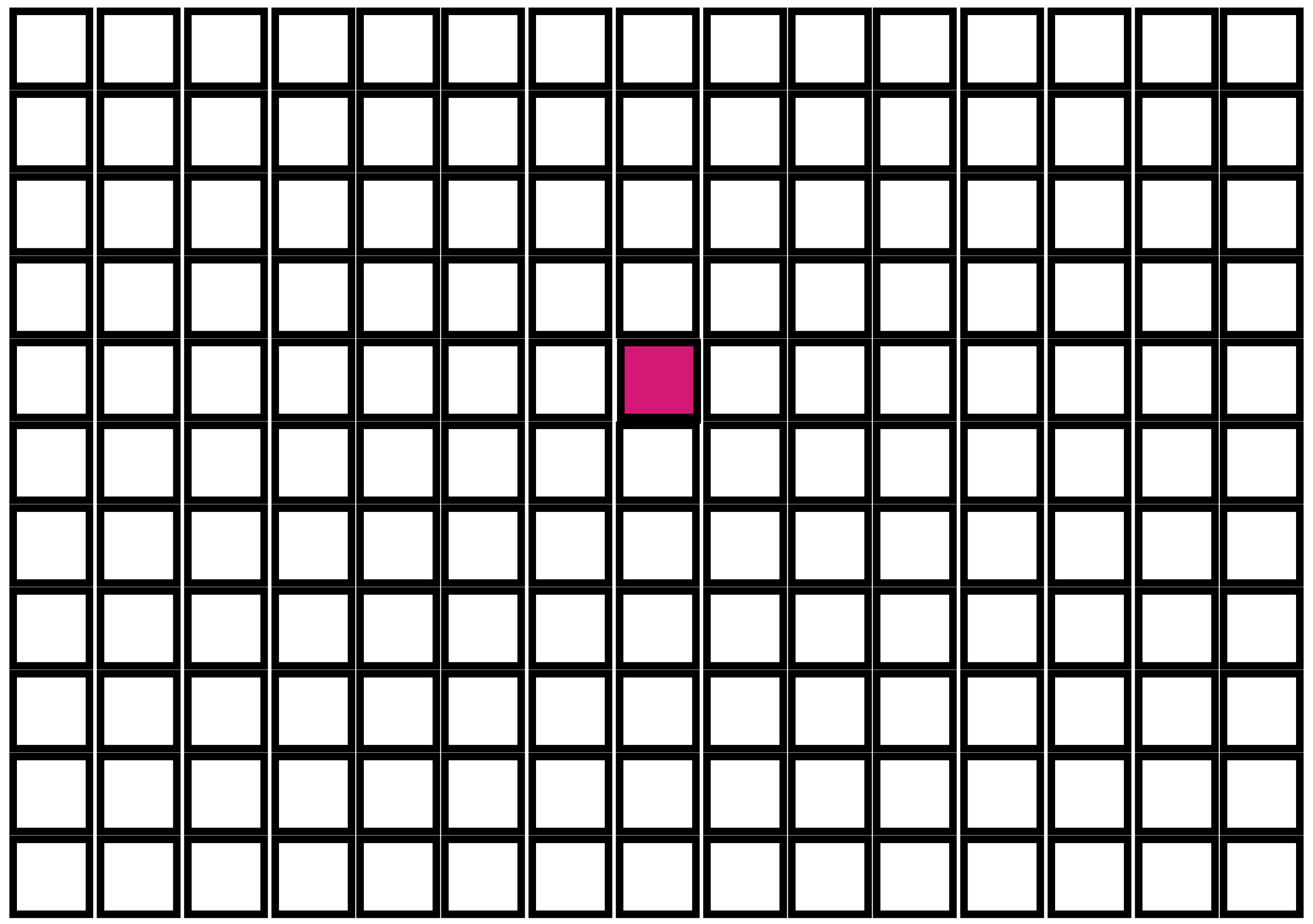
Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that
    - $\mathcal{V}$ satisfies locality

$$\forall x \ \exists \text{ a finite } N^+(x) \ \text{ s.t. } \ \mathcal{V}(\mathsf{A}_x) \subseteq \bigotimes_{y \in N^+(x)} \mathsf{A}_y$$

$$\boxed{\mathcal{V}(\mathsf{A}_L) \subseteq \mathsf{A}_L}$$



    - Translationally invariant

$$\mathscr{V}[\mathcal{T}_z(\mathsf{A}_x)] = \mathcal{T}_z[\mathscr{V}(\mathsf{A}_x)]$$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

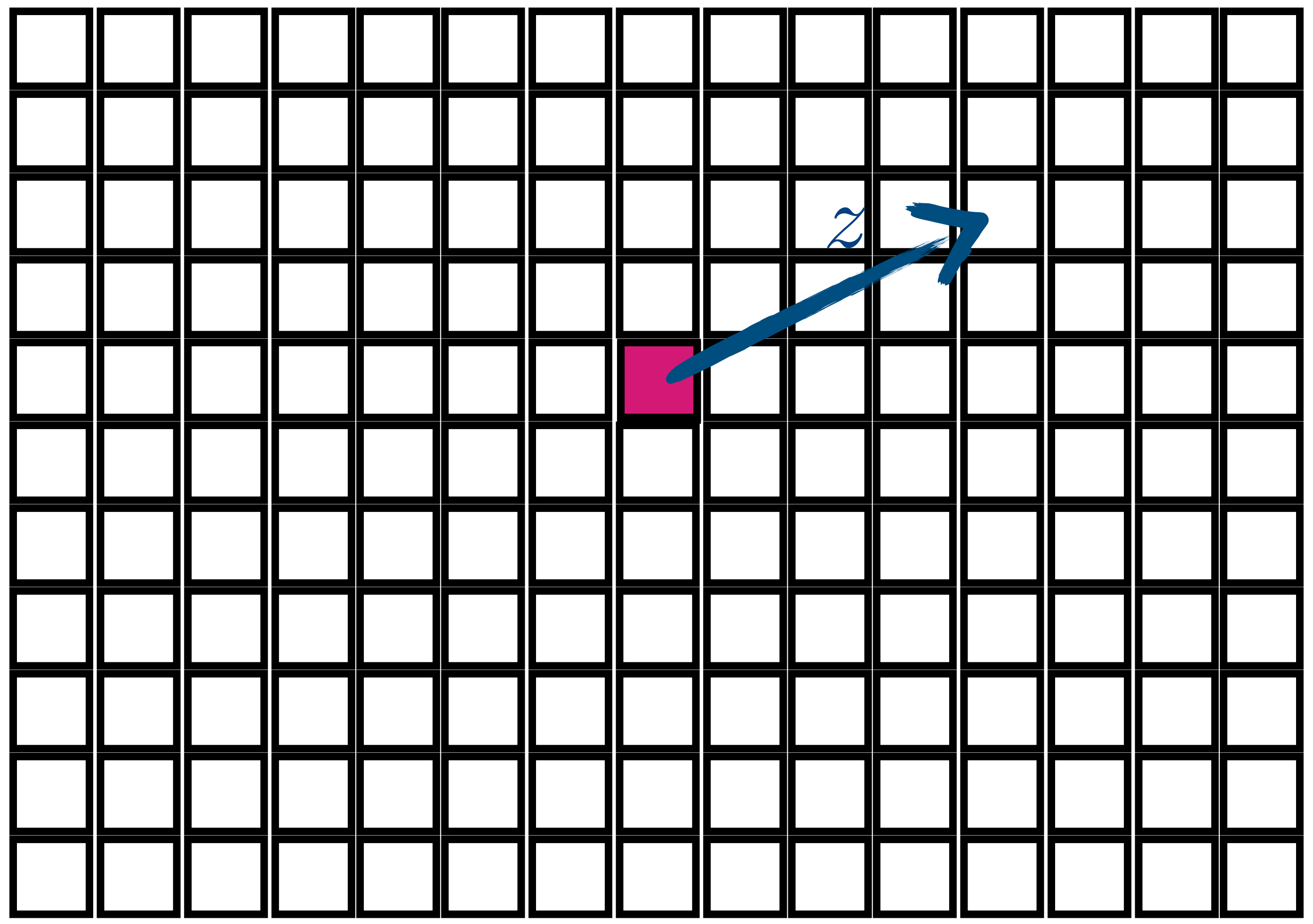
Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that
    - $\mathcal{V}$ satisfies locality

$$\forall x \ \exists \text{ a finite } N^+(x) \ \text{ s.t. } \ \mathcal{V}(\mathsf{A}_x) \subseteq \bigotimes_{y \in N^+(x)} \mathsf{A}_y$$

$$\boxed{\mathcal{V}(\mathsf{A}_L) \subseteq \mathsf{A}_L}$$



    - Translationally invariant

$$\mathscr{V}[\mathcal{T}_z(\mathsf{A}_x)] = \mathcal{T}_z[\mathscr{V}(\mathsf{A}_x)]$$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

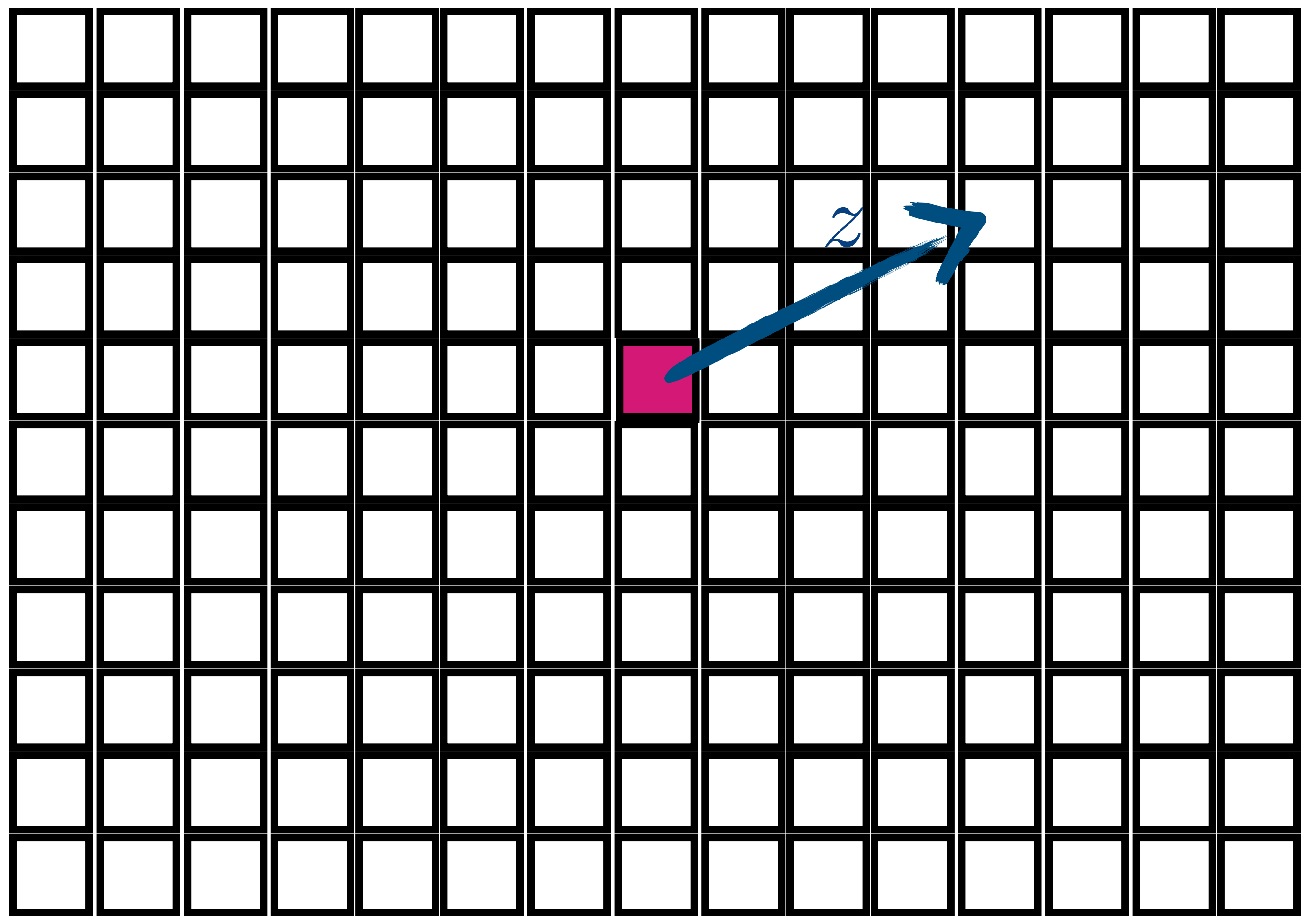
Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that
    - $\mathcal{V}$ satisfies locality

$$\forall x\ \exists \text{ a finite } N^+(x) \text{ s.t. } \mathcal{V}(\mathsf{A}_x) \subseteq \bigotimes_{y \in N^+(x)} \mathsf{A}_y$$

$$\mathcal{V}(\mathsf{A}_L) \subseteq \mathsf{A}_L$$



- Translationally invariant

$$\mathscr{V}[\mathcal{T}_z(\mathsf{A}_x)] = \mathcal{T}_z[\mathscr{V}(\mathsf{A}_x)]$$

$$\mathcal{T}_z(\mathsf{A}_x) = \mathsf{A}_{z+x}$$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Quantum Cellular Automaton

Infinite case

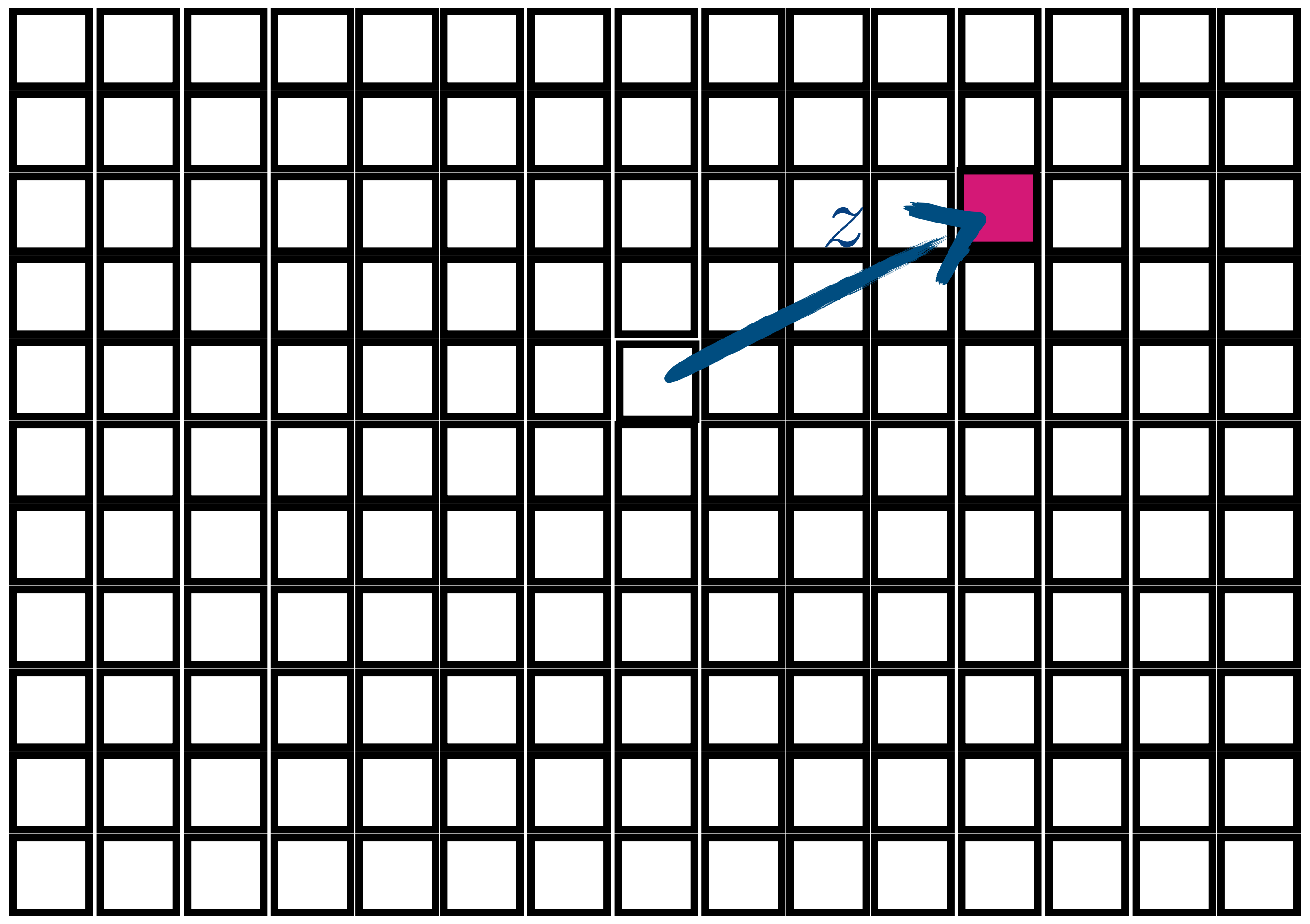
- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that
    - $\mathcal{V}$ satisfies locality

$$\forall x \; \exists \text{ a finite } N^+(x) \;\; \text{s.t.} \;\; \mathcal{V}(\mathsf{A}_x) \subseteq \bigotimes_{y \in N^+(x)} \mathsf{A}_y$$

$$\mathcal{V}(\mathsf{A}_L) \subseteq \mathsf{A}_L$$

    - Translationally invariant

$$\mathscr{V}[\mathcal{T}_z(\mathsf{A}_x)] = \mathcal{T}_z[\mathscr{V}(\mathsf{A}_x)]$$



$$\mathcal{T}_z(\mathsf{A}_x) = \mathsf{A}_{z+x}$$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

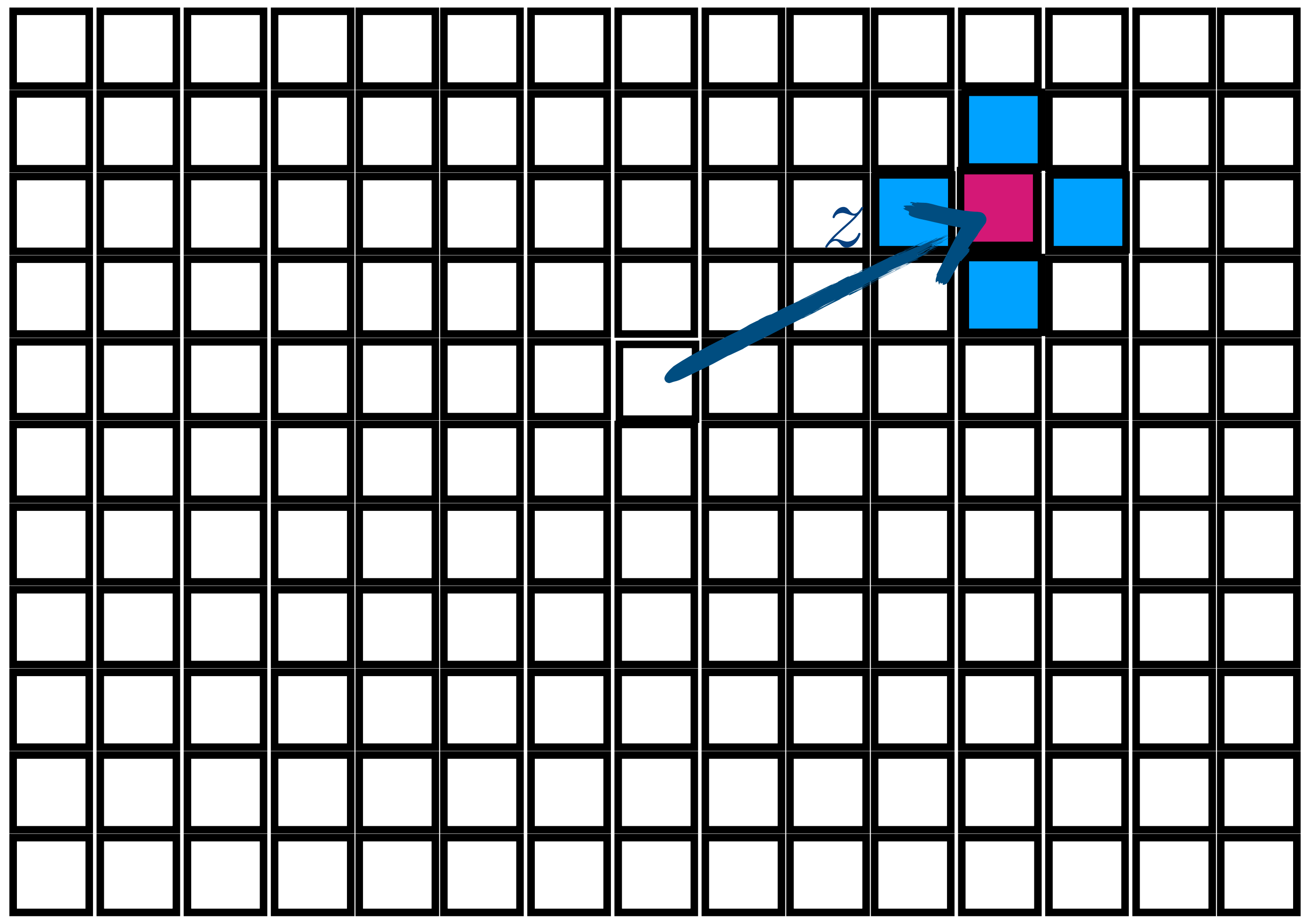
Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that
    - $\mathcal{V}$ satisfies locality

$$\forall x \ \exists \text{ a finite } N^+(x) \ \text{ s.t. } \ \mathcal{V}(\mathsf{A}_x) \subseteq \bigotimes_{y \in N^+(x)} \mathsf{A}_y$$

$$\mathcal{V}(\mathsf{A}_L) \subseteq \mathsf{A}_L$$



$$\mathcal{T}_z(\mathsf{A}_x) = \mathsf{A}_{z+x}$$

- Translationally invariant

$$\mathscr{V}[\mathcal{T}_z(\mathsf{A}_x)] = \mathcal{T}_z[\mathscr{V}(\mathsf{A}_x)]$$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

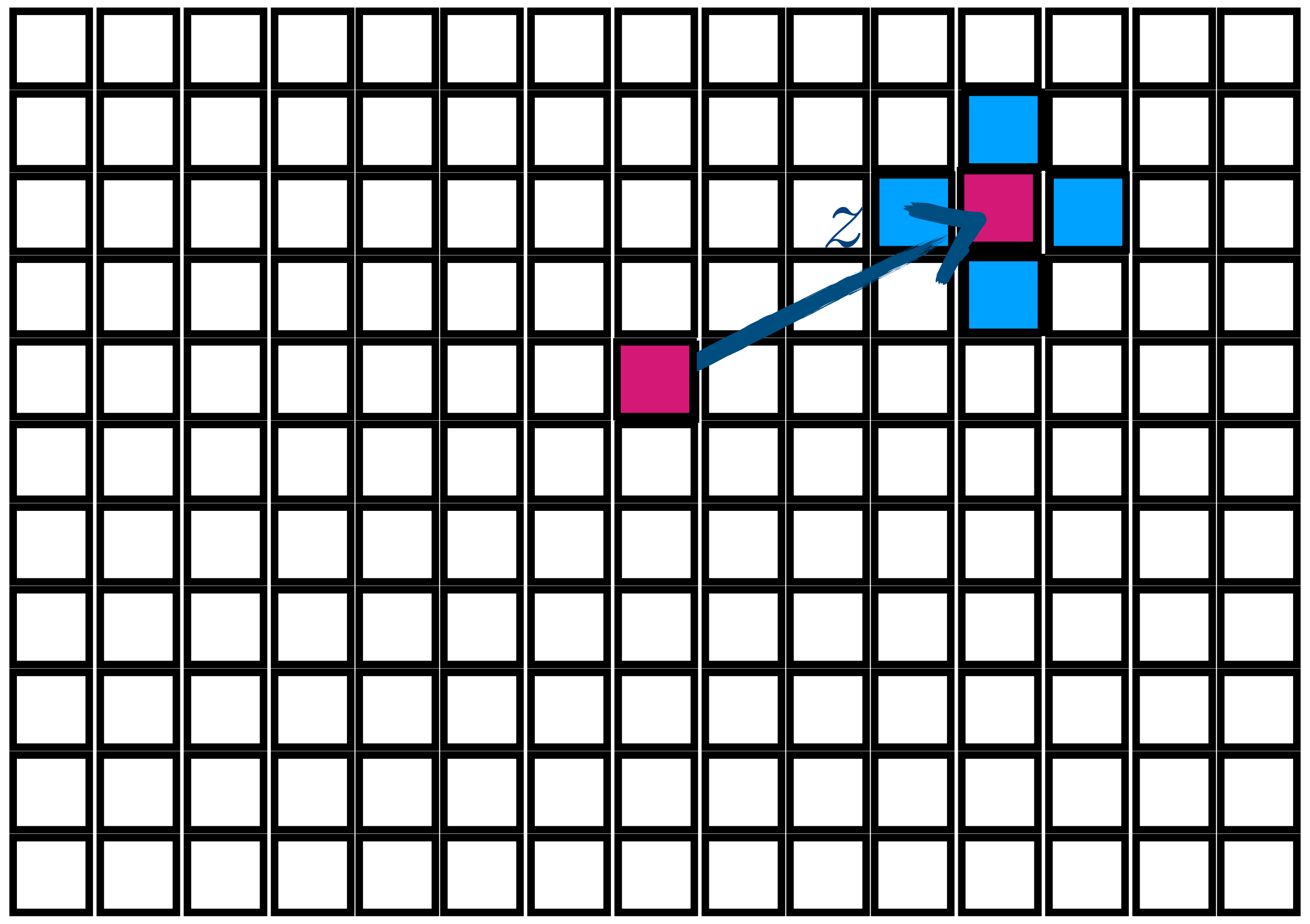
Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that
    - $\mathcal{V}$ satisfies locality

$$\forall x \ \exists \text{ a finite } N^+(x) \ \text{ s.t. } \ \mathcal{V}(\mathsf{A}_x) \subseteq \bigotimes_{y \in N^+(x)} \mathsf{A}_y$$

$$\boxed{\mathcal{V}(\mathsf{A}_L) \subseteq \mathsf{A}_L}$$



$$\mathcal{T}_z(\mathsf{A}_x) = \mathsf{A}_{z+x}$$

    - Translationally invariant

$$\mathscr{V}[\mathcal{T}_z(\mathsf{A}_x)] = \mathcal{T}_z[\mathscr{V}(\mathsf{A}_x)]$$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

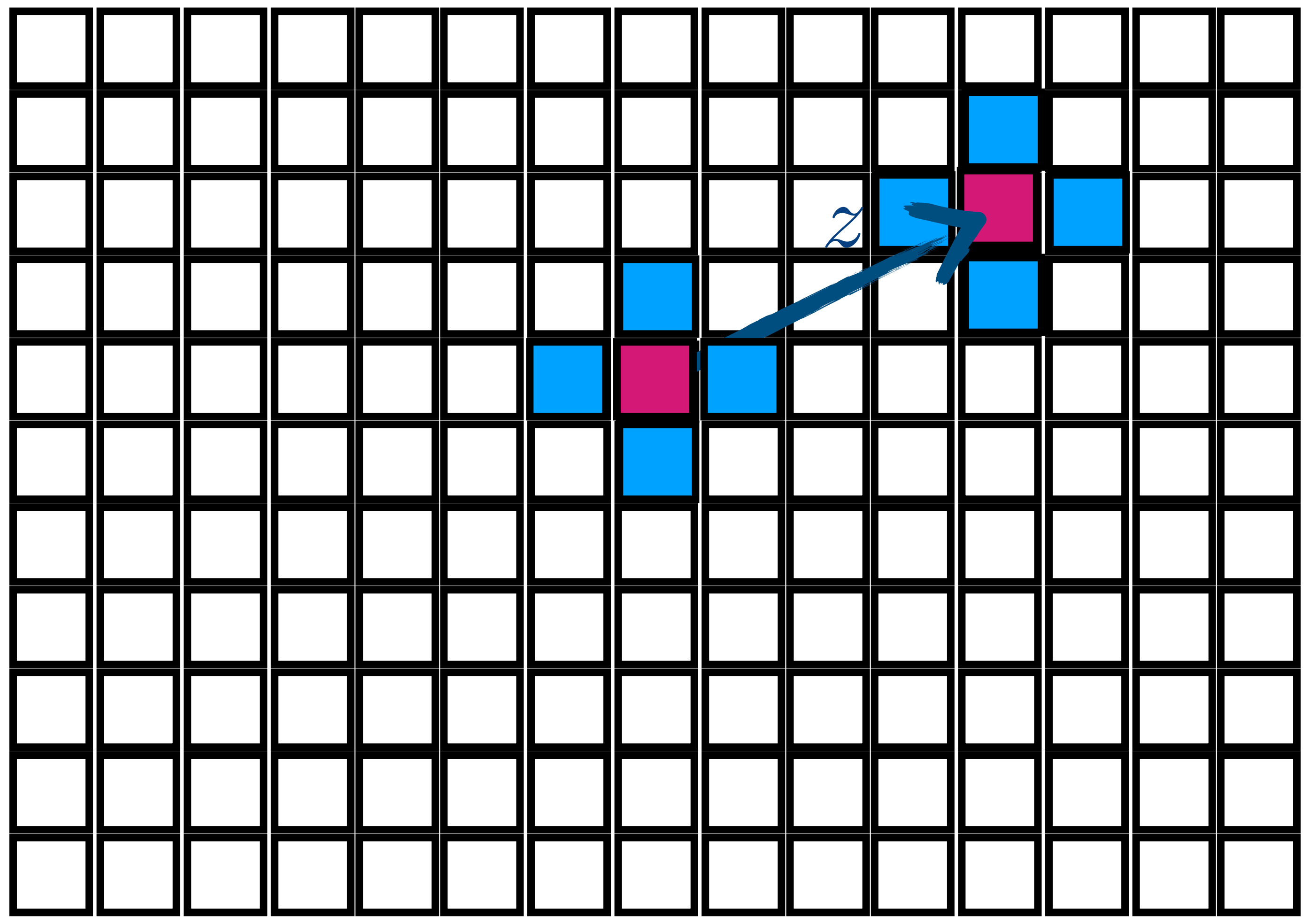
Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that
    - $\mathcal{V}$ satisfies locality

$$\forall x \ \exists \text{ a finite } N^+(x) \ \text{ s.t. } \ \mathcal{V}(\mathsf{A}_x) \subseteq \bigotimes_{y \in N^+(x)} \mathsf{A}_y$$

$$\mathcal{V}(\mathsf{A}_L) \subseteq \mathsf{A}_L$$



$$\mathcal{T}_z(\mathsf{A}_x) = \mathsf{A}_{z+x}$$

    - Translationally invariant

$$\mathscr{V}[\mathcal{T}_z(\mathsf{A}_x)] = \mathcal{T}_z[\mathscr{V}(\mathsf{A}_x)]$$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

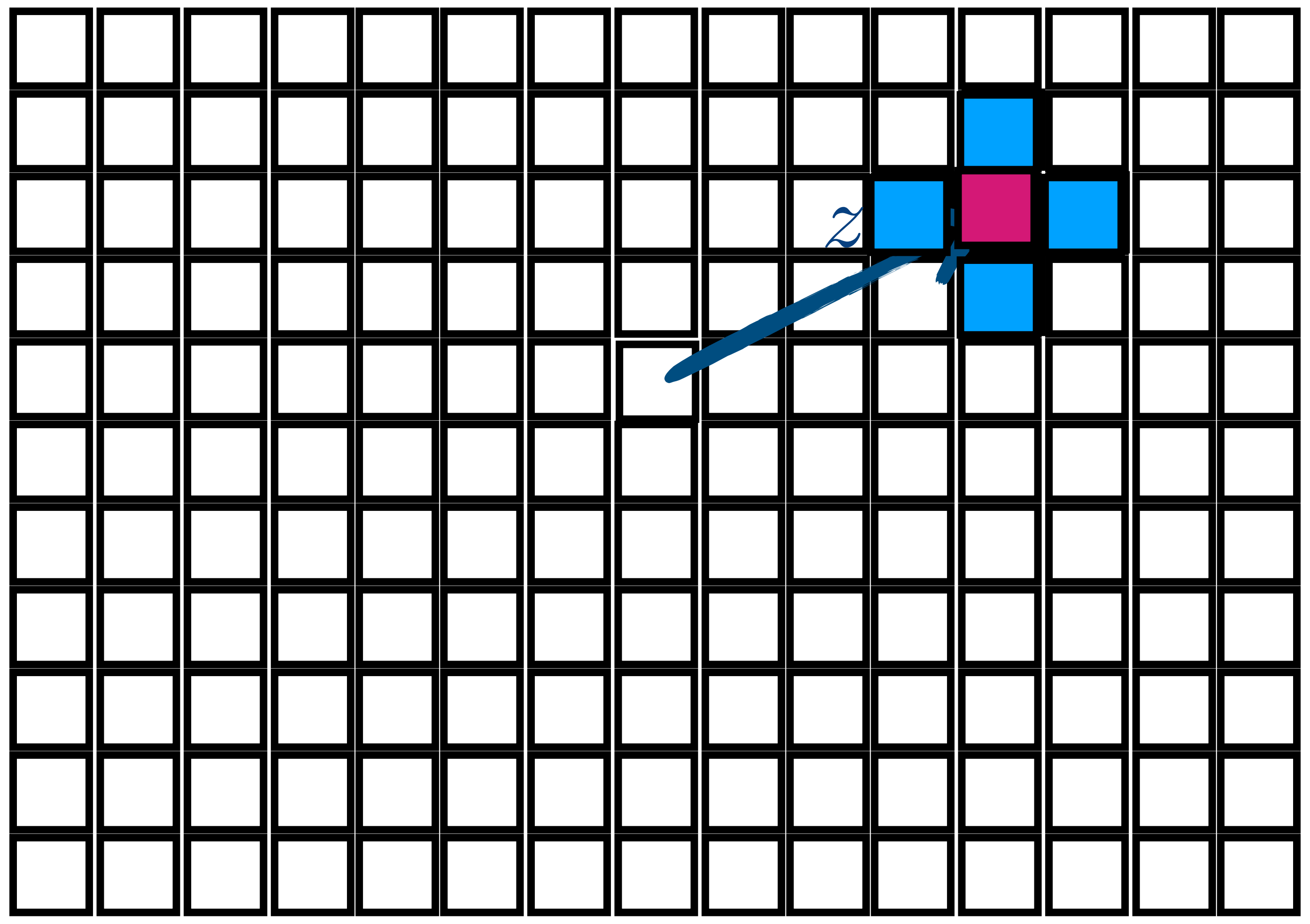
Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$
  - A QCA is an homomomorphism $\mathcal{V}$ of the quasi local algebra A such that
    - $\mathcal{V}$ satisfies locality

$$\forall x \ \exists \text{ a finite } N^+(x) \ \text{ s.t. } \ \mathcal{V}(\mathsf{A}_x) \subseteq \bigotimes_{y \in N^+(x)} \mathsf{A}_y$$

$$\mathcal{V}(\mathsf{A}_L) \subseteq \mathsf{A}_L$$



$$\mathcal{T}_z(\mathsf{A}_x) = \mathsf{A}_{z+x}$$

- Translationally invariant

$$\mathscr{V}[\mathcal{T}_z(\mathsf{A}_x)] = \mathcal{T}_z[\mathscr{V}(\mathsf{A}_x)]$$

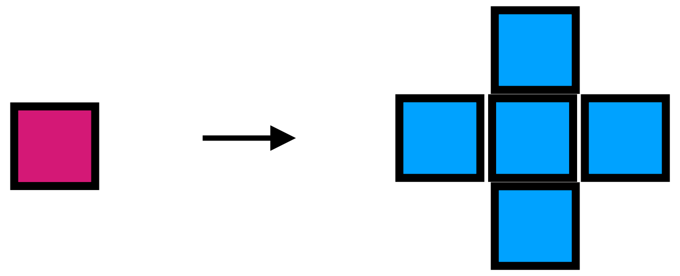
B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Local update rule

Consequence of the definition

- The QCA induces a local rule $\mathcal{V}_x : \mathsf{A}_x \to \mathsf{A}_{N^+(x)}$

$$\mathcal{V}_x(B_x) := \mathcal{V}(B_x)$$

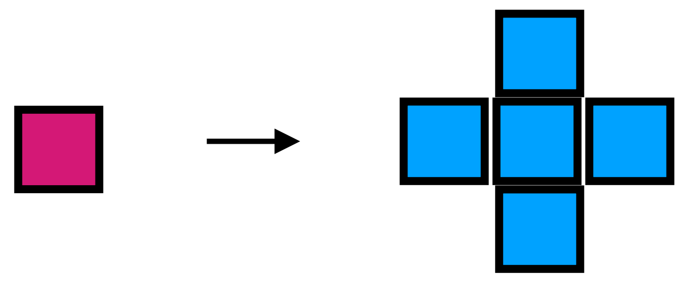


B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Local update rule

Consequence of the definition

- The QCA induces a local rule $\mathcal{V}_x : \mathsf{A}_x \to \mathsf{A}_{N+(x)}$
  $\mathcal{V}_x(B_x) := \mathcal{V}(B_x)$



- The local rule at x=0 completely specifies the QCA

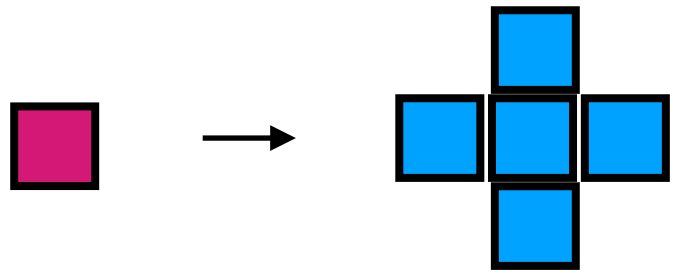
$$
\begin{aligned}
\mathcal{V}(B_x) &= \mathcal{T}_x\{\mathcal{V}[\mathcal{T}_x^{-1}(B_x)]\} \\
&= \mathcal{T}_x\{\mathcal{V}_0[\mathcal{T}_x^{-1}(B_x)]\}
\end{aligned}
$$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Local update rule

Consequence of the definition

- The QCA induces a local rule $\mathcal{V}_x : \mathsf{A}_x \to \mathsf{A}_{N^+(x)}$
  $\mathcal{V}_x(B_x) := \mathcal{V}(B_x)$



- The local rule at x=0 completely specifies the QCA

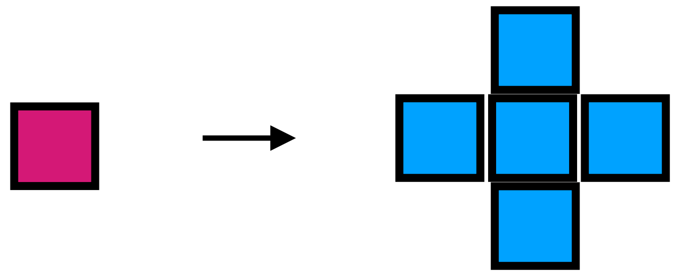
$$\begin{aligned}\mathcal{V}(B_x) &= \mathcal{T}_x\{\mathcal{V}[\mathcal{T}_x^{-1}(B_x)]\} \\ &= \mathcal{T}_x\{\mathcal{V}_0[\mathcal{T}_x^{-1}(B_x)]\}\end{aligned} \quad \Rightarrow \quad \mathcal{V}_x = \mathcal{T}_x \mathcal{V}_0 \mathcal{T}_x^{-1}$$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Local update rule

Consequence of the definition

- The QCA induces a local rule $\mathcal{V}_x : \mathsf{A}_x \to \mathsf{A}_{N^+(x)}$
  $\mathcal{V}_x(B_x) := \mathcal{V}(B_x)$



- The local rule at x=0 completely specifies the QCA

$$\begin{aligned}\mathcal{V}(B_x) &= \mathcal{T}_x\{\mathcal{V}[\mathcal{T}_x^{-1}(B_x)]\} \\ &= \mathcal{T}_x\{\mathcal{V}_0[\mathcal{T}_x^{-1}(B_x)]\}\end{aligned} \quad \Rightarrow \quad \mathcal{V}_x = \mathcal{T}_x \mathcal{V}_0 \mathcal{T}_x^{-1}$$

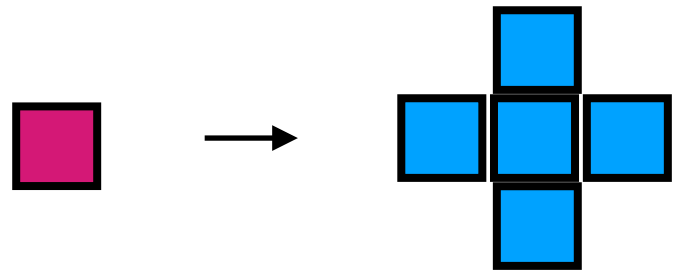
  - The image of every local effect can be obtained using $\mathcal{V}_0$ since $B_R = \sum_i \bigotimes_{x \in R} B_x^{(i)}$

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Local update rule

Consequence of the definition

- The QCA induces a local rule $\mathcal{V}_x : \mathsf{A}_x \to \mathsf{A}_{N^+(x)}$
  $$\mathcal{V}_x(B_x) := \mathcal{V}(B_x)$$



- The local rule at x=0 completely specifies the QCA

$$\begin{aligned}\mathcal{V}(B_x) &= \mathcal{T}_x\{\mathcal{V}[\mathcal{T}_x^{-1}(B_x)]\}\\ &= \mathcal{T}_x\{\mathcal{V}_0[\mathcal{T}_x^{-1}(B_x)]\}\end{aligned} \qquad \Rightarrow \qquad \mathcal{V}_x = \mathcal{T}_x \mathcal{V}_0 \mathcal{T}_x^{-1}$$

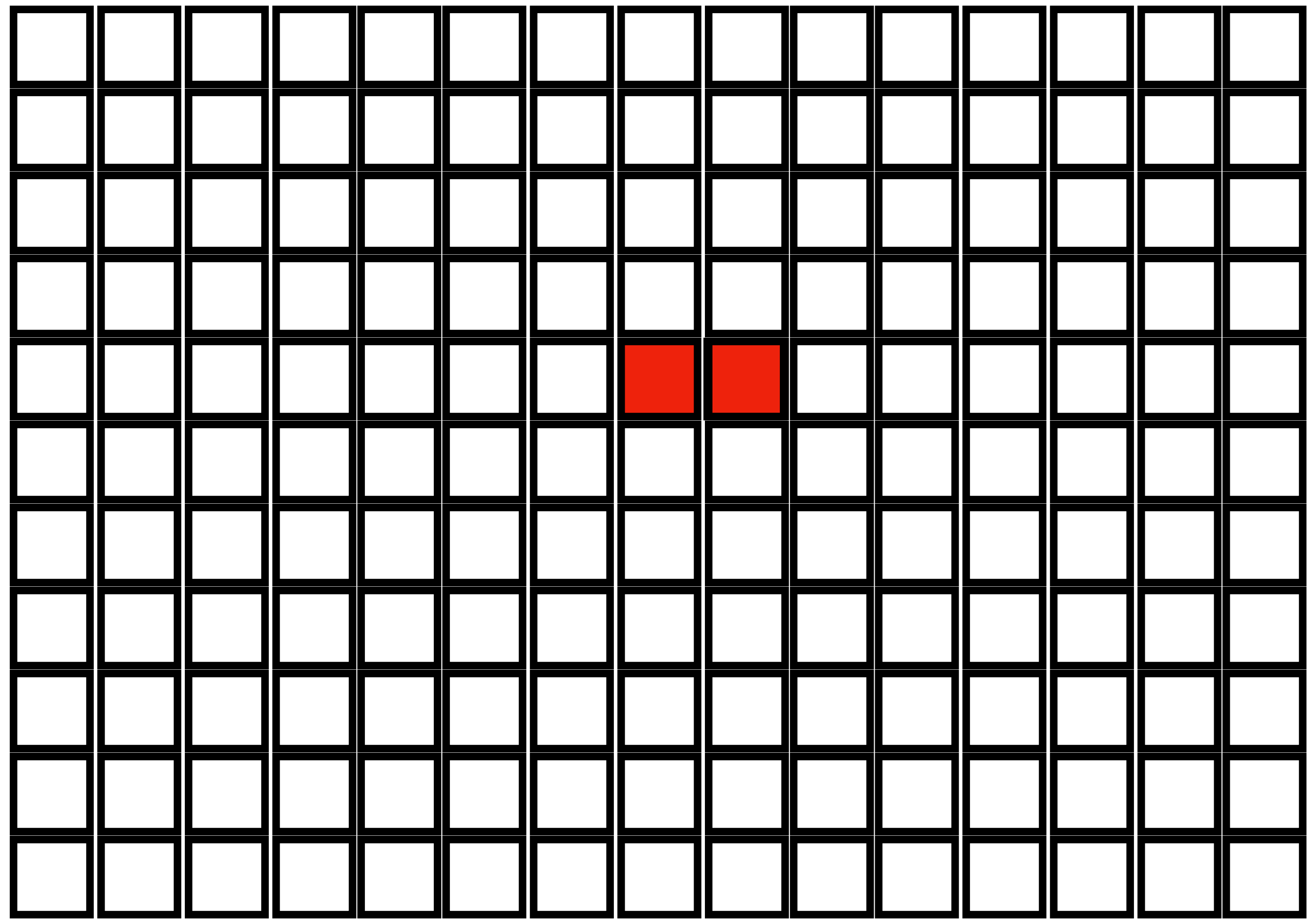
  - The image of every local effect can be obtained using $\mathcal{V}_0$ since $B_R = \sum_i \bigotimes_{x \in R} B_x^{(i)}$
  - Boundedness: image of limits (of the quasi-local algebra) is determined

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Wrapping lemma

Reduces the local rule to that of a finite QCA

- Local algebras commute and so must do their images under $\mathcal{V}$

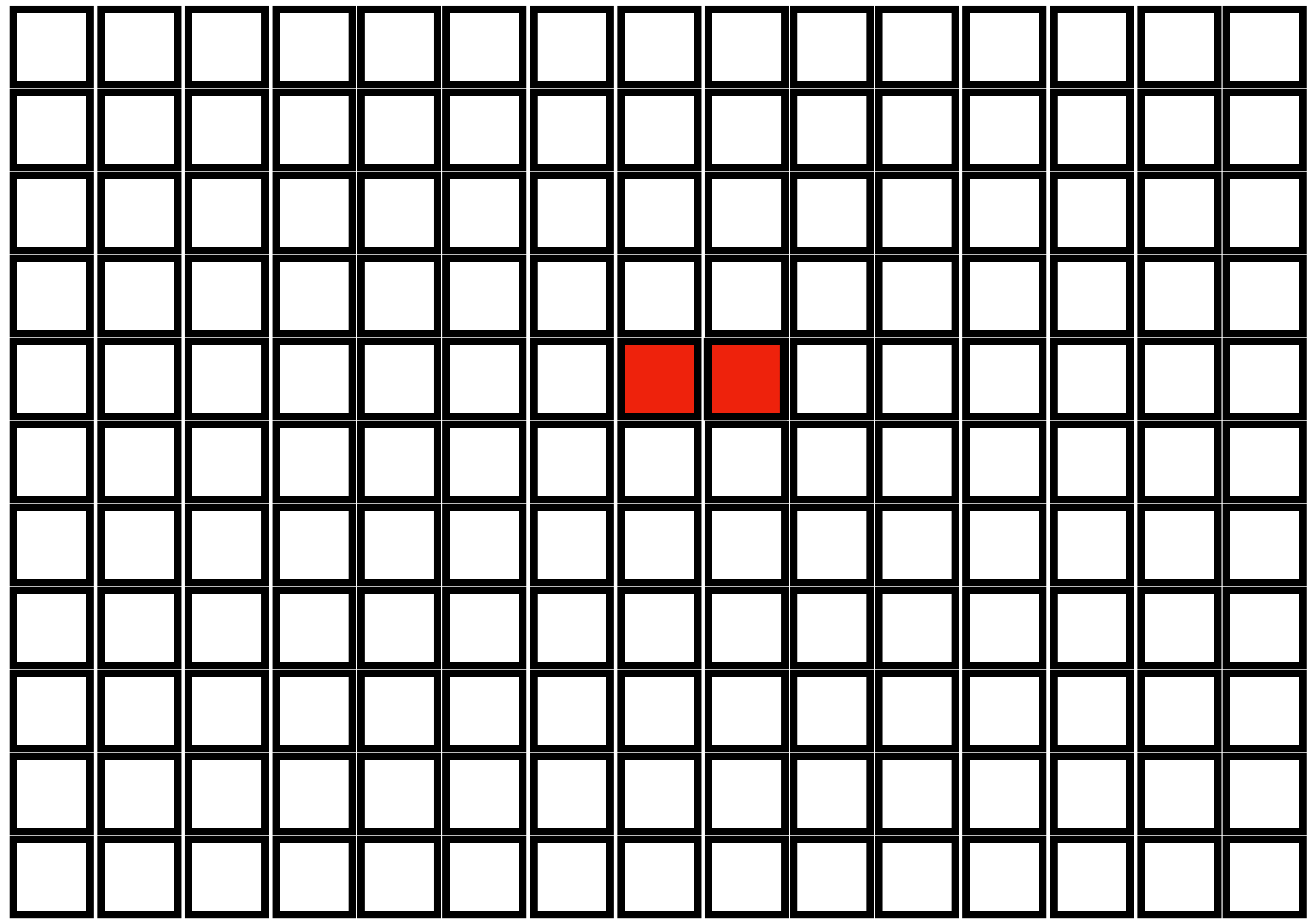


B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Wrapping lemma

Reduces the local rule to that of a finite QCA

- Local algebras commute and so must do their images under $\mathcal{V}$
- Necessarily $[\mathcal{V}_x(B_x), \mathcal{V}_y(B_y)] = 0$

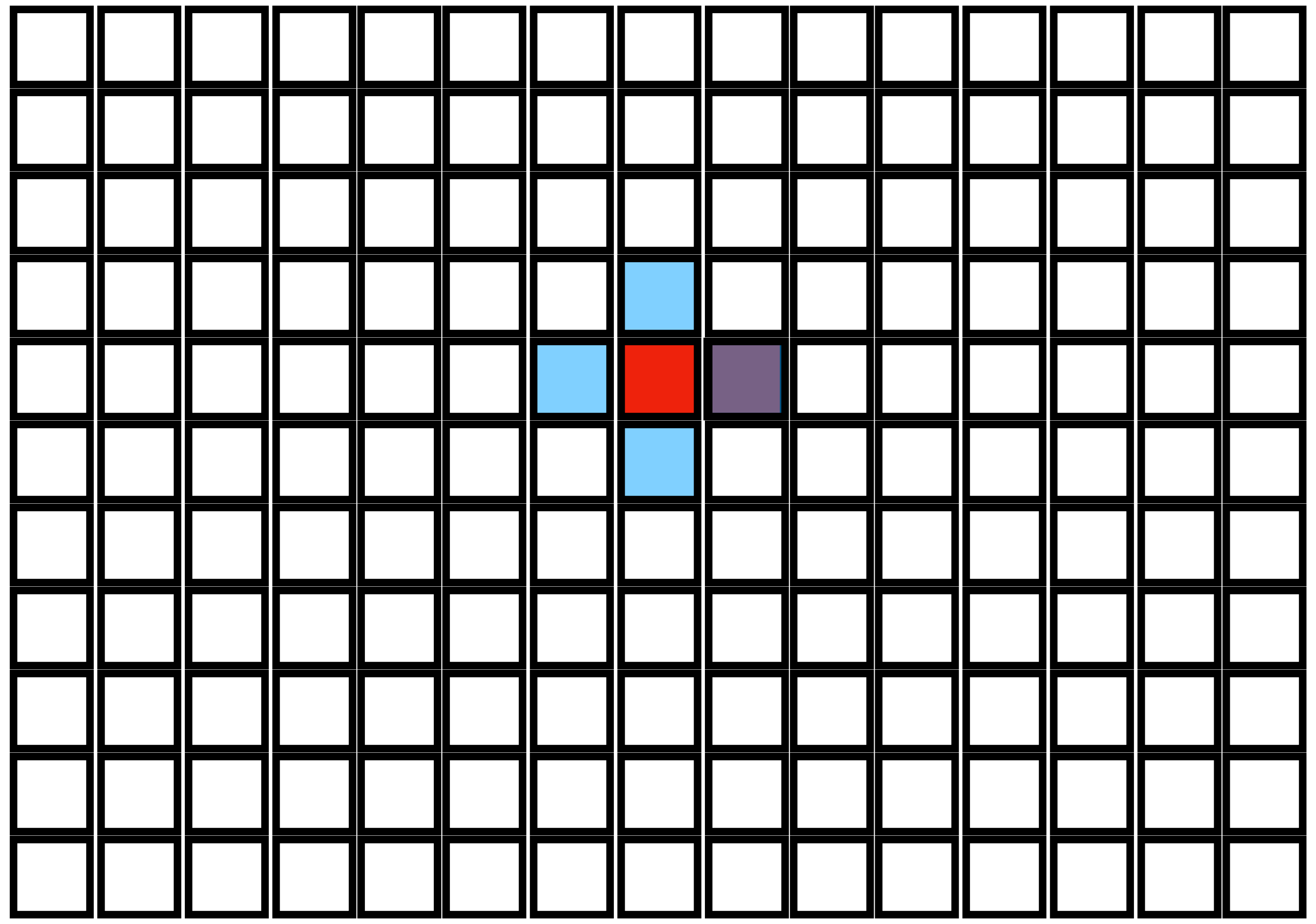


B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Wrapping lemma

Reduces the local rule to that of a finite QCA

- Local algebras commute and so must do their images under $\mathcal{V}$
- Necessarily $[\mathcal{V}_x(B_x), \mathcal{V}_y(B_y)] = 0$

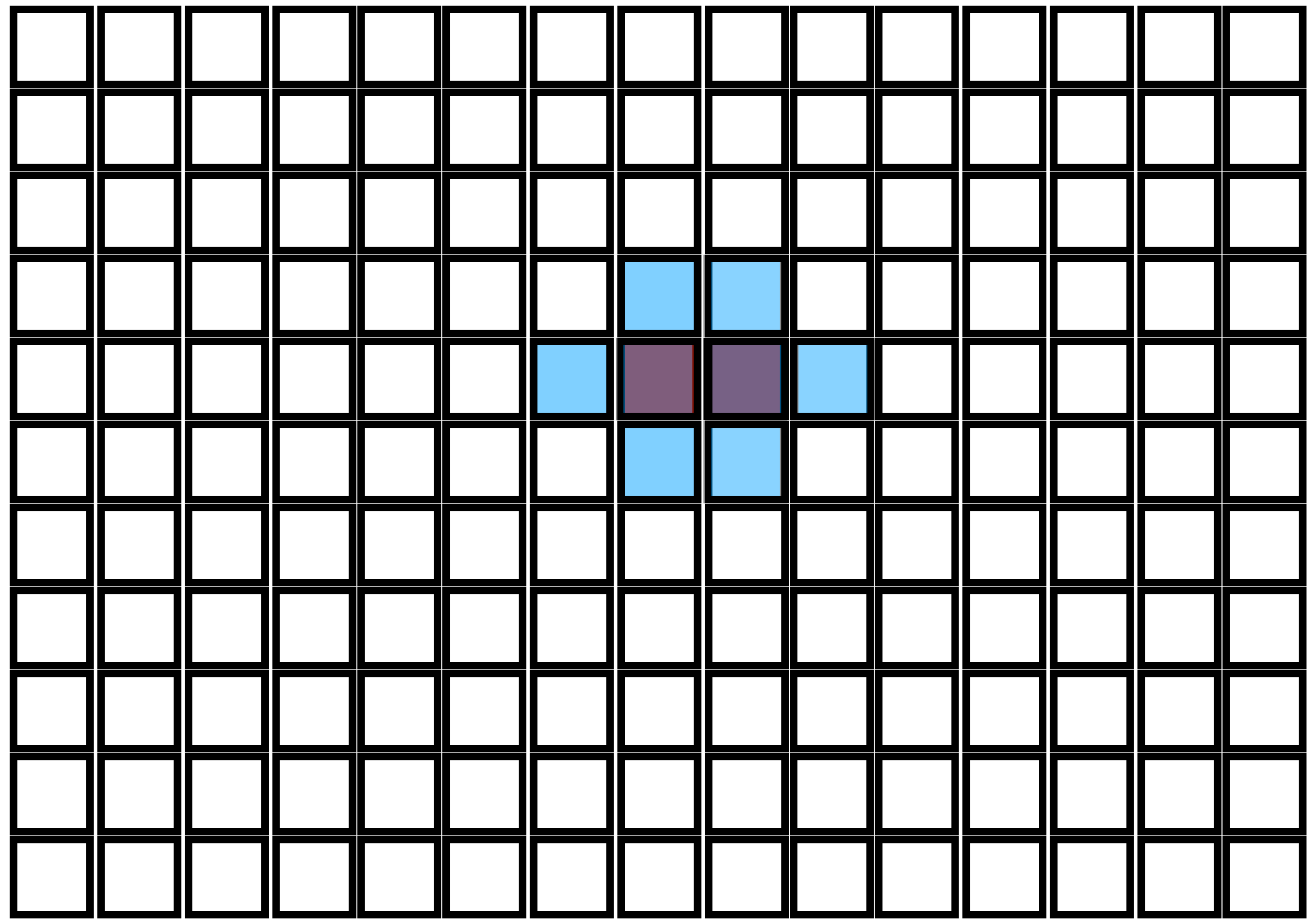


B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Wrapping lemma

Reduces the local rule to that of a finite QCA

- Local algebras commute and so must do their images under $\mathcal{V}$
- Necessarily $[\mathcal{V}_x(B_x), \mathcal{V}_y(B_y)] = 0$



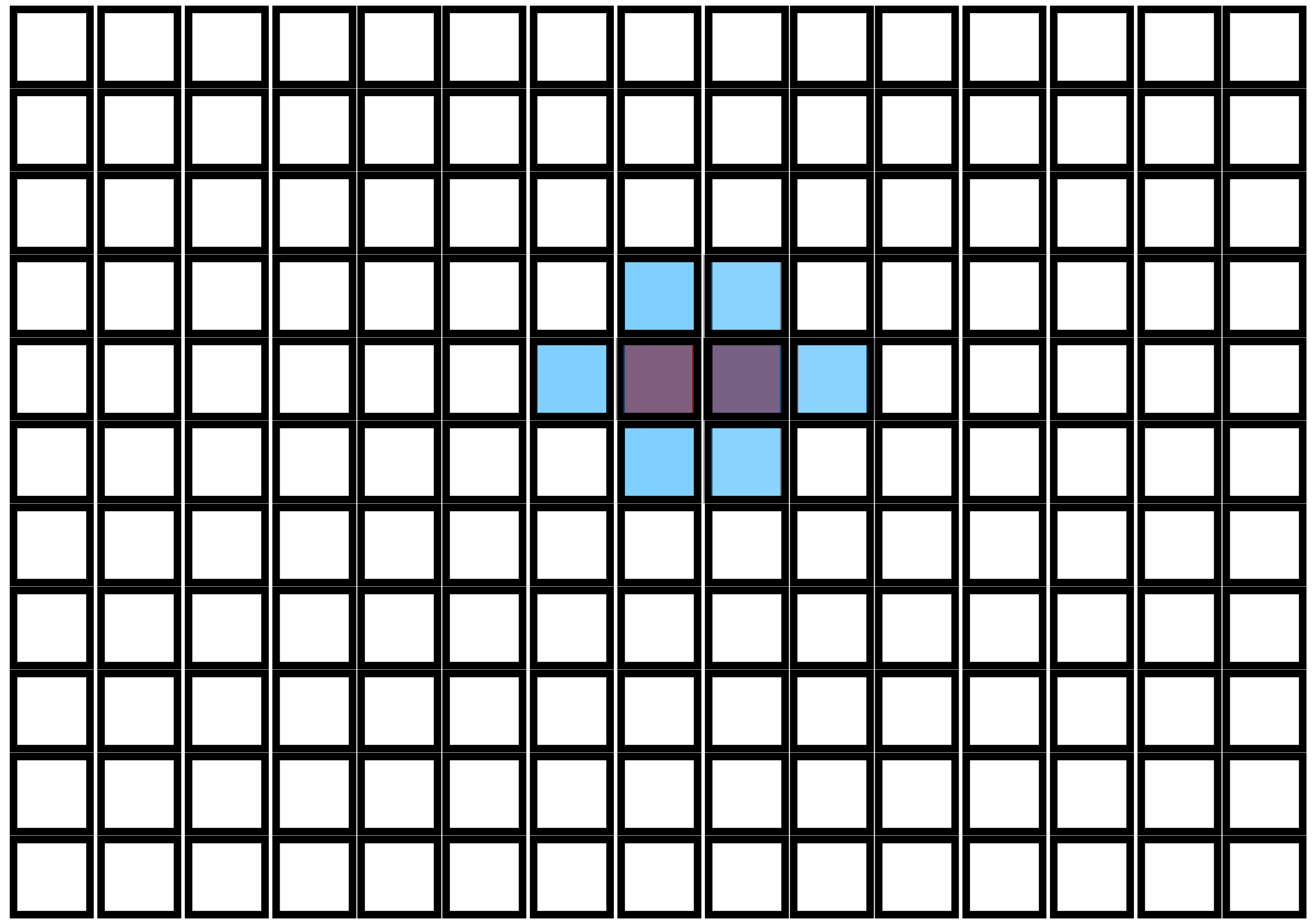
B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Wrapping lemma

Reduces the local rule to that of a finite QCA

- Local algebras commute and so must do their images under $\mathcal{V}$
- Necessarily $[\mathcal{V}_x(B_x), \mathcal{V}_y(B_y)] = 0$
- Also the converse is true

$$\mathscr{V}\left(\bigotimes_{x\in R} B_x\right) := \prod_{x\in R} \mathcal{V}_x(B_x)$$



B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

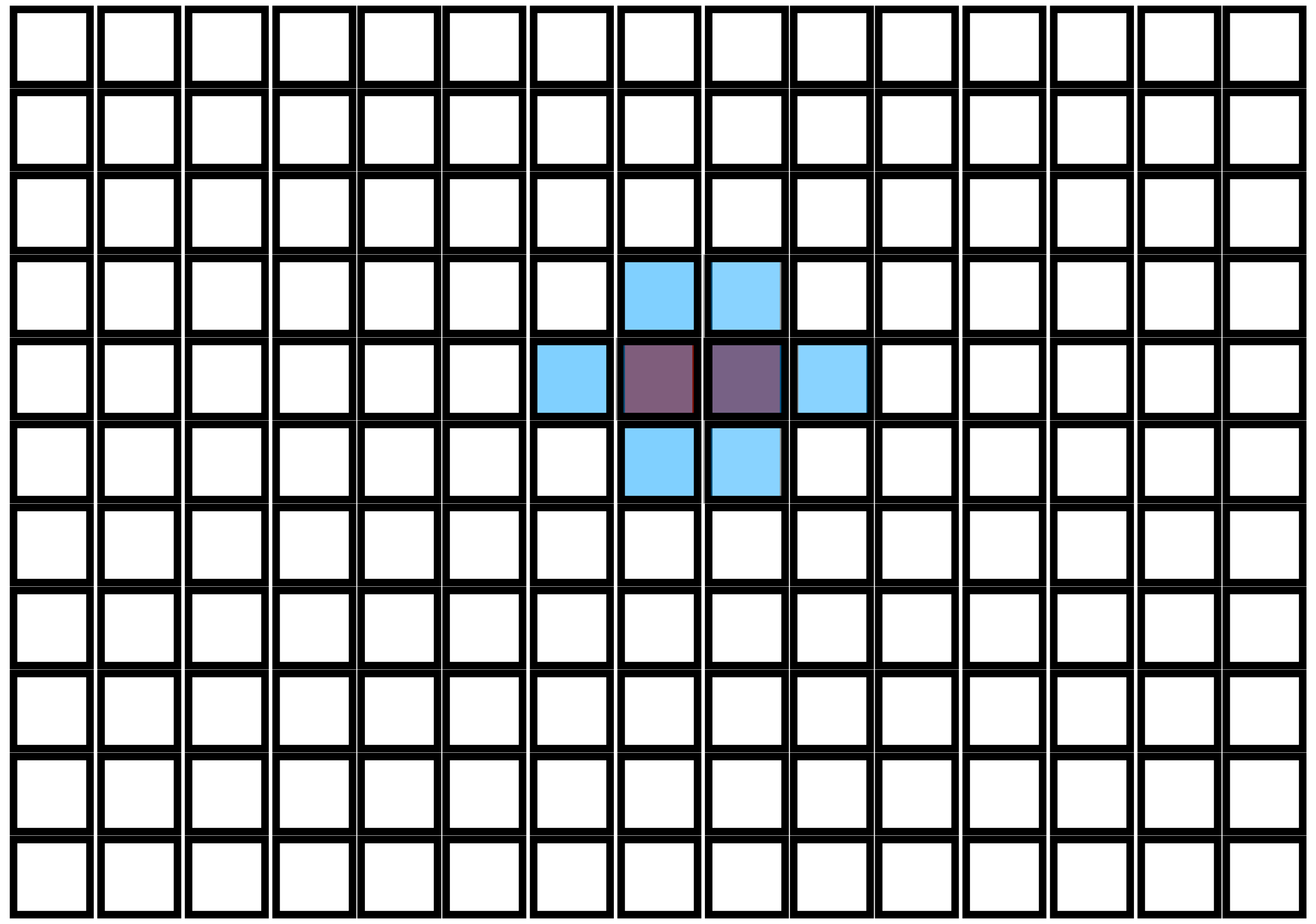
Wrapping lemma

Reduces the local rule to that of a finite QCA

- Local algebras commute and so must do their images under $\mathcal{V}$
- Necessarily $[\mathcal{V}_x(B_x), \mathcal{V}_y(B_y)] = 0$
- Also the converse is true

$$\mathscr{V}\left(\bigotimes_{x\in R} B_x\right) := \prod_{x\in R} \mathcal{V}_x(B_x)$$

- The commutation condition is local



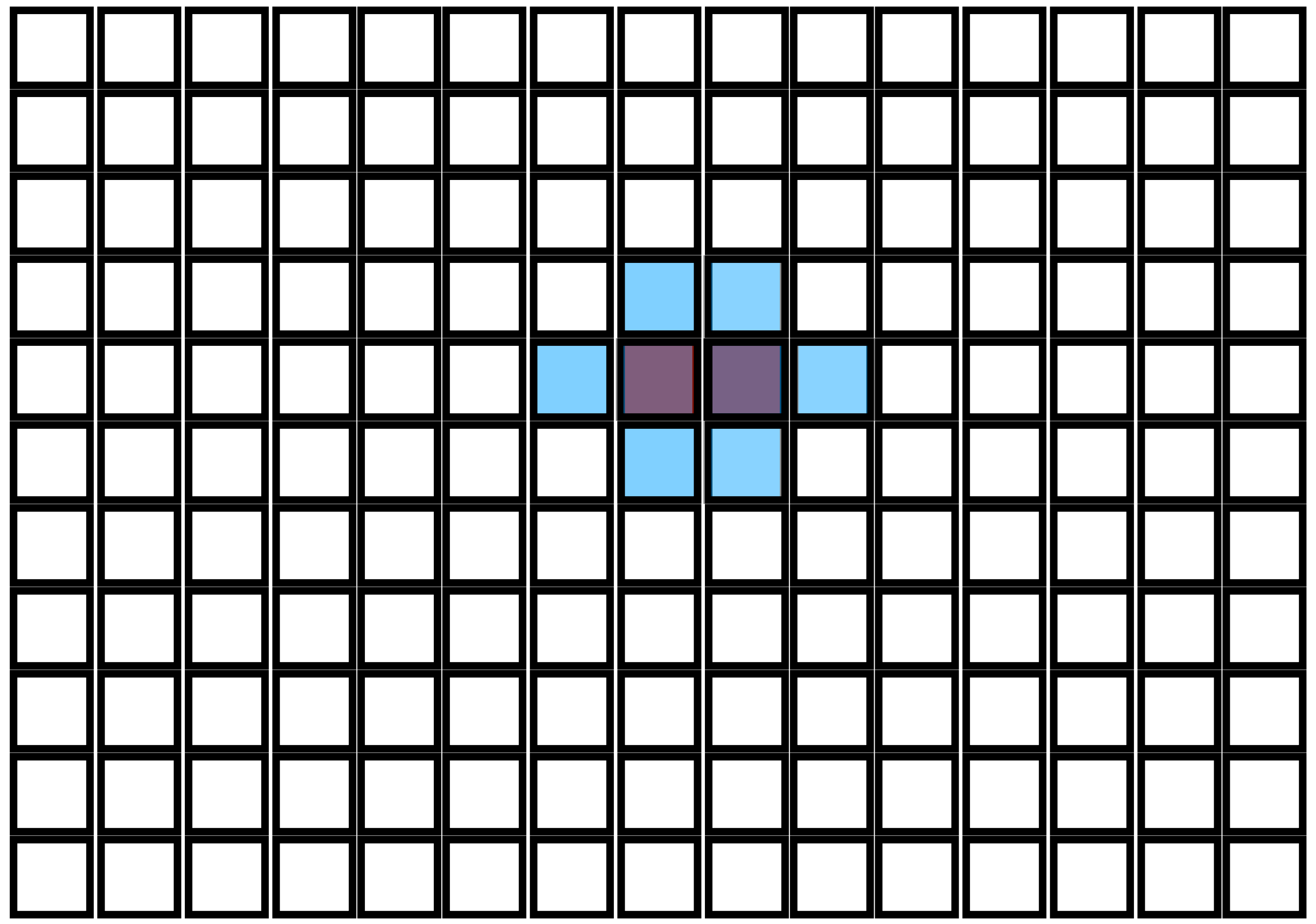
B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Wrapping lemma

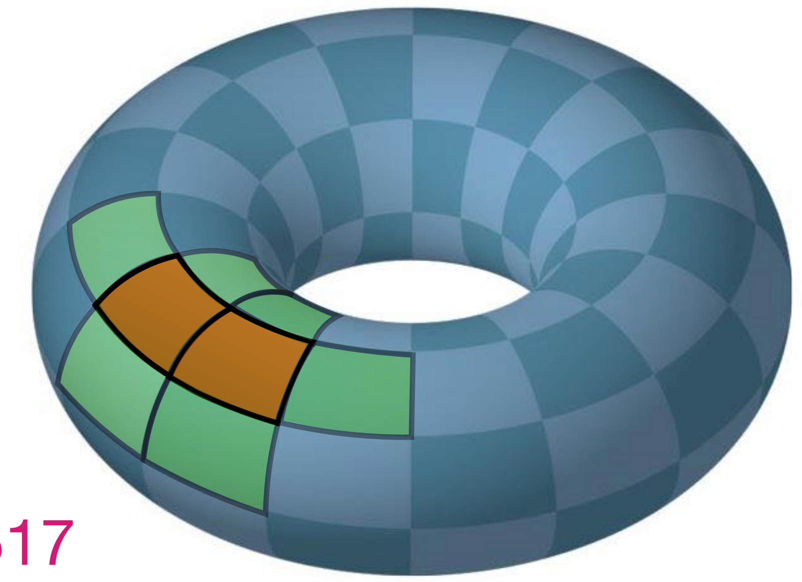
Reduces the local rule to that of a finite QCA

- Local algebras commute and so must do their images under $\mathcal{V}$
- Necessarily $[\mathcal{V}_x(B_x), \mathcal{V}_y(B_y)] = 0$
- Also the converse is true

$$\mathscr{V}\left(\bigotimes_{x\in R} B_x\right) := \prod_{x\in R} \mathcal{V}_x(B_x)$$



- The commutation condition is local
  - Wrapping lemma

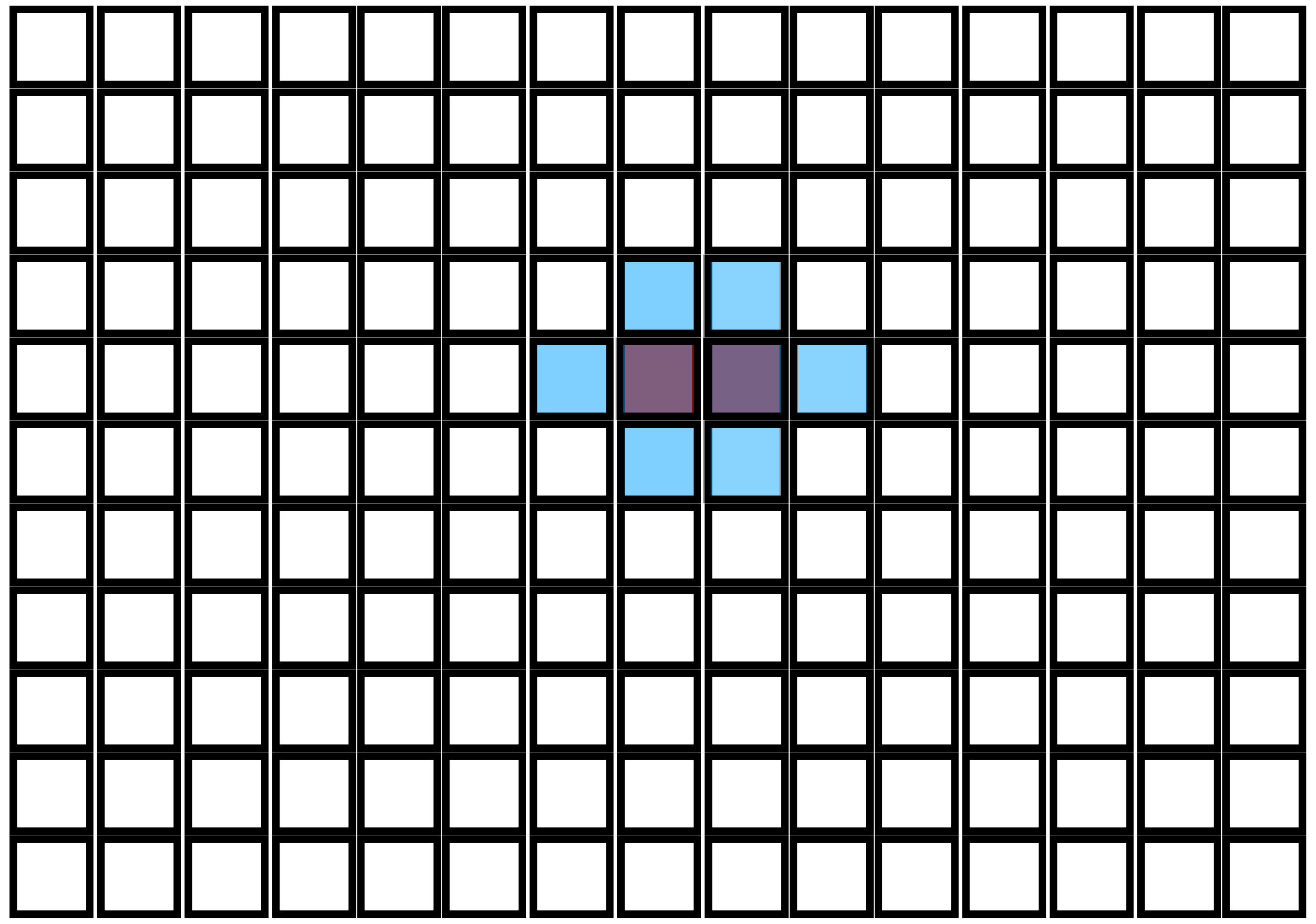


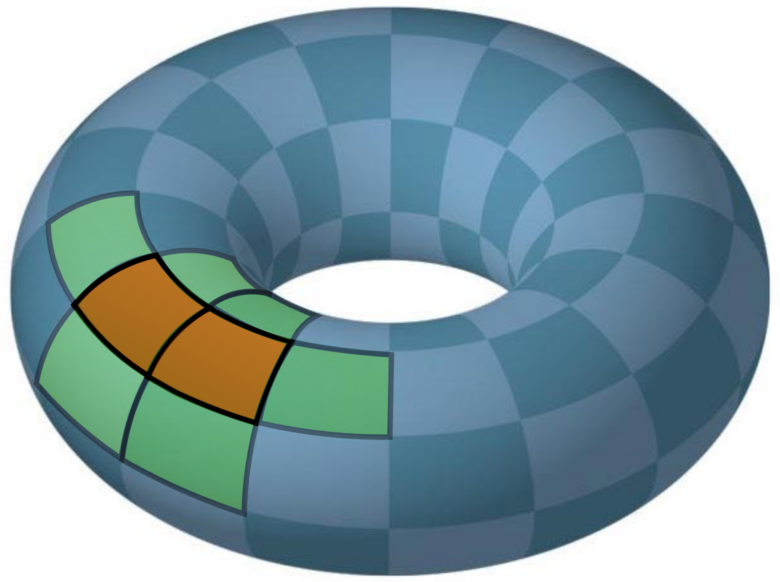
B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Wrapping lemma

Unitarity of QCAs

- The commutation condition is local
  - The same for infinite/finite QCA





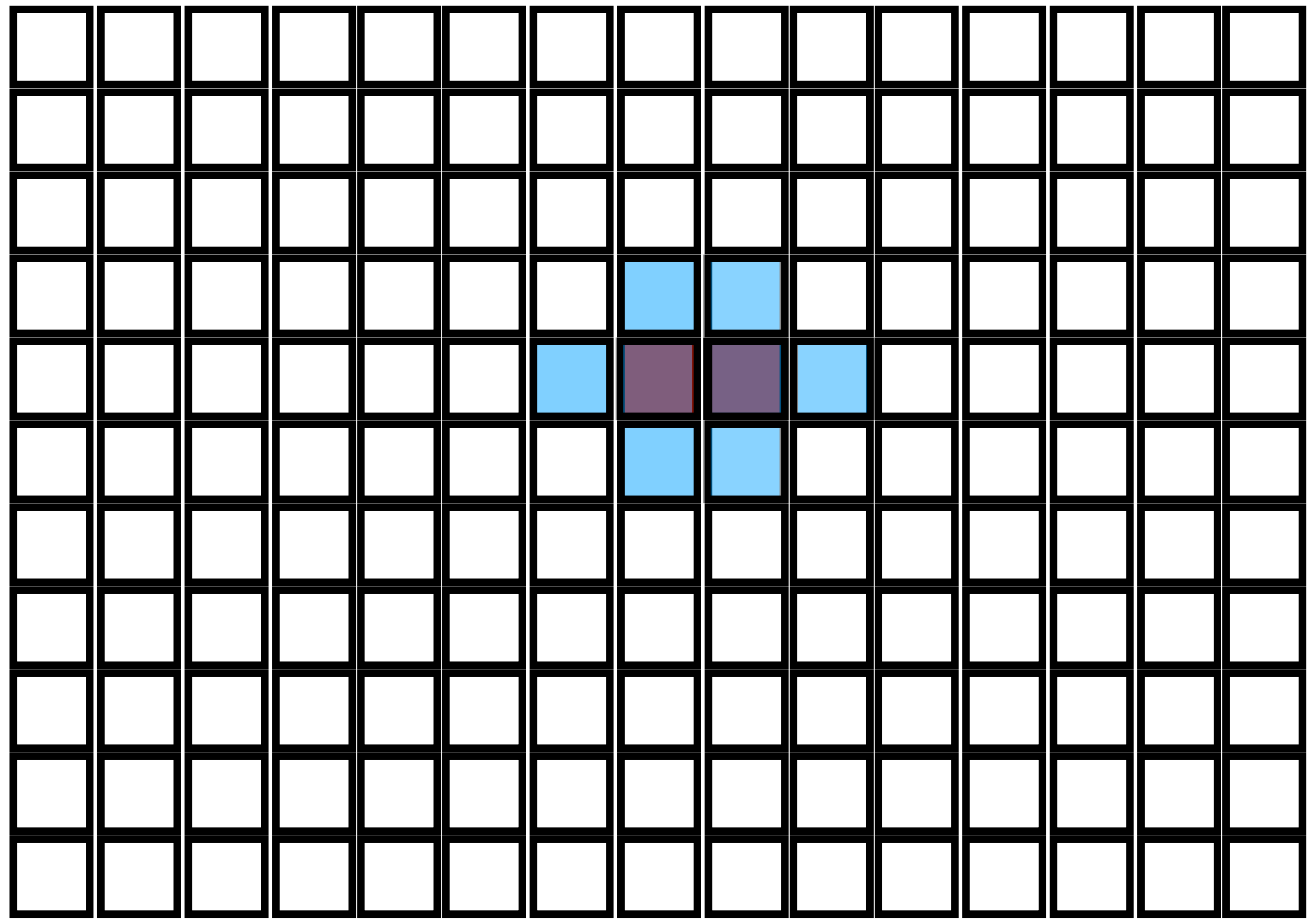
B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

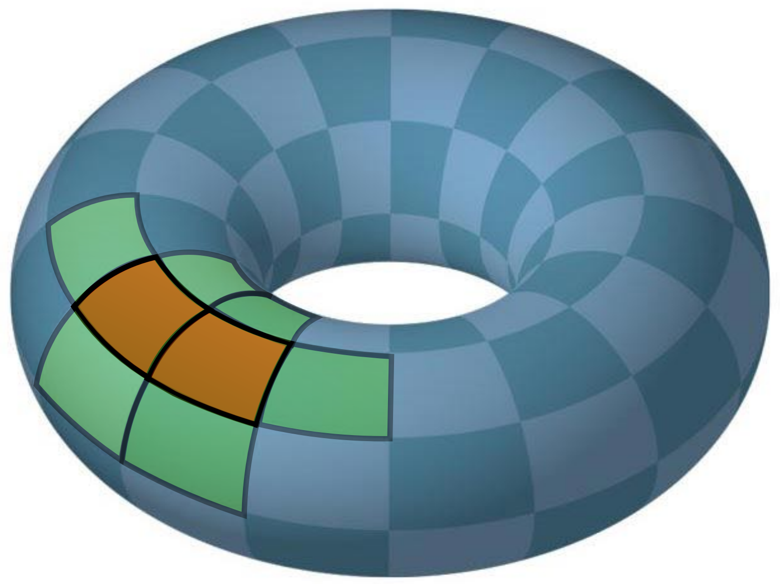
Wrapping lemma

Unitarity of QCAs

- The commutation condition is local
  - The same for infinite/finite QCA
- By Stinespring theorem in the finite case

$$\mathcal{V}(B_R) = U(I \otimes B_R)U^\dagger$$
$$= \sum_i \prod_{x \in R} U_x(I \otimes B_x^{(i)})U_x^\dagger$$





B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Wrapping lemma

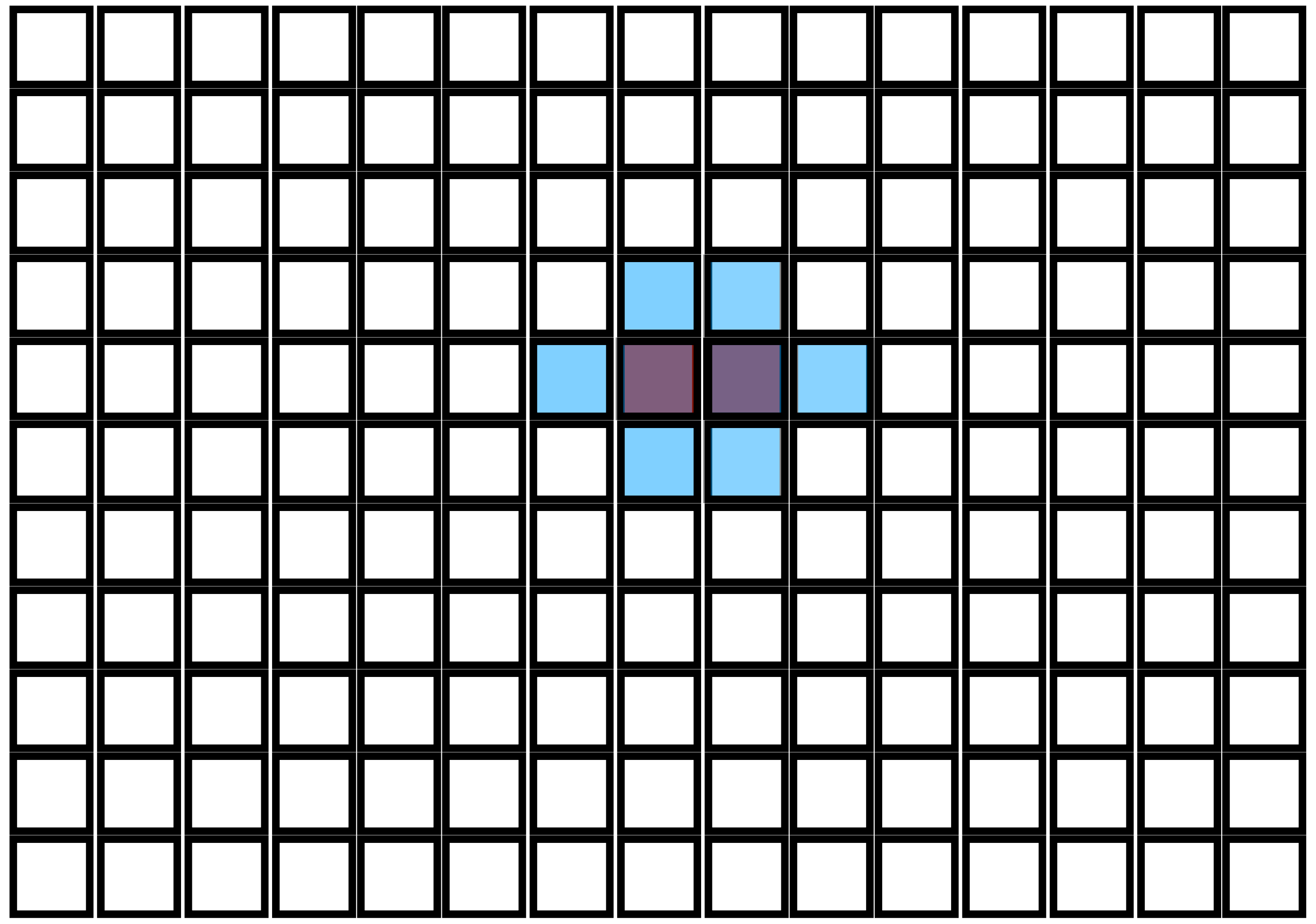
Unitarity of QCAs

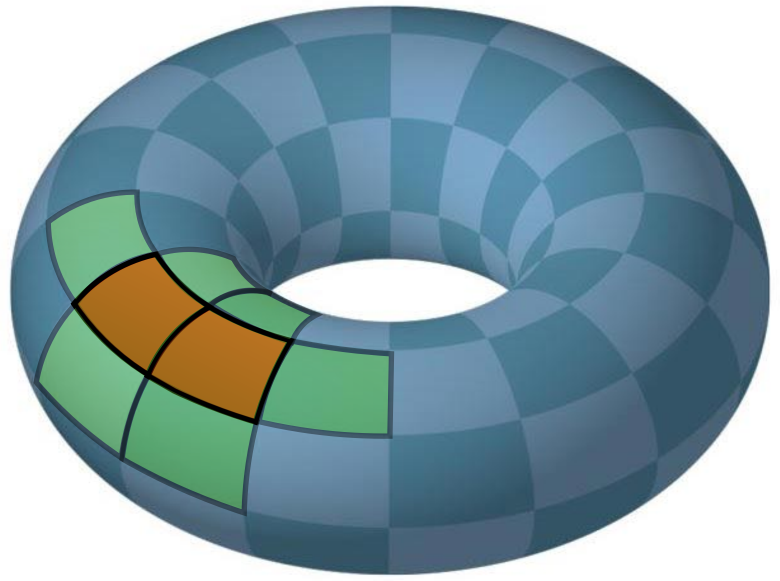
- The commutation condition is local
  - The same for infinite/finite QCA
- By Stinespring theorem in the finite case

$$\mathcal{V}(B_R) = U(I \otimes B_R)U^\dagger$$
$$= \sum_i \prod_{x \in R} U_x(I \otimes B_x^{(i)})U_x^\dagger$$

- In the infinite case the local action is the same

$$\mathcal{V}_\infty(B_R) = \sum_i \prod_{x \in R} U_x(I \otimes B_x^{(i)})U_x^\dagger$$
$$= W(I \otimes B_R)W^\dagger$$



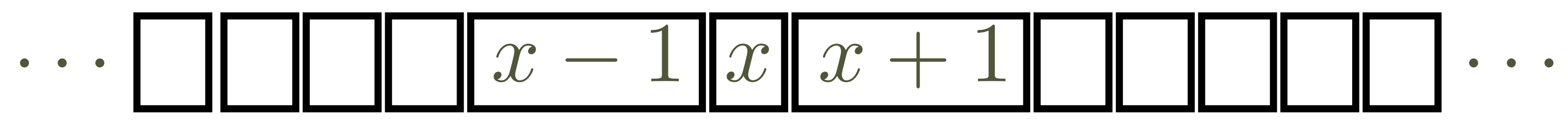


B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Example

Qubit QCA in 1 dimension

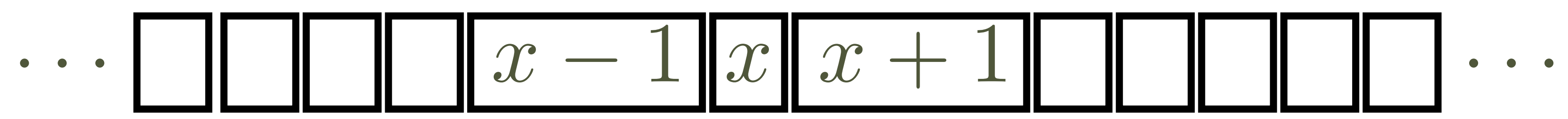
- Every cell is a qubit
- The cells can be labelled by integers $x \in \mathbb{Z}$



Example

Qubit QCA in 1 dimension

- Every cell is a qubit
- The cells can be labelled by integers $x \in \mathbb{Z}$

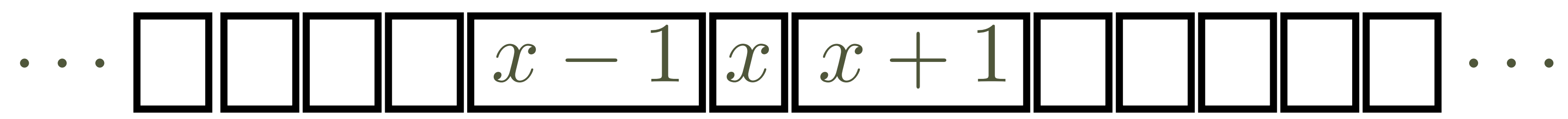


- Local rule $\mathscr{V}_x : \mathsf{A}_x \to \mathsf{A}_{x-1} \otimes \mathsf{A}_x \otimes \mathsf{A}_{x+1}$

Example

Qubit QCA in 1 dimension

- Every cell is a qubit
- The cells can be labelled by integers $x \in \mathbb{Z}$

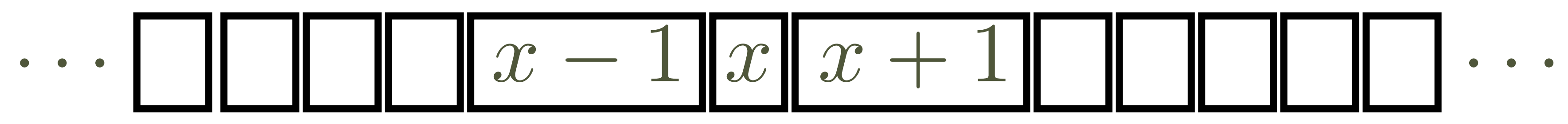


- Local rule $\mathscr{V}_x : \mathsf{A}_x \to \mathsf{A}_{x-1} \otimes \mathsf{A}_x \otimes \mathsf{A}_{x+1}$
  - $\mathscr{V}_x(\sigma_1^{(x)}) = I^{(x-1)} \otimes \sigma_3^{(x)} \otimes I^{(x+1)}, \quad \mathscr{V}_x(\sigma_3^{(x)}) = \sigma_3^{(x-1)} \otimes \sigma_1^{(x)} \otimes \sigma_3^{(x+1)}$

Example

Qubit QCA in 1 dimension

- Every cell is a qubit
- The cells can be labelled by integers $x \in \mathbb{Z}$



- Local rule $\mathscr{V}_x : \mathsf{A}_x \to \mathsf{A}_{x-1} \otimes \mathsf{A}_x \otimes \mathsf{A}_{x+1}$
    - $\mathscr{V}_x(\sigma_1^{(x)}) = I^{(x-1)} \otimes \sigma_3^{(x)} \otimes I^{(x+1)}, \quad \mathscr{V}_x(\sigma_3^{(x)}) = \sigma_3^{(x-1)} \otimes \sigma_1^{(x)} \otimes \sigma_3^{(x+1)}$
    - $\Rightarrow \ \mathscr{V}_x(\sigma_2^{(x)}) = -\sigma_3^{(x-1)} \otimes \sigma_2^{(x)} \otimes \sigma_3^{(x+1)}$ [indeed $\mathscr{V}_x(\sigma_2^{(x)}) = -i\mathscr{V}_x(\sigma_3^{(x)}\sigma_1^{(x)}) = -i\mathscr{V}_x(\sigma_3^{(x)})\mathscr{V}_x(\sigma_1^{(x)})$]

Example

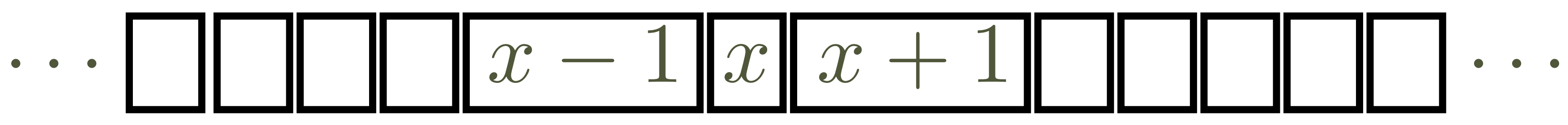
Qubit QCA in 1 dimension

- Every cell is a qubit
- The cells can be labelled by integers $x \in \mathbb{Z}$ $\quad \cdots \square\square\square\square\boxed{x-1}\boxed{x}\boxed{x+1}\square\square\square\square\cdots$
- Local rule $\mathscr{V}_x : \mathsf{A}_x \to \mathsf{A}_{x-1} \otimes \mathsf{A}_x \otimes \mathsf{A}_{x+1}$
  - $\mathscr{V}_x(\sigma_1^{(x)}) = I^{(x-1)} \otimes \sigma_3^{(x)} \otimes I^{(x+1)}, \quad \mathscr{V}_x(\sigma_3^{(x)}) = \sigma_3^{(x-1)} \otimes \sigma_1^{(x)} \otimes \sigma_3^{(x+1)}$
  - $\Rightarrow \; \mathscr{V}_x(\sigma_2^{(x)}) = -\sigma_3^{(x-1)} \otimes \sigma_2^{(x)} \otimes \sigma_3^{(x+1)}$ [indeed $\mathscr{V}_x(\sigma_2^{(x)}) = -i\mathscr{V}_x(\sigma_3^{(x)}\sigma_1^{(x)}) = -i\mathscr{V}_x(\sigma_3^{(x)})\mathscr{V}_x(\sigma_1^{(x)})$]
    - Check that $[\mathscr{V}_x(A_x), \mathscr{V}_y(B_y)] = 0, \quad x \neq y$

Example

Qubit QCA in 1 dimension

- Every cell is a qubit
- The cells can be labelled by integers $x \in \mathbb{Z}$



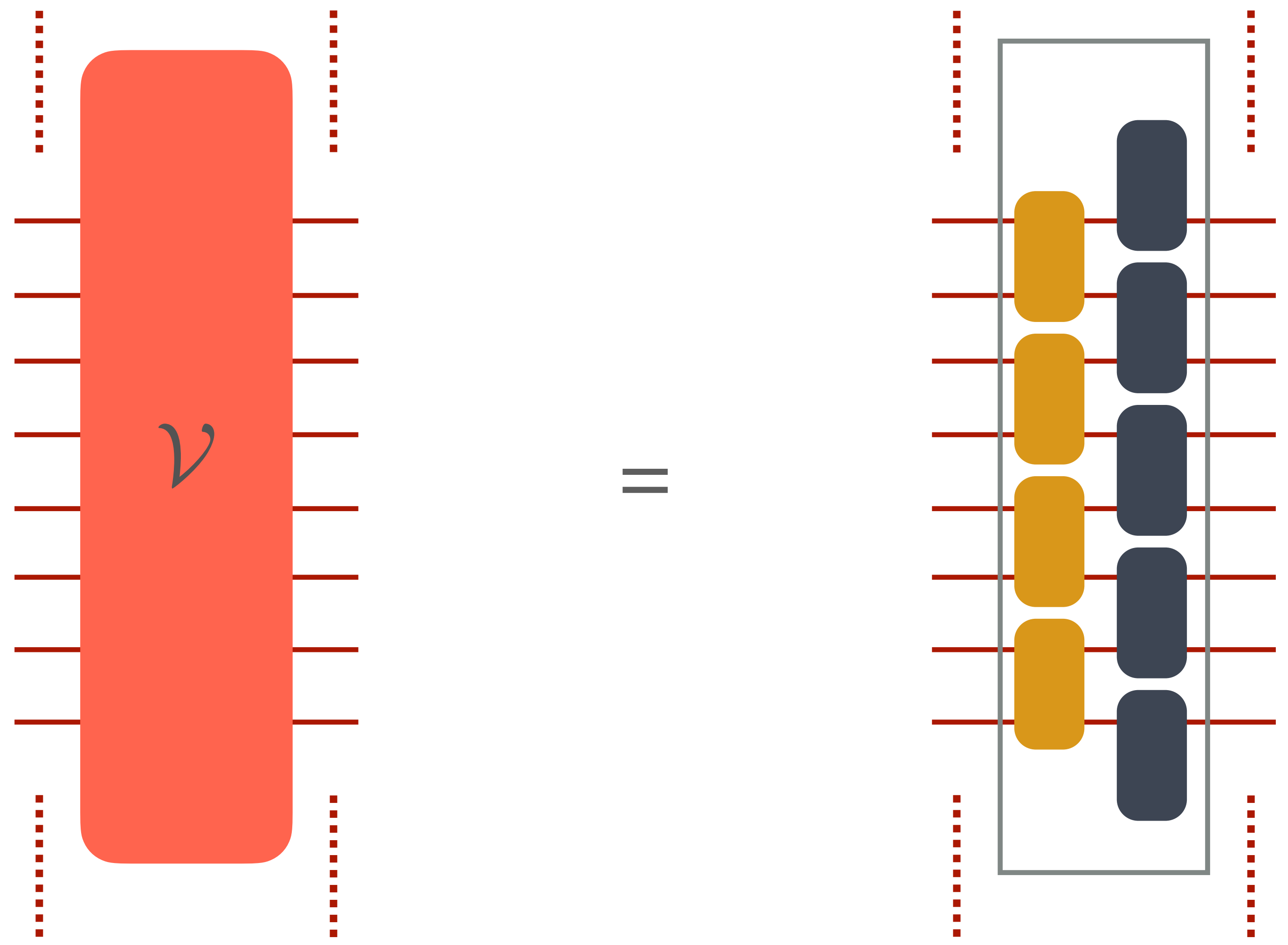
- Local rule $\mathscr{V}_x : \mathsf{A}_x \to \mathsf{A}_{x-1} \otimes \mathsf{A}_x \otimes \mathsf{A}_{x+1}$
  - $\mathscr{V}_x(\sigma_1^{(x)}) = I^{(x-1)} \otimes \sigma_3^{(x)} \otimes I^{(x+1)}, \quad \mathscr{V}_x(\sigma_3^{(x)}) = \sigma_3^{(x-1)} \otimes \sigma_1^{(x)} \otimes \sigma_3^{(x+1)}$
  - $\Rightarrow \; \mathscr{V}_x(\sigma_2^{(x)}) = -\sigma_3^{(x-1)} \otimes \sigma_2^{(x)} \otimes \sigma_3^{(x+1)}$ [indeed $\mathscr{V}_x(\sigma_2^{(x)}) = -i\mathscr{V}_x(\sigma_3^{(x)}\sigma_1^{(x)}) = -i\mathscr{V}_x(\sigma_3^{(x)})\mathscr{V}_x(\sigma_1^{(x)})$]
    - Check that $[\mathscr{V}_x(A_x), \mathscr{V}_y(B_y)] = 0, \quad x \neq y$
    - Problem: what is U_x (such that $\mathscr{V}_x(A) = U_x(I_{x-1} \otimes A \otimes I_{x+1})U_x^\dagger$)?

Margolus decomposition

- Let us consider a QCA ν on $\mathbb{Z}$

Margolus decomposition

- Let us consider a QCA $\mathcal{V}$ on $\mathbb{Z}$
- It admits a Margolus decomposition if it corresponds to a FDQC



Unitarity and locality imply Margolus

- Let $\mathcal{V}$ be a QCA on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r$

P. Arrighi, V. Nesme, R.F. Werner, J. Comput. Syst. Sci. **77**, 372–378 (2011)

Unitarity and locality imply Margolus

- Let $\mathcal{V}$ be a QCA on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r$
- Consider the QCA $\mathcal{V} \otimes \mathcal{V}^{-1}$ on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r \otimes \mathbb{C}^r$

P. Arrighi, V. Nesme, R.F. Werner, J. Comput. Syst. Sci. **77**, 372–378 (2011)

Unitarity and locality imply Margolus

- Let $\mathcal{V}$ be a QCA on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r$
- Consider the QCA $\mathcal{V} \otimes \mathcal{V}^{-1}$ on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r \otimes \mathbb{C}^r$
- Since $V \otimes V^{-1} = (V \otimes I)S(V^{-1} \otimes I)S$, where $S|\phi\rangle|\psi\rangle := |\psi\rangle|\phi\rangle$, and $S = \bigotimes_x S_x$

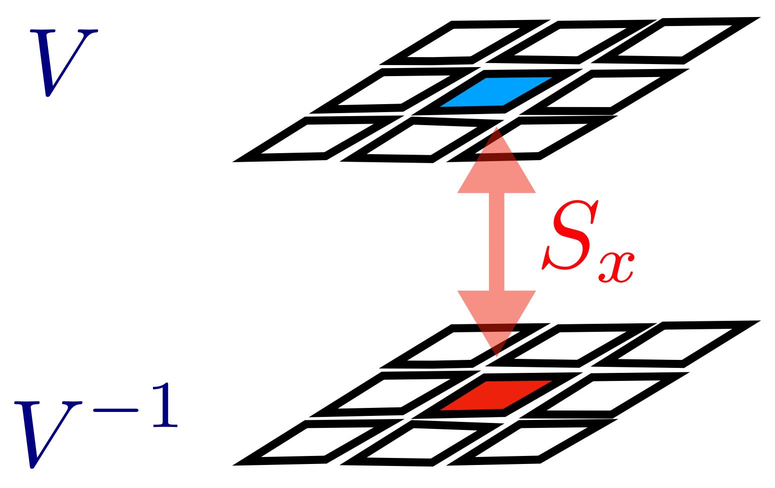
$$V \otimes V^{-1} = \prod_x (S'_x) \bigotimes_y S_y \qquad S'_x := (V \otimes I) S_x (V^{-1} \otimes I)$$

P. Arrighi, V. Nesme, R.F. Werner, J. Comput. Syst. Sci. **77**, 372–378 (2011)

Unitarity and locality imply Margolus

- Let $\mathcal{V}$ be a QCA on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r$
- Consider the QCA $\mathcal{V} \otimes \mathcal{V}^{-1}$ on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r \otimes \mathbb{C}^r$
- Since $V \otimes V^{-1} = (V \otimes I)S(V^{-1} \otimes I)S$, where $S|\phi\rangle|\psi\rangle := |\psi\rangle|\phi\rangle$, and $S = \bigotimes_x S_x$

$$V \otimes V^{-1} = \prod_x (S'_x) \bigotimes_y S_y \qquad S'_x := (V \otimes I)S_x(V^{-1} \otimes I)$$

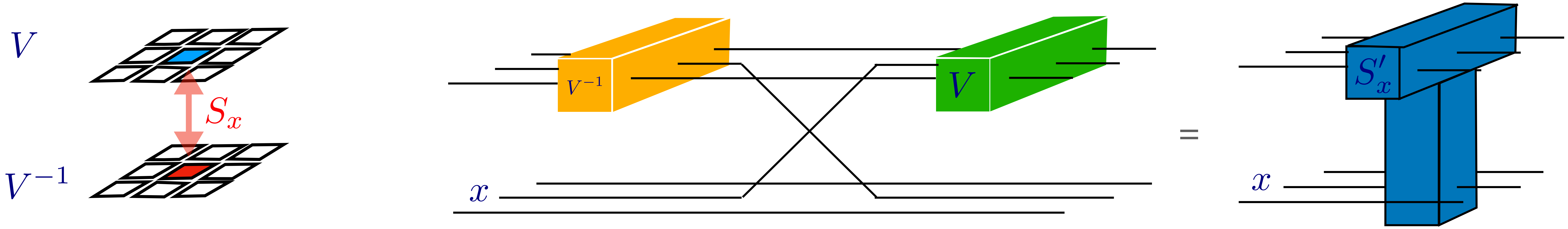


P. Arrighi, V. Nesme, R.F. Werner, J. Comput. Syst. Sci. **77**, 372–378 (2011)

Unitarity and locality imply Margolus

- Let $\mathcal{V}$ be a QCA on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r$
- Consider the QCA $\mathcal{V} \otimes \mathcal{V}^{-1}$ on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r \otimes \mathbb{C}^r$
- Since $V \otimes V^{-1} = (V \otimes I)S(V^{-1} \otimes I)S$, where $S|\phi\rangle|\psi\rangle := |\psi\rangle|\phi\rangle$, and $S = \bigotimes_x S_x$

$$V \otimes V^{-1} = \prod_x (S'_x) \bigotimes_y S_y \qquad S'_x := (V \otimes I)S_x(V^{-1} \otimes I)$$



P. Arrighi, V. Nesme, R.F. Werner, J. Comput. Syst. Sci. **77**, 372–378 (2011)

QCA and quantum field theories

Two purposes

Foundations

- QCA: microscopic model, QFT **and** space-time emerge
- QCAs underpin spacetime
  - Revisit homogeneity
  - Revisit the geometry of cell array

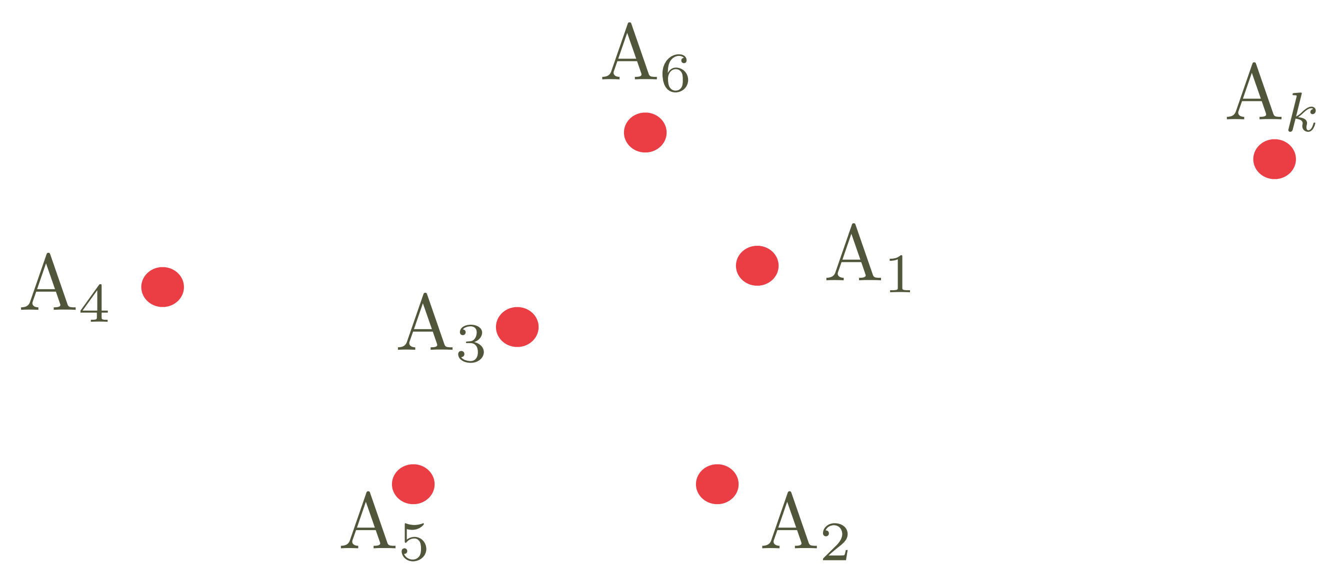
Simulations

- QCAs are approximations
- “Hamiltonian” model vs. Lagrangian or Action
- Find appropriate “discretization”
- Tools for the analysis

Causal influence and neighbourhood

For a fundamental approach

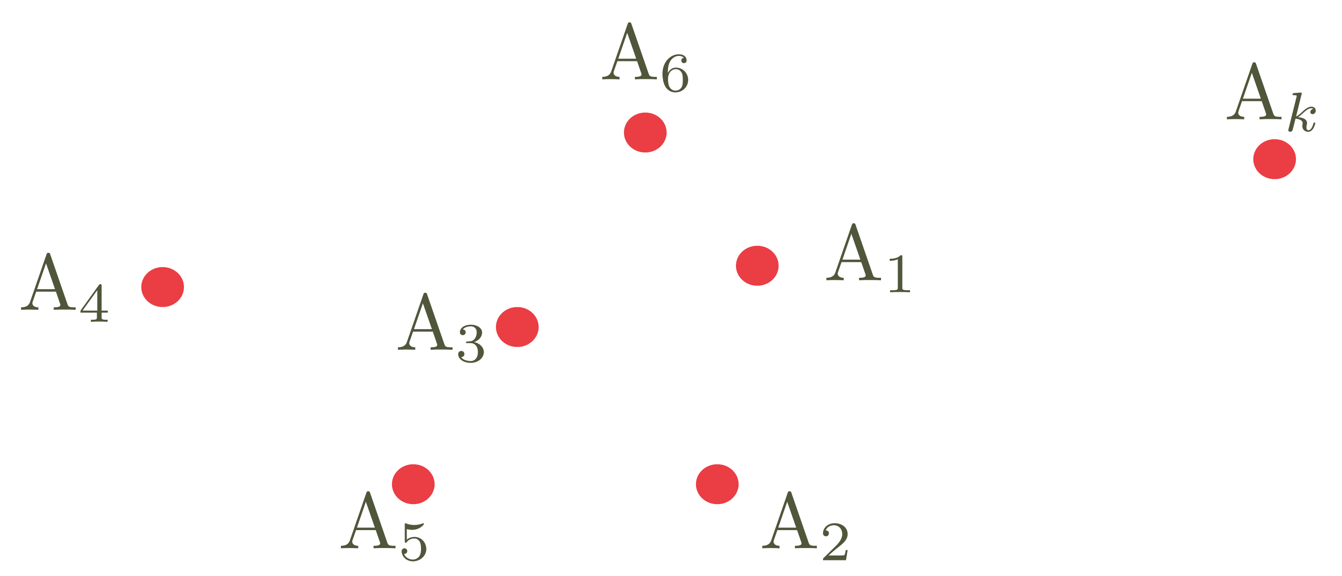
- We start with a denumerable collection of systems (cells)



Causal influence and neighbourhood

For a fundamental approach

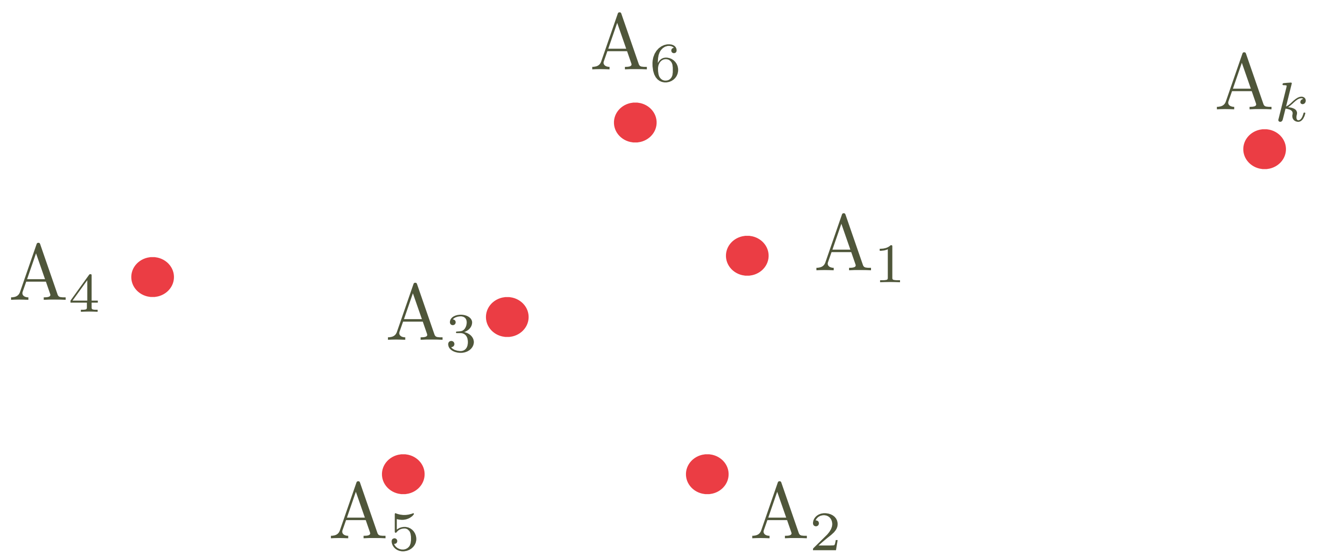
- We start with a denumerable collection of systems (cells)
- Given an automorphism of the quasi-local algebra



Causal influence and neighbourhood

For a fundamental approach

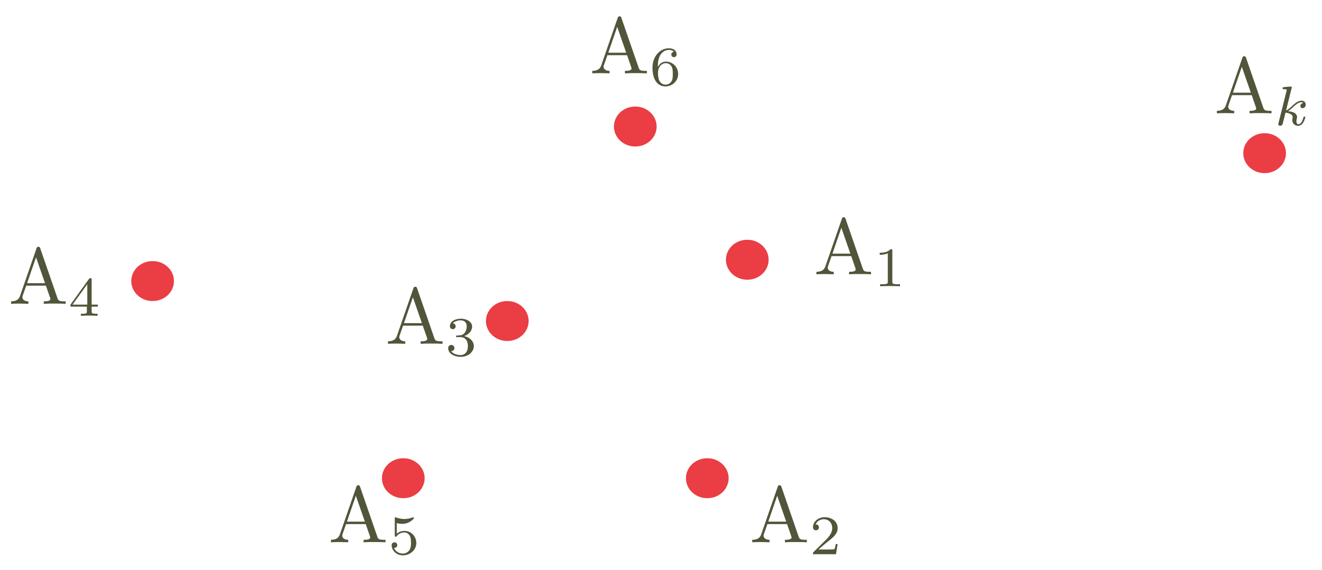
- We start with a denumerable collection of systems (cells)
- Given an automorphism of the quasi-local algebra
    - For every cell we check its neighbourhood



Causal influence and neighbourhood

For a fundamental approach

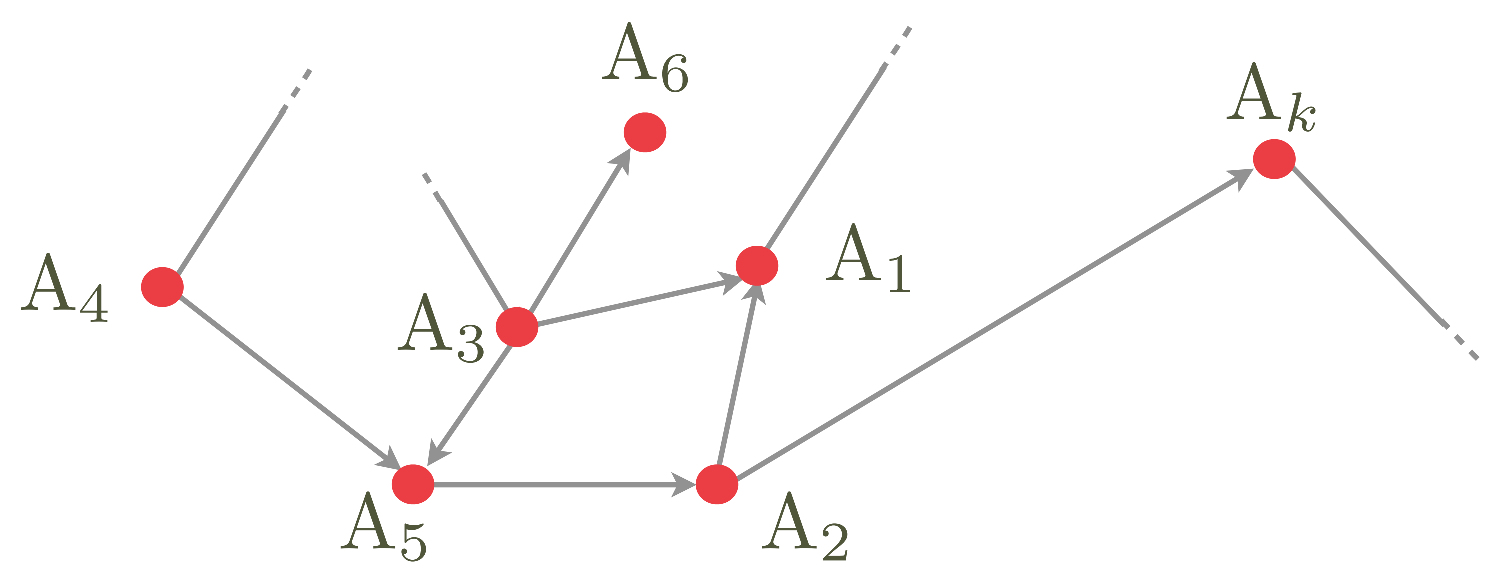
- We start with a denumerable collection of systems (cells)
- Given an automorphism of the quasi-local algebra
  - For every cell we check its neighbourhood
    - $N^-(x_0)$ is the smallest set of cells such that $U^{-1} \mathsf{A}_{x_0} U = \mathsf{A}_{N^-(x_0)} \otimes I_{\bar{x}_0}$



Causal influence and neighbourhood

For a fundamental approach

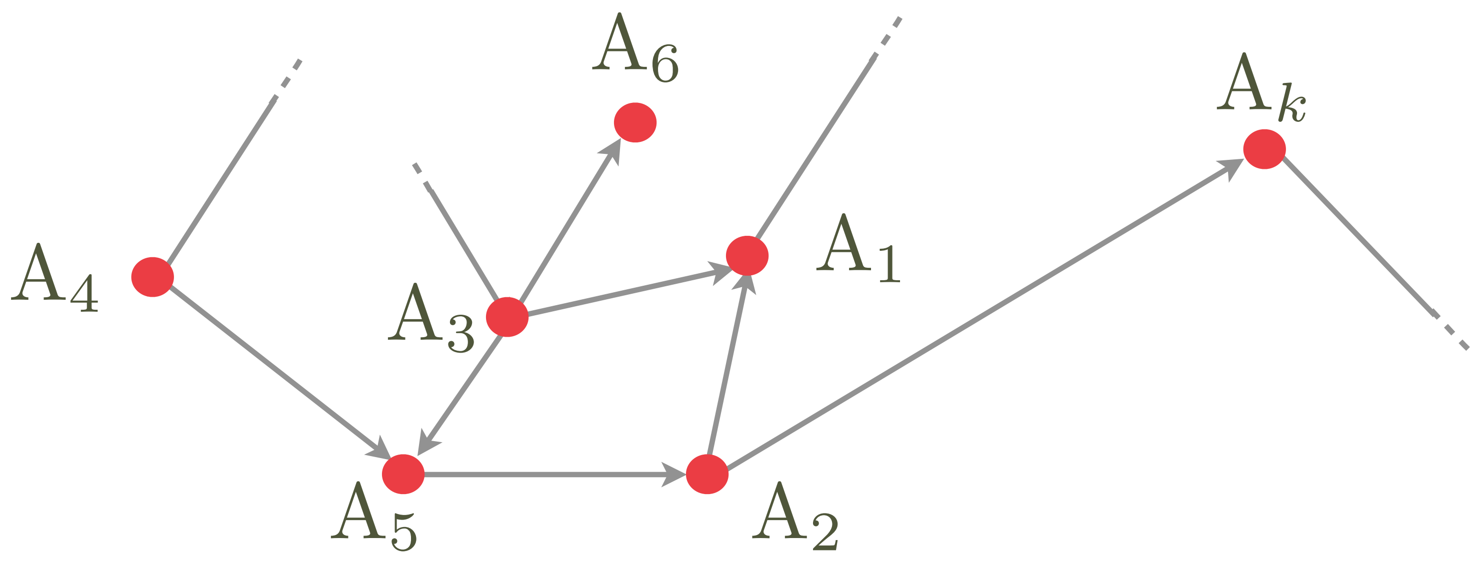
- We start with a denumerable collection of systems (cells)
- Given an automorphism of the quasi-local algebra
  - For every cell we check its neighbourhood
    - $N^-(x_0)$ is the smallest set of cells such that $U^{-1}\mathsf{A}_{x_0}U = \mathsf{A}_{N^-(x_0)} \otimes I_{\bar{x}_0}$
  - For every “neighbourhood” relation we draw an arrow



Causal influence and neighbourhood

For a fundamental approach

- We start with a denumerable collection of systems (cells)
- Given an automorphism of the quasi-local algebra
  - For every cell we check its neighbourhood
    - $N^{-}(x_0)$ is the smallest set of cells such that $U^{-1}\mathsf{A}_{x_0}U = \mathsf{A}_{N^{-}(x_0)} \otimes I_{\bar{x}_0}$
  - For every "neighbourhood" relation we draw an arrow



- What we obtain is a graph of causal relations dictated by the evolution rule

Causal influence graph

Homogeneity and Cayley graphs

- Starting from the evolution rule we can construct a metric space—the graph

Causal influence graph

Homogeneity and Cayley graphs

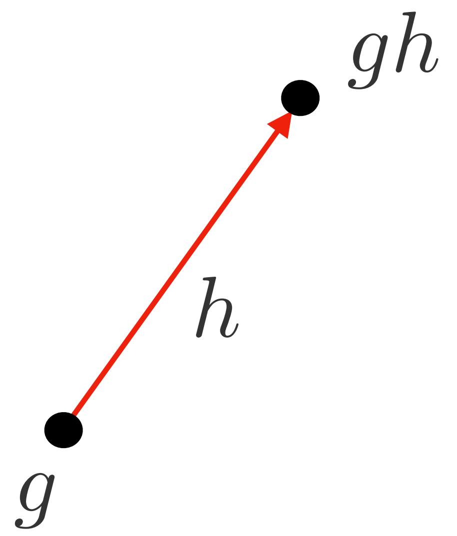
- Starting from the evolution rule we can construct a metric space—the graph
- Homogeneity: “every cell is treated in the same way” (made rigorous)

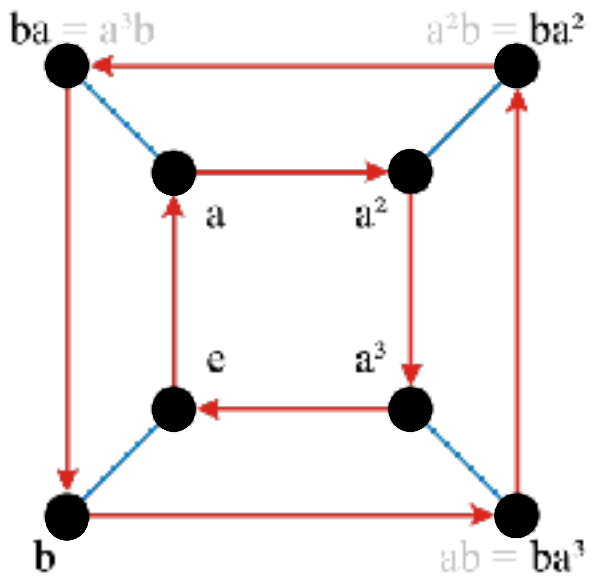
PP, Quantum **4**, 294 (2020)

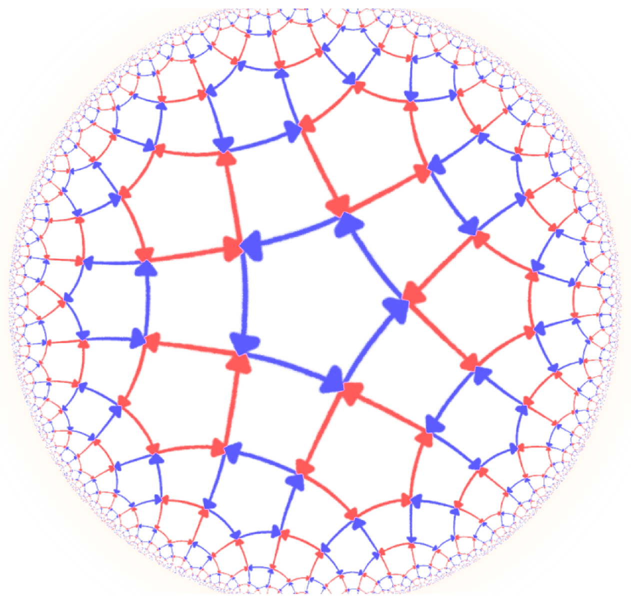
Causal influence graph

Homogeneity and Cayley graphs

- Starting from the evolution rule we can construct a metric space—the graph
- Homogeneity: "every cell is treated in the same way" (made rigorous)
- If the FCA is homogeneous the graph is the Cayley graph of a group







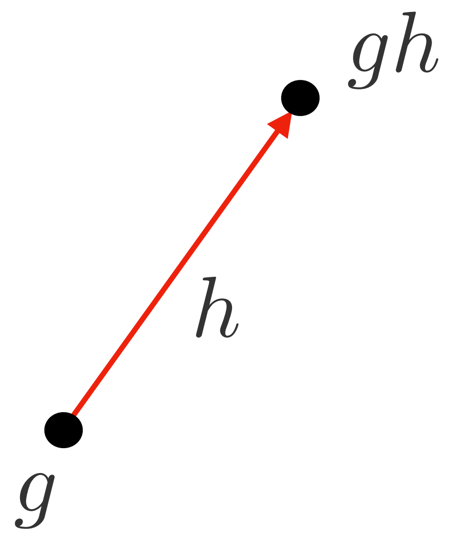
$$\langle r, b | r^5, b^5, (rb)^2 \rangle$$

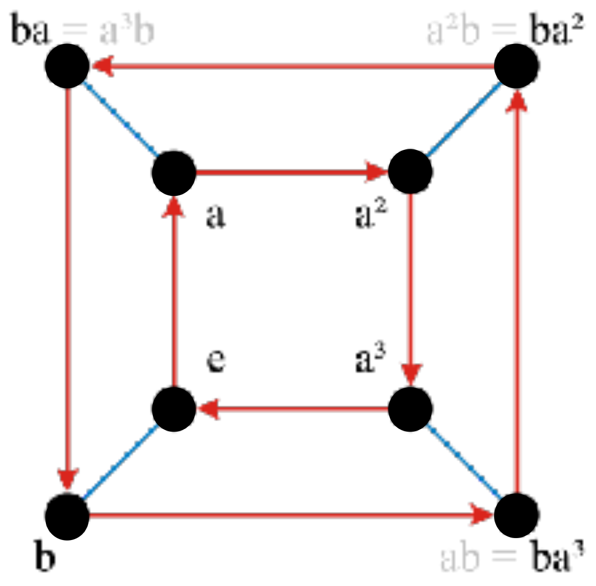
PP, Quantum **4**, 294 (2020)

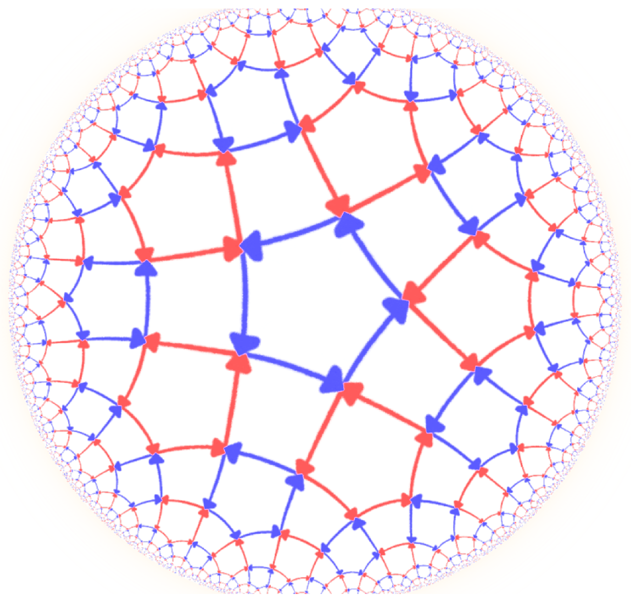
Causal influence graph

Homogeneity and Cayley graphs

- Starting from the evolution rule we can construct a metric space—the graph
- Homogeneity: “every cell is treated in the same way” (made rigorous)
- If the FCA is homogeneous the graph is the Cayley graph of a group
  - The group of translations on the graph itself







$$\langle r, b | r^5, b^5, (rb)^2 \rangle$$

PP, Quantum **4**, 294 (2020)

Emergent space

Geometric group theory

- Metric spaces can be grouped in equivalence classes modulo quasi-isometry

$$\frac{1}{a} d_R(\mathsf{E}(g), \mathsf{E}(g')) - b \leq d_\Gamma(g, g') \leq a\, d_R(\mathsf{E}(g), \mathsf{E}(g')) + b$$

Emergent space

Geometric group theory

- Metric spaces can be grouped in equivalence classes modulo quasi-isometry

$$\frac{1}{a} d_R(\mathsf{E}(g), \mathsf{E}(g')) - b \leq d_\Gamma(g, g') \leq a\, d_R(\mathsf{E}(g), \mathsf{E}(g')) + b$$

- Every Cayley graph of the same group is in the same class

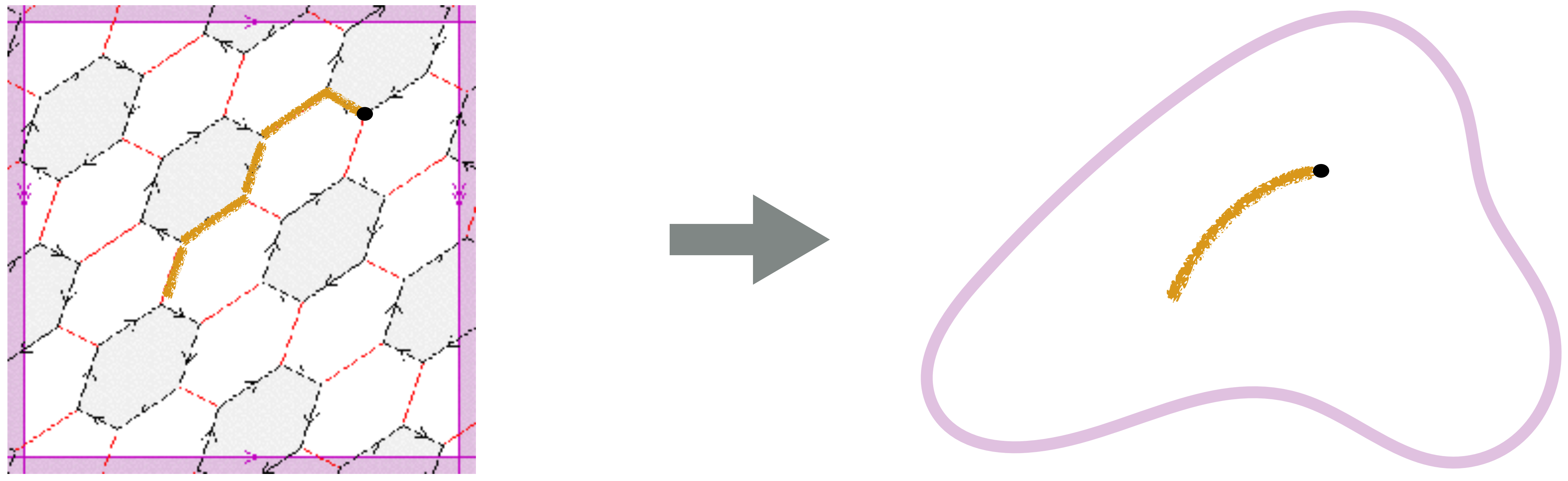
Emergent space

Geometric group theory

- Metric spaces can be grouped in equivalence classes modulo quasi-isometry

$$\frac{1}{a} d_R(\mathsf{E}(g), \mathsf{E}(g')) - b \leq d_\Gamma(g, g') \leq a\, d_R(\mathsf{E}(g), \mathsf{E}(g')) + b$$

- Every Cayley graph of the same group is in the same class
- In every class there is an essentially unique smooth manifold



Fermionic theory

- The theory is meant to provide a realisation of the fermion (CAR) algebra

$$\{\varphi_i^\dagger, \varphi_j\} = \delta_{ij} I, \quad \{\varphi_i, \varphi_j\} = 0$$

for an arbitrary finite number N of Local Fermionic Modes (LFM) $1 \leq i, j \leq N$

Bravy and Kitaev, Annals of Physics **298**, 210–226 (2002)
G. M. D'Ariano, F. Manessi, PP, and A. Tosini, Int. J. Mod. Phys. A **29**, 1430025 (2014)

Fermionic theory

- The theory is meant to provide a realisation of the fermion (CAR) algebra

$$\{\varphi_i^\dagger, \varphi_j\} = \delta_{ij} I, \quad \{\varphi_i, \varphi_j\} = 0$$

  for an arbitrary finite number N of Local Fermionic Modes (LFM) $1 \leq i, j \leq N$

- One can prove that the fermion algebra can be represented on $\mathscr{H} := \mathbb{C}^{2N}$

$$\varphi_i^\dagger \varphi_i |00\dots0\rangle = 0, \ \forall i \qquad |s\rangle = {\varphi_1^\dagger}^{s_1} \dots {\varphi_N^\dagger}^{s_N} |00\dots0\rangle$$

Bravy and Kitaev, Annals of Physics **298**, 210–226 (2002)
G. M. D'Ariano, F. Manessi, PP, and A. Tosini, Int. J. Mod. Phys. A **29**, 1430025 (2014)

Fermionic theory

- The theory is meant to provide a realisation of the fermion (CAR) algebra

$$\{\varphi_i^\dagger, \varphi_j\} = \delta_{ij} I, \quad \{\varphi_i, \varphi_j\} = 0$$

  for an arbitrary finite number N of Local Fermionic Modes (LFM) $1 \leq i, j \leq N$

- One can prove that the fermion algebra can be represented on $\mathscr{H} := \mathbb{C}^{2N}$

$$\varphi_i^\dagger \varphi_i |00\ldots 0\rangle = 0, \ \forall i \qquad\qquad |s\rangle = {\varphi_1^\dagger}^{s_1} \ldots {\varphi_N^\dagger}^{s_N} |00\ldots 0\rangle$$

- Example of basis state: $|0010110\rangle = \varphi_3^\dagger \varphi_5^\dagger \varphi_6^\dagger |0000000\rangle$

Bravy and Kitaev, Annals of Physics **298**, 210–226 (2002)
G. M. D'Ariano, F. Manessi, PP, and A. Tosini, Int. J. Mod. Phys. A **29**, 1430025 (2014)

Jordan-Wigner representation

$$\varphi_i \leftrightarrow I_1 \otimes I_2 \otimes \ldots \otimes I_{i-1} \otimes \sigma_i^- \otimes \sigma_{i+1}^z \otimes \sigma_{i+2}^z \otimes \ldots \otimes \sigma_N^z$$

Jordan-Wigner representation

$$\varphi_i \leftrightarrow I_1 \otimes I_2 \otimes \ldots \otimes I_{i-1} \otimes \sigma_i^- \otimes \sigma_{i+1}^z \otimes \sigma_{i+2}^z \otimes \ldots \otimes \sigma_N^z$$

E.g. $N = 5$

$$\begin{aligned}\{\varphi_3, \varphi_4\} &= \{I_1 \otimes I_2 \otimes \sigma_3^- \otimes \sigma_4^z \otimes \sigma_5^z, I_1 \otimes I_2 \otimes I_3 \otimes \sigma_4^- \otimes \sigma_5^z\} \\ &= I_1 \otimes I_2 \otimes \sigma_3^- \otimes \{\sigma_4^z, \sigma_4^-\} \otimes I_5^z = 0\end{aligned}$$

$$\begin{aligned}\{\varphi_3, \varphi_3^\dagger\} &= \{I_1 \otimes I_2 \otimes \sigma_3^- \otimes \sigma_4^z \otimes \sigma_5^z, I_1 \otimes I_2 \otimes \sigma_3^+ \otimes \sigma_4^z \otimes \sigma_5^z\} \\ &= I_1 \otimes I_2 \otimes \{\sigma_3^-, \sigma_3^+\} \otimes I_4 \otimes I_5 = I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5\end{aligned}$$

Fermionic CA

- Local algebra of cell x is generated by $\varphi_{x,s},\ \varphi^{\dagger}_{x,t}$

$$\varphi_i = \begin{pmatrix} \varphi_{i,1} \\ \varphi_{i,2} \\ \vdots \\ \varphi_{i,s} \end{pmatrix}$$

Fermionic CA

- Local algebra of cell x is generated by $\varphi_{x,s},\ \varphi^\dagger_{x,t}$
- Any reversible evolution must preserve CAR

$$\mathscr{T}(\varphi_i) = \varphi'_i$$

$$\varphi_i = \begin{pmatrix} \varphi_{i,1} \\ \varphi_{i,2} \\ \vdots \\ \varphi_{i,s} \end{pmatrix}$$

Fermionic CA

- Local algebra of cell x is generated by $\varphi_{x,s},\ \varphi^\dagger_{x,t}$
- Any reversible evolution must preserve CAR

$$\mathscr{T}(\varphi_i) = \varphi'_i$$

$$\{\varphi'_i, \varphi'^\dagger_j\} = \delta_{ij} I$$
$$\{\varphi'_i, \varphi'_j\} = 0$$

$$\varphi_i = \begin{pmatrix} \varphi_{i,1} \\ \varphi_{i,2} \\ \vdots \\ \varphi_{i,s} \end{pmatrix}$$

Fermionic CA

- Local algebra of cell x is generated by $\varphi_{x,s},\ \varphi^\dagger_{x,t}$
- Any reversible evolution must preserve CAR

$$\mathscr{T}(\varphi_i) = \varphi'_i$$

$$\{\varphi'_i, \varphi'^\dagger_j\} = \delta_{ij} I$$
$$\{\varphi'_i, \varphi'_j\} = 0$$

- In the finite case this is an iff condition

$$\varphi_i = \begin{pmatrix} \varphi_{i,1} \\ \varphi_{i,2} \\ \vdots \\ \varphi_{i,s} \end{pmatrix}$$

Fermionic CA

- Local algebra of cell x is generated by $\varphi_{x,s},\ \varphi^{\dagger}_{x,t}$
- Any reversible evolution must preserve CAR

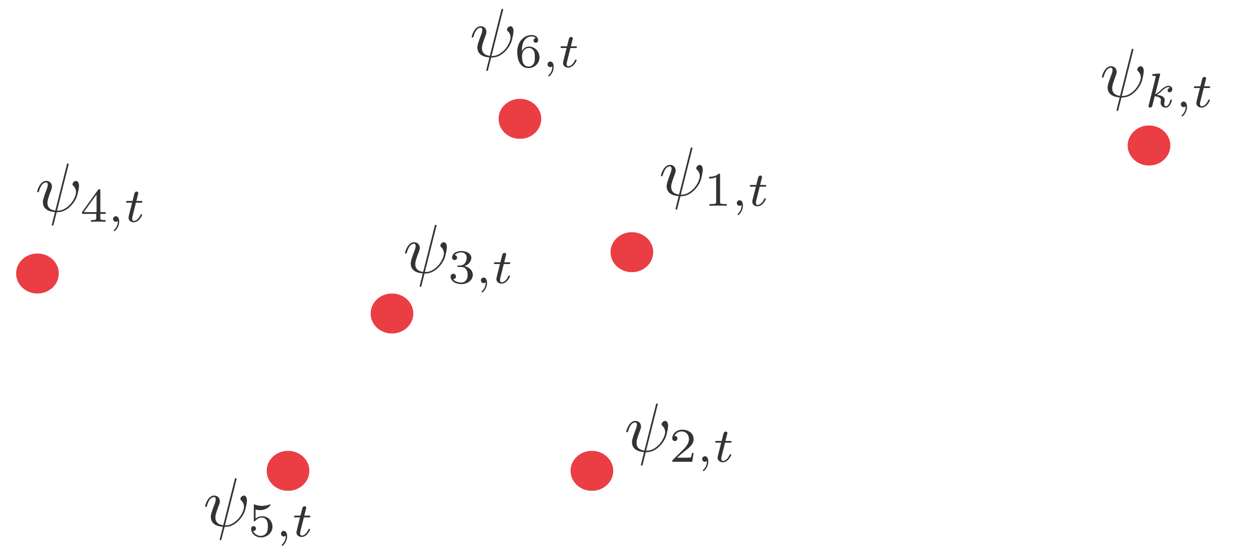
$$\mathscr{T}(\varphi_i) = \varphi'_i$$

$$\{\varphi'_i, \varphi'^{\dagger}_j\} = \delta_{ij} I$$
$$\{\varphi'_i, \varphi'_j\} = 0$$

$$\varphi_i = \begin{pmatrix} \varphi_{i,1} \\ \varphi_{i,2} \\ \vdots \\ \varphi_{i,s} \end{pmatrix}$$

- In the finite case this is an iff condition
  - Wrapping lemma: the iff condition holds also in the infinite case

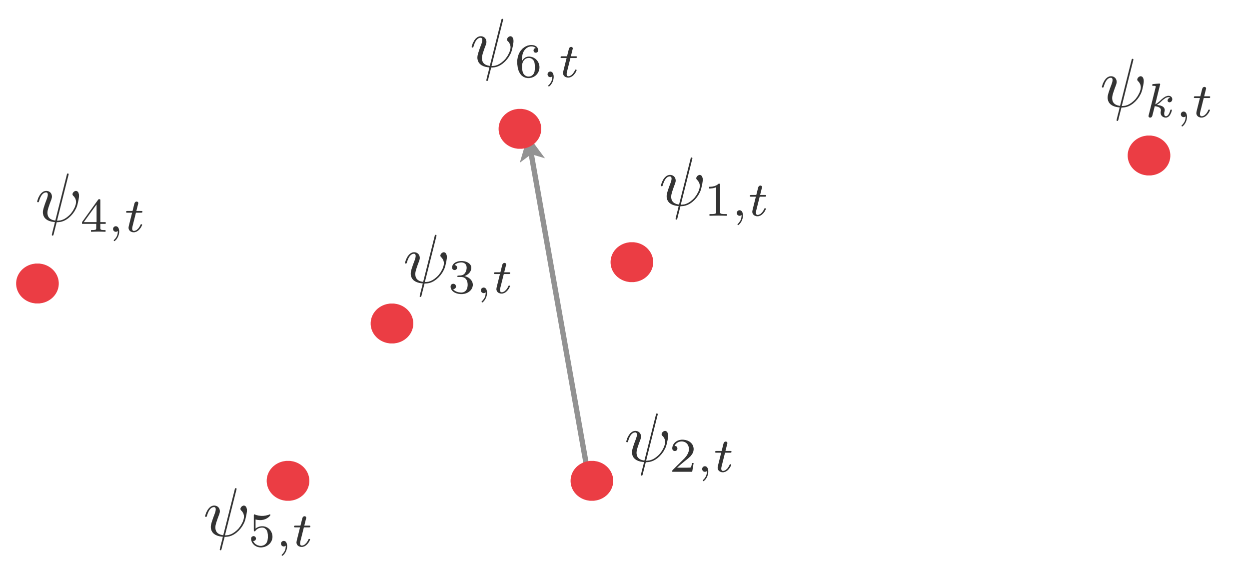
Linear FCAs



$$\psi_i = \begin{pmatrix} \psi_i^1 \\ \psi_i^2 \\ \vdots \\ \psi_i^s \end{pmatrix}$$

$$\psi_{i,t+1} = \sum_{j \in N_i} A_{i,j} \psi_{j,t} \qquad A_{ij} \in M_{s_i \times s_j}$$

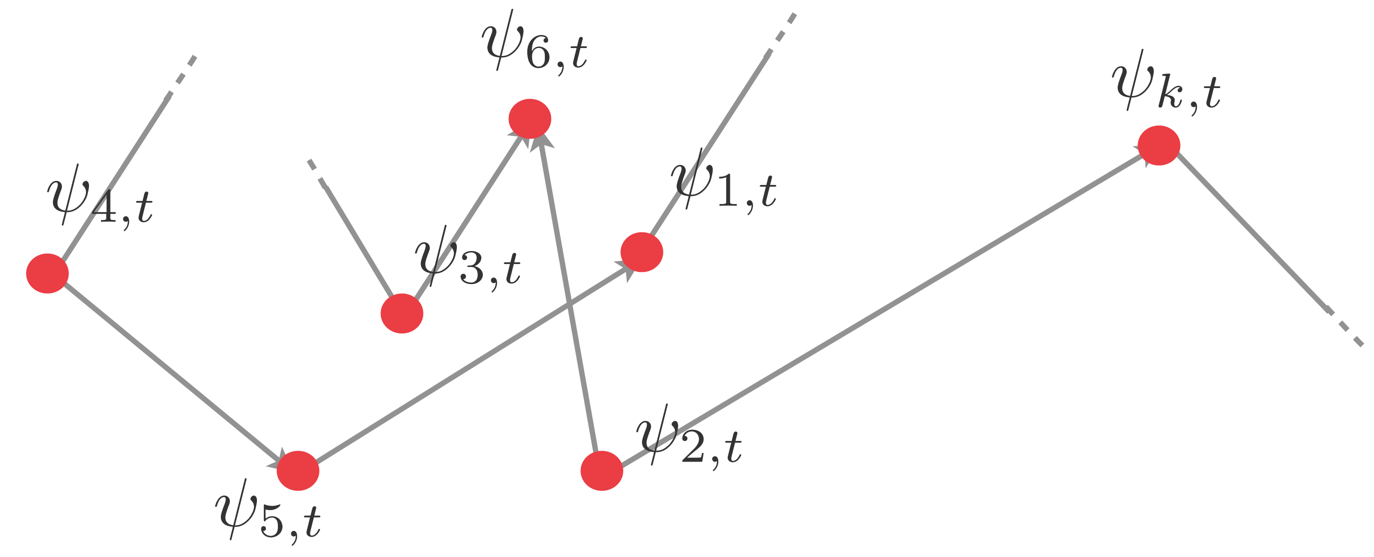
Linear FCAs



$$\psi_i = \begin{pmatrix} \psi_i^1 \\ \psi_i^2 \\ \vdots \\ \psi_i^s \end{pmatrix}$$

$$\psi_{i,t+1} = \sum_{j \in N_i} A_{i,j} \psi_{j,t} \qquad A_{ij} \in M_{s_i \times s_j}$$

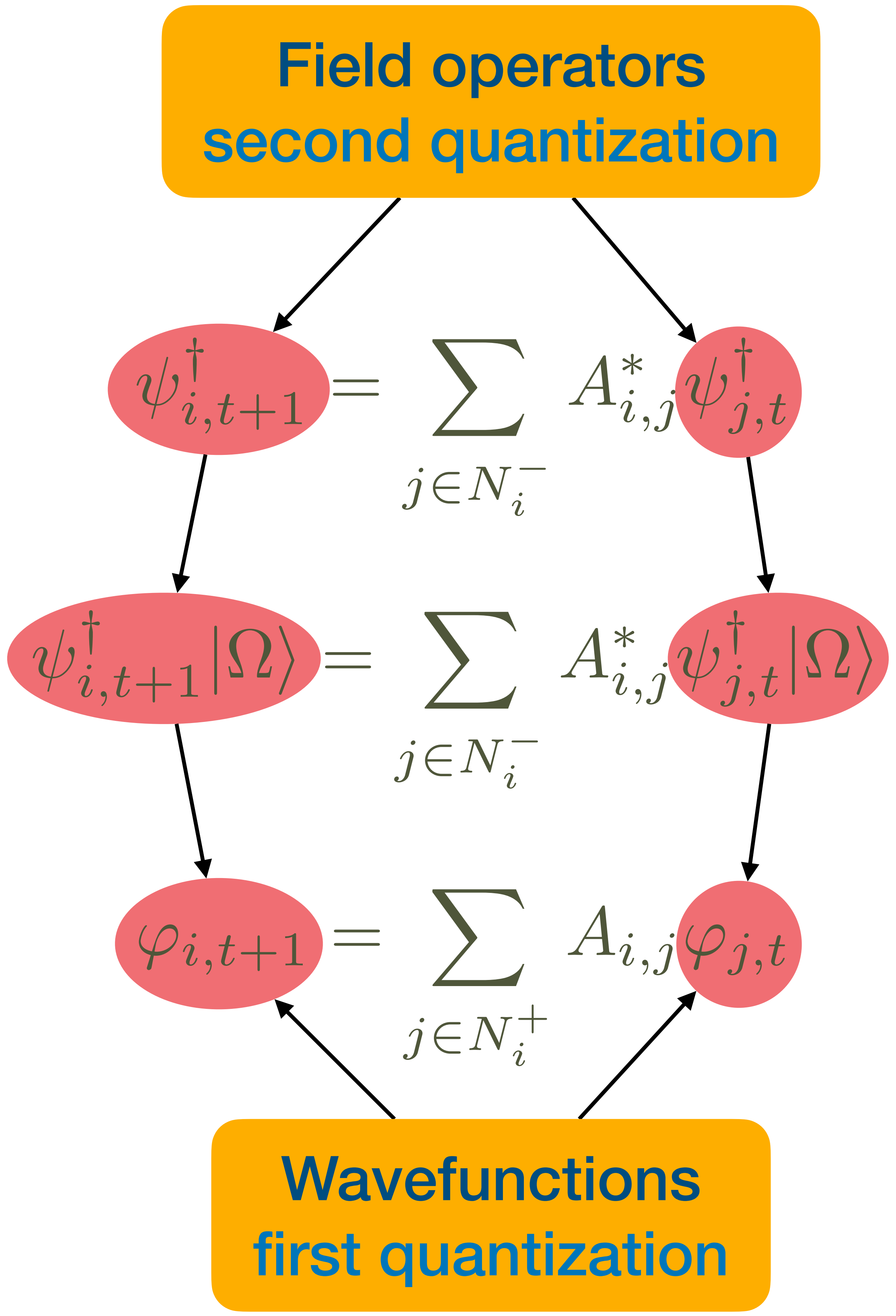
Linear FCAs



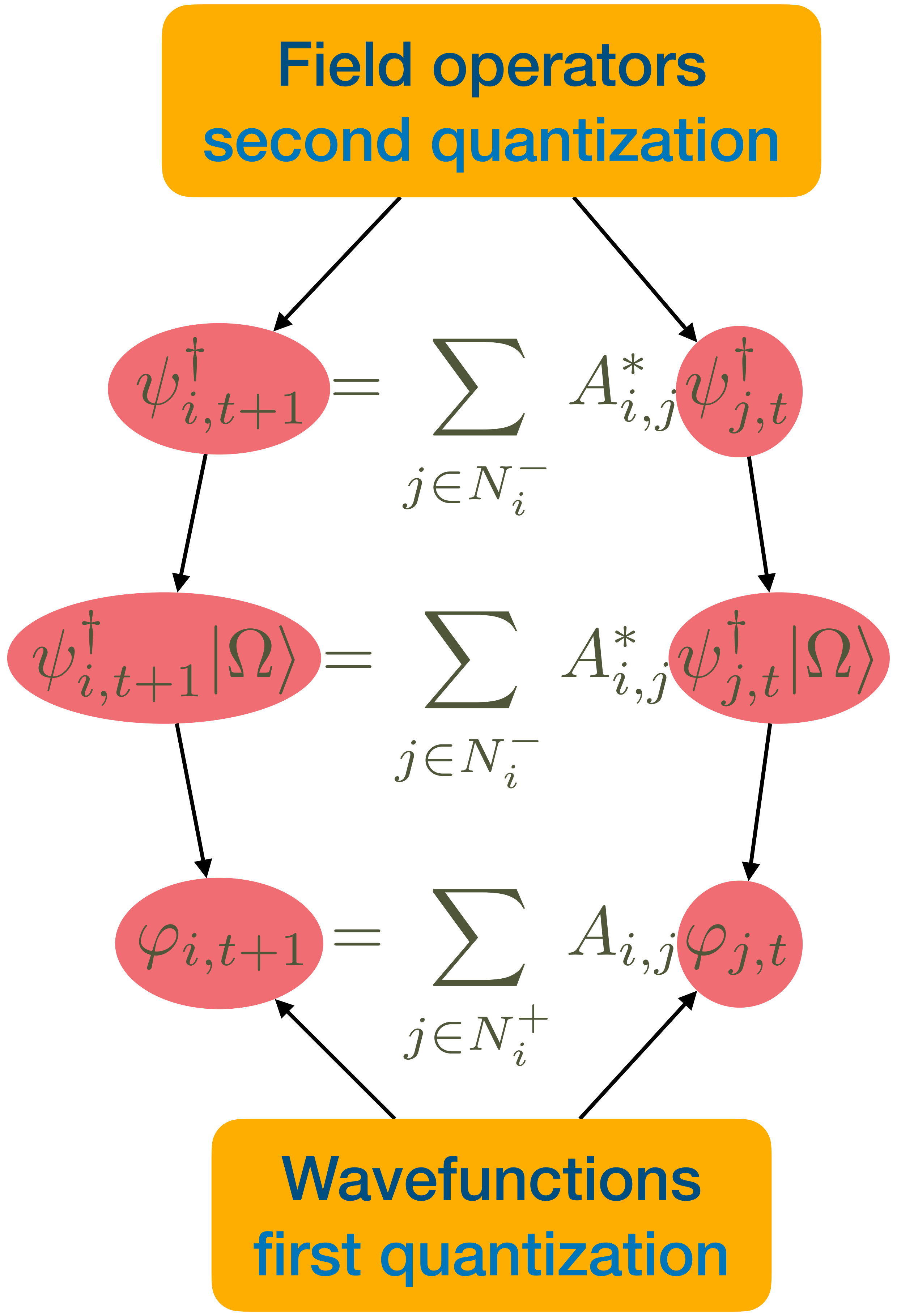
$$\psi_i = \begin{pmatrix} \psi_i^1 \\ \psi_i^2 \\ \vdots \\ \psi_i^s \end{pmatrix}$$

$$\psi_{i,t+1} = \sum_{j \in N_i} A_{i,j} \psi_{j,t} \qquad A_{ij} \in M_{s_i \times s_j}$$

Quantum walks



Quantum walks

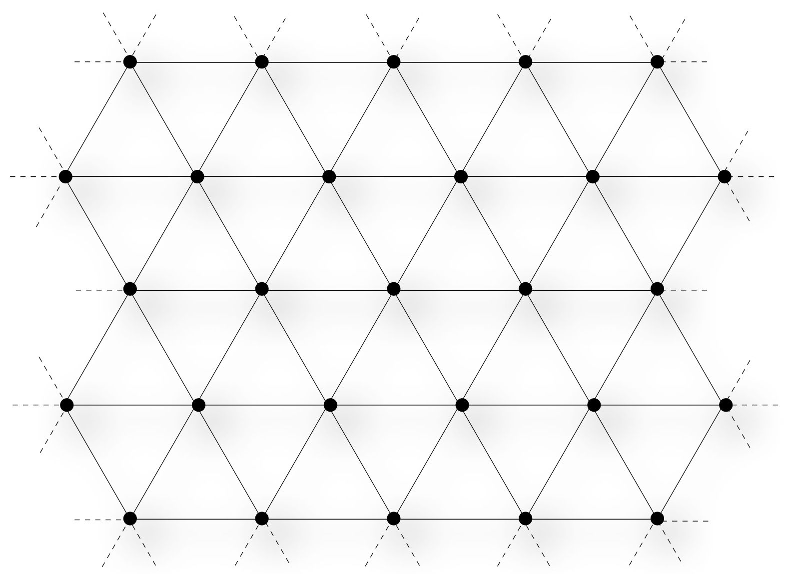


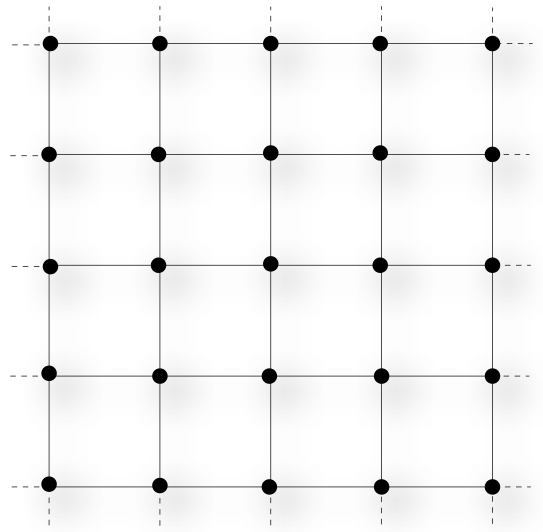
$A_{i,j}$'s: blocks of a "matrix" representing a
unitary operator on the space of wavefunctions
(single-particle excitations):

Quantum Walk

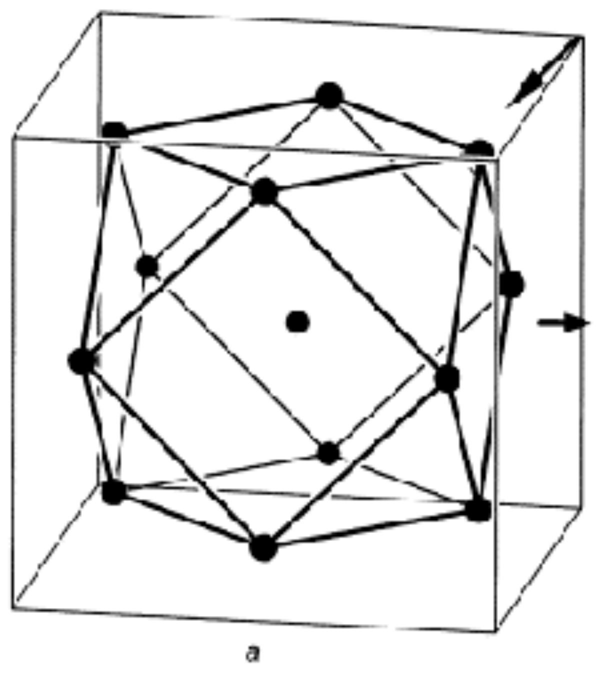
Bravais lattices

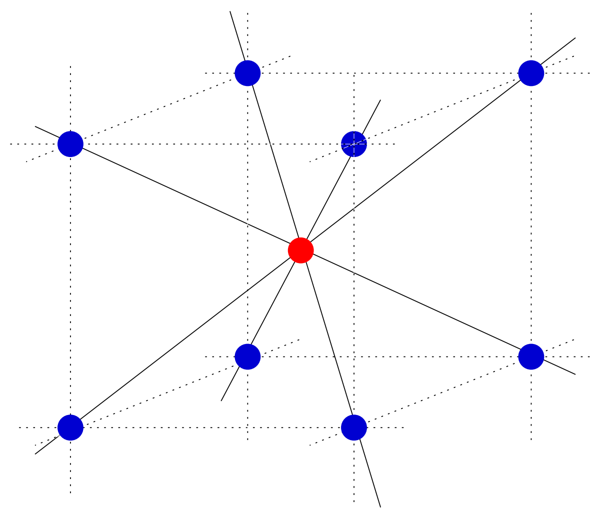
- Cayley graphs in Euclidean geometry: abelian group
- 2-d: square, hexagonal

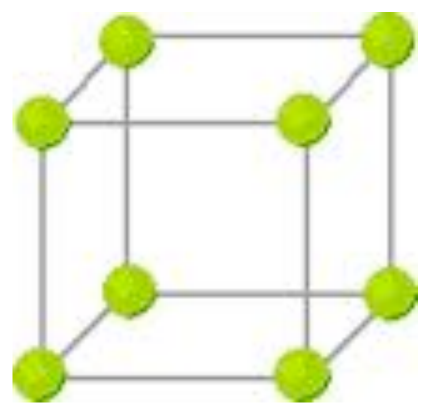




- 3-d: orthorhombic, PC, BCC





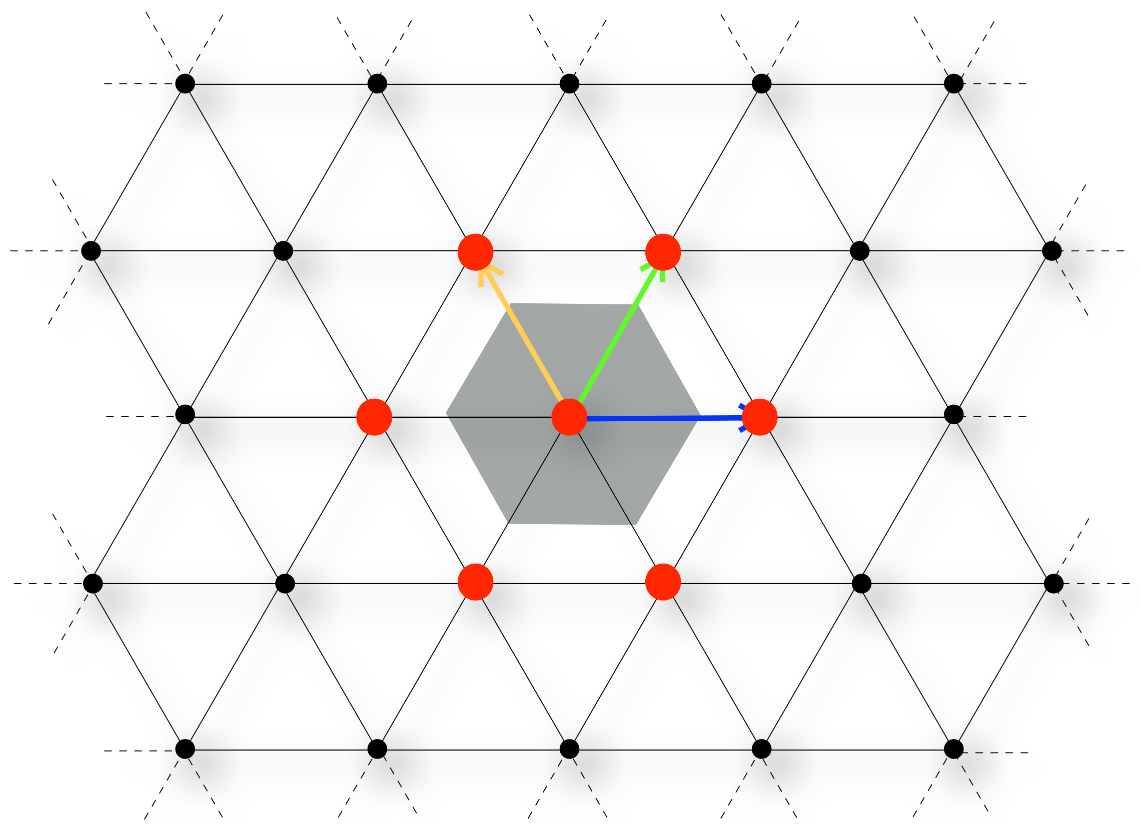


Diagonalising the walk

- From translation invariance $[T_{\mathbf{h}} \otimes I, W] = 0$

$$W = \int_B \mathrm{d}^d \mathbf{k} \; |\mathbf{k}\rangle\langle\mathbf{k}| \otimes W_{\mathbf{k}}$$

$$W_{\mathbf{k}} := \sum_{\mathbf{h} \in S} e^{-i\mathbf{k}\cdot\mathbf{h}} A_{\mathbf{h}}$$



Brillouin zone

Unitarity conditions for a QW

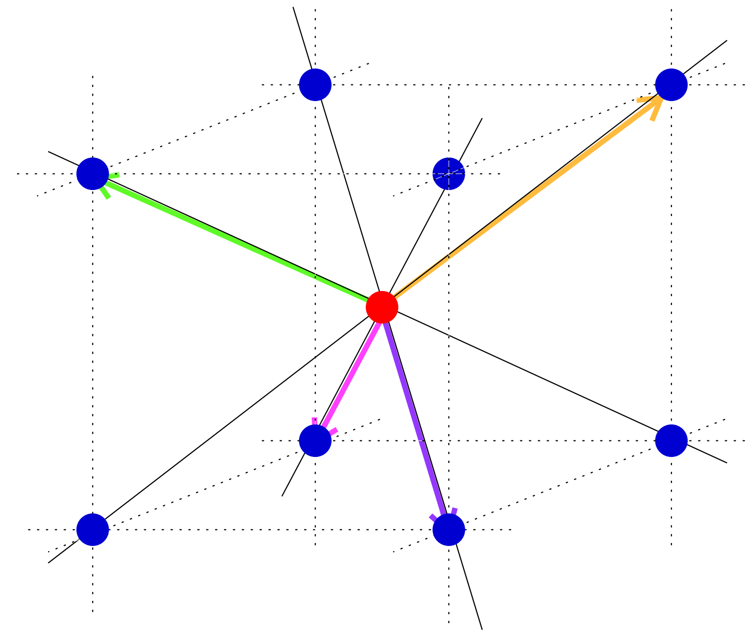
- Locality: $W_{\mathbf{k}} := \sum_{\mathbf{h}\in S} e^{-i\mathbf{h}\cdot\mathbf{k}} A_{\mathbf{h}}$
- Unitarity:
  - in the reciprocal representation $W_{\mathbf{k}}^{\dagger} W_{\mathbf{k}} = W_{\mathbf{k}} W_{\mathbf{k}}^{\dagger} = I_s$
  - in the direct representation

$$\begin{cases} \sum_{\mathbf{h}\in S} A_{\mathbf{h}}^{\dagger} A_{\mathbf{h}} = \sum_{\mathbf{h}\in S} A_{\mathbf{h}} A_{\mathbf{h}}^{\dagger} = I_s \\ \sum_{\mathbf{h}-\mathbf{h}'=\mathbf{h}''} A_{\mathbf{h}}^{\dagger} A_{\mathbf{h}'} = \sum_{\mathbf{h}-\mathbf{h}'=\mathbf{h}''} A_{\mathbf{h}} A_{\mathbf{h}'}^{\dagger} = 0 \end{cases}$$

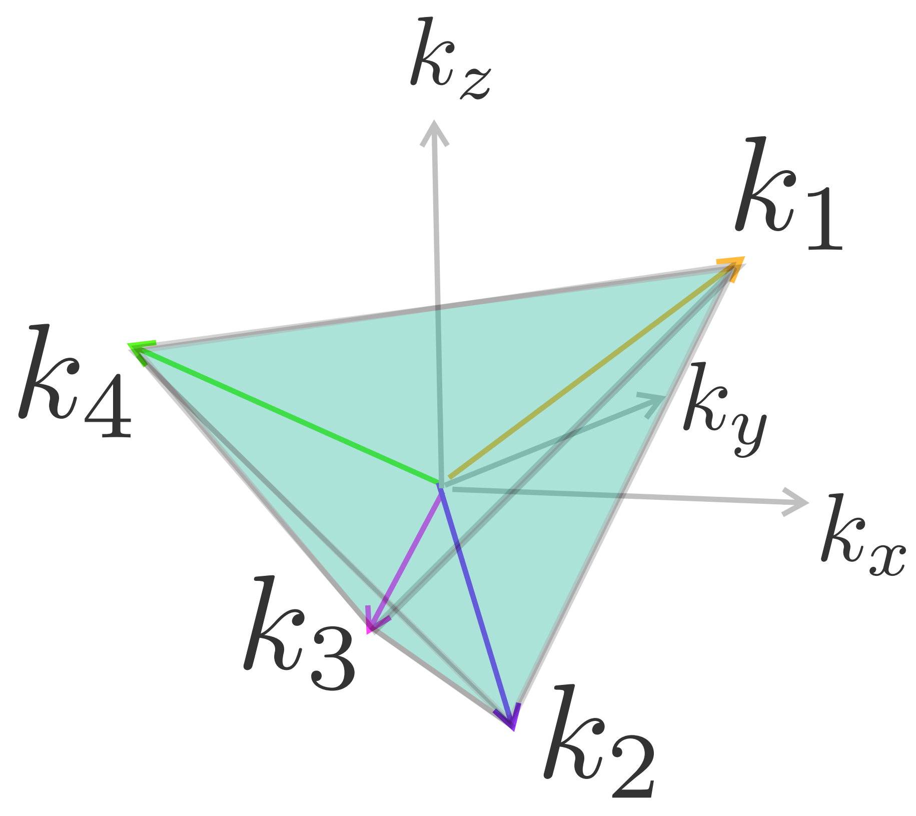
Solving the equations

For d=3

- Minimal complexity $s = 2$
- Unitarity + locality constraints can be solved only for BCC



$$W_{\mathbf{k}}^{\pm} = e^{-i\frac{k_x}{\sqrt{3}}\sigma_x} e^{\mp i\frac{k_y}{\sqrt{3}}\sigma_y} e^{-i\frac{k_z}{\sqrt{3}}\sigma_z}$$

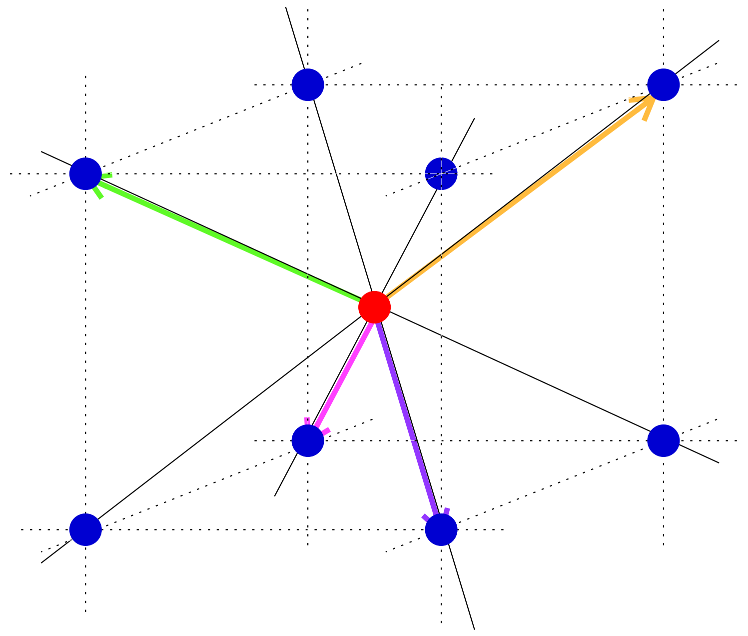


G. M. D'Ariano and PP, Phys. Rev. A **90**, 062106 (2014)

Solving the equations

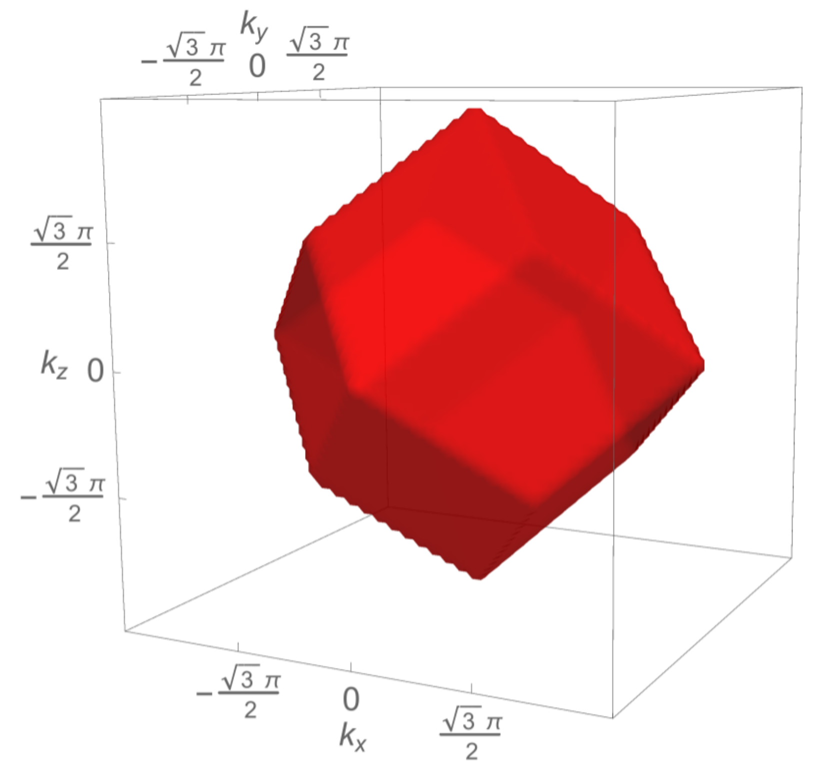
For d=3

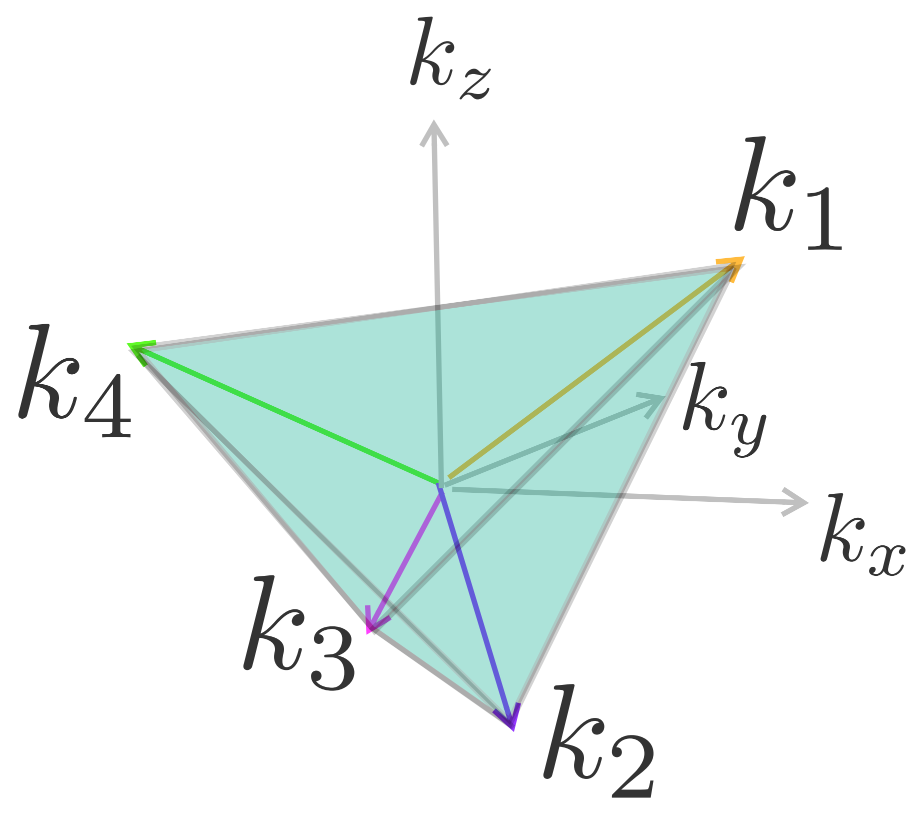
- Minimal complexity $s = 2$
- Unitarity + locality constraints can be solved only for BCC



$$W_{\mathbf{k}}^{\pm} = e^{-i\frac{k_x}{\sqrt{3}}\sigma_x} e^{\mp i\frac{k_y}{\sqrt{3}}\sigma_y} e^{-i\frac{k_z}{\sqrt{3}}\sigma_z}$$

- Brillouin zone





G. M. D’Ariano and PP, Phys. Rev. A **90**, 062106 (2014)

Dispersion relations

Eigenvalue equation for the quantum walk

$$W_{\mathbf{k}}^{\pm}|\psi_{\mathbf{k}}^{\pm}\rangle = e^{-i\omega_{\mathbf{k}}^{\pm}}|\psi_{\mathbf{k}}^{\pm}\rangle$$

Dispersion relation

$$\omega_{\mathbf{k}}^{\pm} = \arccos(c_x c_y c_z \mp s_x s_y s_z)$$

$$s_i := \sin\frac{k_i}{\sqrt{3}}$$

$$c_i := \cos\frac{k_i}{\sqrt{3}}$$

G. M. D'Ariano and PP, Phys. Rev. A **90**, 062106 (2014)

Interpolation

- Interpolating Hamiltonian

$$W^{\pm}_{\mathbf{k}} = e^{-iH^{\pm}_{\mathbf{k}}}$$
$$H^{\pm}_{\mathbf{k}} = \mathbf{n}^{\pm}_{\mathbf{k}} \cdot \boldsymbol{\sigma}$$

$$W^{\pm}_{\mathbf{k}}(t) := e^{-iH^{\pm}_{\mathbf{k}} t}$$
$$H_{\mathbf{k}} = i\partial_t W^{\pm}_{\mathbf{k}}(t)$$
$$W^{\pm}_{\mathbf{k}} = W^{\pm}_{\mathbf{k}}(1)$$

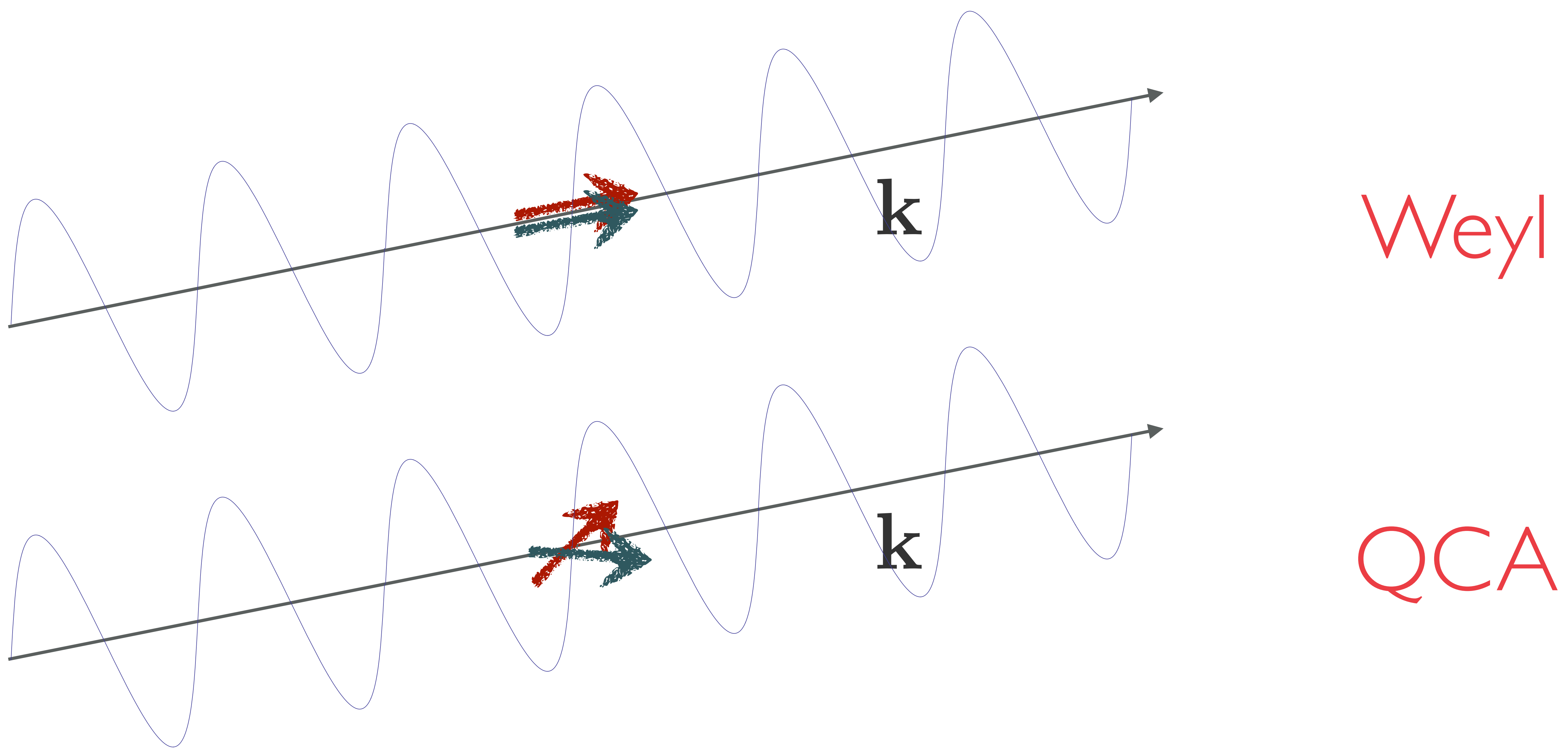
- Differential equation $\quad i\partial_t \psi(\mathbf{k}) = H_{\mathbf{k}} \psi(\mathbf{k})$
- Small wave-vector $\quad i\partial_t \psi(\mathbf{k}) \simeq \mathbf{k} \cdot \boldsymbol{\sigma} \psi(\mathbf{k})$

Weyl's equation

G. M. D'Ariano and PP, Phys. Rev. A **90**, 062106 (2014)

Weyl's equation

Weyl QCA	Helicity	Group velocity
Exact QCA	$\mathbf{n}_{\mathbf{k}} \cdot \boldsymbol{\sigma}$	$v_g(\mathbf{k}) = \nabla_{\mathbf{k}} \omega(\mathbf{k})$
Small $\|\mathbf{k}\|$ limit	$\mathbf{k} \cdot \boldsymbol{\sigma}$	$\frac{\mathbf{k}}{\|\mathbf{k}\|}$



G. M. D'Ariano and PP, Phys. Rev. A **90**, 062106 (2014)

Combining Weyl QCA

$$Z_{\mathbf{k}}^{\pm} = \begin{pmatrix} nW_{\mathbf{k}}^{\pm} & imI \\ imI & nW_{\mathbf{k}}^{\pm\dagger} \end{pmatrix}$$

$$n^2 + m^2 = 1$$

Dirac

$$M_{\mathbf{k}}^{\pm} = A_{\mathbf{k}}^{\pm} \otimes A_{\mathbf{k}}^{\pm *}$$

Maxwell

Dirac's equation

- Interpolating Hamiltonian for $Z^{\pm}_{\mathbf{k}} = \begin{pmatrix} nW^{\pm}_{\mathbf{k}} & imI \\ imI & nW^{\pm\dagger}_{\mathbf{k}} \end{pmatrix}$

$$H^{\pm}_{\mathbf{k}} = n\boldsymbol{\alpha} \cdot \mathbf{n}^{\pm}_{\mathbf{k}} + m\beta$$

- In the limit $|\mathbf{k}| \ll \pi$

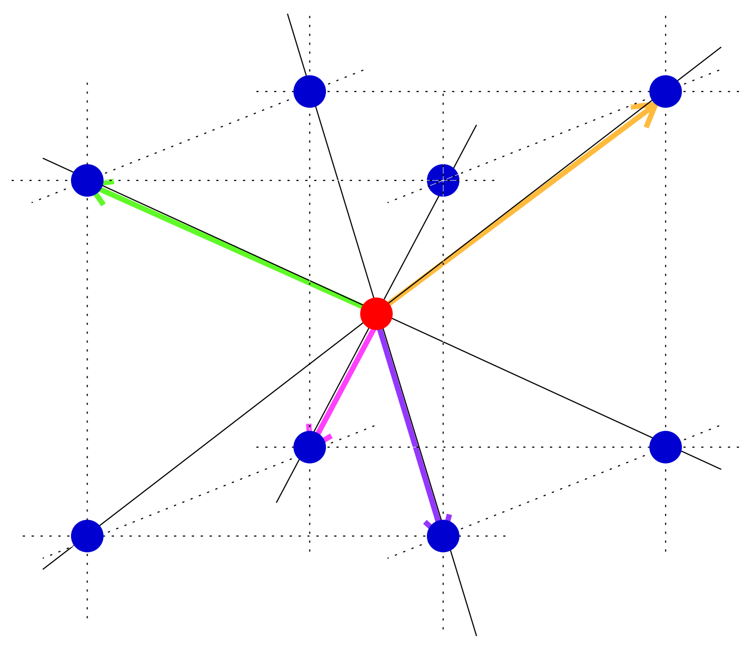
$$i\partial_t \varphi(\mathbf{k}, t) = (n\boldsymbol{\alpha} \cdot \mathbf{k} + m\beta)\varphi(\mathbf{k}, t)$$

G. M. D'Ariano and PP, Phys. Rev. A **90**, 062106 (2014)

Two Weyl fields

- Consider two Fermionic fields $\varphi_i(\boldsymbol{x}), \quad \psi_i(\boldsymbol{x}) \qquad i = R, L$

$$\psi(\boldsymbol{x}) := \begin{pmatrix} \psi_R(\boldsymbol{x}) \\ \psi_L(\boldsymbol{x}) \end{pmatrix} \qquad \varphi(\boldsymbol{x}) := \begin{pmatrix} \varphi_R(\boldsymbol{x}) \\ \varphi_L(\boldsymbol{x}) \end{pmatrix}$$

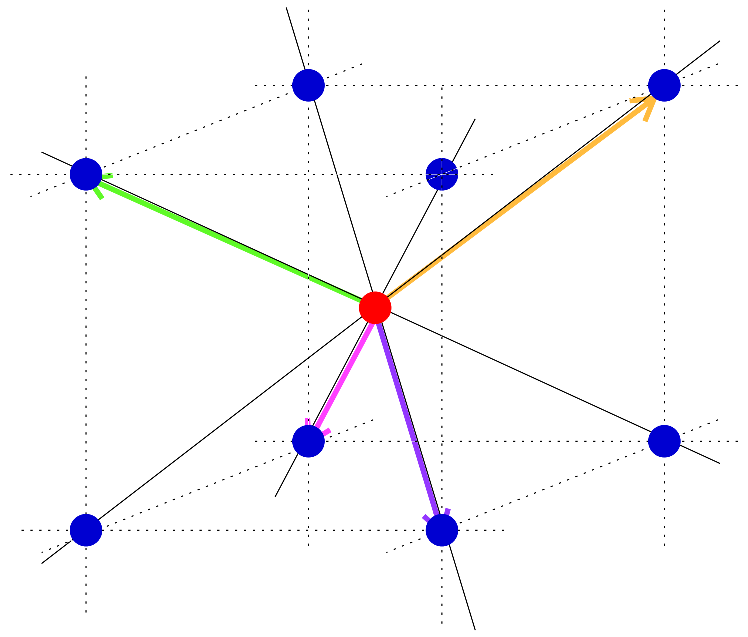


A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Two Weyl fields

- Consider two Fermionic fields $\varphi_i(\boldsymbol{x}), \quad \psi_i(\boldsymbol{x}) \qquad i = R, L$

$$\psi(\boldsymbol{x}) := \begin{pmatrix} \psi_R(\boldsymbol{x}) \\ \psi_L(\boldsymbol{x}) \end{pmatrix} \qquad \varphi(\boldsymbol{x}) := \begin{pmatrix} \varphi_R(\boldsymbol{x}) \\ \varphi_L(\boldsymbol{x}) \end{pmatrix}$$



- Evolution in the Fourier representation

$$\varphi(\boldsymbol{k}, t+1) = A^*_{\boldsymbol{k}} \varphi(\boldsymbol{k}, 0)$$
$$\psi(\boldsymbol{k}, t+1) = A_{\boldsymbol{k}} \psi(\boldsymbol{k}, t)$$

$$A_{\boldsymbol{k}} = A^{\pm}_{\boldsymbol{k}}$$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Tensor Product of Weyl QCA

- Consider the functions $H^{\mu}(\boldsymbol{k}) := \varphi\left(\frac{\boldsymbol{k}}{2}\right)\sigma^{\mu}\psi\left(\frac{\boldsymbol{k}}{2}\right)$ $\sigma^0 := I$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Tensor Product of Weyl QCA

- Consider the functions $H^{\mu}(\boldsymbol{k}) := \varphi\left(\frac{\boldsymbol{k}}{2}\right)\sigma^{\mu}\psi\left(\frac{\boldsymbol{k}}{2}\right)$ $\sigma^0 := I$
- Their evolution is $H^0(\boldsymbol{k}, t+1) = H^0(\boldsymbol{k}, t)$ $\boldsymbol{H}(\boldsymbol{k}, t+1) = M_{\boldsymbol{k}}\boldsymbol{H}(\boldsymbol{k}, t)$

$$M_{\boldsymbol{k}} := \exp\left(-2i\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}} \cdot \boldsymbol{J}\right)$$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Tensor Product of Weyl QCA

- Consider the functions $H^\mu(\boldsymbol{k}) := \varphi\left(\frac{\boldsymbol{k}}{2}\right)\sigma^\mu\psi\left(\frac{\boldsymbol{k}}{2}\right)$ $\sigma^0 := I$
- Their evolution is $H^0(\boldsymbol{k}, t+1) = H^0(\boldsymbol{k}, t)$ $\boldsymbol{H}(\boldsymbol{k}, t+1) = M_{\boldsymbol{k}}\boldsymbol{H}(\boldsymbol{k}, t)$

$$M_{\boldsymbol{k}} := \exp\left(-2i\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}} \cdot \boldsymbol{J}\right)$$

- For small |**k**|, interpolating the discrete space-time lattice

$$\partial_t \operatorname{Re}\boldsymbol{H}(\boldsymbol{x}, t) = \nabla \times \operatorname{Im}\boldsymbol{H}(\boldsymbol{x}, t),$$
$$\partial_t \operatorname{Im}\boldsymbol{H}(\boldsymbol{x}, t) = -\nabla \times \operatorname{Re}\boldsymbol{H}(\boldsymbol{x}, t),$$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Tensor Product of Weyl QCA

- Consider the functions $H^\mu(\boldsymbol{k}) := \varphi\left(\frac{\boldsymbol{k}}{2}\right)\sigma^\mu\psi\left(\frac{\boldsymbol{k}}{2}\right)$ $\sigma^0 := I$
- Their evolution is $H^0(\boldsymbol{k}, t+1) = H^0(\boldsymbol{k}, t)$ $\boldsymbol{H}(\boldsymbol{k}, t+1) = M_{\boldsymbol{k}}\boldsymbol{H}(\boldsymbol{k}, t)$

$$M_{\boldsymbol{k}} := \exp\left(-2i\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}} \cdot \boldsymbol{J}\right)$$

- For small |**k**|, interpolating the discrete space-time lattice

$$\partial_t \operatorname{Re} \boldsymbol{H}(\boldsymbol{x}, t) = \nabla \times \operatorname{Im} \boldsymbol{H}(\boldsymbol{x}, t),$$
$$\partial_t \operatorname{Im} \boldsymbol{H}(\boldsymbol{x}, t) = -\nabla \times \operatorname{Re} \boldsymbol{H}(\boldsymbol{x}, t),$$

- The functions do not satisfy Bosonic commutation relations

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Electromagnetic field

- Slightly modify the bilinear functions to get

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Electromagnetic field

- Slightly modify the bilinear functions to get
  - Maxwell's equations with small corrections

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Electromagnetic field

- Slightly modify the bilinear functions to get
  - Maxwell's equations with small corrections
  - Bosonic commutation relations with small corrections

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Electromagnetic field

- Slightly modify the bilinear functions to get
  - Maxwell's equations with small corrections
  - Bosonic commutation relations with small corrections
- Is this possible?

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Electromagnetic field

- Slightly modify the bilinear functions to get
  - Maxwell's equations with small corrections
  - Bosonic commutation relations with small corrections
- Is this possible?
  - Composite particles (e.g. nuclei)

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Electromagnetic field

- Slightly modify the bilinear functions to get
  - Maxwell's equations with small corrections
  - Bosonic commutation relations with small corrections
- Is this possible?
  - Composite particles (e.g. nuclei)
  - Neutrino theory of light (De Broglie, Jordan, Pauli, Kronig)

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Evolution

- The new bilinear functions are

$$F^{\mu}(\boldsymbol{k}) := \int_{B} \frac{\mathrm{d}\,\boldsymbol{q}}{(2\pi)^3} f_{\boldsymbol{k}}(\boldsymbol{q})\, \varphi^{T}\left(\tfrac{\boldsymbol{k}}{2} - \boldsymbol{q}\right) \sigma^{\mu} \psi\left(\tfrac{\boldsymbol{k}}{2} + \boldsymbol{q}\right)$$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Evolution

- The new bilinear functions are

$$F^{\mu}(\boldsymbol{k}) := \int_B \frac{\mathrm{d}\,\boldsymbol{q}}{(2\pi)^3} f_{\boldsymbol{k}}(\boldsymbol{q})\, \varphi^T\left(\tfrac{\boldsymbol{k}}{2} - \boldsymbol{q}\right) \sigma^{\mu} \psi\left(\tfrac{\boldsymbol{k}}{2} + \boldsymbol{q}\right)$$

- Their evolution is given by

$$\boldsymbol{F}_T(\boldsymbol{k}, t) = \exp\left(-2i\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}} \cdot \boldsymbol{J}t\right) \boldsymbol{F}_T(\boldsymbol{k}, 0) + O\Big(\frac{\bar{q}(\boldsymbol{k})}{|\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}|}\Big).$$

$$\boldsymbol{F}_T(\boldsymbol{k}, t) := \boldsymbol{F}(\boldsymbol{k}, t) - \left(\frac{\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}}{|\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}|} \cdot \boldsymbol{F}(\boldsymbol{k}, t)\right) \frac{\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}}{|\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}|}$$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Maxwell's equations

- Electric and magnetic field operators

$$\boldsymbol{E} := |\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}|(\boldsymbol{F}_T + \boldsymbol{F}_T^\dagger), \quad \boldsymbol{B} := i|\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}|(\boldsymbol{F}_T^\dagger - \boldsymbol{F}_T),$$

$$2|\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}|\boldsymbol{F}_T = \boldsymbol{E} + i\boldsymbol{B}.$$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Maxwell's equations

- Electric and magnetic field operators

$$\boldsymbol{E} := |\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}|(\boldsymbol{F}_T + \boldsymbol{F}_T^\dagger), \quad \boldsymbol{B} := i|\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}|(\boldsymbol{F}_T^\dagger - \boldsymbol{F}_T),$$

$$2|\boldsymbol{n}_{\frac{\boldsymbol{k}}{2}}|\boldsymbol{F}_T = \boldsymbol{E} + i\boldsymbol{B}.$$

- Free e.m. field dynamics in the relativistic limit $|\mathbf{k}| \ll 1$

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\partial_t \boldsymbol{E} = c\nabla \times \boldsymbol{B} \qquad \partial_t \boldsymbol{B} = -c\nabla \times \boldsymbol{E} \quad .$$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Bosonic commutation relations

- We now consider elementary bilinear functions

$$\gamma_{\alpha,\beta}(\boldsymbol{k}) := \frac{1}{N_{\boldsymbol{k}}} \sum_{\boldsymbol{q}\in\Omega_{\boldsymbol{k}}} \varphi_\alpha\left(\tfrac{\boldsymbol{k}}{2} - \boldsymbol{q}\right) \psi_\beta\left(\tfrac{\boldsymbol{k}}{2} + \boldsymbol{q}\right)$$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Bosonic commutation relations

- We now consider elementary bilinear functions

$$\gamma_{\alpha,\beta}(\boldsymbol{k}) := \frac{1}{N_{\boldsymbol{k}}} \sum_{\boldsymbol{q}\in\Omega_{\boldsymbol{k}}} \varphi_\alpha\left(\tfrac{\boldsymbol{k}}{2} - \boldsymbol{q}\right) \psi_\beta\left(\tfrac{\boldsymbol{k}}{2} + \boldsymbol{q}\right)$$

- Their commutation relations are

$$[\gamma_{\alpha,\beta}(\boldsymbol{k}), \gamma_{\alpha',\beta'}(\boldsymbol{k}')]_- = 0$$

$$[\gamma_{\alpha,\beta}(\boldsymbol{k}), \gamma^\dagger_{\alpha',\beta'}(\boldsymbol{k}')]_- = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\delta_{\boldsymbol{k},\boldsymbol{k}'} - \Delta_{\alpha,\alpha',\beta,\beta',\boldsymbol{k},\boldsymbol{k}'}$$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Bosonic commutation relations

- We now consider elementary bilinear functions

$$\gamma_{\alpha,\beta}(\boldsymbol{k}) := \frac{1}{N_{\boldsymbol{k}}} \sum_{\boldsymbol{q}\in\Omega_{\boldsymbol{k}}} \varphi_\alpha\left(\tfrac{\boldsymbol{k}}{2} - \boldsymbol{q}\right) \psi_\beta\left(\tfrac{\boldsymbol{k}}{2} + \boldsymbol{q}\right)$$

- Their commutation relations are

$$[\gamma_{\alpha,\beta}(\boldsymbol{k}), \gamma_{\alpha',\beta'}(\boldsymbol{k}')]_- = 0$$

$$[\gamma_{\alpha,\beta}(\boldsymbol{k}), \gamma^\dagger_{\alpha',\beta'}(\boldsymbol{k}')]_- = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\delta_{\boldsymbol{k},\boldsymbol{k}'} - \Delta_{\alpha,\alpha',\beta,\beta',\boldsymbol{k},\boldsymbol{k}'}$$

- The correction is small when we satisfy

$$\frac{1}{N_{\boldsymbol{k}}} \sum_{\boldsymbol{q}\in\Omega_{\boldsymbol{k}}} \left\langle \psi^\dagger_\alpha\left(\tfrac{\boldsymbol{k}}{2} + \boldsymbol{q}\right) \psi_\alpha\left(\tfrac{\boldsymbol{k}}{2} + \boldsymbol{q}\right)\right\rangle \ll 1 \qquad \frac{1}{N_{\boldsymbol{k}}} \sum_{\boldsymbol{q}\in\Omega_{\boldsymbol{k}}} \left\langle \varphi^\dagger_\alpha\left(\tfrac{\boldsymbol{k}}{2} + \boldsymbol{q}\right) \varphi_\alpha\left(\tfrac{\boldsymbol{k}}{2} + \boldsymbol{q}\right)\right\rangle \ll 1$$

A. Bisio, G. M. D'Ariano, and PP, Ann. Phys. **368**, pagg.177-190 (2016)

Interactions

Non-linear evolution

A. Bisio, N. Mosco, and PP, Phys. Rev. Lett. **126**, 250503 (2021)

Interactions

Non-linear evolution

- Relevant interactions introduce non-linearity

  1. The analysis is much harder

     - We need approximation techniques

  2. Defining the continuum limit is harder

     - E.g. the small wave-vector regime might not be invariant

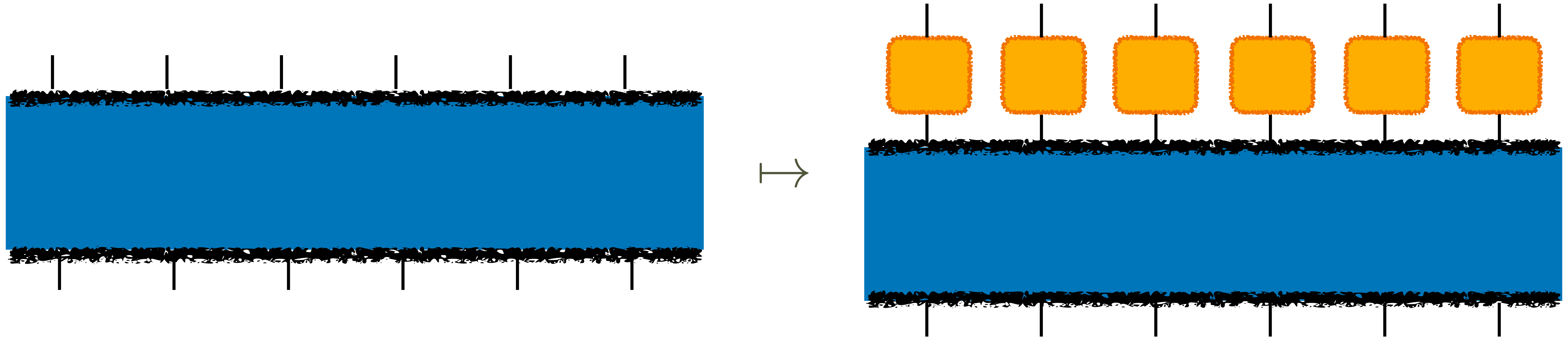
- Both problems can be faced in a unified treatment

  - Discrete time perturbation theory

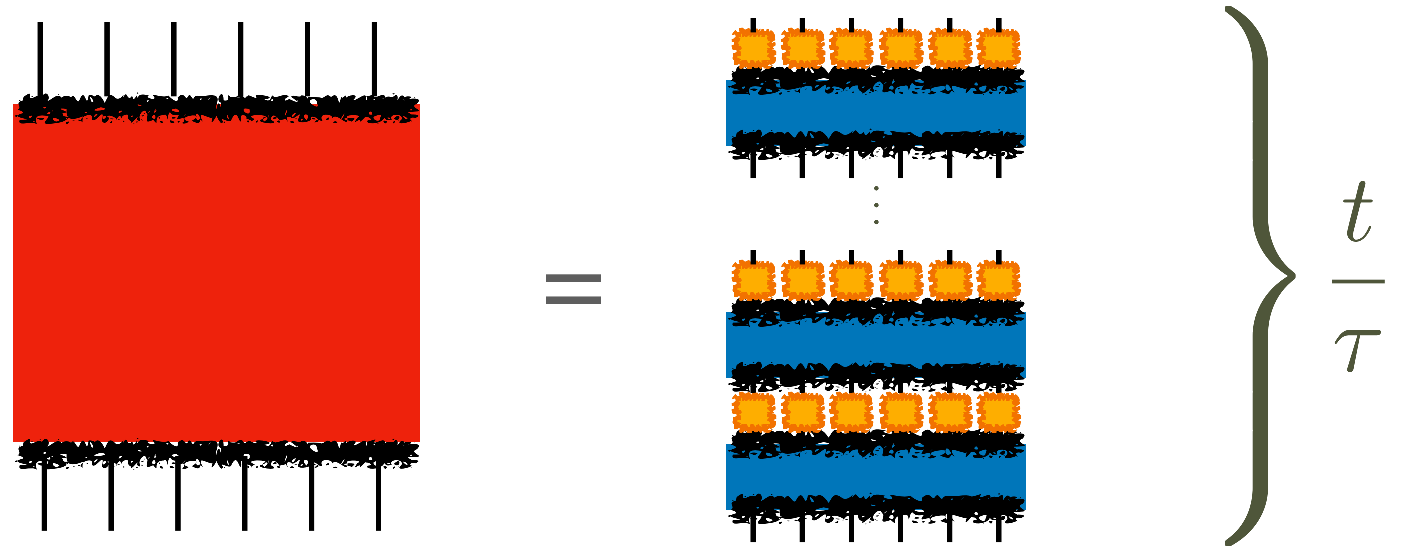
A. Bisio, N. Mosco, and PP, Phys. Rev. Lett. **126**, 250503 (2021)

Local nonlinear evolution

- From a fundamental point of view
  - Local gauge theory
  - $U \mapsto \prod_x V_x \, U, \qquad V_x = v(\varphi_x, \varphi_x^\dagger)$

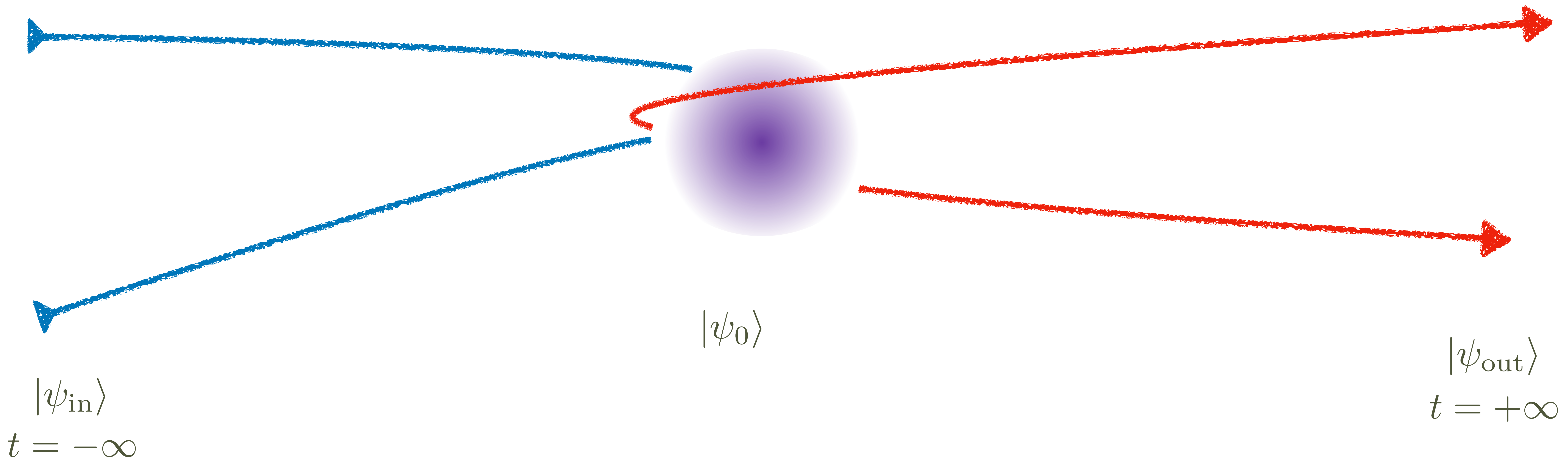


- Quantum simulations: Lie-Trotter
  - Interaction hamiltonian $H = H_0 + H_I$
  - $e^{-i(H_0+H_I)t} = \lim_{\tau \to 0} \left\{ e^{-iH_0\tau} e^{-iH_I\tau} \right\}^{t/\tau}$



A. Bisio, N. Mosco, and PP, Phys. Rev. Lett. **126**, 250503 (2021)

Scattering



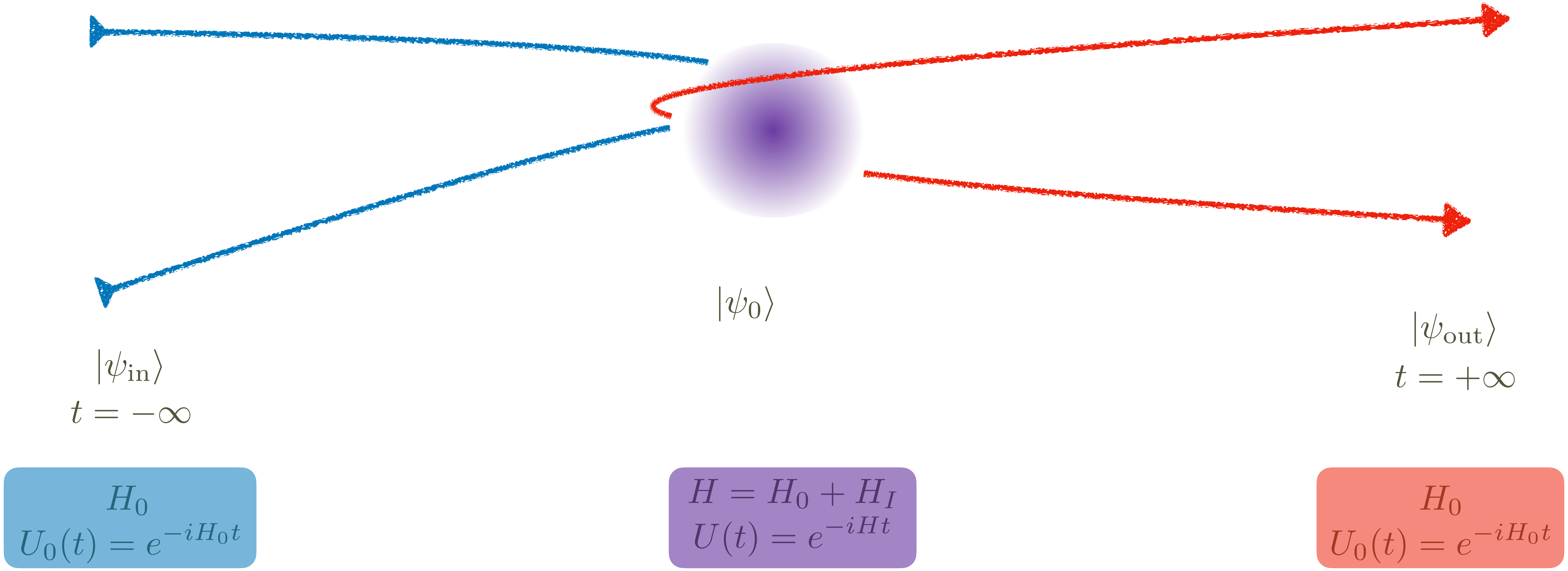
H_0
$U_0(t) = e^{-iH_0 t}$

$H = H_0 + H_I$
$U(t) = e^{-iHt}$

H_0
$U_0(t) = e^{-iH_0 t}$

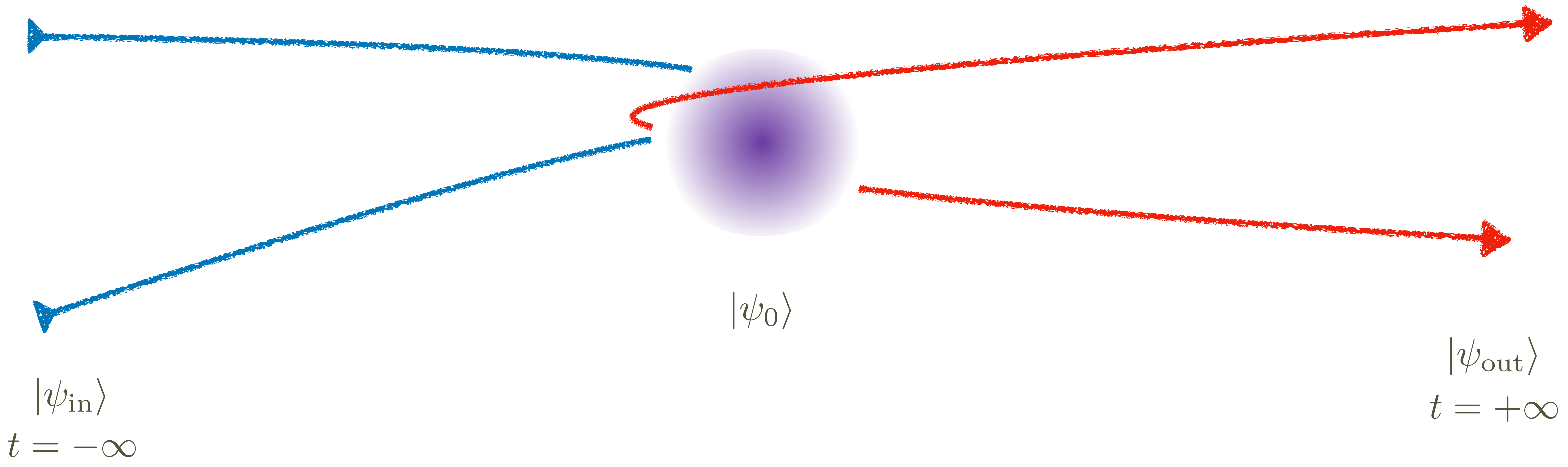
Scattering

- Ignore bound states—focus on scattering



Scattering

Working hypothesis



$$\lim_{t \to -\infty} \| U_0^t |\psi_{\text{in}}\rangle - U^t |\psi_0\rangle \| = 0$$

$$\lim_{t \to +\infty} \| U_0^t |\psi_{\text{out}}\rangle - U^t |\psi_0\rangle \| = 0$$

Scattering theory

Møller operators

Continuous time $\Omega_\pm := \lim_{t\to\pm\infty} U^{\dagger t} U_0^t P_{ac}(U_0)$

Discrete time $\Omega_\pm := \lim_{n\to\pm\infty} U^{\dagger n} U_0^n P_{ac}(U_0)$

Removes bound states

$\Rightarrow$ $|\psi_{out}\rangle = \Omega_+^\dagger \Omega_- |\psi_{in}\rangle$

Scattering matrix $S := \Omega_+^\dagger \Omega_-$ $\quad |\psi_{out}\rangle = S|\psi_{in}\rangle$

A. Bisio, N. Mosco, and PP, Phys. Rev. Lett. **126**, 250503 (2021)

Scattering theory

Møller operators

- Continuous time: $\Omega_\pm := \lim_{t \to \pm\infty} U^{\dagger t} U_0^t P_{ac}(U_0)$
- Discrete time: $\Omega_\pm := \lim_{n \to \pm\infty} U^{\dagger n} U_0^n P_{ac}(U_0)$

Removes bound states

$$|\psi_{out}\rangle = \Omega_+^\dagger \Omega_- |\psi_{in}\rangle$$

Scattering matrix $S := \Omega_+^\dagger \Omega_-$ $\quad |\psi_{out}\rangle = S|\psi_{in}\rangle$

- Check conditions for the existence of Møller operators

A. Bisio, N. Mosco, and PP, Phys. Rev. Lett. **126**, 250503 (2021)

Scattering theory

Møller operators

Continuous time $\Omega_\pm := \lim_{t\to\pm\infty} U^{\dagger t} U_0^t P_{ac}(U_0)$

Discrete time $\Omega_\pm := \lim_{n\to\pm\infty} U^{\dagger n} U_0^n P_{ac}(U_0)$

Removes bound states

$|\psi_{out}\rangle = \Omega_+^\dagger \Omega_- |\psi_{in}\rangle$

Scattering matrix $S := \Omega_+^\dagger \Omega_-$ $|\psi_{out}\rangle = S|\psi_{in}\rangle$

- Check conditions for the existence of Møller operators
- Under suitable conditions *S* is unitary

A. Bisio, N. Mosco, and PP, Phys. Rev. Lett. **126**, 250503 (2021)

Scattering theory

Møller operators

Continuous time $\Omega_{\pm} := \lim_{t\to\pm\infty} U^{\dagger t} U_0^t P_{ac}(U_0)$

Discrete time $\Omega_{\pm} := \lim_{n\to\pm\infty} U^{\dagger n} U_0^n P_{ac}(U_0)$

Removes bound states

$|\psi_{out}\rangle = \Omega_+^\dagger \Omega_- |\psi_{in}\rangle$

Scattering matrix $S := \Omega_+^\dagger \Omega_-$ $\quad |\psi_{out}\rangle = S|\psi_{in}\rangle$

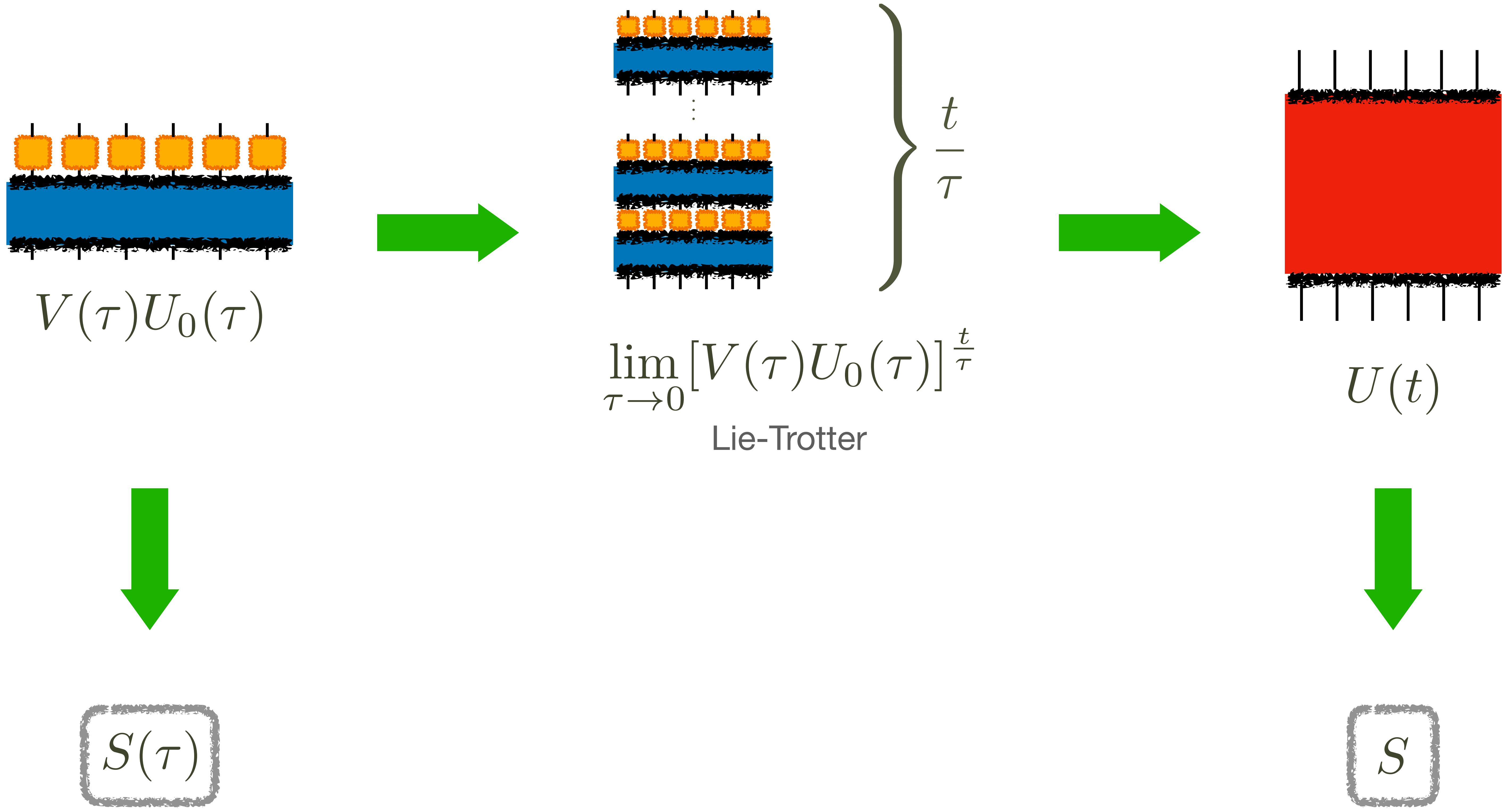
- Check conditions for the existence of Møller operators
- Under suitable conditions *S* is unitary
- Under the same conditions $[S, U_0] = 0$

A. Bisio, N. Mosco, and PP, Phys. Rev. Lett. **126**, 250503 (2021)

Convergence

Comparison of discrete and continuous time theories

- $U_0(\tau) = e^{-iH_0\tau}$
- $V(\tau) = e^{-iH_I\tau}$
- $U(\tau) = e^{-i(H_0+H_I)\tau}$

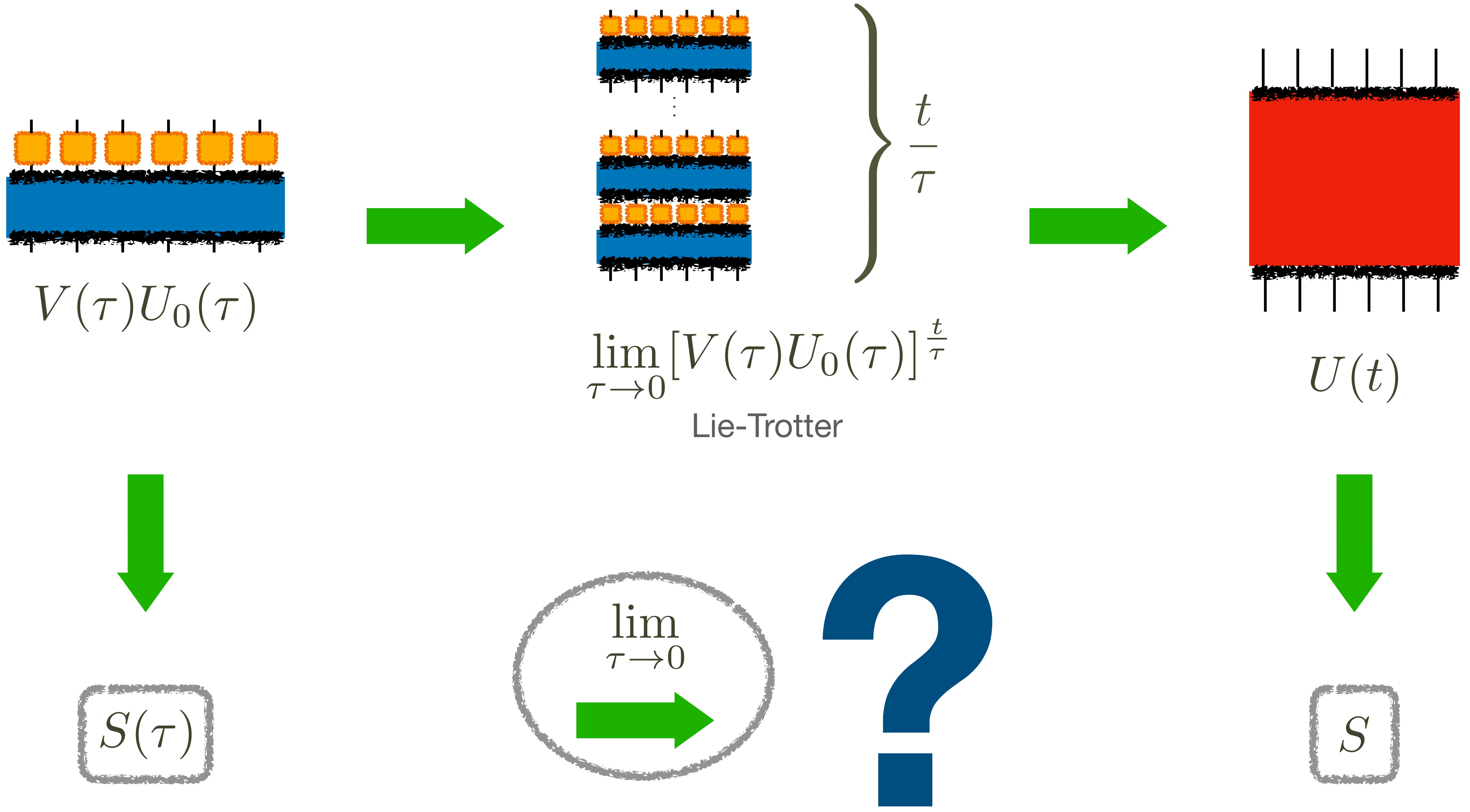


A. Bisio, N. Mosco, and PP, Phys. Rev. Lett. **126**, 250503 (2021)

Convergence

Comparison of discrete and continuous time theories

- $U_0(\tau) = e^{-iH_0\tau}$
- $V(\tau) = e^{-iH_I\tau}$
- $U(\tau) = e^{-i(H_0+H_I)\tau}$



A. Bisio, N. Mosco, and PP, Phys. Rev. Lett. **126**, 250503 (2021)

Comparison

Perturbation theory for QCA

- Target: compare the scattering matrices
  - Tool: perturbation theory for QCA (beyond Hamiltonian framework)
  - Perturbative expansion for $S(\tau)$ allows for the comparison
  - Sufficient conditions for convergence of $S(\tau)$ to S
    - Bounded energy spectrum for $i \ln U_0$
    - $|G_0^{(c)}(\omega_k + i\varepsilon)H_I| < 1$ [where $G_0(z) = (zI - H_0)^{-1}$]

A. Bisio, N. Mosco, and PP, Phys. Rev. Lett. **126**, 250503 (2021)

Overview

- QCA
- FCA
- Quantum Walks and free QFTs
  - Weyl/Dirac
  - Maxwell
- Interactions
  - Perturbation theory for discrete time