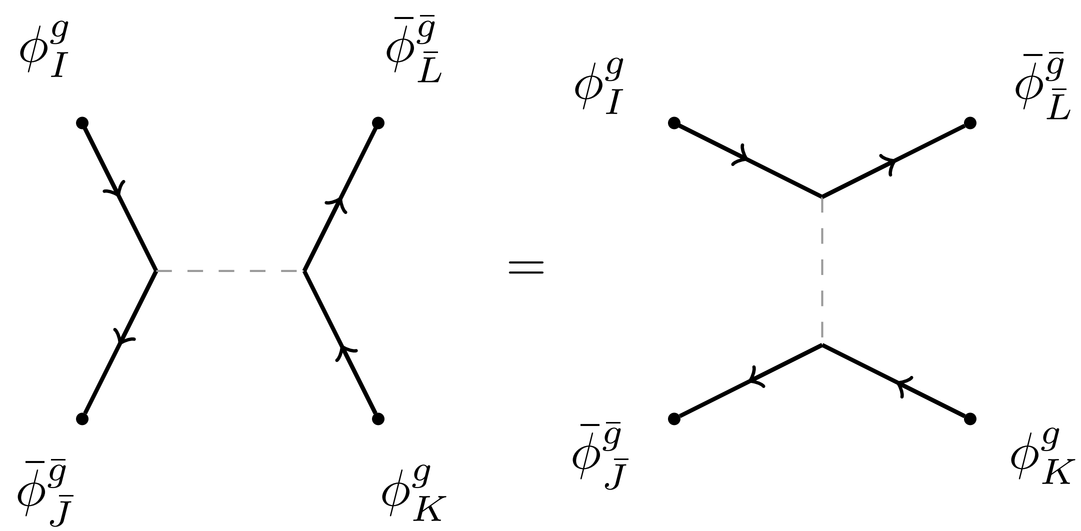
Bootstrapping Lieb-Schultz-Mattis anomalies

arXiv:2207.05092 with Lukasz Fidkowski



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Plan

I. Lieb-Schultz-Mattis on the lattice

II. Lieb-Schultz-Mattis in CFT

III. Symmetries and anomalies in CFT

IV. Numerical bootstrap technique

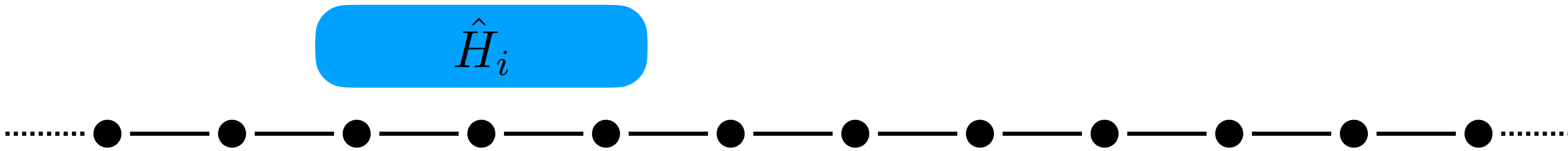
V. Bounds

VI. Outlook & Summary

Basic setup

- Bosonic tensor product Hilbert space $\mathcal{H} = \bigotimes_i \mathcal{H}_i$
- Hamiltonian sum of local terms

$$\hat{H} = \sum_i \hat{H}_i$$



- May have global symmetries

$$[U(g), \hat{H}] = 0 \qquad \forall g \in G$$

$$G = G_{\text{int}} \times G_{\text{lat}}$$

In the thermodynamic limit, $\hat{H}$

- *Gapped* (trivial)
- *Spontaneously breaks (discrete) symmetry*
- *Gapless*

Generalized Lieb-Schultz Mattis Theorems

- Assume $\hat{H}$ is translation-invariant

$$\hat{H}_i = \hat{H}_{i+1}$$

- Internal global symmetry $G_{\text{int}} = \mathbb{Z}_N^2$

Generators: $X = \bigotimes_i X_i \qquad Z = \bigotimes_i Z_i$

- Internal symmetry realized *projectively* at each site

$$U_i(g)U_i(h) = \alpha(g,h)U_i(gh) \qquad [\alpha] \in H^2(G, U(1))$$

$$X_i Z_i = e^{2\pi i/N} Z_i X_i \qquad [\alpha]^N = [0]$$
$$X_i^N = Z_i^N = I$$

[Lieb, Schultz, Mattis, 1961]
[Affleck, 1989]
[Oshikawa, 2000]
[Hastings, 2004]
[Chen,Gu,Wen, 2011]
[Else, Thorngren, 2019]
[Prakash, 2020]
[Ogata, Tachikawa, Tasaki, 2020]
[Gioia, Wang, 2022]

(Generalized) Lieb-Schultz-Mattis (LSM) Theorem:

$\hat{H}$ cannot have a unique, symmetric, gapped ground state.

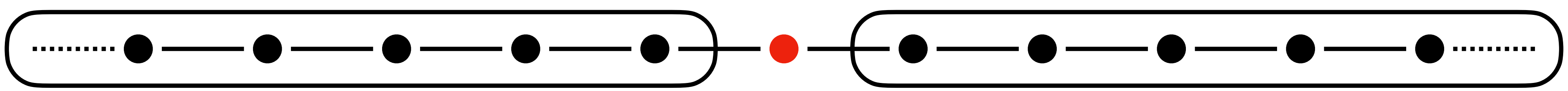
Generalized Lieb-Schultz Mattis Theorems

Intuition behind LSM ingappability/anomaly:

Take system to obey:

- $U(g)U(h) = U(gh)$
- Assume ground state $|\psi_0\rangle$ is unique
- At most $U(g)|\psi_0\rangle = e^{i\alpha}|\psi_0\rangle$

Adding a site amounts to adding translation symmetry defect



- Dangling site transforms in projective representation
- Irreducible projective representations have dimension > 1

- Symmetry spoiled in presence of non-trivial translation background
- Signal of mixed 't Hooft anomaly

Need non-trivial low-energy theory to saturate
i.e. CFT

LSM at $c = 1$ and beyond

Simplest examples of $1{+}1d$ CFTs with discrete LSM anomaly are $c = 1$ compact bosons

Known to describe e.g. anti-ferromagnetic Heisenberg chain ($\mathfrak{su}(2)_1$) and related models

$$\hat{H} = \sum_i J_x X_i X_{i+1} + J_y Y_i Y_{i+1} + J_z Z_i Z_{i+1}$$

- These models have $\mathbb{Z}_2^2 \subset G_{\text{int}}$
- Gapped (SSB) unless microscopic $U(1)$ present (i.e. $|J_x| = |J_y|$ etc.)
- $\mathbb{Z}_2^2 \times \mathbb{Z}_{\text{trans}}$ LSM anomaly matched in lattice and continuum

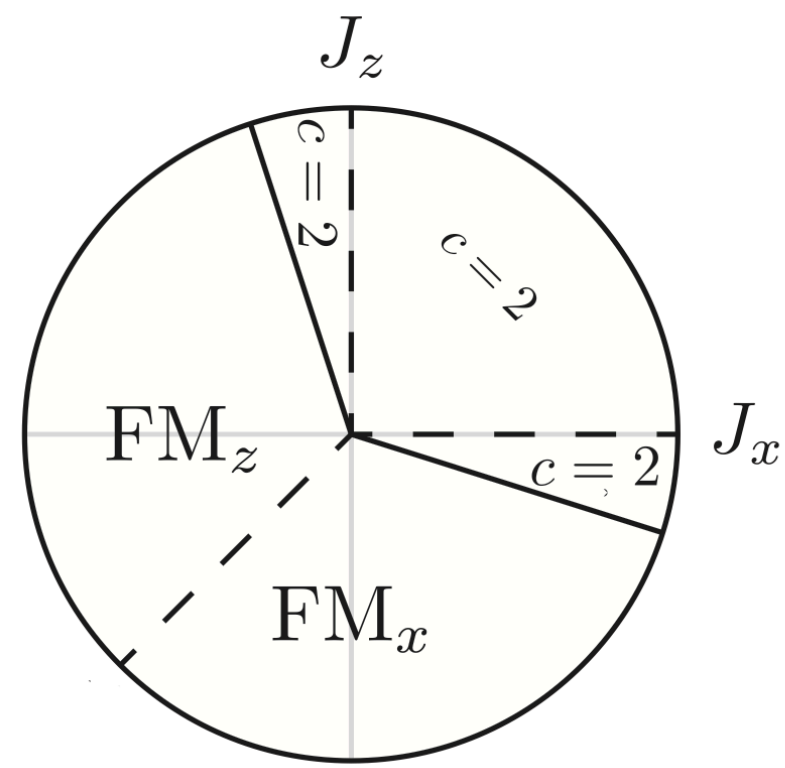
LSM at $c = 1$ and beyond

For $N = 3$ the story is more rich

Many gapless models have $c = 2$ compact boson descriptions

$$\hat{H} = \sum_i \left(J_x X_i X_{i+1}^\dagger + J_z Z_i Z_{i+1}^\dagger \right) + h.c.$$

[Qin, Leinaas, Ryu, Ardonne, Xiang, Lee, 2012]



Taken from [Alavirad, Barkeshli, 2019]

- Above models gapless with only $\mathbb{Z}_3^2 \times \mathbb{Z}_{\text{trans}}$ microscopically
- With more terms and enhanced symmetry, described by $\mathfrak{su}(3)_1$

Story not clear beyond $N = 3$, no numerics for models with just $\mathbb{Z}_N^2$

Anomaly matching for $N = 2$

Action

$$S = \int d^2x\, \partial_\mu \varphi \partial^\mu \varphi$$

$$\varphi \sim \varphi + 2\pi R$$

$$\varphi(z,\bar{z}) = X_L(z) + X_R(z)$$

Local operators

$$V_{n,m}(z,\bar{z}) =: e^{i\left(\frac{n}{R}+\frac{mR}{2}\right)X_L(z)+i\left(\frac{n}{R}-\frac{mR}{2}\right)X_R(\bar{z})} :$$

$$n, m \in \mathbb{Z}$$

$$J(z) = \partial X_L(z)$$

$$\bar{J}(\bar{z}) = \bar{\partial} X_R(\bar{z})$$

Global Symmetries

Winding/momentum $U(1)$ symmetries

$U(1)_w$

$$X_L \to X_L + \frac{\alpha}{R}$$

$$X_R \to X_R - \frac{\alpha}{R}$$

$U(1)_m$

$$X_L \to X_L + \frac{\alpha R}{2}$$

$$X_R \to X_R + \frac{\alpha R}{2}$$

Charge conjugation $\mathbb{Z}_2^C$

$$X_{L/R} \to -X_{L/R}$$

Internal symmetry group at generic R: $G = (U(1)_w \times U(1)_m) \rtimes \mathbb{Z}_2^C$

$\mathbb{Z}_2^w \times \mathbb{Z}_2^m \times \mathbb{Z}_2^C \subset G$ **carries LSM anomaly**

Anomaly matching for $N = 2$

To see anomaly, look at twisted sectors

Twist boundary conditions by $\mathbb{Z}_2^m$ symmetry (assume IR action of lattice translation)

$$\varphi(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \varphi(z, \bar{z}) + 2\pi(m + \frac{1}{2})R$$

The winding mode is now is quantized as $m \in \mathbb{Z} + \dfrac{1}{2}$

$$V_{n,m} \xrightarrow{\mathbb{Z}_2^C} V_{-n,-m} \xrightarrow{\mathbb{Z}_2^w} (-1)^{-m} V_{-n,-m}$$

$$V_{n,m} \xrightarrow{\mathbb{Z}_2^w} (-1)^m V_{n,m} \xrightarrow{\mathbb{Z}_2^C} (-1)^m V_{-n,-m}$$

$$(-1)^m \neq (-1)^{-m}$$

Charge conjugation and winding symmetry do not commute in twisted sector!

$\mathbb{Z}_2^w \times Z_2^C$ realized *projectively*

Questions for bootstrap

What is the space of CFTs with LSM anomalies?

Assume in the IR $\mathbb{Z}_{\text{trans}} \to \mathbb{Z}_N \implies$ looking for CFTs with internal $G = \mathbb{Z}_N^3$

For any N odd and prime, abundance of examples at $c = N$ - 1

For N not prime, we point out other free boson theories with lower c

Hard Question: For any N is there any CFT with the LSM anomaly with $c < N$ - 1?

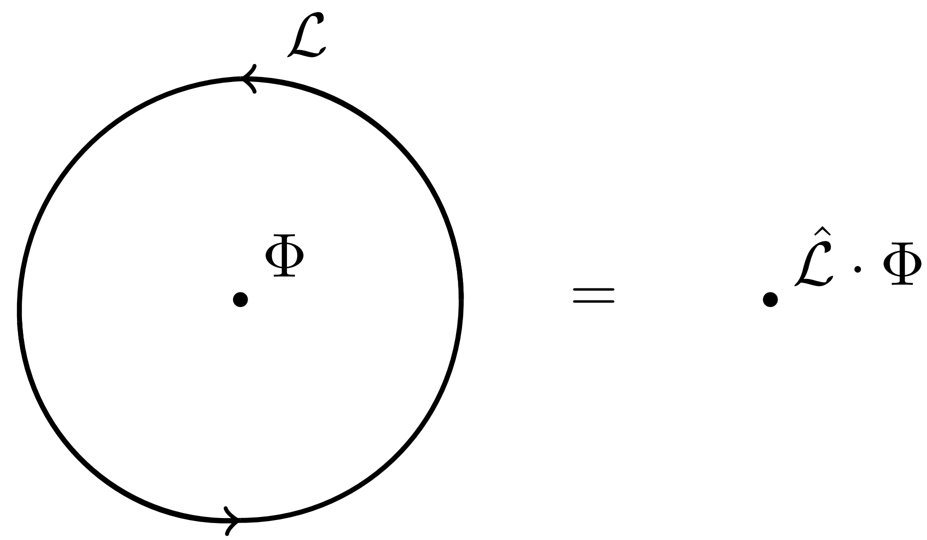
[Alavirad, Barkeshli, 2019]

Easier Question: Is there a universal upper bound on the lightest charged operator in theories with the LSM anomaly as a function of c?

[Lin, Shao, 2019,2021]

Symmetries and topological defect lines

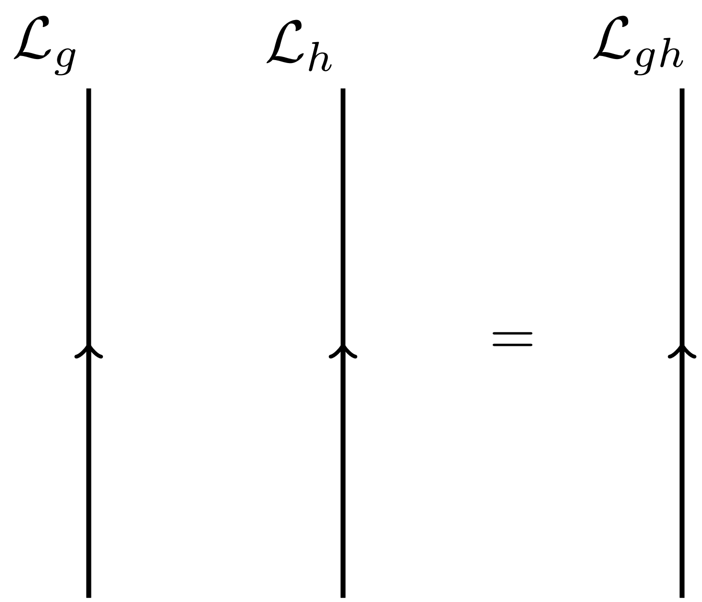
Internal symmetries in (1+1)d CFT implemented via topological line operators called *topological defect lines* (TDLs)

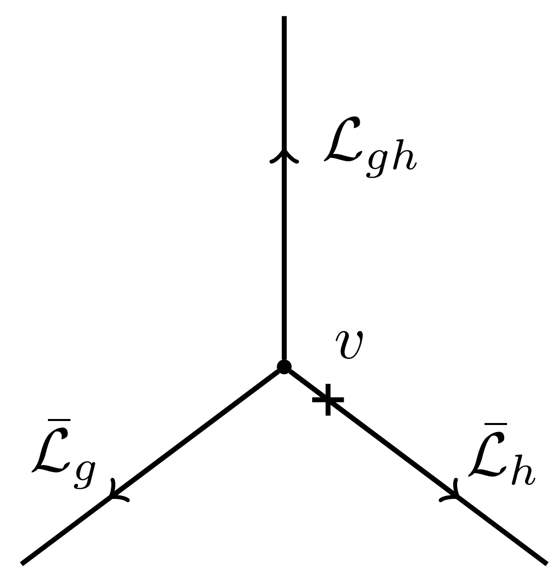


[Frölich, Fuchs, Runkel, Schweigert, 2004]
[Gaiotto, Kapustin, Seiberg, Willett, 2015]
[Chang,Lin,Shao,Wang,Yin,2018]

Assume each TDL corresponds to group element $g \in G$ for finite group G

Lines can fuse and form junctions

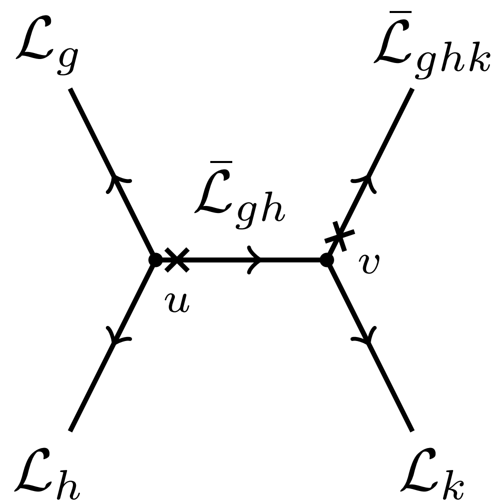




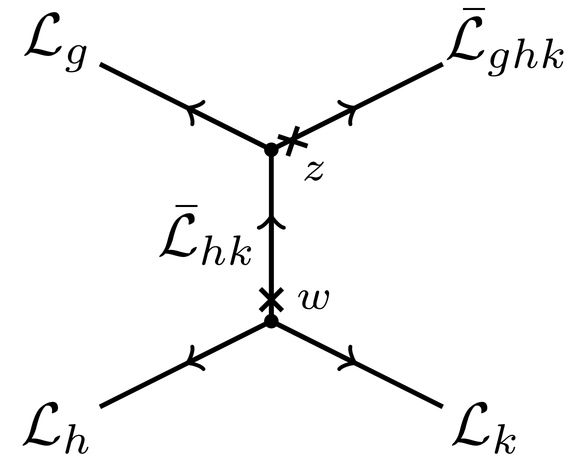
$v \in V_{\mathcal{L}_{gh},\bar{\mathcal{L}}_g,\bar{\mathcal{L}}_h}$

’t Hooft anomalies with TDLs

Different internal configurations of TDLs with four external TDLs related by F-move



$= \omega(g,h,k)$

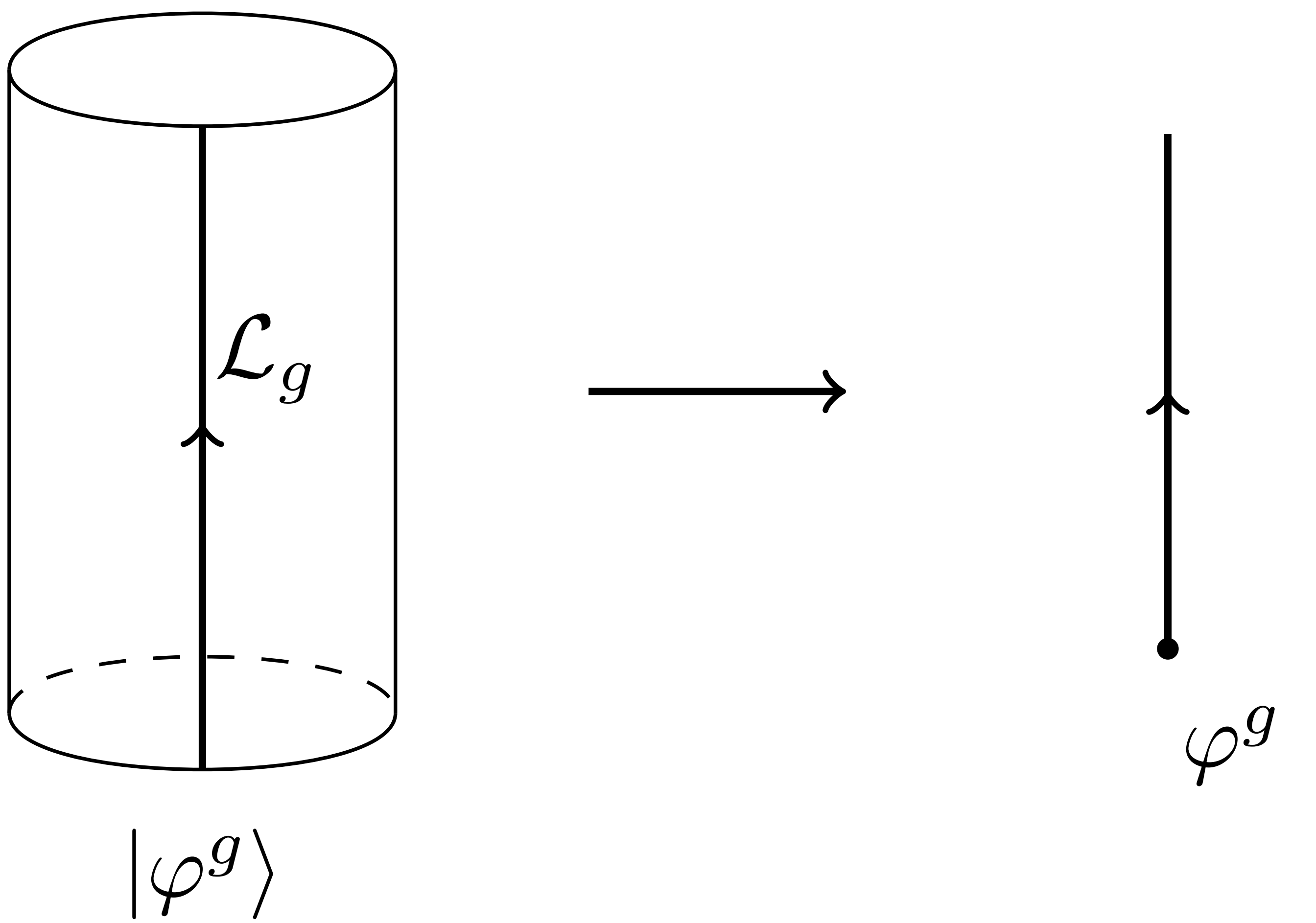


Phase difference is a 3-cocycle $\omega \in H^3(G, U(1))$

G is anomalous if ω cohomologically non-trivial

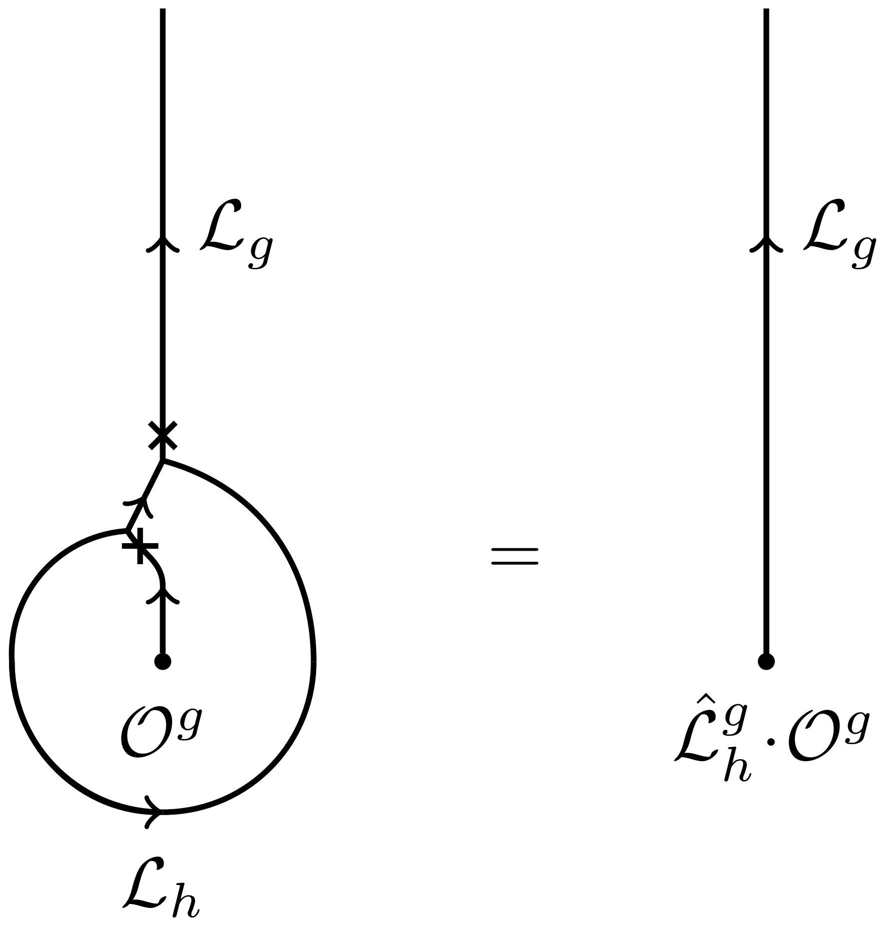
Anomalous discrete symmetries affect symmetry properties of defect operators

Defect operators



- TDLs can terminate on point-like *defect operators*
- Spectrum of defect operators encoded in *defect Hilbert space* $\mathcal{H}_{\mathcal{L}_g}$

Defect operators



- Can act on defect operators with other TDLs via "lasso"
- Anomalies imply failure of the group law in $\mathcal{H}_{\mathcal{L}_g}$
- Defect operators carry fractionalized/projective representations when G is anomalous

$$\hat{\mathcal{L}}_h^g \hat{\mathcal{L}}_k^g = \chi_g(h,k) \hat{\mathcal{L}}_{hk}^g$$

$\mathbb{Z}_M$ anomalies

Simplest possible anomalous symmetry in $(1{+}1)d$ is $\mathbb{Z}_M$

$$H^3(\mathbb{Z}_M, U(1)) = \mathbb{Z}_M$$

$[k] \in H^3(\mathbb{Z}_M, U(1))$ leads defect operators to have restricted spin

For defect operator living on defect generating the $\mathbb{Z}_M$:

$$s \in k/M^2 + \mathbb{Z}/M$$

$$\text{Unitarity} \implies \Delta_{\text{min}} = \min\left(\frac{|k|}{M^2}, \frac{|k-M|}{M^2}\right)$$

Successfully incorporated into modular bootstrap by Lin, Shao [Lin, Shao, 2019,2021]

Note: for general group G and anomaly ω

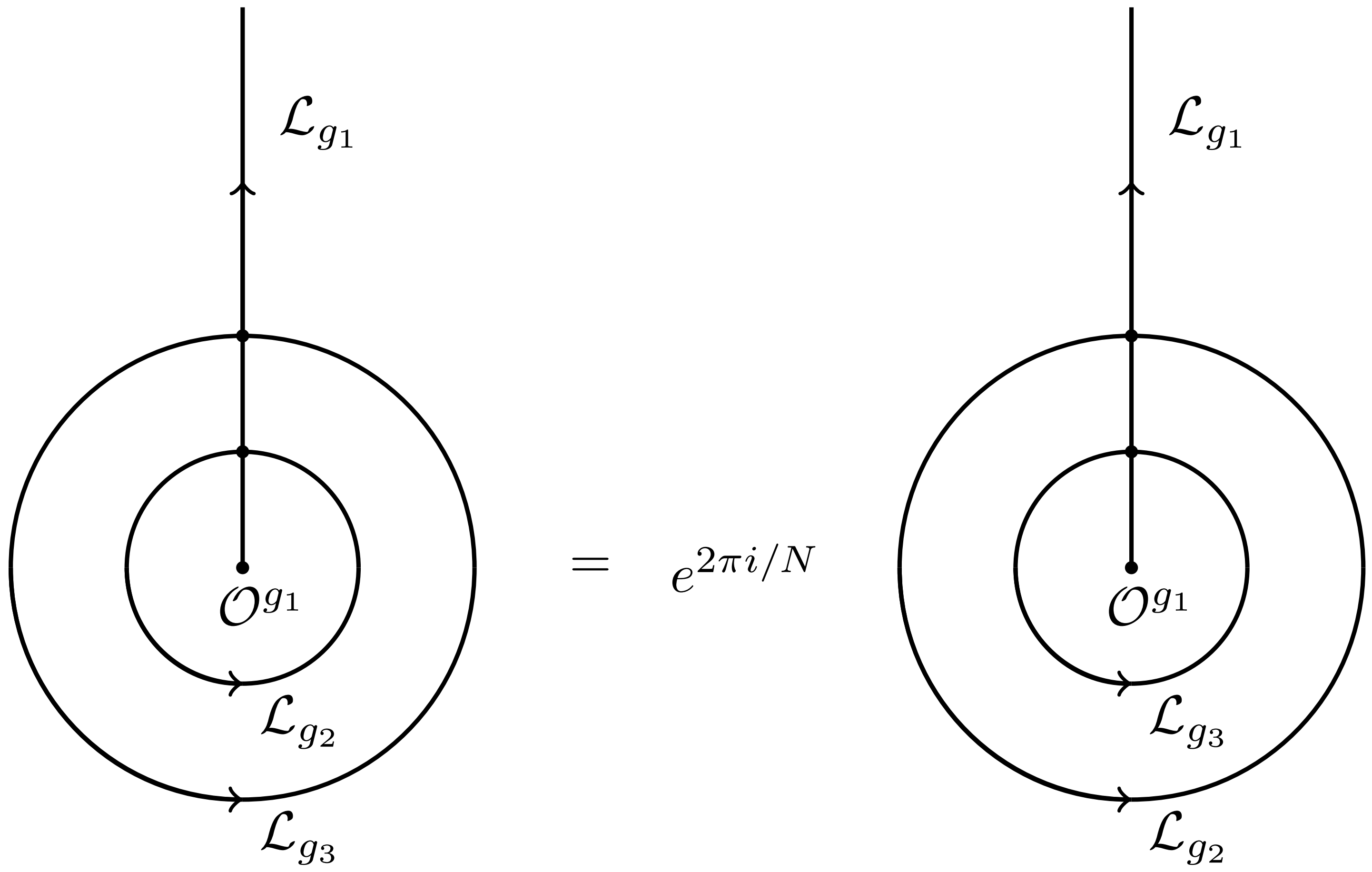
$$\omega\Big|_{\mathbb{Z}_M \times \mathbb{Z}_M \times \mathbb{Z}_M}$$

determines spin selection rule for $\mathbb{Z}_M \subset G$

$\mathbb{Z}_N^3$ LSM anomalies

- Representative cocycle for LSM anomaly:

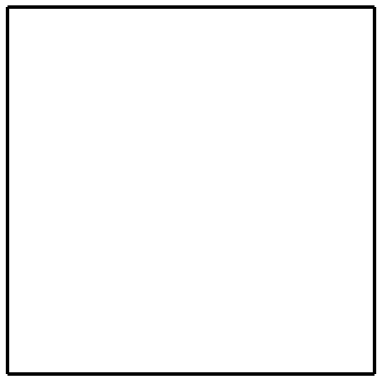
$$\omega(g,h,k) = e^{\frac{2\pi i}{N} g_1 h_2 k_3}$$
$$g = (g_1, g_2, g_3) \in \mathbb{Z}_N^3$$



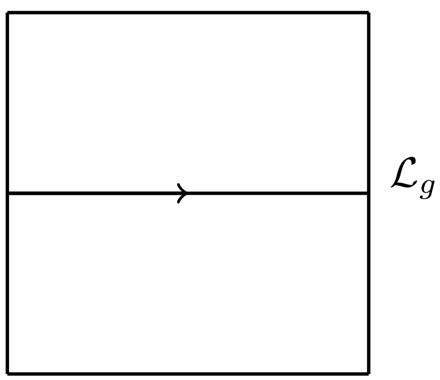
- LSM anomaly leads to projective representations of $\mathbb{Z}_N^2$ for defect operators
- This is essentially the *only* signature for odd N
- Modular bootstrap insensitive to LSM anomaly for odd N
- For even N some defects may have $\mathbb{Z}_M$ anomalies

Why expect a bound?

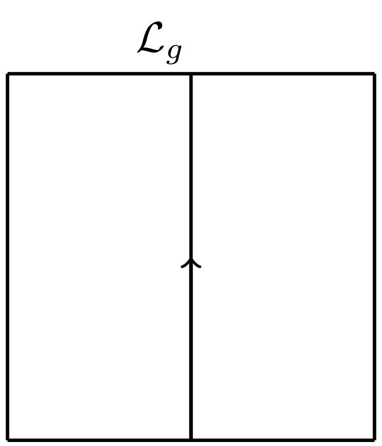
Place theory on a torus with symmetry twists



$$Z(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0-\frac{c}{24}}\,\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) = \sum_{h,\bar{h}} n_{h,\bar{h}}\chi_{h,\bar{h}}(\tau,\bar{\tau})$$



$$Z^{\mathcal{L}_g}(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\left(\hat{\mathcal{L}}_g q^{L_0-\frac{c}{24}}\,\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) = \sum_{h,\bar{h},\rho} n^{\rho}_{h,\bar{h}}\chi_{\rho}(g)\chi_{h,\bar{h}}(\tau,\bar{\tau})$$

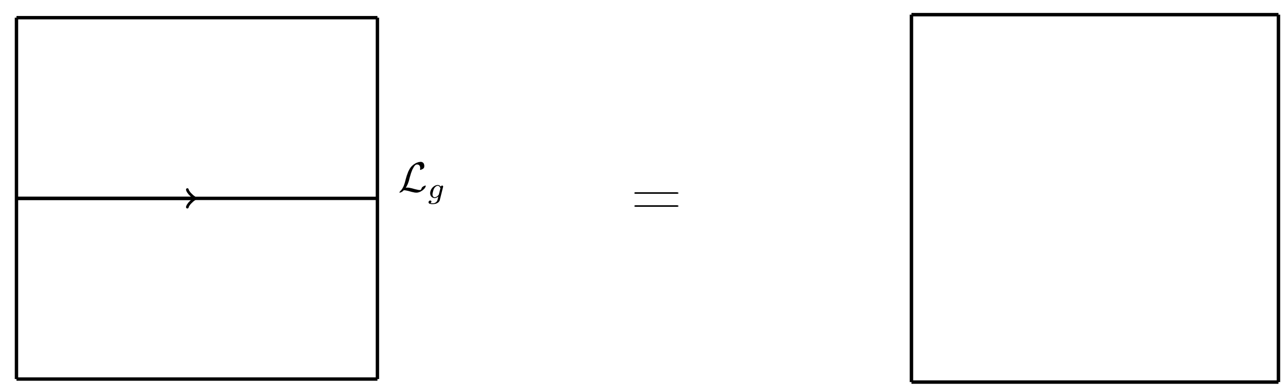


$$Z_{\mathcal{L}_g}(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}_{\mathcal{L}_g}}\left(q^{L_0^g-\frac{c}{24}}\,\bar{q}^{\bar{L}_0^g-\frac{c}{24}}\right) = \sum_{h,\bar{h}} n^{g}_{h,\bar{h}}\chi_{h,\bar{h}}(\tau,\bar{\tau})$$

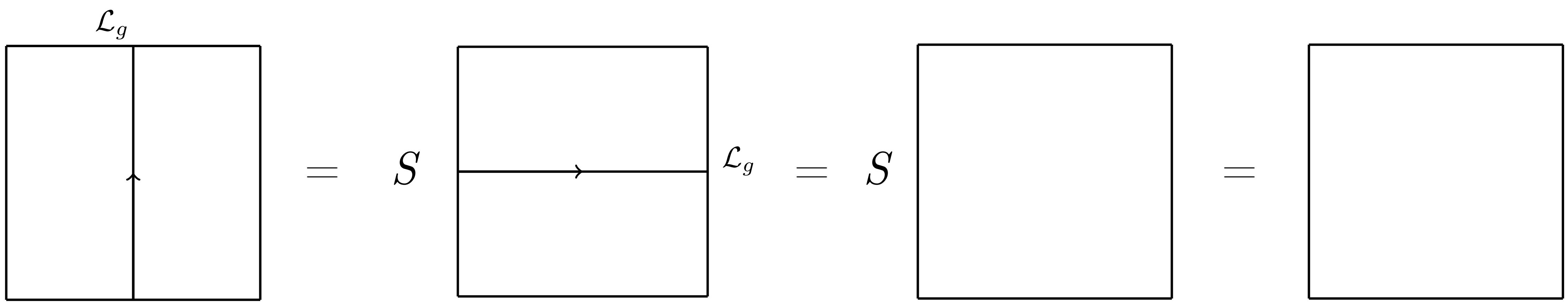
Why expect a bound?

Choose some subgroup $\mathbb{Z}_N \subset \mathbb{Z}_N^3$ and suppose it acts trivially (no charged operator)
Assume we have a CFT with unique vacuum and LSM anomaly

Then



But then



Defect ground state N-fold degenerate but bulk vacuum is unique

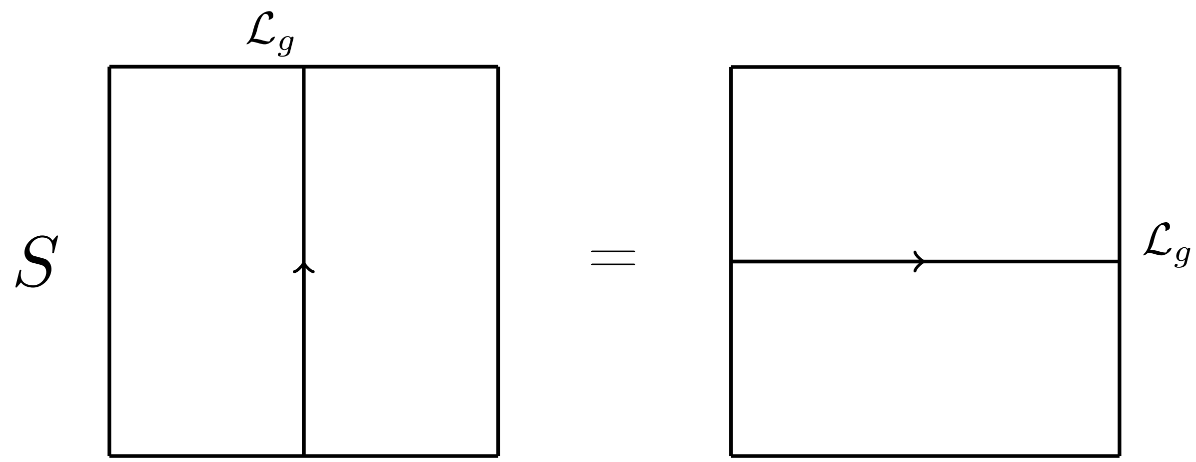
$\implies$ *Must have a charged operator,* $c = 0$ *forbidden*

Constraints on $(1+1)d$ CFTs with global symmetries

Modular *covariance*

[Cardy, 1986]

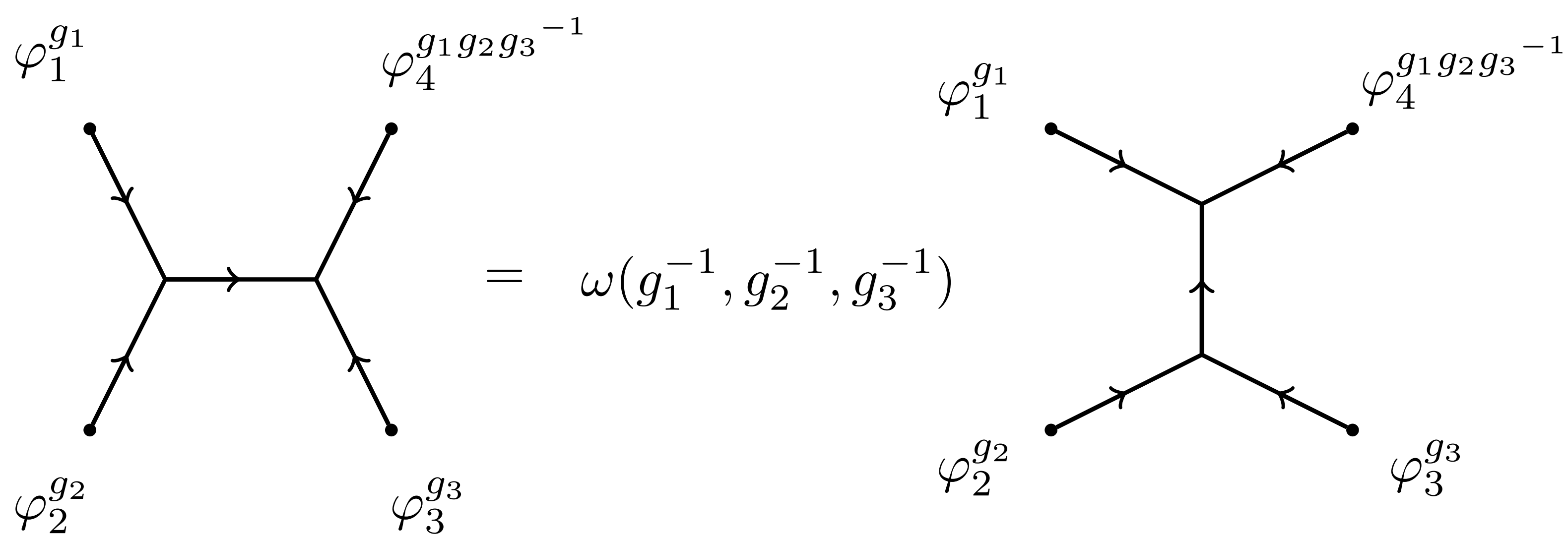
[Lin, Shao, 2019,2021]



Crossing symmetry

[Zamolodchikov, Zamoldchikov, 1989]

[Chang,Lin,Shao,Wang,Yin,2018]



Bootstrapping LSM anomalies

Correlator bootstrap

$$\sum_R \sum_{\mathcal{O}_R} \boldsymbol{\lambda}^T_{\mathcal{O}_R} \cdot \mathbf{V}^{\Delta_D}_{R,\Delta_{\mathcal{O}_R},s_{\mathcal{O}_R}}(x,\bar{x}) \cdot \boldsymbol{\lambda}_{\mathcal{O}_R} = 0$$

[Rattazzi, Rychkov, Tonni, Vichi, 2008]
[El-Showk, Paulos, Poland, Richkov, Simmons-Duffin, Vichi, 2012]
[Chang,Lin,Shao,Wang,Yin,2018]

- Fix some external operators (may be defect or local)
- Obtain upper bounds on dimension of operators appearing in OPE
- May also bound e.g. OPE coefficients, consequently lower bound central charge

Modular bootstrap

$$\sum_{j,\Delta,s} n^j_{h,\bar{h}} \mathbf{M}^j_{\Delta,s}(\tau,\bar{\tau}) = 0$$

[Collier, Lin, Yin, 2016]
[Lin, Shao, 2019,2021]
[Chang,Lin,Shao,Wang,Yin,2018]

- Constraint for each irreducible representation of G *and* each $\mathcal{H}_{\mathcal{L}_g}$
- Positive coefficients encode degeneracy of Virasoro primaries
- Able to rule out combination of gaps in local/defect operator sectors
- I.e. universal *upper bound* on scalar gap

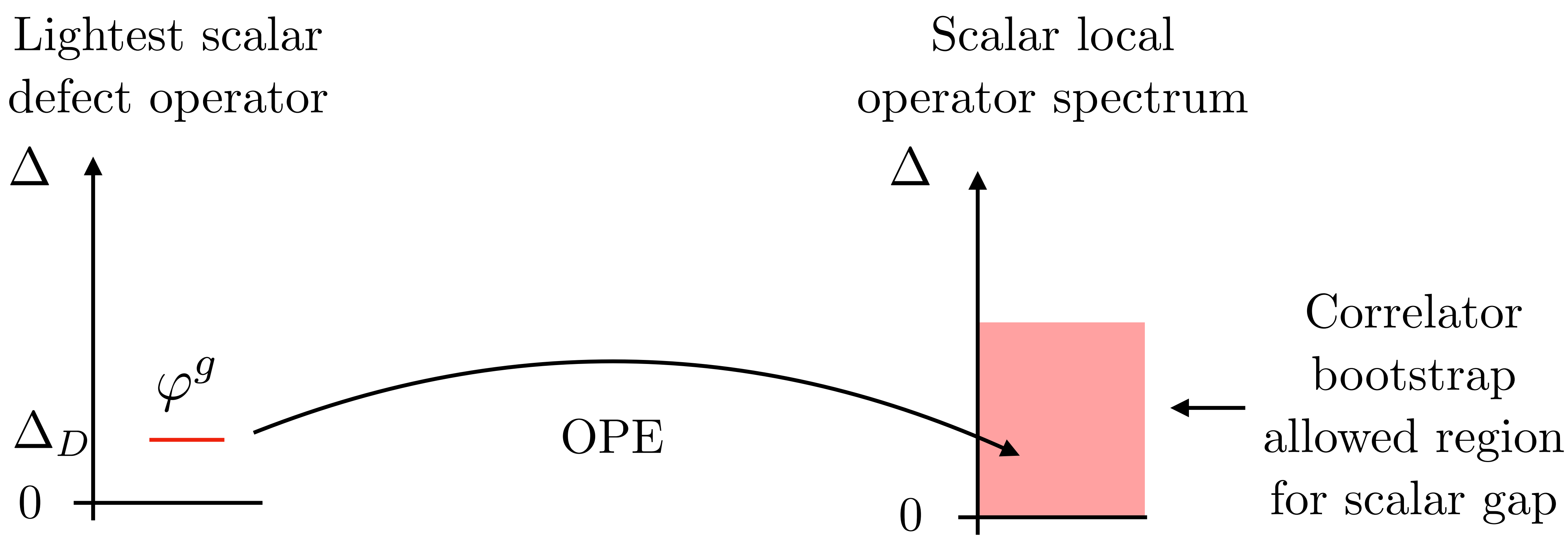
Bootstrapping LSM anomalies

Ultimately want universal *upper bound* on e.g. scalar gap Δ_L

- Can restrict to lightest charged/symmetric operator

Assume we have CFT obeying:

1. Central charge c 2. LSM anomaly 3. Scalar defect operator φ^g



Conversely… *assuming* scalar gap leads to *lower* bound on lightest defect operator

Bootstrapping LSM anomalies

Defect crossing symmetry leads to a stronger assumption ***beyond unitarity*** *on* ***defect spectrum gap*** *when scalar gap is assumed*

<u>Local operators</u>

$$\mathcal{H}_L = \bigoplus_\rho \mathcal{H}_\rho$$

Gaps $\Delta_\rho^{\min}$

<u>Defect operators</u>

$$\mathcal{H}_D = \bigoplus_g \mathcal{H}_{\mathcal{L}_g}$$

Gaps $\Delta_{\mathcal{L}_g}^{\min}$

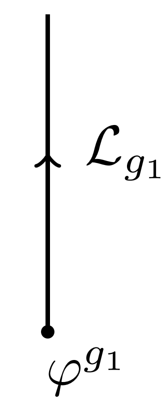
Ask modular bootstrap to rule out combination of defect and local operator gaps

If successful, we have upper bound on local scalar gap

Correlator bootstrap with defect operators

Want to quantify the consequences of a single ***scalar*** defect operator in the spectrum living on an ***order-N***, ***non-anomalous*** defect line

- Two such defect operators mutually local
- Treat as local operators transforming in representations of non-Abelian groups



$$0 \to \mathbb{Z}_N \to \underset{\substack{\uparrow \\ \text{“He}(3,\mathbb{Z}_N)\text{”}}}{\tilde{G}} \to \mathbb{Z}_N^2 \to 0$$

[Bhardwaj, Tachikawa, 2017]

- Take external operators to transform in the N-dimensional rep
- Anomaly *ensures* OPE of defect operators contains *charged* local operators

$$[N] \otimes [\bar{N}] = \bigoplus_{\substack{\rho \in \mathrm{Rep}(\mathrm{He}(3,\mathbb{Z}_N)) \\ \rho \ \mathrm{1d}}} \rho$$

- One-dimensional irreps of “$\mathrm{He}(3,\mathbb{Z}_N)$” are $\mathbb{Z}_N^2$ representations
- Without anomaly, just have i.e.

$$\phi \times \phi^\dagger \sim I + ...$$

Implementation

Used **autoboot** to generate correlator bootstrap constraints [Go, Tachikawa, 2019]

- Assume external operators transforming in N dimensional rep

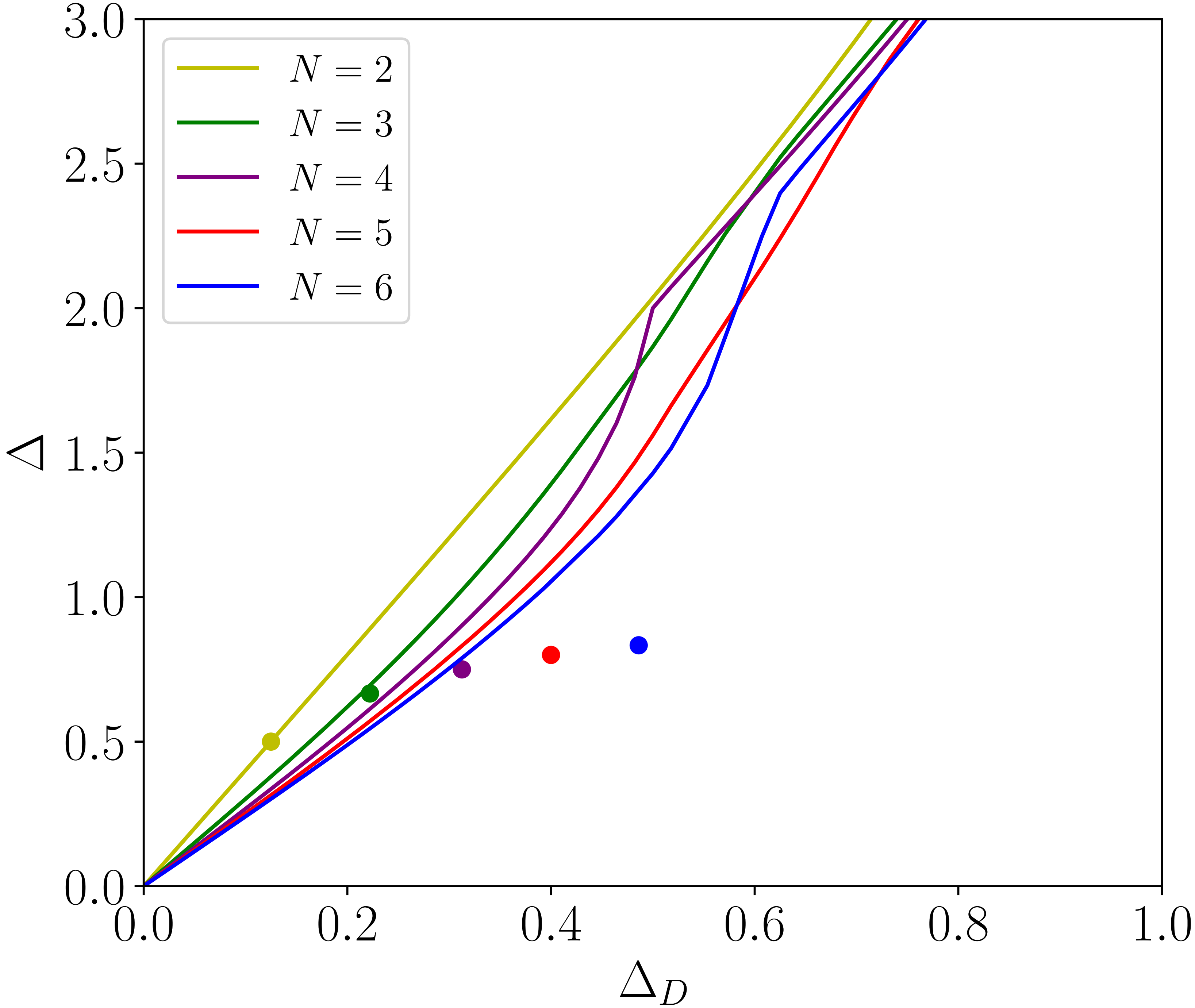
N	# vector constraints	Dimension
2	3	3
3	6	12
4	15	33
5	6	28
6	18	36

Modular bootstrap constraints obtained with character theory

Computations performed on Hyak cluster at UW

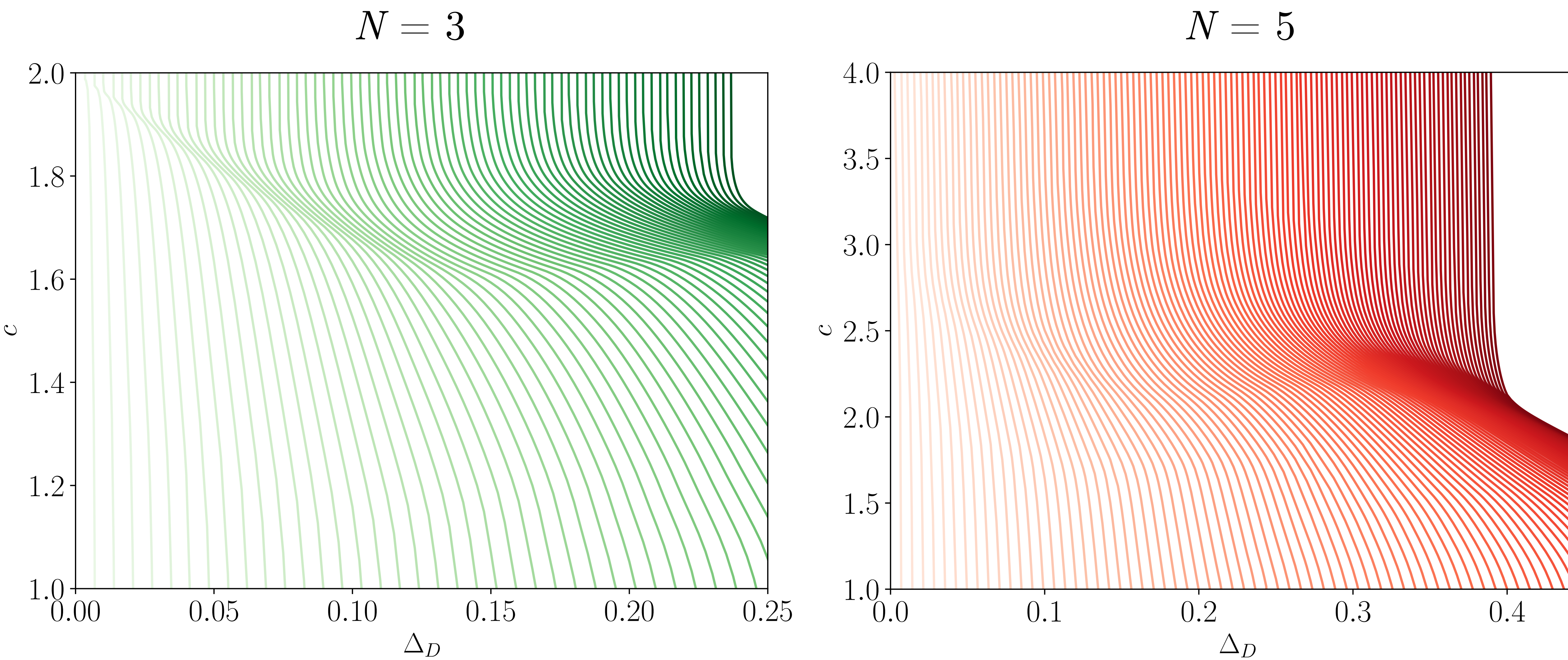
- For central charge bounds at $\Lambda = 15$ each call took 1 hour on 40-core node for $N = 5$
- Thousands of calls to generate plots

Local operators from defect operators



Upper bound on dimension of lightest local operator appearing in OPE of defect operators computed at $\Lambda = 25$ and $S_{\max} = 50$.

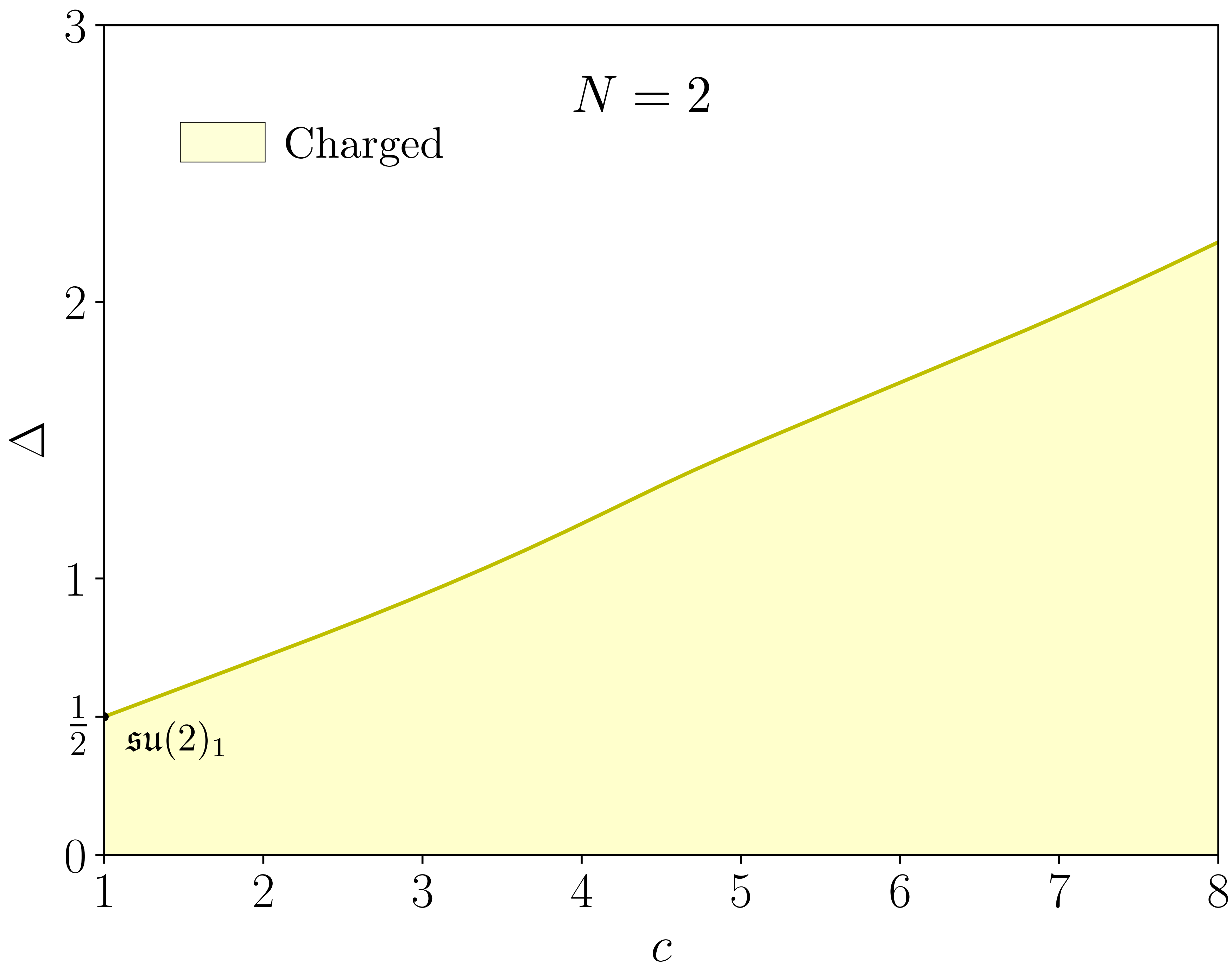
Lower bounds on central charge



Lower bounds on central charge assuming varying gap in all local operators as a function of lightest defect operator dimension. Computed at $\Lambda = 15$ and $S_{\max} = 30$.

- Scan over local, scalar operator gap (light to dark = low to high gap)
- Impose same gap for all charge sectors

Bounds on scalar operators: $\mathbb{Z}_2^3$

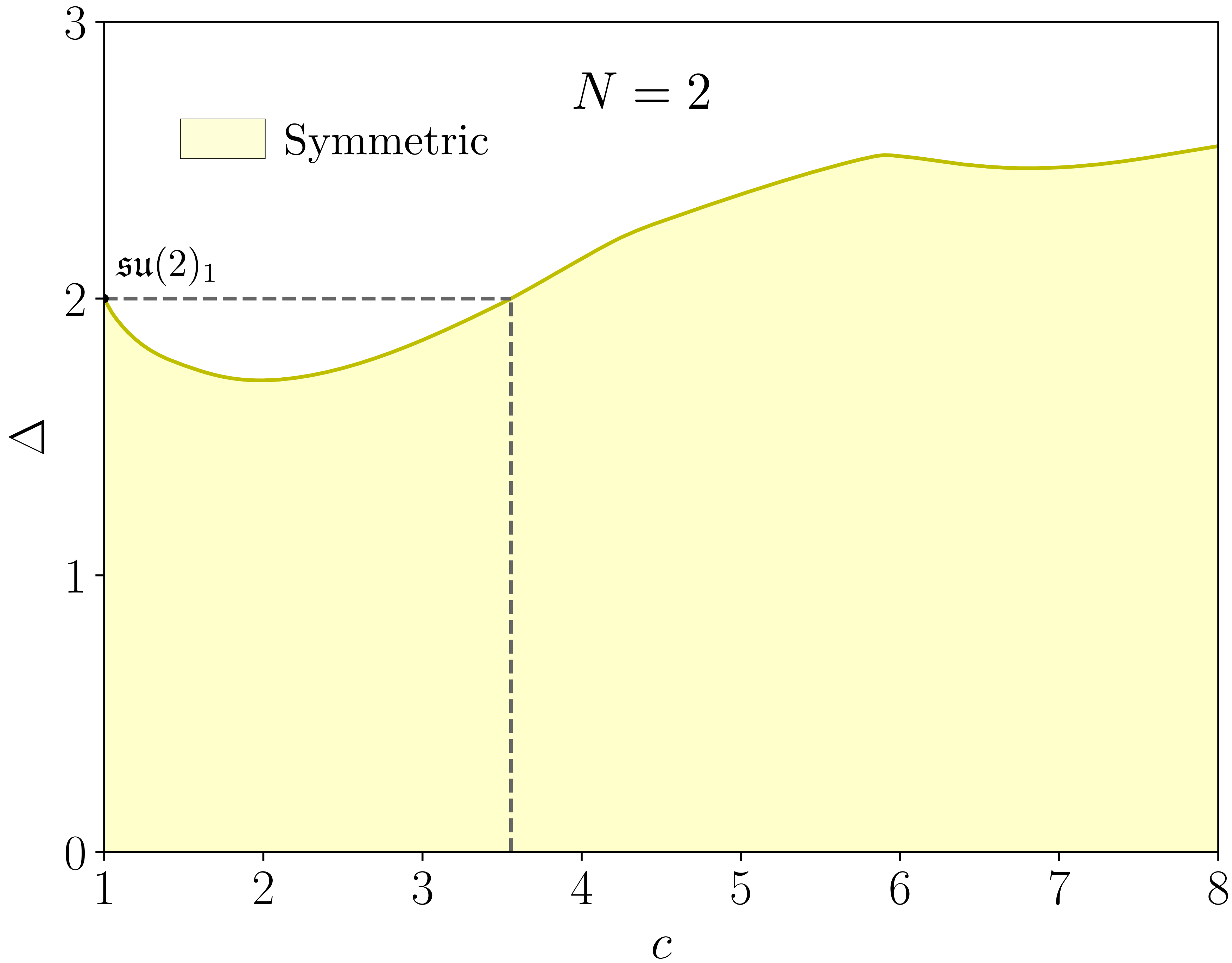


$\Lambda = 25$

$S_{\max} = 50$

- Known theory $\mathfrak{su}(2)_1$ saturates bound at $c = 1$

Bounds on scalar operators: $\mathbb{Z}_2^3$

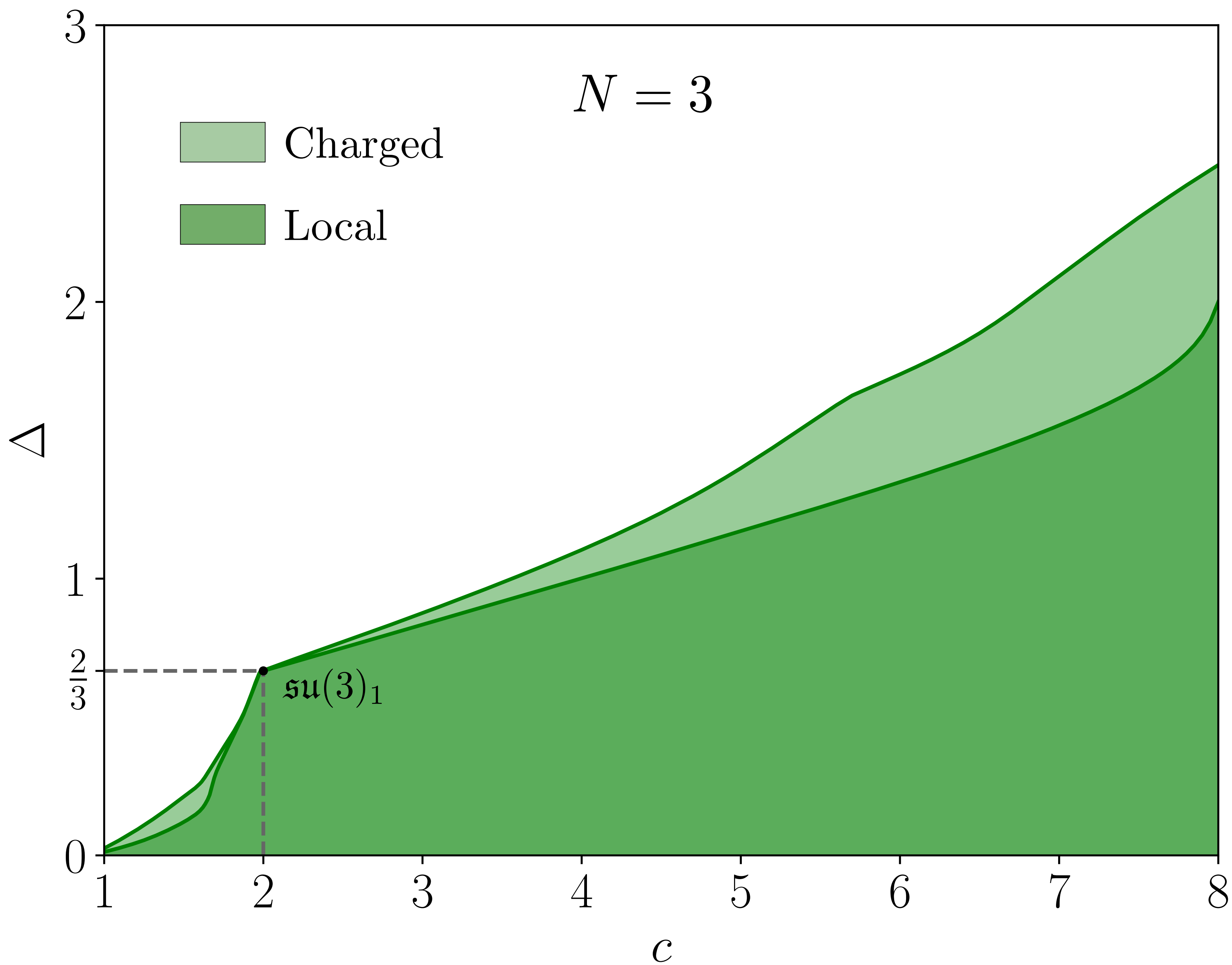


$\Lambda = 25$

$S_{\text{max}} = 50$

- Known theory $\mathfrak{su}(2)_1$ saturates bound at $c = 1$
- Relevant, symmetric scalar if $1 < \mathrm{c} < 3.5565$

Bounds on scalar operators: $\mathbb{Z}_3^3$

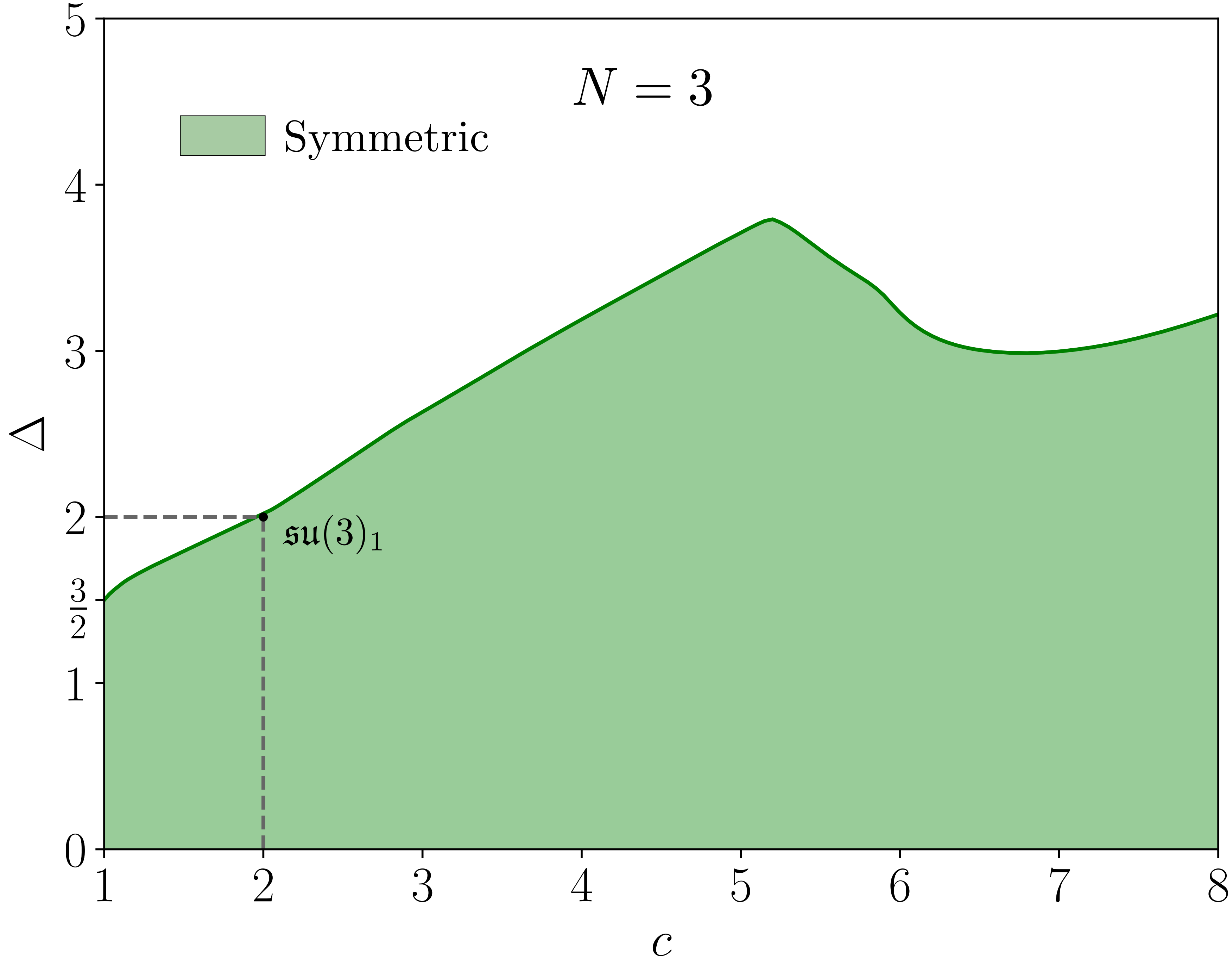


$\Lambda = 25$

$S_{\max} = 50$

- Known theory $\mathfrak{su}(3)_1$ at prominent kink
- Charged operator bound impossible with modular bootstrap alone!

Bounds on scalar operators: $\mathbb{Z}_3^3$

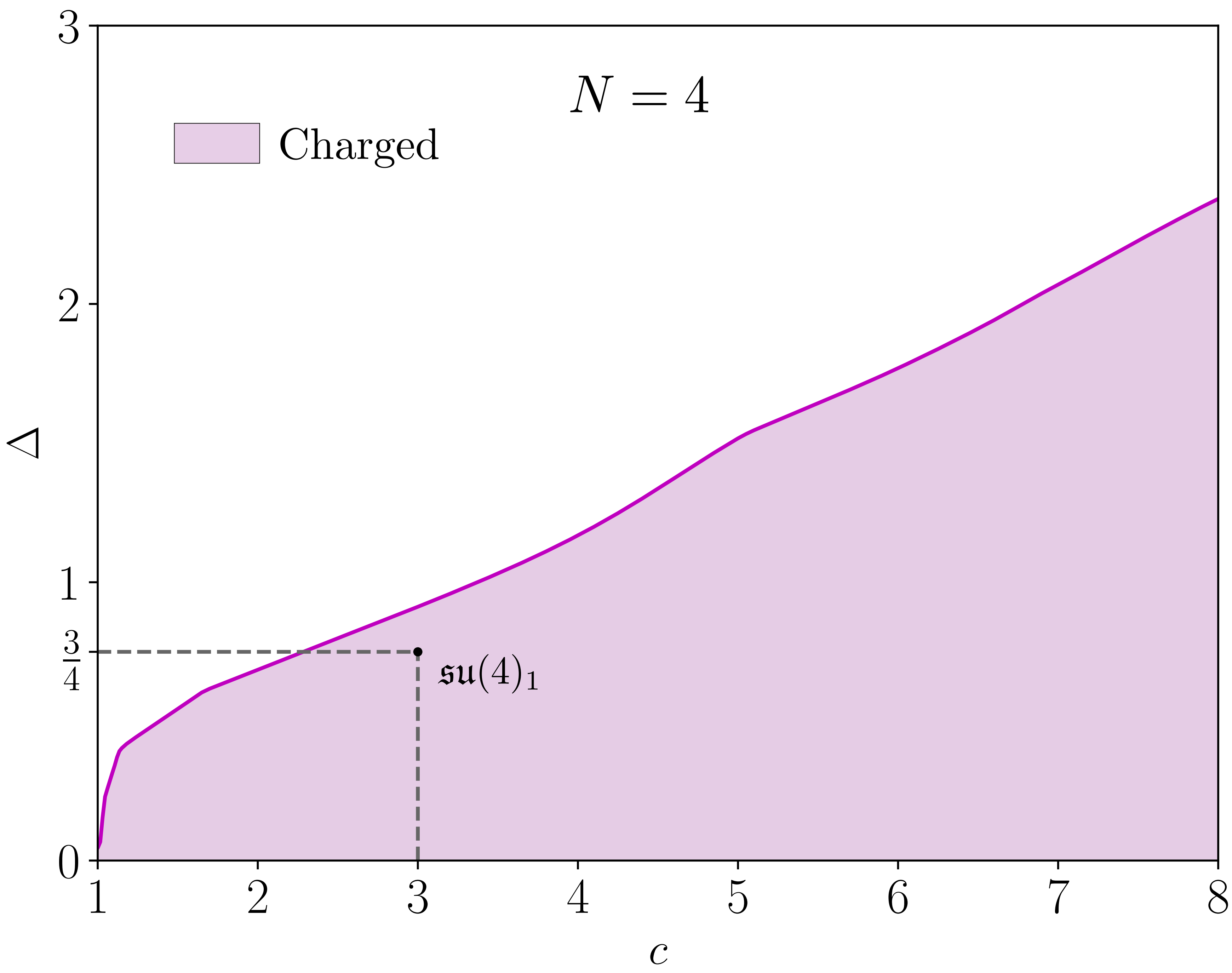


$\Lambda = 25$

$S_{\max} = 50$

- Must contain relevant symmetric scalar if $c < 2$

Bounds on scalar operators: $\mathbb{Z}_4^3$

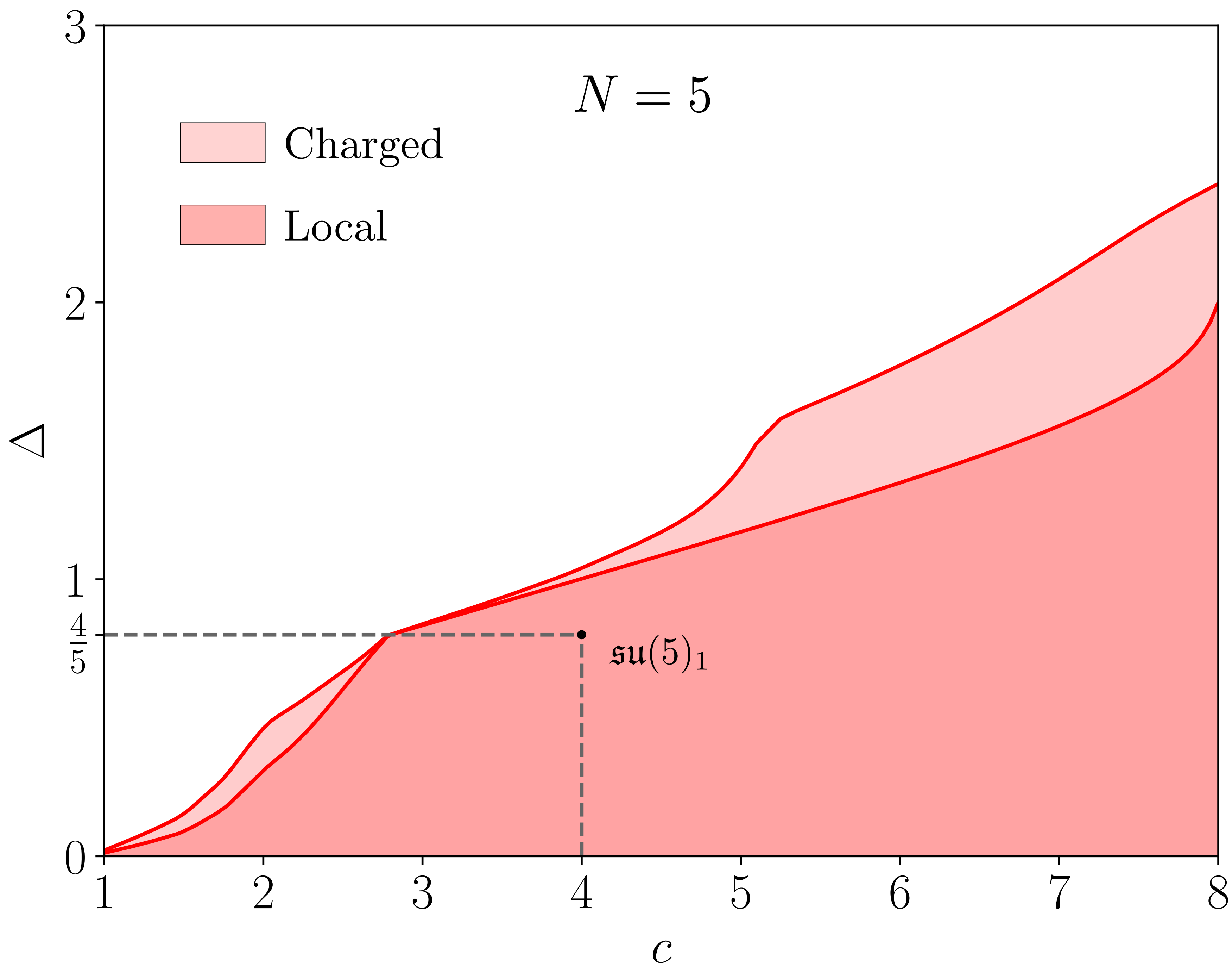


$\Lambda = 25$

$S_{\max} = 50$

- Kinks near c = 1.12,1.65—unexplained

Bounds on scalar operators: $\mathbb{Z}_5^3$

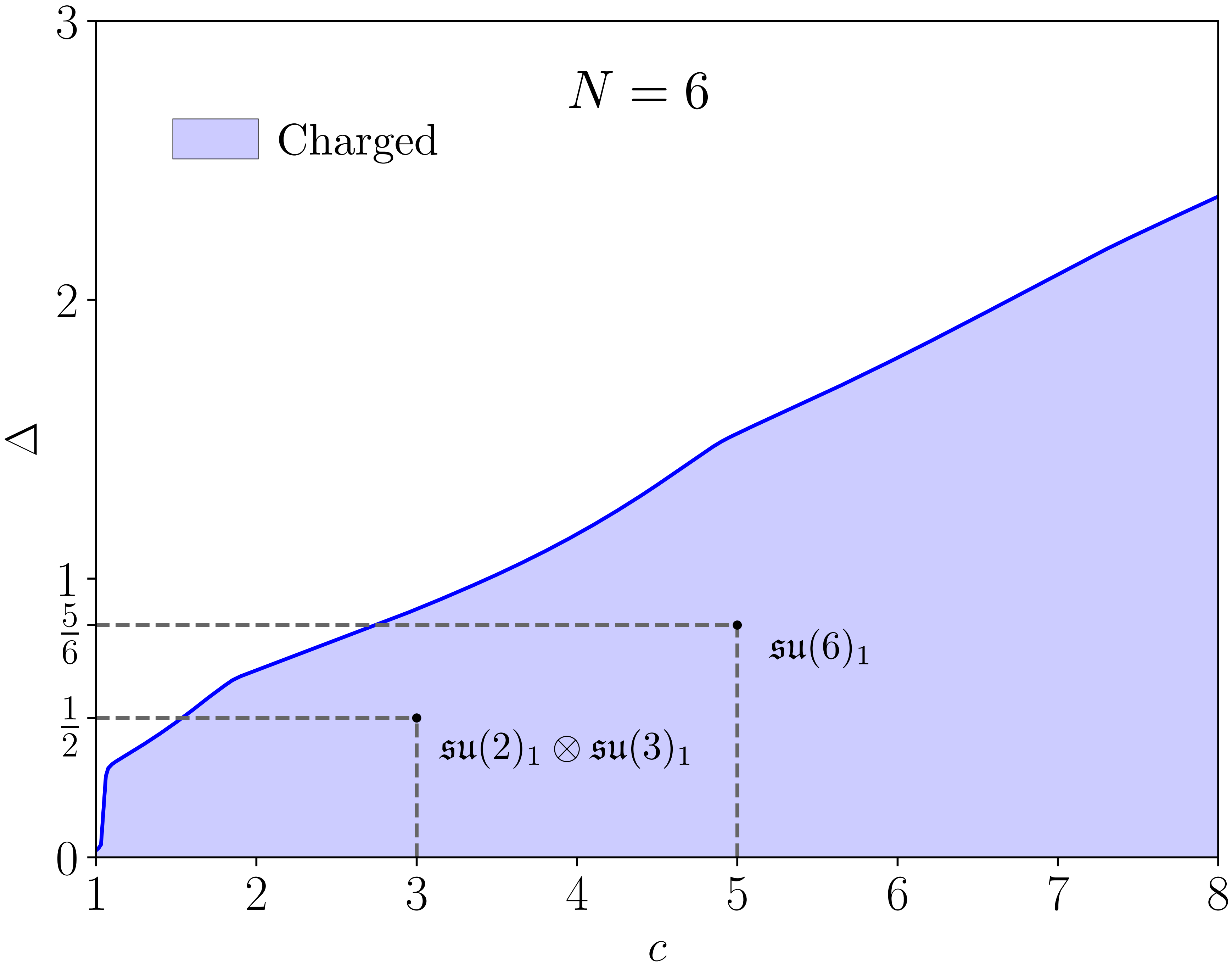


$\Lambda = 25$

$S_{\max} = 50$

- Sharp kink near c = 2.8—unexplained
- $(\mathfrak{g}_2)_1$ ruled out at $\Lambda = 41$ and $S_{\max} = 80$

Bounds on scalar operators: $\mathbb{Z}_6^3$



- Kinks near $c = 1.08$,1.85—unexplained

Summary & Outlook

- LSM anomalies imply bounds on charged operators
- Quantified relationship between central charge (ground state quantity) and scaling of energy gap in gapless spin chains
    - LSM tells us $\sim \mathcal{O}(1)/L$, we make this more precise
- State-of-the-art way of bootstrapping $(1+1)d$ CFTs with global symmetry

- Extend to non-invertible symmetries (ongoing Lin, Shao)
- Study interesting $N=5$ kink
- Anomalous symmetries constrain conformal boundary conditions [Thorngren, Wang, 2020] [Collier, Mazac, Wang, 2021]
    - Leverage this for possible anomaly-dependent c bound?
- Is this a roadmap for anomalies in bootstrap in higher d?
    - Line operators in $(2+1)d$?

Spin-selection rules

	Subgroup generated by $g \in G$				
G	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_5$	$\mathbb{Z}_6$
$\mathbb{Z}_2^3$	1 if $g = (1, 1, 1)$ 0 else	–	–	–	–
$\mathbb{Z}_3^3$	–	0	–	–	–
$\mathbb{Z}_4^3$	0	–	2 if g_i odd 0 else	–	–
$\mathbb{Z}_5^3$	–	–	–	0	–
$\mathbb{Z}_6^3$	1 if $g = (3, 3, 3)$ 0 else	0	–	–	3 if g_i odd 0 else