Recent development in bootstrap numerics

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Conformal bootstrap and Critical Phenomena

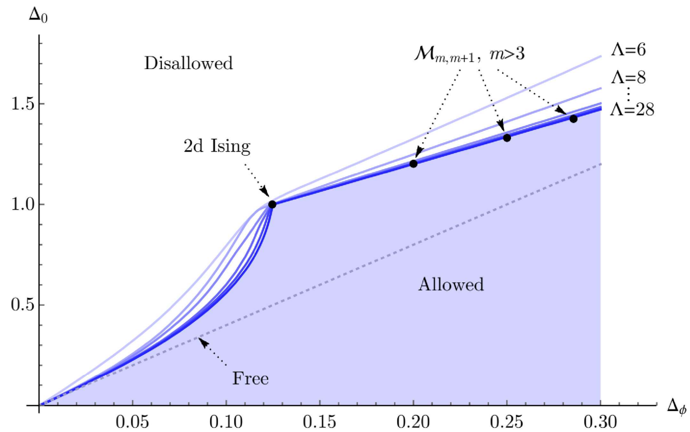
Problems in critical phenomena
↔ Questions on CFT data
↔ Conformal bootstrap constraints
↔ semidefinite program (SDP)
↔ high efficiency SDP numerics (bottleneck!)

In practice : the efficiency of the bootstrap numerics crucially limits our ability to access certain CFT data.

This talk : Latest examples of large scale bootstrap problem. New algorithm for SDP numerics (10+ times faster).

Sophisticated CFTs ↔ Large scale bootstrap constraints

Bootstrapping $\langle\phi\phi\phi\phi\rangle$ in 2D. Plot lowest singlet Δ_0 v.s. Δ_ϕ .



(taken from Simmons-Duffin arXiv:1602.07982)

Why 2D Ising CFT sits at the kink of the bounds? How to make other $\mathcal{M}(m,\ m+1)$ sit on kinks?

Sophisticated CFTs ↔ Large scale bootstrap constraints

Why 2D Ising CFT sits on the kink of the bounds?

[Connor Behan 2017]
There is a series of solutions that saturate $\langle\phi\phi\phi\phi\rangle$: $\Delta_\epsilon = \frac{8}{3}\Delta_\phi + \frac{2}{3}$.
They smoothly connect $m = 3,\ 4,\ \ldots \infty$ for $\mathcal{M}(m,\ m+1)$.

As $m \to 3$, an operator in spin 2 channel T' decouple from spectrum : $\lambda_{\phi\phi\mathrm{T'}} \to 0$.

T' : level 2 null state in Virasoro primary

Sharp features in the allowed region (kinks, pikes) ↔ operators decoupling/disappear/recombine in the primary spectrum

[Belavin, Polykov, Zamolodchikov 1984] For c<1 to be unitary, there must be certain null states in the Virasoro primary

Sophisticated CFTs ↔ Large scale bootstrap constraints

In general, to access more and more sophisticated CFTs, we have to mix more and more operators.

Our setup must be able to access enough gap features in the spectrum that discriminate the target CFT from all other theories (solutions) with the same symmetry.

more correlators, more parameters to scan → large scale SDP and challenging numerics

2D Potts and tricritical Potts model

Consider 2D square lattice of random spins with Hamiltonian for $s_i \in \{1, 2, \ldots q\}$ [Potts 1952]

$$Z = \sum_{\{s_i\}} e^{-\beta \sum_{\langle i j \rangle} \delta_{s_i, s_j}}$$

Symmetry : S_q

Tuning β : at low temperature (small β), ordered phase with broken S_q. At high temperature, disordered phase.

At $\beta = \beta_{\text{crit}}$: Second order phase transition, described by critical Potts $\subset \mathcal{M}(5, 6)$

Assuming some sites are vacant, turning both β and chemical potential of vacancies :
Second order phase transition, described by tricritical Potts $\subset \mathcal{M}(6, 7)$

2+ϵ dimensional $q = 3$ Potts and tricritical Potts model

Symmetry : S_3 has 3 irreps : singlet **1**, sign rep $1'$, standard rep **2**

Critical Potts : two relevant rep **2** operators σ, σ' , one relevant singlet ϵ.
2D values: $\Delta_\sigma = \frac{2}{15}$, $\Delta_{\sigma'} = \frac{4}{3}$, $\Delta_\epsilon = \frac{4}{5}$

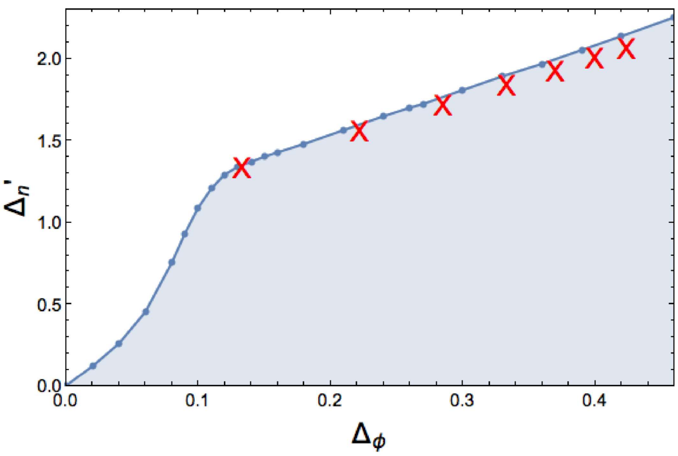
Tricritical Potts : two relevant rep **2** operators σ, σ', two relevant singlet ϵ, ϵ'.
2D values: $\Delta_\sigma = \frac{2}{21}$, $\Delta_{\sigma'} = \frac{20}{21}$, $\Delta_\epsilon = \frac{2}{7}$, $\Delta_{\epsilon'} = \frac{10}{7}$

In $2+\epsilon$ dimension, $q = 3$ critical Potts and tricritical Potts get closer, merge and become complex CFT at d_{crit} (the merger/annihilation scenario [Gorbenko, Rychkov, Zan 2018])
Note : for $q = 2$ (Ising), $d_{\text{crit}} = 4$; For $q = 4$, $d_{\text{crit}} = 2$.

Two predictions around d_{crit} :
(1), Potts & tricritical Potts merges with $\Delta_\epsilon = \Delta_{\epsilon'} = 3$ at d_{crit}
(2), $\Delta_O - \Delta_{O,\text{crit}} \propto \sqrt{d_{\text{crit}} - d}$

Bootstrap 2+ϵ dimensional $q = 3$ Potts, tricritical Potts

Consider correlator $\langle\sigma\sigma\sigma\sigma\rangle$ in 2D, bound $\Delta_{\sigma'}$ $v.s.$ Δ_σ :

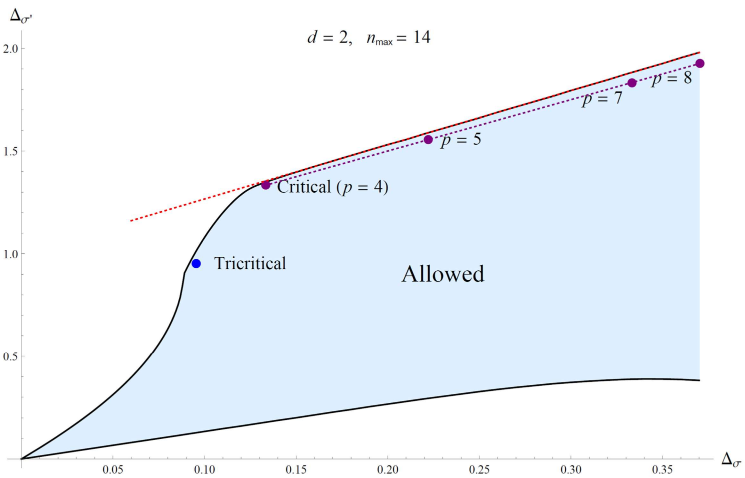


[Junchen Rong, NS, 2017]

Bootstrap 2+ϵ dimensional $q = 3$ Potts, tricritical Potts

[Shai Chester, NS, to appear]

Consider correlator $\langle\sigma\sigma\sigma\sigma\rangle$ in 2D, assuming only σ, σ' are relevant operators



Bootstrap 2+ϵ dimensional $q = 3$ Potts, tricritical Potts

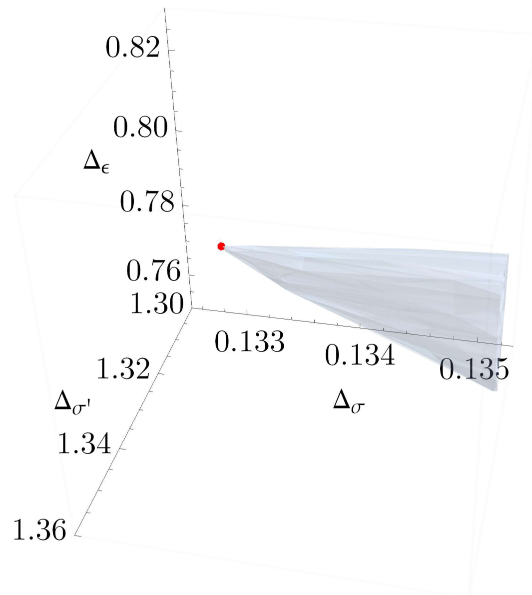
Consider all correlator of σ, σ', ϵ :
$\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma'\sigma'\sigma'\sigma'\rangle$, $\langle\sigma\sigma'\sigma\sigma'\rangle$, $\langle\sigma\sigma\sigma'\sigma'\rangle$, $\langle\sigma\sigma'\sigma'\sigma'\rangle$, $\langle\sigma'\sigma\sigma\sigma\rangle$,
$\langle\epsilon\epsilon\epsilon\epsilon\rangle$, $\langle\epsilon\sigma\epsilon\sigma\rangle$, $\langle\epsilon\sigma'\epsilon\sigma'\rangle$, $\langle\epsilon\sigma\epsilon\sigma'\rangle$, $\langle\epsilon\epsilon\sigma\sigma\rangle$, $\langle\epsilon\epsilon\sigma'\sigma'\rangle$, $\langle\epsilon\epsilon\sigma\sigma'\rangle$, $\langle\epsilon\sigma\sigma\sigma\rangle$, $\langle\epsilon\sigma'\sigma'\sigma'\rangle$,
$\langle\epsilon\sigma\sigma'\sigma'\rangle$, $\langle\epsilon\sigma'\sigma\sigma\rangle$, $\langle\sigma\epsilon\sigma\sigma'\rangle$, $\langle\sigma'\epsilon\sigma'\sigma\rangle$

Totally 39 crossing equations.

Bootstrap 2+ϵ dimensional $q=3$ Potts, tricritical Potts

Consider all correlators of σ, σ', ϵ. Assume they are the only relevant scalars in $\mathbf{2}$ and $\mathbf{0}$.

Bootstrap at 2D with truncation order : $\Lambda = 11$:



Bootstrap 2+ϵ dimensional $q=3$ Potts, tricritical Potts

$2+\epsilon$ bootstrap :

For Potts : all correlators of σ, σ', ϵ. Scan

$\Delta_\sigma,\ \Delta_{\sigma'},\ \Delta_\epsilon,\ \lambda_{\epsilon\epsilon\epsilon},\ \lambda_{\sigma\sigma\epsilon},\ \lambda_{\sigma\sigma'\epsilon},\ \lambda_{\sigma\sigma\sigma},\ \lambda_{\epsilon\sigma'\sigma'},\ \lambda_{\sigma\sigma'\sigma'},\ \lambda_{\sigma'\sigma'\sigma'}$ (10D)

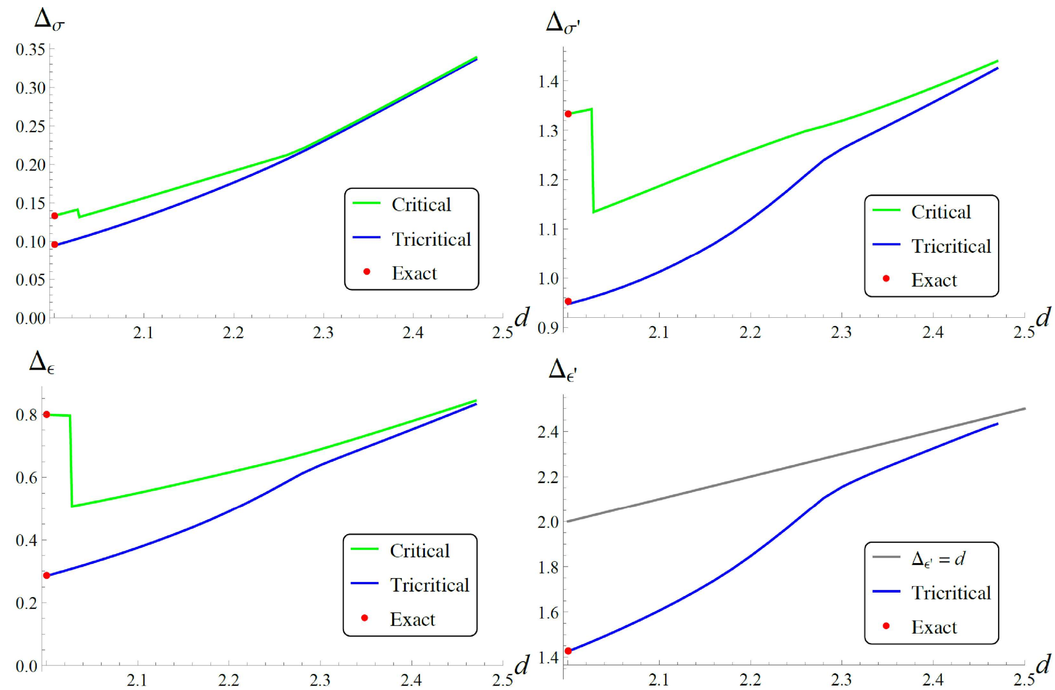
For tri-Potts : all correlators of σ, σ'. Scan

$\Delta_\sigma,\ \Delta_{\sigma'},\ \Delta_\epsilon,\ \Delta_{\epsilon'},\ \lambda_{\sigma\sigma\sigma},\ \lambda_{\sigma\sigma'\sigma'},\ \lambda_{\sigma'\sigma'\sigma'},\ \lambda_{\sigma\sigma\epsilon},\ \lambda_{\sigma\sigma'\epsilon},\ \lambda_{\epsilon\sigma'\sigma'},\ \lambda_{\sigma\sigma\epsilon'},\ \lambda_{\sigma\sigma'\epsilon'},\ \lambda_{\sigma'\sigma'\epsilon'}$ (13D)

Gigantic SDP and scan problem. We need the navigator method [Reehorst, Rychkov, Simmons-Duffin, Sirois, SN, van Rees 2021]. See Reehorst Marteen’s talk.

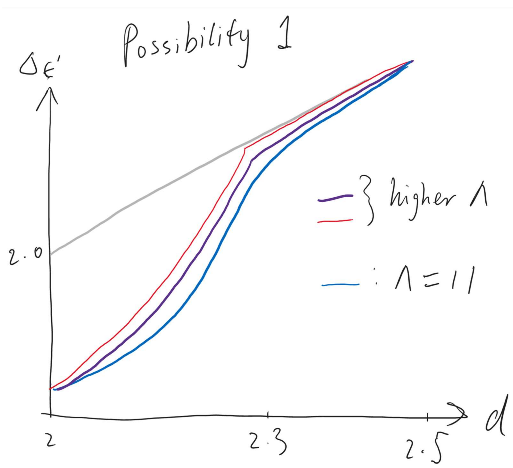
In both setups, we minimize Δ_σ in the allowed region, i.e. go to the tip of the dagger.

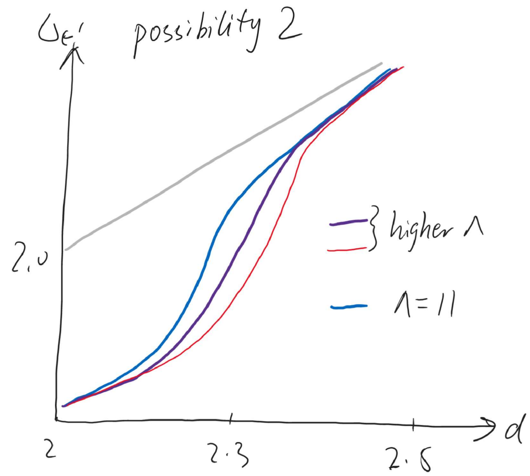
Bootstrap 2+ϵ dimensional $q = 3$ Potts, tricritical Potts Λ=11



Bootstrap 2+ϵ dimensional $q = 3$ Potts, tricritical Potts

Conclusion : $d_{\text{crit}} \lessapprox 2.5$. Approximately correct square root behavior $\Delta_{\epsilon'} - d_{\text{crit}} \propto \sqrt{d_{\text{crit}} - d}$, but not conclusive. Two possibilities in higher Λ :





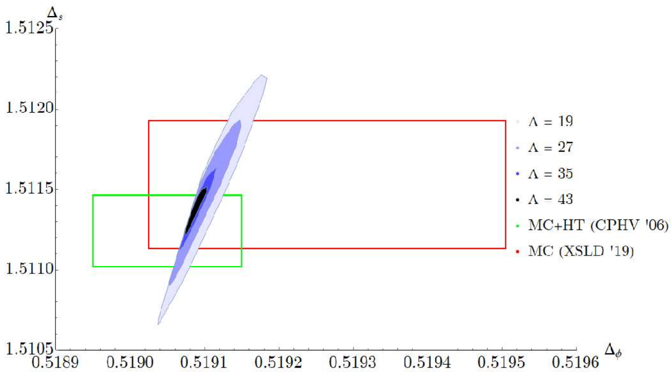
We need much higher Λ bootstrap study (not possible with current techniques)

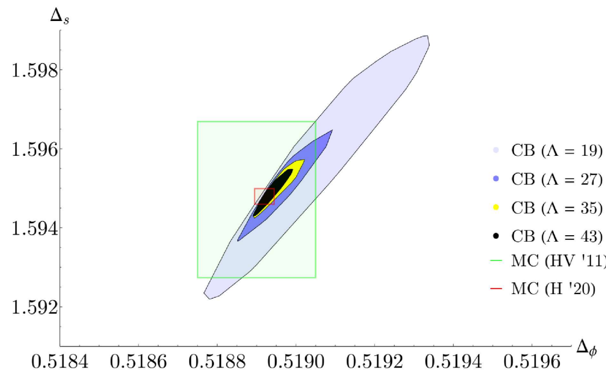
(Higher Λ might even turn the dagger plot into an island)

O(N) vector model bootstrap

Lagrangian : $\mathcal{L} = \partial\phi_i\,\partial\phi_i + m^2\,(\phi_i\,\phi_i) + \lambda\,(\phi_i\,\phi_i)^2$

Bootstrapping all 4pt involves $\{v = \phi_i,\; s = \phi^2,\; t = \phi_i\,\phi_j\}$:





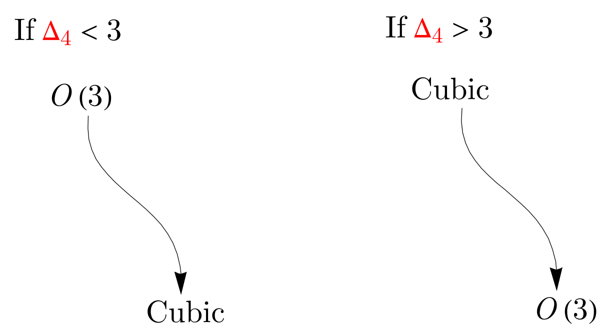
[Chester, Landry, Liu, Poland, Simmons-Duffin, SN, Vichi 2019, 2020]

Gigantic bootstrap problem. We used XSEDE super-cluster.

3D O(3) model

O(3) model : $\mathcal{L} = \partial\phi_i\,\partial\phi_i + \lambda_1\,(\phi_i\,\phi_i)^2$

Cubic model : $\mathcal{L} = \partial\phi_i\,\partial\phi_i + \lambda_1\,(\phi_i\,\phi_i)^2 + \lambda_2 \sum_i \phi_i{}^4$ (symmetry: $S_3 \times Z_2{}^3$)



A final verdict : $\Delta_4 \leq 2.99056$. $O(3)$ is unstable against cubic perturbation!

($\mathrm{LaAlO_3}$ has structural phase transition in Cubic universality, not O(3) universality)

3D cubic model : conformal perturbation theory

(Junchen Rong, NS, to appear. See Junchen's talk for more details)

$S_{\text{cubic}} = S_{O(3)} + g \int d^3 x\, t_4$ where $\Delta_{t_4} \approx 2.988$. Perturbation parameter : $\delta = 3 - 2.988 = 0.012$

$\beta_g = -\delta\, g - \frac{1}{2} S_{d-1}\, \lambda_{t_4\, t_4\, t_4}\, g^2$

$\Delta_O = \Delta_0 + \delta\, \frac{2\, \lambda_{OO\, t_4}}{\lambda_{t_4\, t_4\, t_4}}$

$\lambda_{OO\, t_4} / \lambda_{t_4\, t_4\, t_4}$ can be accessed by bootstrap correlators of $\{v,\ s,\ t,\ t_4\}$

$\lambda_{OO\, t_4} / \lambda_{t_4\, t_4\, t_4} \approx\ 0.8$ from one allowed point at $\Lambda = 11$.

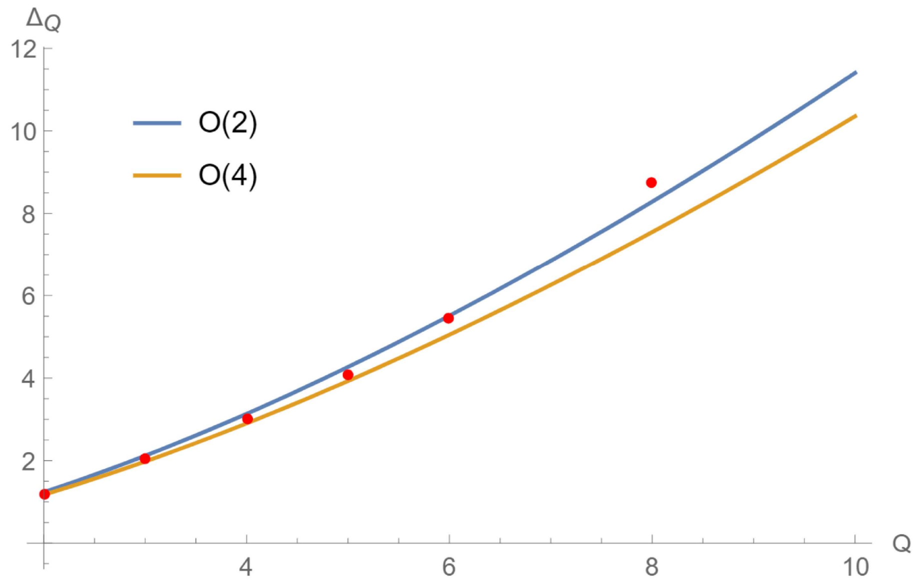
The O(3) rank 2 tensor (dim=5) ($\Delta_t = 1.2096$) split to cubic $\mathbb{2} \oplus \mathbb{3}$

$\Delta_{\mathbb{2}} \approx 1.2200 \quad , \ \gamma_{\mathbb{2}} \approx 0.01043$

$\Delta_{\mathbb{3}} \approx 1.2091 \quad , \ \gamma_{\mathbb{3}} \approx -0.00044$

We couldn't know error bar of $\lambda_{OOX} / \lambda_{XXX} \approx\ 0.8$. Need larger Λ bootstrap study (not possible with current techniques).

large charge operators in O(3)



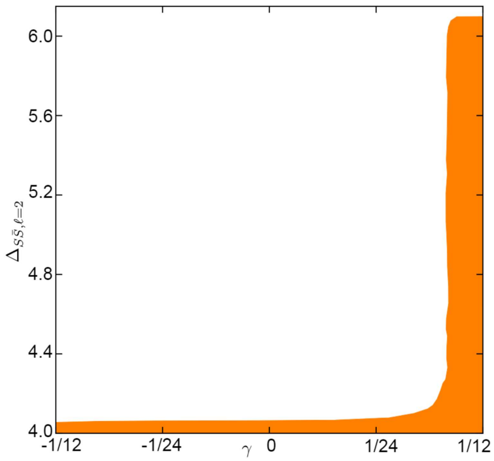
Non-abelian currents bootstrap

[Yin-Chen He, Junchen Rong, NS, Alessandro Vichi, ongoing work]
Consider $\langle J_\mu\, J_\nu\, J_\rho\, J_\sigma \rangle$ where $(J_\mu)_a^b$ is the current of global symmetry $\mathrm{SU}(N_f)$

$S\,\overline{S}\;:\, A^{(a\,b)}_{(c\,d)}$, an example of decoupling operator that can detect color group (See Yinchen's discussion talk last week). $\Delta_{S\,\overline{S},L=2} \approx 4$ for $U(N)_{N\geq 2}$ gauge theory, $\Delta_{S\,\overline{S},L=2} \approx 6$ for QED.

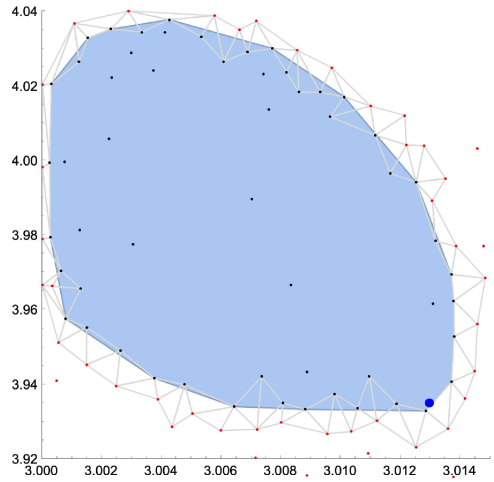
Non-abelian currents bootstrap

For $N_f = 100$ at $\Lambda = 19$:



$\langle J\ J\ T_{\mu\nu}\rangle$ has 4 tensor structures, depending on two parameters : $C,\ \gamma$. For free fermion $\gamma = \frac{1}{12}$

Non-abelian currents bootstrap



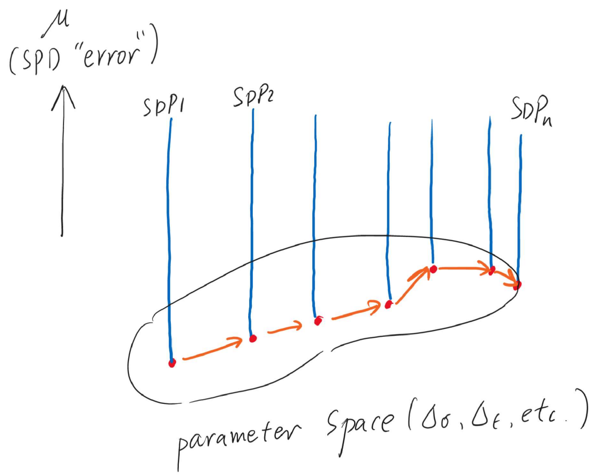
To shrink the QED3 island and get precise CFT data, we might need to mix J^f with J^t and/or a monopole operator and do calculation at larger Λ. (not possible with current techniques)

A common challenge

To get sharp results for some problems in critical phenomena, we often have to bootstrap a large scale problem that beyond current limits.

Current bootstrap numerics

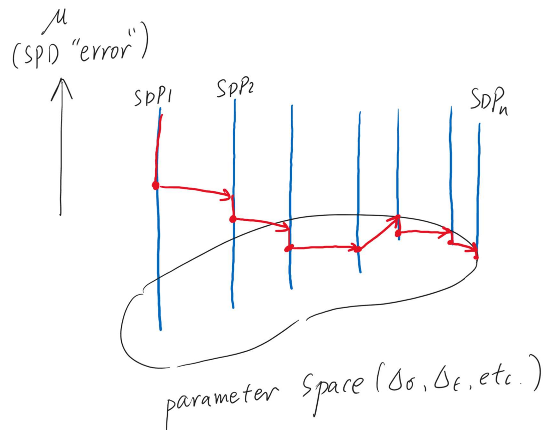
The old numerical method: a section of SDPs over a manifold. We solve SDPs one by one.



Typical Newtonian iterations per SDP : 100 to 300
(A well-known issue in SDP : for optimality runs, hotstart is not efficient. Best solution before : save a middle checkpoint.)

New algorithm for SDP fibre bundle

[Aike Liu, David Simmons-Duffin, NS, Balt van Rees, ongoing work]
The new numerical method: treat the section of SDPs over a manifold as a single optimization problem.
Hopping to a new SDP without completely solving current SDP.



Simultaneously solves the optimization in the parameter manifold (maximize Δ_σ) and solves the optimization of SDP ($\mu \to 0$)

New algorithm for SDP fibre bundle

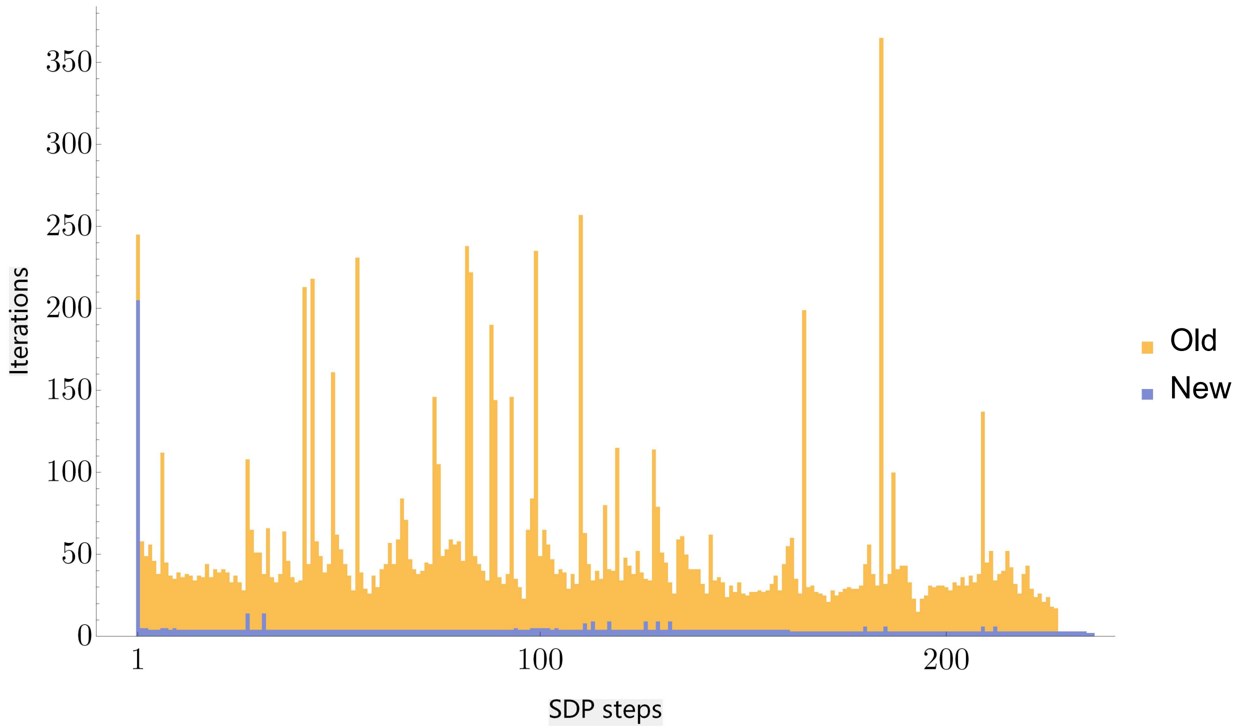
A novel algorithm:

1, Determine a good hopping point
2, Determine a good hopping direction in both SDP internal variables (x, y, X, Y) and external parameters $(\Delta_\sigma, \ldots)$

Key feature : only a few Newtonian iterations per SDP.

New algorithm for SDP fibre bundle

Example : O(3) bootstrapping correlators of $v,\ s,\ t$ at $\Lambda = 19$. Maximize Δ_{t_4} in the allowed region of $\{\Delta_v,\ \Delta_s,\ \Delta_t,\ \Delta_{t_4},\ \lambda_{vtv},\ \lambda_{tts},\ \lambda_{ttt},\ \lambda_{sss}\}$



New algorithm for SDP fibre bundle

Example : O(3) bootstrapping correlators of $v,\ s,\ t$ at $\Lambda = 19$. Maximize Δ_{t_4} in the allowed region of $\{\Delta_v,\ \Delta_s,\ \Delta_t,\ \Delta_{t_4},\ \lambda_{vtv},\ \lambda_{tts},\ \lambda_{ttt},\ \lambda_{sss}\}$

Average Newtonian iterations per SDP (excluding 1st SDP):
New algorithm : ~4
Old algorithm : ~ 52

As the scale of the SDP get larger, the different is even bigger.

We invented a bag of tricks. Some of our tricks works for SDP itself. For example, in some cases, in one Newtonian iteration, we could make μ go from 10^{-15} to 10^{-33}.

(I will give a technical talk regarding the algorithm in the last week)

Future goals

Problems in critical phenomena
↔ Questions on CFT data
↔ Bootstrap constraints
↔ semidefinite program (SDP) (bottleneck)
↔ high efficiency SDP numerics

Right now : Parameters $\xleftrightarrow{\text{iterate Casimir equ.}}$ numerical conformal blocks $\xleftrightarrow{\text{some algebra}}$ SDP

If we have a iterative process : Parameters $\xleftrightarrow{\text{iterate}}$ SDP and $d(\text{Parameters}) \longleftrightarrow d(\text{SDP})$

We may make the two iterative processes play against each other.

Future goals

Given a set of operators $\{\phi_1, \phi_2, \phi_3, \ldots\}$, we assign different derivative truncation Λ to $\langle\phi_1\,\phi_1\,\phi_1\,\phi_1\rangle$, $\langle\phi_2\,\phi_2\,\phi_2\,\phi_2\rangle$, $\langle\phi_1\,\phi_1\,\phi_2\,\phi_2\rangle$. (Already implemented in simpleboot package)

Adding more derivatives to $\langle\phi_1\,\phi_1\,\phi_1\,\phi_1\rangle$ or adding a new operator ϕ_n ? Should be decided by their contribution to the navigator function.

Technical issues : non-uniform polynomial degrees. Create some degeneracies in SDP.

Thank you