Line Defects, Renormalization Group Flows, Magnets, and Wilson Lines

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October 18, 2022

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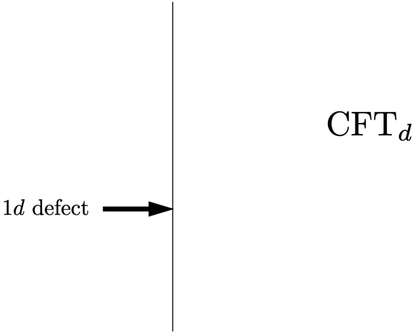
- w/ Gabriel Cuomo, Mark Mezei: 2112.10634
- w/ Gabriel Cuomo, Mark Mezei, Avia Raviv-Moshe: 2202.00040
- w/ Ofer Aharony, Gabriel Cuomo, Mark Mezei, Avia Raviv-Moshe: to appear

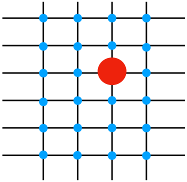
The subject of line defects has been historically extremely productive. The Kondo line defect in 2d has led to the **renormalization group** [Wilson...], to substantial progress on **integrability** [Andrei, Tsvelick-Wiegmann...], and of course to the development of **conformal symmetry** at the end points of the RG flow. The topic of this talk is to explore **line defects in higher dimensions**.

While a lot is known about the space of *local* operators, much less is known about the space of conformal *line* operators. Much like local operators, a line operator does not have to be charged (under a one-form symmetry) to be non-trivial.

Our setup would be a conformal theory in d space-time dimensions with a one-dimensional defect.



$$H = H_{bulk} + H_{imp}$$



$$H = J_0 \vec{T} \cdot \vec{S} + H_{bulk}$$

We are already familiar with many constructions of line defects in $d > 2$:

- Wilson/'t Hooft loops.
- Twist (symmetry) defects in 2+1 dimensions
- SPT defects
- Worldlines of anyons in 2+1 dimensions
- Pinning Field Defects
- $\cdots$

We will touch briefly upon several subjects:

- RG flows on line defects
- Magnetic field defects
- Spin impurities
- Wilson lines

Consider a straight line in a d-dimensional CFT. It can be conformal or non-conformal. A conformal line preserves

$$SL(2, \mathbb{R}) \times SO(d-1)$$

(we assume the line has no transverse spin). A non-conformal line preserves

$$\mathbb{R} \times SO(d-1) \ .$$

It describes a point-like impurity in space at zero temperature, with a critical bulk. At long distances, the impurity becomes critical (and may or may not be non-trivial).

In this setup there are bulk operators, which are the usual ones, and defect operators, which are local operators acting on the line defect. At the defect fixed point (DCFT), operators are classified by their $SL(2,\mathbb{R}) \times SO(d-1)$ quantum numbers. In general, the space of defect operators has nothing to do with the bulk operators.

There is a bulk-defect OPE, where we expand bulk operators in terms of defect operators

$$O(x_\perp, t) \sim \sum a_k x_\perp^{\Delta_{\hat{O}_k} - \Delta_O} \hat{O}_k(t) \ .$$

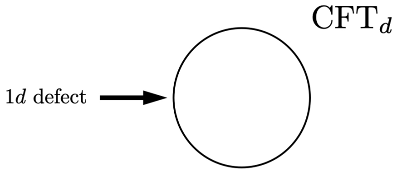
This expansion is useful at short distances from the defect.

The trivial line defect is just the unit line operator. It is completely transparent. The defect operators then coincide with the bulk operators restricted to the line.

An interesting observable for such a line is its "defect entropy." We make the line into a Euclidean circle and compute the expectation value of the circle.

$$s = \left(1 - R\frac{\partial}{\partial R}\right)\log\langle L\rangle \equiv \log g \ .$$

The differential operator $\left(1 - R\frac{\partial}{\partial R}\right)$ cancels a scheme dependent linear in R term in $\log\langle L\rangle$ (mass renormalization of the impurity).



Therefore s is a scheme-independent intrinsic observable. At the fixed point of the line defect the value of s is also called $\log g$.

It is hard to directly measure $\log g$.

For line defects in a 2+1 dimensional topological theory, g is called the "quantum dimension." Unlike line defects in topological theories, it is not necessarily true that $g \geq 1$ for general conformal defects, as we will see.

In the event that relevant defect operators exist ($\Delta_{\hat{O}} < 1$), we can deform by $M_0^{1-\Delta_{\hat{O}}} \int dt \hat{O}(t)$. M_0 becomes the physical scale of the flow.

The defect entropy $s = \left(1 - R\frac{\partial}{\partial R}\right) \log\langle L\rangle$ becomes a nontrivial function

$$s = s(M_0 R)$$

We have

$$s(M_0 R) \to \begin{cases} \log g_{UV} \text{ as } R \to 0 \\ \log g_{IR} \text{ as } R \to \infty \end{cases}$$

The renormalization group flow is implemented by changing the radius of the circle of the defect worldline.

One basic result is the following identity (with $\hat{T}_D$ the energy localized at the defect)

$$R\frac{\partial s}{\partial R} = -R^2 \int d\phi_1 d\phi_2 \langle \hat{T}_D(\phi_1)\hat{T}_D(\phi_2)\rangle_c \, (1-\cos(\phi_1-\phi_2)) \ .$$

Since $\langle \hat{T}_D(\phi_1)\hat{T}_D(\phi_2)\rangle_c \geq 0$ at separated points and since $(1-\cos(\phi_1-\phi_2)) \geq 0$ we have that

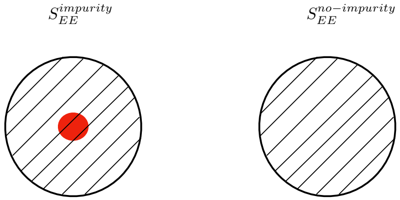
$$R\frac{\partial s}{\partial R} \leq 0 \ ,$$

and therefore also $g_{UV} \geq g_{IR}$.

This generalizes the familiar results of [Affleck-Ludwig, Friedan-Konechny] to line defects/impurities in higher dimensions.

Note that it follows that g is independent of exactly marginal defect couplings. We will soon see an example of that.

This higher-dimensional g theorem should be connected somehow to entanglement entropy. Our g function is not the same as the additional entanglement entropy of the vacuum with the impurity. See [Casini – Salazar-Landea – Torroba] for the $d = 2$ case, where these two quantities coincide.

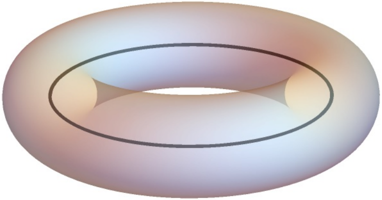


The main idea of the derivation is to promote the massive perturbation to a time dependent one

$$M_0^{1-\Delta}\int dt\hat{O}(t) \longrightarrow M_0^{1-\Delta}\int dte^{(1-\Delta)\Phi(t)}\hat{O}(t)\ .$$

$\Phi(t)$ is usually called the dilaton though it has nothing to do with the string theory dilaton. $\Phi(t)$ is a classical field (background field).

We then fuse appropriate bulk topological surfaces $\int d\Sigma^\nu \xi^\mu T_{\mu\nu}$ with the line defect and obtain new, but equivalent defects. This leads to infinitely many new identities of which one is the gradient formula.



The defect entropy inequality $s_{UV} \geq s_{IR}$ provides a general non-perturbative constraint on RG flows on line defects. Another non-perturbative constraint arises from one-form symmetry. The statement is that if a line defect is charged under an end-able one-form symmetry then it cannot flow to a trivial or topological line in the infrared.

The constrain from defect entropy applies in relativistic systems but the constraint from one-form symmetry should be more general.

Three ways to construct line defects:

- Start from the trivial line defect ($g_{UV} = 1$). If there is a bulk operator with $\Delta < 1$ then we can integrate it on the line:

$$S = S_{bulk} + M_0^{1-\Delta_O} \int dt O(t)$$

This is called a “pinning field” defect or an external field defect. Physically this is an impurity created by applying external fields in a manner localized in space, independent of time. Example: applying a magnetic field in a critical magnet, but only at a few lattice sites.

- Start from a QM model on the line with d states. Couple some operators acting on these states to the bulk operators:

$$S = S_{bulk} + M_0^{1-\Delta_O-\Delta_T} \int dt T_{QM}(t) O(t)$$

Example: a qubit coupled to some bulk CFT.

- Start from a QM model on the line with d states in a representation of G and consider a bulk CFT with G gauge symmetry. Couple the two systems by gauging the symmetry G in QM.

$$S_{bulk} + \int dt J^a_{QM}(t) A^a(t)$$

In this case there is no free defect coupling constant since the coefficient of $J^a_{QM}(t)A^a(t)$ is fixed. Wilson lines are constructed in this manner.

The most abstract point of view on line defects which would include all the preceding examples (and include new ones) is that line defects are defined by prescribing the behavior of all bulk fields (operators) near the line defect.

For the remaining part of this talk we will discuss some examples and quote results about their properties.

The Pinning Field in $O(N)$ Models

Consider the $O(N)$ model in $2 \leq d \leq 4$ with an external localized magnetic field:

$$S = S_{O(N)} + h \int dt \phi_1(t)$$

where $S_{O(N)}$ stands for the critical bulk $O(N)$ model in d space-time dimensions and ϕ_1 is the first component of $\vec{\phi}$.

This is a relevant perturbation in $2 \leq d \leq 4$. By the g theorem, this must flow to a nontrivial ($g < 1$) infrared DCFT in any $2 \leq d < 4$. Hence, the external magnetic field cannot be "screened" and cannot disappear.

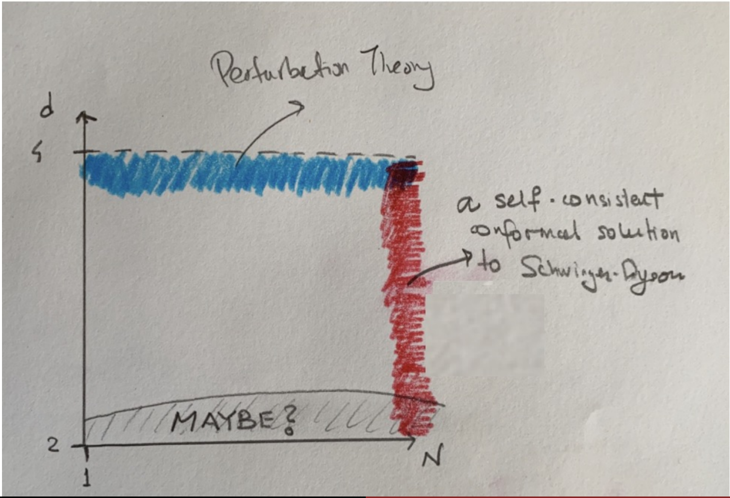
The Pinning Field in $O(N)$ Models

This is physically realizable as a localized magnetic field at zero temperature at a bulk quantum critical point and it can be tested in quantum critical points and also in Monte Carlo [....Assaad, Herbut; Parisen Toldin, Assaad, Wessel....]

This infrared DCFT will have no nontrivial relevant operators whatsoever.

The Pinning Field in $O(N)$ Models

In principle, understanding the infrared is a strongly coupled problem.



The Pinning Field in $O(N)$ Models

Many concrete predictions can be made in the epsilon expansion. For instance,

$$\log g_{IR} = -\frac{N+8}{16}\epsilon + \cdots \ .$$

$$\Delta(\hat{\phi}_1) = 1 + \epsilon - \epsilon^2 \frac{3N^2 + 49N + 194}{2(N+8)^2} + \cdots \ ,$$

Note: the infrared value of h is NOT small in the ϵ expansion. However, the ϵ expansion makes sense since the bulk is weakly coupled.

The Pinning Field in $O(N)$ Models

The line operator

$$e^{-h \int dt \phi_1(t)}$$

has a smooth large N limit which is manifested if we define a 't Hooft coupling $\lambda = h/\sqrt{N}$. The claim is that in the large N limit the coupling λ flows to some $\lambda_* \sim \mathcal{O}(1)$ in the infrared.

There is a saddle point that determines the DCFT observables, e.g. the g function:

$$g = e^{-NS_{classical}}$$

For more analytic work from recently see [Rodriguez Gomez, Popov, Wang, Grau, Lauria, Liendo]

The Pinning Field in $O(N)$ Models

It should be in principle possible to solve for the whole RG flow of the 't Hooft coupling λ in the large N limit. Here, we will try to go after the infrared directly by looking for a self-consistent conformal line defect solution.

The Pinning Field in $O(N)$ Models

At the conformal point we can map the problem by a Weyl transformation to $\mathrm{AdS}_2 \times S^{d-2}$ with standard boundary conditions in AdS_2. So the problem reduces to the $O(N)$ model on this space with standard (source) boundary conditions familiar from the AdS/CFT correspondence.

The Pinning Field in $O(N)$ Models

After the transformation to $\mathrm{AdS}_2 \times S^{d-2}$, a Hubbard Stratonovich transformation, and a guess for self-consistent conformal boundary conditions, the problem turns out to reduce to the following Schwinger-Dyson type action

$$S = \frac{N}{2}\left[Tr\log(-\Box + s + \frac{d-2}{4(d-1)}R) - JG_{\partial\partial}[s]J\right] .$$

$G_{\partial\partial}$ is the boundary-to-boundary propagator, J is a standard non-normalizable source, and $s(x)$ is a bulk field.

The Pinning Field in $O(N)$ Models

We can find the critical point of this action analytically, and we can analyze fluctuations about this critical point using recent progress on loop diagrams in AdS_2 [see especially Carmi, Di Pietro, Komatsu].
This allows us to obtain many exact predictions for the spectrum of the defect in the large N limit and the dimensions of defect operators.

The Pinning Field in $O(N)$ Models

Here is a sample of results at large N and $d = 3$:

$$\log g = -0.1536N + \mathcal{O}(N^0)$$

$$\Delta(\hat{\phi}_1) = 1.542 + \mathcal{O}(N^{-1})$$

Note: g is exponentially small at large N.

The Pinning Field in $O(N)$ Models

Combining all the data we amassed suggests that in $d = 3$ one should expect $\Delta(\hat{\phi}_1) \sim 1.5$ with rather weak N dependence. This is the first nontrivial $O(N-1)$ singlet operator. It is roughly consistent with Monte Carlo simulations and this along with several other predictions should be testable.

Spin Impurities

Another important line defect especially for the $O(3)$ model comes about as follows: We begin with QM with a spin s representation of $SO(3)$, so just a QM system with Hilbert space of dimension $2s+1$. We then couple the $SO(3)$ generators S^a to the interacting bulk:

$$S = S_{O(3)} - \gamma \int dt S^a(t)\phi^a(t) \ .$$

This is the line operator

$$Tr_s P e^{\gamma \int dt S^a \phi^a} \ .$$

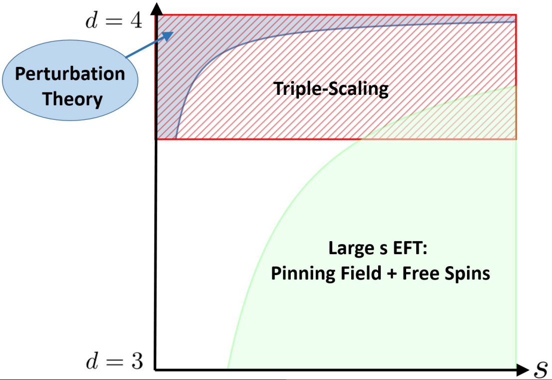
It is similar to Wilson lines but it is just a line defect in a magnet.

Spin Impurities

Physically this is realizable by putting an external atom of spin s in a quantum anti-ferromagnet at the critical point. While there is a lot to say about this problem here I will mention one general result.

Spin Impurities

At $s \to \infty$ the spin impurity breaks up into two almost-decoupled DCFTs, one being the pinning field DCFT we studied above and the other being just the theory of a free spin s. There is a systematic $1/s$ expansion. This statement leads to many predictions that can be checked in the future.



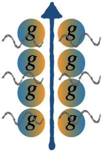
Spin Impurities

For additional recent work on this subject see [Beccaria-Giombi-Tseytlin, Rodriguez Gomez-Russo, Nahum, Weber-Vojta, Grau]

Now we will consider gauge theories in 3 or 4 space-time dimensions. An important line operator is the Wilson line:

$$W_R = Tr_R \left[P \ exp \left(i \int_\gamma dx^\mu A^a_\mu T^a_R \right) \right]$$

This describes the insertion of a probe particle moving on the worldline γ.



In confining theories Wilson lines serve as order parameters for confinement. Wilson lines are order parameters only if there are no dynamical fields with the same quantum numbers.

But today we are interested in de-confining theories (conformal theories). The Wilson lines are potentially interesting whether or not there are dynamical particles with the same quantum numbers.

Massless scalar QED_4 is not a conformal theory but the beta function and generated mass are so small that it would be an excellent playground to explain almost all the main points of this talk.

$$S = \int d^4x \left[-\frac{1}{4e^2} F^2 + |D\phi|^2 - \lambda |\phi|^4 \right] + q \int dt A_t \ .$$

$D\phi = \partial\phi - iA\phi$.

This leads to the classical saddle point

$$A_t = \frac{e^2 q}{4\pi r} \ , \quad \phi = 0 \ .$$

To justify the saddle point treatment mathematically and to neglect the beta function of QED_4 we can take $e^2 q \sim const$, $e^2 \sim \lambda \to 0$.

An important computation to do is the measure the dimension of the defect operator $\Delta(\hat{\phi}^\dagger\hat{\phi}) =$? For small q it has to be close to the bulk dimension so we expect

$$\Delta(\hat{\phi}^\dagger\hat{\phi}) = 2 + \#e^4q^2 + \#(e^4q^2)^2 + \cdots$$

The answer can be found exactly by studying the Green's function around the saddle point. One finds:

$$\Delta(\hat{\phi}^\dagger\hat{\phi}) = 1 + \sqrt{1 - \frac{e^4q^2}{4\pi^2}} \ .$$

Clearly for $\frac{e^2|q|}{2\pi} > 1$ there is some sickness.

Digression

The square root may ring a bell. To understand its origin one can think of Wilson lines in radial quantization [e.g. Kapustin 05]:

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega_2^2 = r^2 \left(\frac{dt^2 - dr^2}{r^2} - d\Omega_2^2 \right) .$$

Therefore Wilson lines can be studied as boundary conditions for $AdS_2 \times S^2$. The Coulomb field in these coordinates stays $A_t = \frac{e^2 q}{4\pi r}$. From the $AdS_2 \times S^2$ point of view, this is a constant electric field since $F^2 = const$.

The field Φ moves in a constant electric field in $AdS_2 \times S^2$. The propagation in AdS_2 in a constant electric field is quite similar mathematically to a shift in the AdS_2 mass. This is why we have

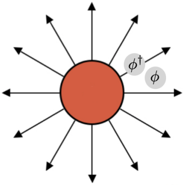
$$\frac{\Delta}{2} \left(\frac{\Delta}{2} - 1 \right) = -\frac{e^4 q^2}{16\pi^2} ,$$

which is like the effective AdS_2 mass due to the electric field.

Continuation of Digression

The instability due to an electric field in AdS_2 is familiar from the work of [Gubser; Hartnoll-Herzog-Horowitz] on holographic super-conductors, but the fate of the instability is completely different here.

Intuitively, for $\frac{e^2|q|}{2\pi} > 1$ the electric field is so strong that it leads to pair creation and the saddle point is destablized.



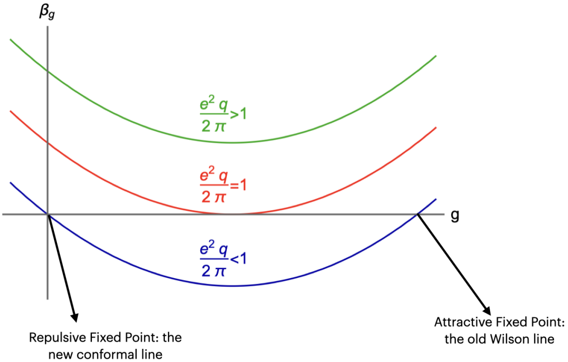
$$A_0 = \frac{e^2 q}{4\pi r}$$

As we approach $\frac{e^2|q|}{2\pi} = 1$ from below, the defect operator $\hat{\phi}^\dagger \hat{\phi}$ becomes closer and closer to being marginal. So we must consider the more general Wilson line

$$W_q^g = P \; exp\left(i\int_\gamma dt \; (q\frac{dx^\mu}{dt}A_\mu - g\hat{\phi}^\dagger\hat{\phi})\right) \; .$$

q cannot be renormalized, being an integer. But g can be renormalized.

The deformation by $\hat{\phi}^\dagger\hat{\phi}$ is analogous to a double-trace deformation in $AdS_2 \times S^2$. The old (new) Wilson line corresponds to standard (alternative) quantization.



Therefore something drastic happens $\frac{e^2|q|}{2\pi} = 1$ because there are two fixed points that annihilate and past that point, for $\frac{e^2|q|}{2\pi} > 1$, the Wilson line must define a new type of defect in the infrared.

Additionally, if we fine tune to the new fixed point, a deformation by $\hat{\phi}^\dagger\hat{\phi}$ with a positive coefficient leads to the old Wilson line while the deformation by $\hat{\phi}^\dagger\hat{\phi}$ with a negative coefficient, leads to some long flow to a new infrared DCFT which we have to understand.

It is not too hard to guess what happens when $g \to -\infty$; this drives the scalar field to condense on the defect which subsequently triggers a condensate in the bulk.

The annihilation of the two fixed points we have seen above is reminiscent of how QCD exits the conformal phase and also of the BKT transition. It leads to Miransky scaling [Kaplan-Lee-Son-Stephanov].

It is then not surprising that the scalar cloud that forms around the Wilson loop is actually exponentially large in units of the cutoff (e.g. the impurity radius). There is therefore dimensional transmutation!

Also we will see that the cloud completely screens the Wilson line and the infrared DCFT is entirely trivial. That is, at distances much bigger than the scale of the scalar cloud, all the bulk one point functions vanish and the space of defect operators is trivial.

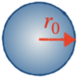
The naive guess for the extent of the cloud should be

$$\Lambda = \mu_0 e^{\frac{-2\pi}{\sqrt{\frac{e^4 q^2}{4\pi^2}-1}}}$$

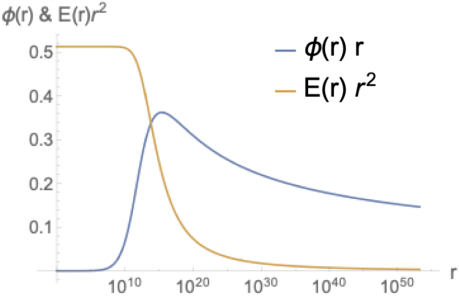
This is the spread of the wave functions of the bosons which are tachyonic around the original saddle point.

In reality we find a larger extent due to the non-linearities of the condensate.

$$F_{0r}|_{r=r_0} = \frac{e^2 q}{4\pi r^2}$$



$$|\phi|^2|_{r=r_0} = 0$$



$$\frac{e^2 q}{2\pi} = 1.02$$

If there is a one-form symmetry, such as in scalar QED with a charge-2 scalar field, the screening cannot be complete as there has to be a left-over nontrivial DCFT to transform under the one-form symmetry. Naturally one should guess that the screening reduces the charge all the way to 1 rather than 0 for lines with odd q.

It is not entirely obvious that this is what happens or how to prove it.

A few words about the new conformal Wilson line:

$$\Delta_{new}(\hat{\phi}^\dagger\hat{\phi}) = 1 - \sqrt{1 - \frac{e^4 q^2}{4\pi^2}}\ .$$

This must be positive which is satisfied for all $0 < \frac{e^2|q|}{2\pi} < 1$. But that does not mean that the new conformal line indeed exists for this whole range. For instance, the operator $|\hat{\phi}|^4$ becomes marginal as

$$\frac{e^2 q}{2\pi} \to \frac{\sqrt{3}}{2}$$

from above. Then we need to re-analyze the space of lines.

To conclude:

- Scalar QED_4 has two conformal lines which annihilate for $e^2|q| = 2\pi$. (A similar story happens for some spin-boson impurities in magnets, as we have shown recently. See also [Beccaria-Giombi-Tseytlin, Nahum, Weber-Vojta].)
- The new conformal line can be understood as alternative quantization of the boson field Φ near the probe charge.
- For $e^2|q| > 2\pi$ the theory develops a low-energy scale (dimensional transmutation) and flows to a trivial line defect. There is interesting physics at that new scale, where we have a boson cloud.
- The full phase diagram of Wilson lines is not yet understood as at some sufficiently small $e^2|q|$ there are quartics in the new Wilson line.

For massless fermionic QED_4 the story is similar

$$S = \int d^4x \left[-\frac{1}{4e^2} F^2 + i\bar{\Psi}\gamma^\mu D_\mu \Psi \right] + q \int dt A_0 \ .$$

Now one finds an instability at $\frac{e^2|q|}{4\pi} = 1$.

For $\frac{e^2|q|}{4\pi} < 1$ there are two conformal Wilson lines, given by two fixed points for the coupling g in (schematic)

$$W_q^g = P \ exp \left(i \int_\gamma dt \ (q \frac{dx^\mu}{dt} A_\mu - g \hat{\bar{\Psi}} \gamma^5 \hat{\Psi}) \right) \ .$$

The fact that something dramatic happens at $\frac{e^2|q|}{4\pi} > 1$ is not limited to massless fermionic QED_4. A similar instability exists for massive QED_4. The Wilson line is a nucleus of charge q. Since $e^2/4\pi = 1/137$, we see that

- For nuclei with $q > 137$ there is an instability to electron-positron production in the vacuum.
- Of course there are practical difficulties to measure it, but it is observable in graphene where the inverse fine structure constant is much smaller [Pereira-Nilsson-Castro Neto; Shytov-Katsnelson-Levitov; Wang-Wong-Shytov et al.].

For a range of $119 < q < 137$ there are two conformal boundary conditions for the electron wave function at the nucleus and there is an RG flow between them. These are our two line defects which annihilate at $q = 137$. This flow could be observable in the phase shift of electron scattering. A similar phenomenon should exist in graphene.

Note: perhaps various fermion quartics become marginal in the UV fixed point in the range $119 < q < 137$.

Just as an elementary remark: to see why $q = 137$ emerges for hydrogen-like atoms [Pomeranchuk-Smorodinsky], consider the energy of the ground state orbital

$$E_{Rydberg} = -\frac{1}{2}\alpha^2 mq^2$$

We can compare it to the electron rest mass m and find that the ground state of the hydrogen atom facilitates pair creation far from the nucleus if $-\frac{1}{2}\alpha^2 mq^2 = 2m$ i.e. $\alpha q = 2$. This is almost the correct answer. If one includes corrections from relativity in the hydrogen-like atom, one find $\alpha q = 1$.

Another interesting problem is to understand what happens for $\frac{e^2|q|}{4\pi} > 1$ in fermionic QED_4.

For bosons we saw a cloud forming with an exponentially large scale and the infrared was a trivial DCFT.

For fermions, instead, we only have a rigorous solution for $\frac{e^2|q|}{4\pi} = 1 + \epsilon$ for $0 < \epsilon \ll 1$.

Again a charge cloud is forming, namely a bulk condensate

$$\langle \bar{\Psi}\gamma_0\Psi(x_\perp)\rangle \neq 0$$

The size of the cloud is again exponentially large

$$\log(\Lambda/\mu_0) \sim -\frac{1}{e^2}$$

but the crucial difference from the boson case is that it only screens the defect down to $\frac{e^2|q|}{4\pi} = 1$.

So far we have covered bosonic and fermionic QED_4. However, the story is similar in any conformal gauge theory in 4d.

The case of $\mathcal{N} = 4$ SYM is quite interesting to contemplate. Let us pick the gauge group to be $SU(2)$ and denote the spin of the representation of the Wilson line by s

$$W_s = Tr_s \left[P \ exp \left(i \int_\gamma dt A_0^a T_s^a \right) \right]$$

Then in the limit of

$$g_{YM}^2 s \sim \text{fixed} \ , \quad g_{YM}^2 \to 0$$

the results are essentially identical to what we found above. Wilson lines cease to exist as distinct DCFTs for $g_{YM}^2 s \sim 1$.

By S-duality it implies that there are many distinct 't Hooft lines at strong coupling. But it *suggests* that perhaps they disappear at weak coupling! Are there 't Hooft lines at weak coupling?! It turns out that in $SU(2)$ gauge theory there are none!!

Let us now make some comments about conformal gauge theories in 3d. Other than purely theoretical motivation, such theories appear as quantum phase transitions in the lab and one can study charged impurities.

One interesting setup is a 3d multi-critical boson charged under a 4d gauge field

$$S = \int_{z\geq 0} dz d^3x \frac{1}{4e^2} F^2 + \int_{z=0} d^3x \left(|D_A \Phi|^2 + \rho(|\Phi|^2)^3 \right) \qquad (1)$$

There is a fixed point $\rho_* = \frac{8}{15} e^2 + O(e^4)$ [Di Pietro-Gaiotto-Lauria-Wu].

The electric potential due to a Wilson line at z_0 is

$$A_t = \frac{e^2 q}{4\pi} \left(\frac{1}{\sqrt{r^2 + (z - z_0)^2}} + \frac{1}{\sqrt{r^2 + (z + z_0)^2}} \right) .$$

One finds that, regardless of e, already for small values of q the line leads to an instability! So all charged impurities should develop a scalar cloud that screens the charge.

The cloud will develop only on the three-dimensional boundary, where Φ is allowed to condense.

Some obvious questions we did not discuss

- Chern-Simons Matter theories, $U(1)_0 + N_f \Psi$ – here we find that lines exist up to charge $\sim N_f/2$.
- A large N and holographic understanding?
- Which Wilson lines exist beyond weak coupling?
- The effects of the quartic $|\hat{\Phi}|^4$ and possible fermion quartics.
- $\cdots$

Thank You!