# Uncovering conformal symmetry of 3D Ising: a gift from quantum fuzziness

Yin-Chen He
(何寅琛)
Perimeter Institute

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PERIMETER INSTITUTE

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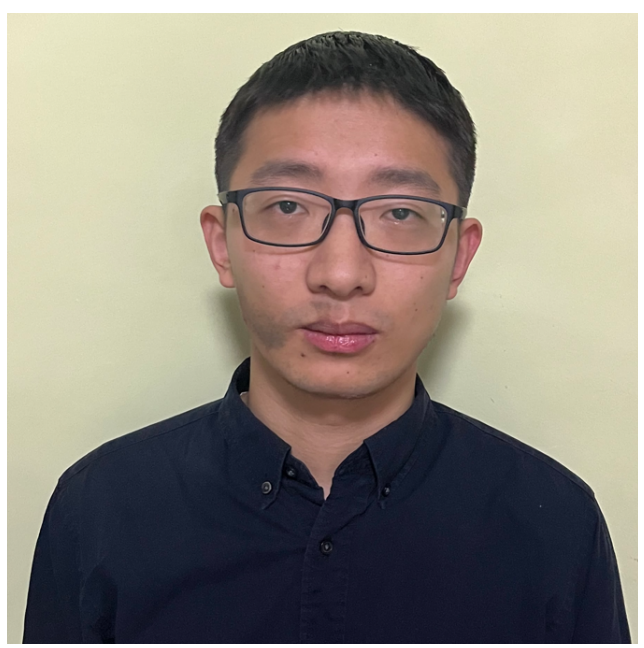

Wei Zhu Chao Han
(Westlake University)

Emilie Huffman
(Perimeter)

Johannes Hofmann
(Weizmann Inst.)

# Overview of main results

- We numerically simulated 3D Ising transition on the sphere S^2 x R.
- We observed almost perfect state-operator correspondence on an incredibly small system size, i.e. 16 spins. (In Sandvik's talk 250,000 spins were simulated.)
- We identified 13 parity even primaries and 2 parity odd primaries.

| | CB | 16 spins | Error |
|---|---|---|---|
| $\sigma$ | 0.518 | 0.524 | 1.2% |
| $\sigma'$ | 5.291 | 5.303 | 0.2% |
| $\sigma_{\mu_1\mu_2}$ | 4.180 | 4.214 | 0.8% |
| $\sigma'_{\mu_1\mu_2}$ | 6.987 | 7.048 | 0.9% |
| $\sigma_{\mu_1\mu_2\mu_3}$ | 4.638 | 4.609 | 0.6% |
| $\sigma_{\mu_1\mu_2\mu_3\mu_4}$ | 6.113 | 6.069 | 0.7% |
| $\sigma^{P-}$ | NA | 11.19 | – |

Bootstrap data from Simmons-Duffin, 2017

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| $\epsilon$ | 1.413 | 1.414 | 0.07% |
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| $\epsilon''$ | 6.896 | 6.908 | 0.2% |
| $T_{\mu\nu}$ | 3 | 3 | – |
| $T'_{\mu\nu}$ | 5.509 | 5.583 | 1.3% |
| $\epsilon_{\mu_1\mu_2\mu_3\mu_4}$ | 5.023 | 5.103 | 1.6% |
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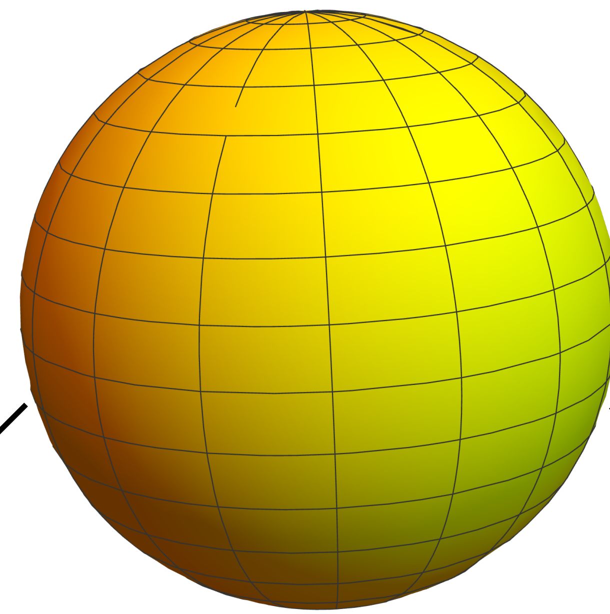
# The magic recipe: make it fuzzy!

Sphere is a curved space.

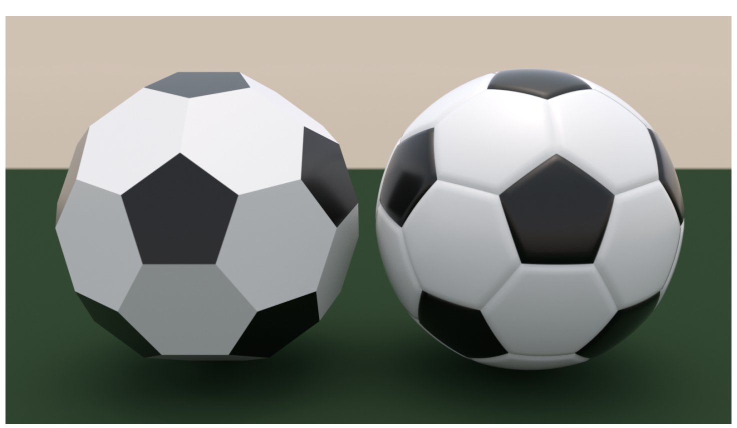


Discretize

Fuzzify

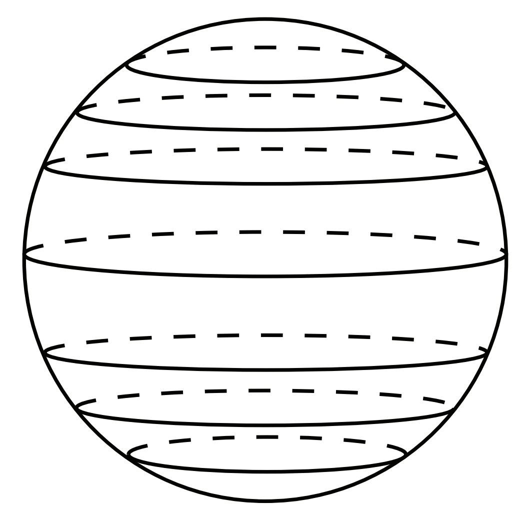
Lowest Landau level projection

Spherical tiling



Spherical rotation is broken badly.

fuzzy (non-commutative) sphere



Spherical rotation is kept exactly.

# Outline

- Introduction
- Spherical Landau level regularization
- Numerical results: state-operator correspondence of 3D Ising CFT
- Outlook and discussion

# Conformal symmetry and lattice models

2D classical Ising model: $H = -J \sum_{ij} \sigma_i \sigma_j$

1+1D quantum Ising model: $H = -\sum_i \sigma_i^z \sigma_{i+1}^z + h \sum_i \sigma_i^x$

Polyakov 1970

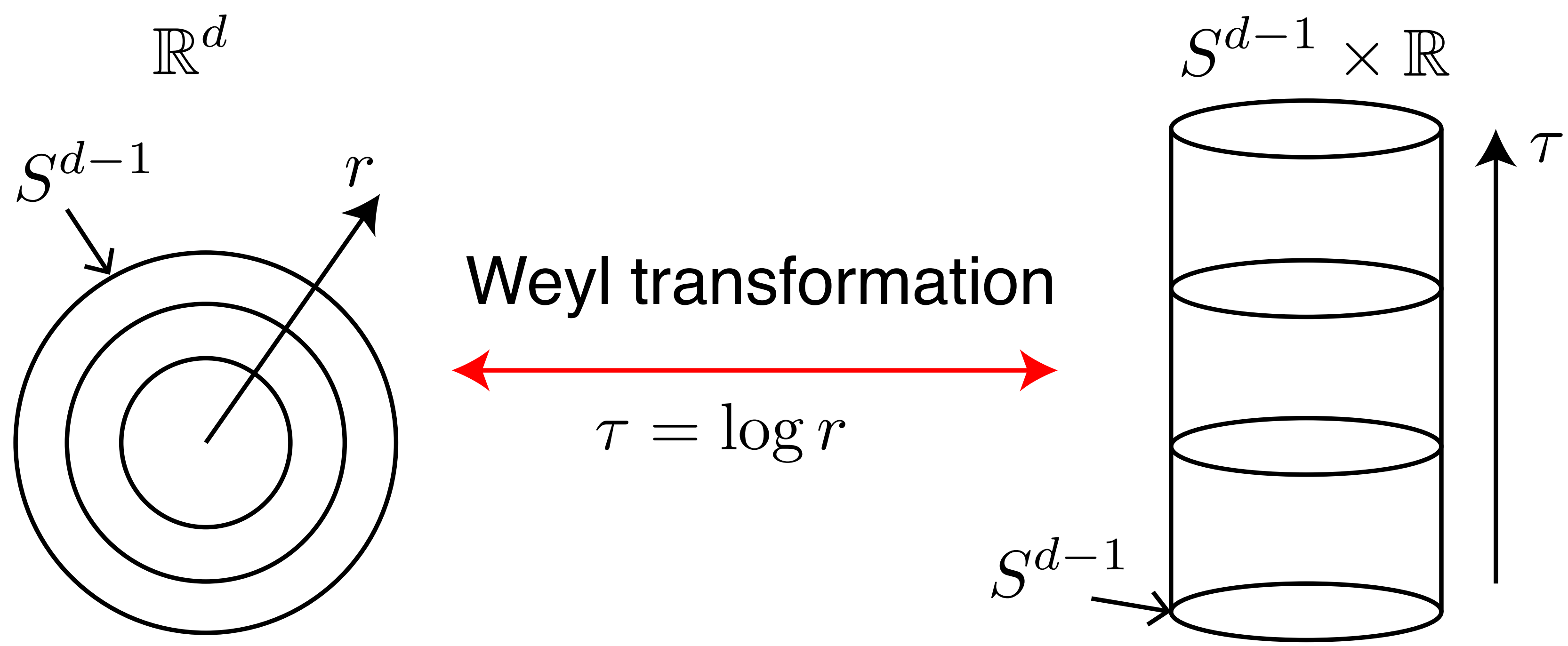
Conformal symmetry

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{\lambda_{ijk}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

3D Ising transition is conjectured to be conformal.

# State-operator correspondence

Radial quantization




Eigenstates of the quantum Hamiltonian defined on $S^{d-1}$ are in one-to-one correspondence with CFT's scaling operators.

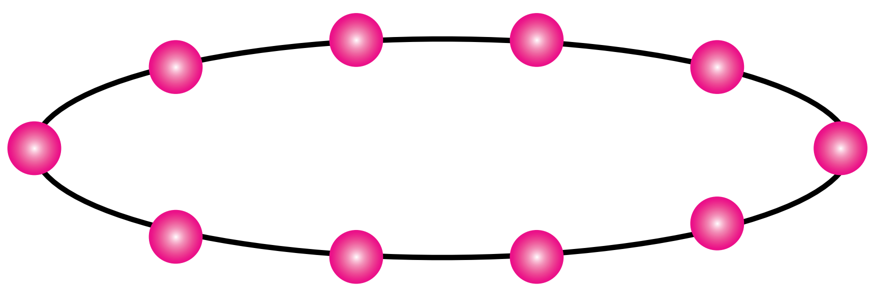
Energy gaps$\sim$scaling dimensions: $\delta E_n = E_n - E_0 = \frac{v}{R}\Delta_n$

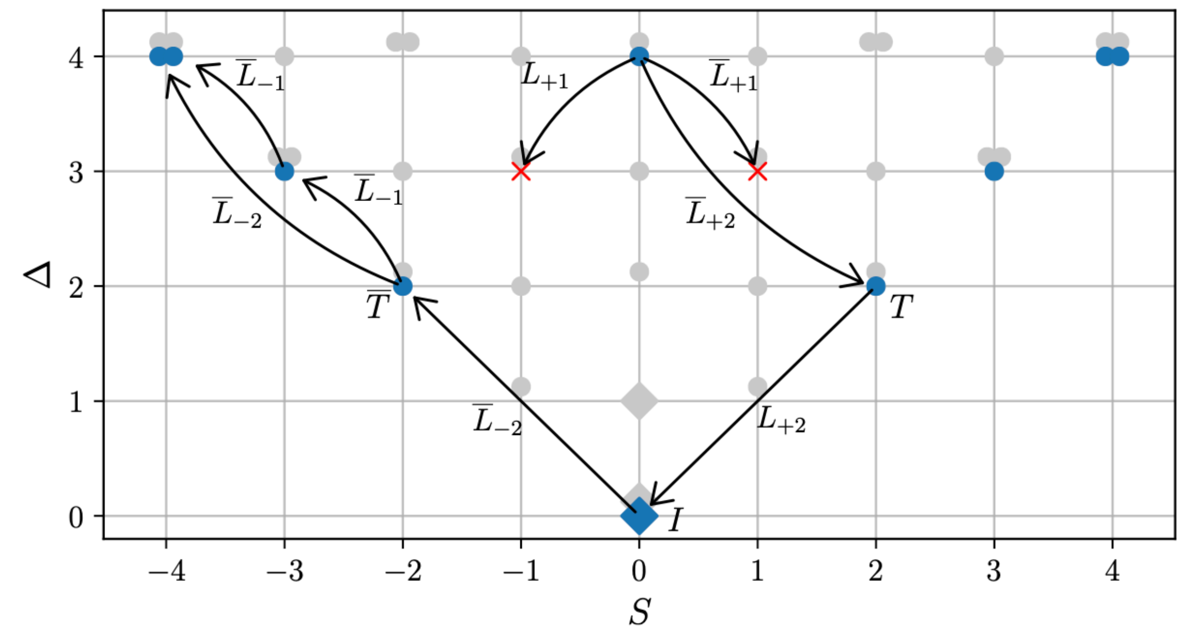
# Radial quantization on a lattice

2D CFT: We can just study a quantum Hamiltonian on a circle.

Most conformal data can be extracted.

Cardy

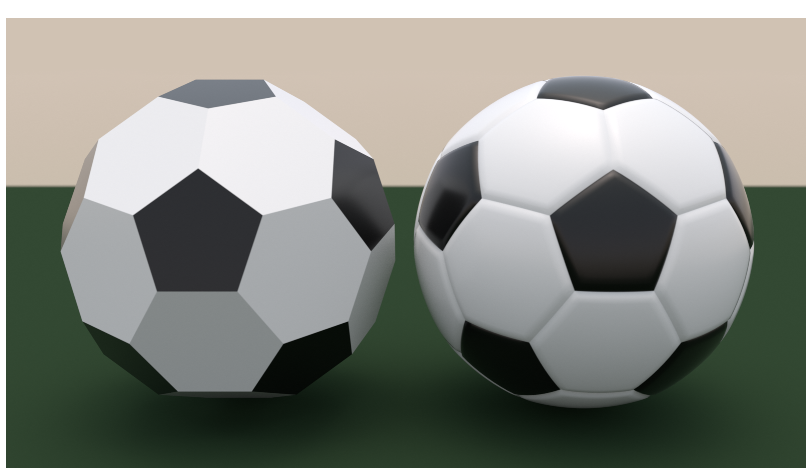


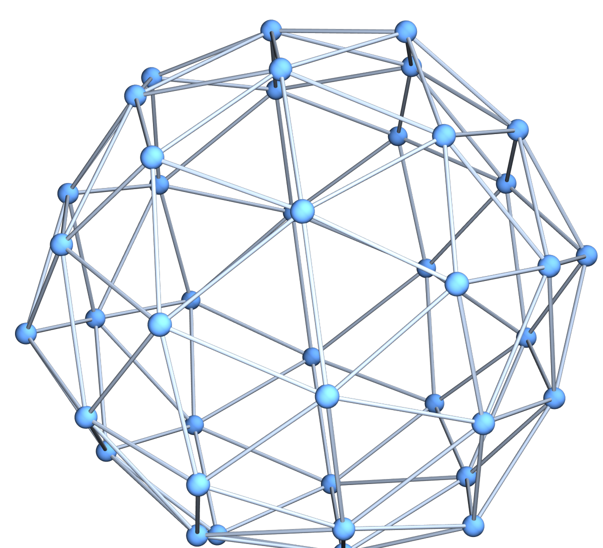


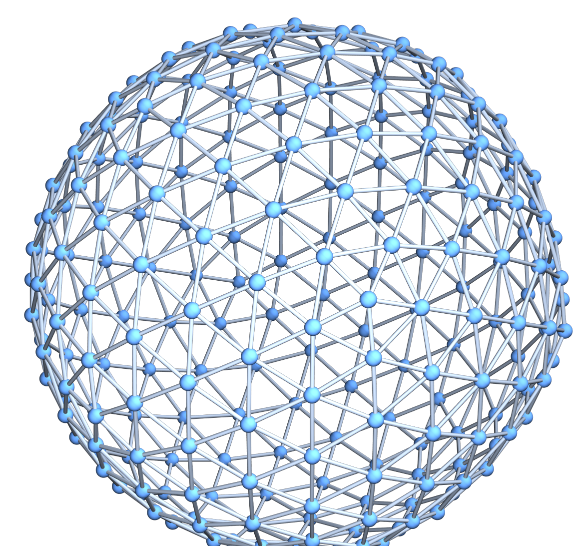

Milsted and Vidal 2017

3D CFT: We need to put a quantum Hamiltonian on a two-sphere.

But a regular lattice won't fit since two-sphere has a curvature…



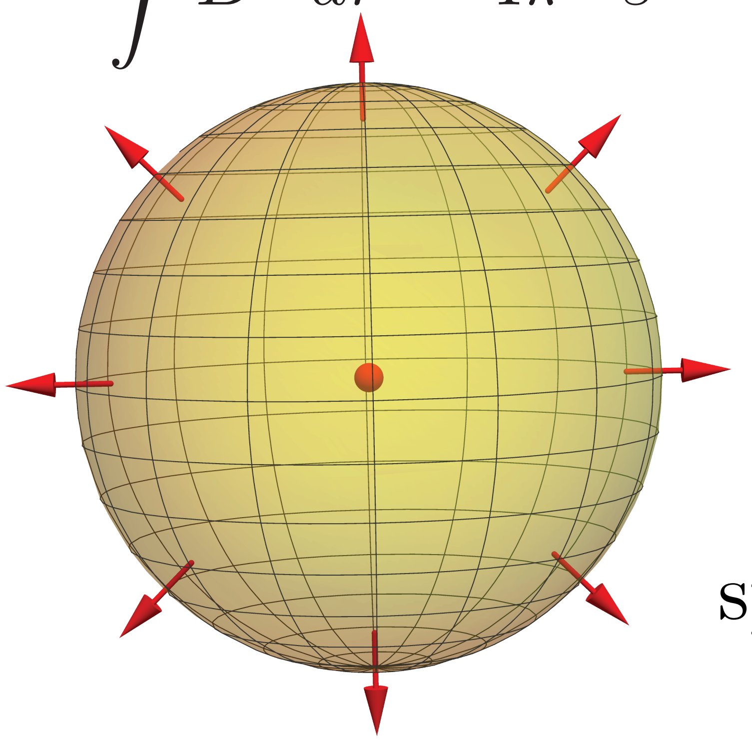




# Spherical Landau levels

Electrons moving under a magnetic monopole.

$$\int \vec{B} \cdot d\vec{r} = 4\pi \cdot s$$



$$H = \frac{1}{2Mr^2}(\partial_\mu + iA_\mu)^2$$

Landau levels: $n = 0, 1, \cdots$

Energy: $E_n = \dfrac{n(n+1) + (2n+1)s}{2Mr^2}$

Degeneracy: $2n + 2s + 1$

spin-$n+s$ irreducible representation of $SO(3)$ rotation

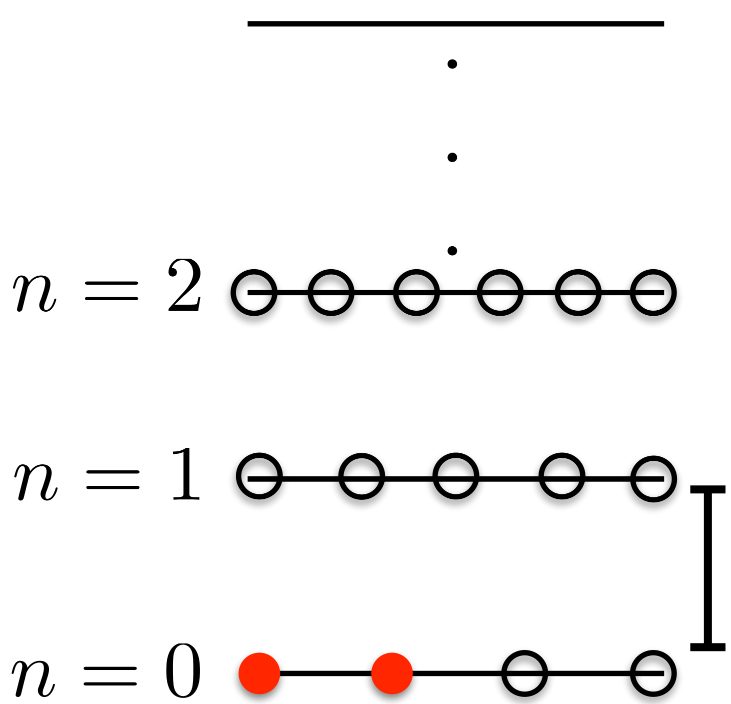
Lowest Landau level wave-function (monopole Harmonics)

$$m = -s, -s+1, \cdots, s$$

$$\Phi_{m,s}(\theta, \varphi) = \sqrt{\frac{(2s+1)!}{4\pi(s+m)!(s-m)!}} e^{im\varphi} \cos^{s+m}\left(\frac{\theta}{2}\right) \sin^{s-m}\left(\frac{\theta}{2}\right)$$

# Lowest Landau level (LLL) projection

Landau levels




$$H = \frac{1}{2Mr^2}(\partial_\mu + iA_\mu)^2 + H_{int}$$

E.g. $H_{int} = U \int d\Omega_a d\Omega_b \, \delta(\Omega_{ab}) n(\theta_a, \varphi_a) n(\theta_b, \varphi_b)$

$\text{gap} \gg H_{int} \longrightarrow$ LLL projection is valid

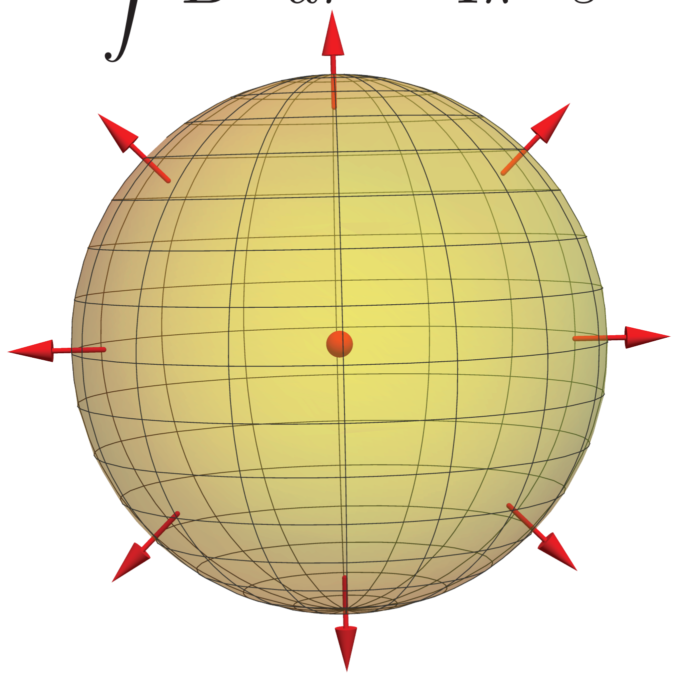
LLL $\Phi_{m,s}(\theta,\varphi) = \sqrt{\frac{(2s+1)!}{4\pi(s+m)!(s-m)!}} e^{im\varphi} \cos^{s+m}\left(\frac{\theta}{2}\right) \sin^{s-m}\left(\frac{\theta}{2}\right)$

Fermion annihilation: $\hat{\psi}(\theta,\varphi) = \sum_{m=-s}^{s} \Phi_m^* \hat{c}_m.$

Fermion density: $n(\theta,\varphi) = \sum_{m_1,m_2} \Phi_{m_1} \Phi_{m_2}^* c_{m_1}^\dagger c_{m_2}.$

# Physics on the LLL

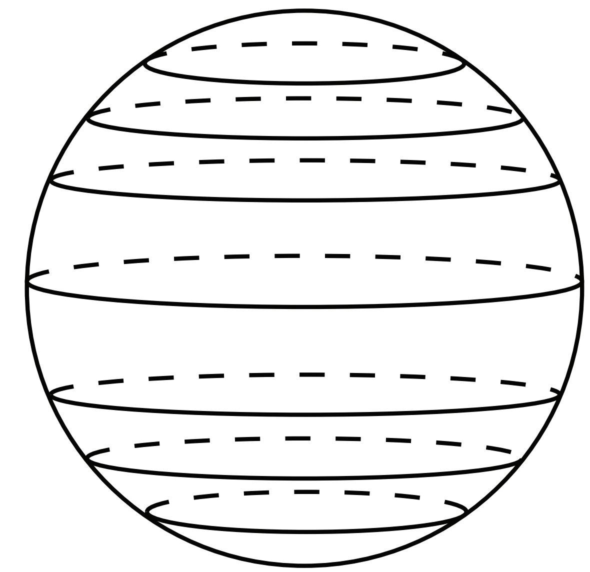
$$\int \vec{B} \cdot d\vec{r} = 4\pi \cdot s$$



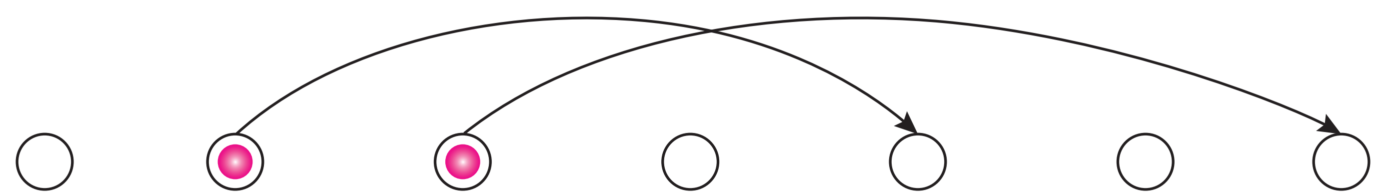
LLL projection

$$\hat{\psi}(\theta, \varphi) = \sum_{m=-s}^{s} \Phi_m^* \hat{c}_m.$$

$2s + 1$ orbitals



$2s+1$-site fermionic chain



$m = -s, -s+1, \cdots, s$ spin-$s$ rep of $SO(3)$

**2-body**

$$\sum_{m=-s}^{s} c_m^\dagger c_m$$

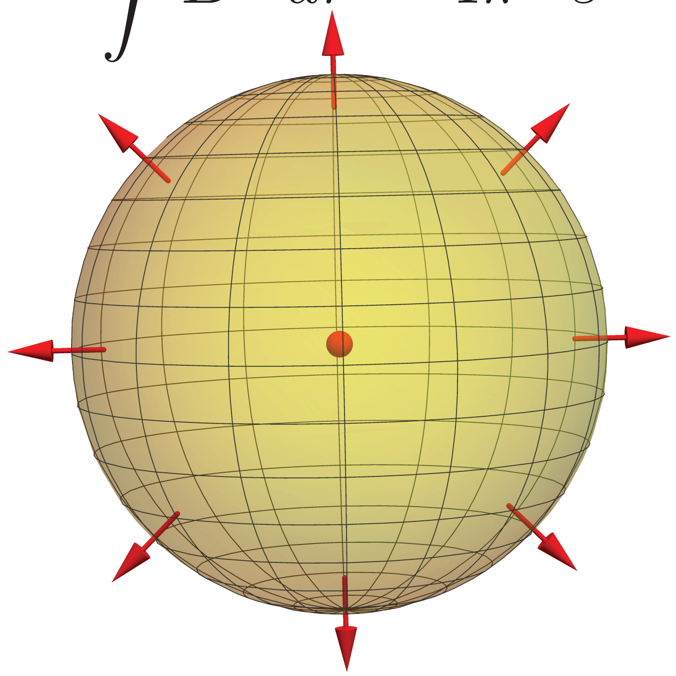
**4-body interaction** Haldane

$$V_l \sum_{m_1, m_2, m_3, m_4} F(m_1, m_2, m_3, m_4, s, l) c_{m_1}^\dagger c_{m_2}^\dagger c_{m_3} c_{m_4}$$

$$F(m_1, m_2, m_3, m_4, s, l) = \delta_{m_1+m_2, m_3+m_4} \begin{pmatrix} s & s & 2s-l \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} \begin{pmatrix} s & s & 2s-l \\ m_3 & m_4 & -m_3-m_4 \end{pmatrix}$$

# Physics on the LLL

$$\int \vec{B}\cdot d\vec{r} = 4\pi\cdot s$$



LLL projection

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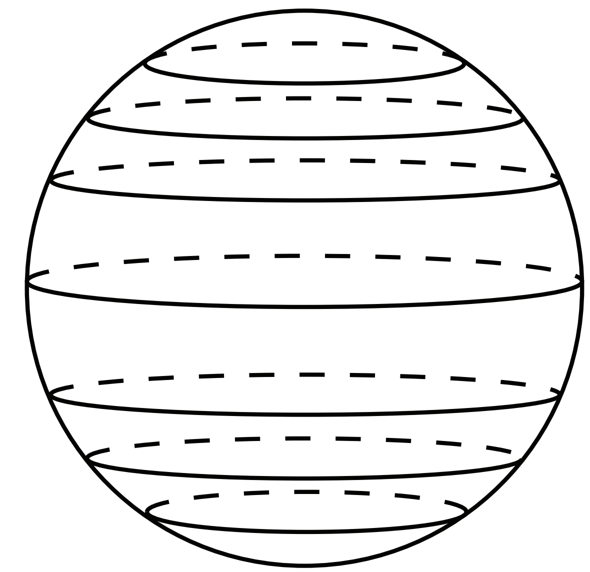
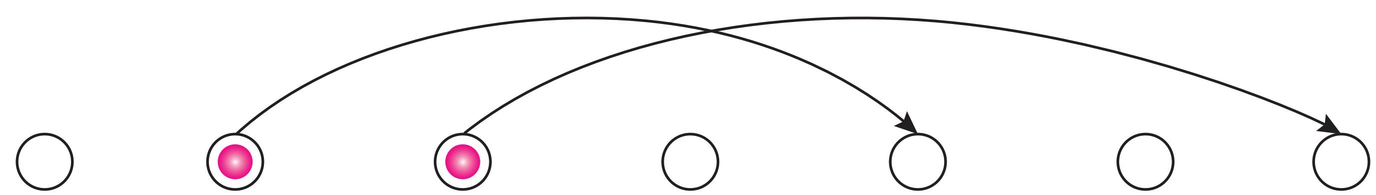


$2s+1$-site fermionic chain



$m = -s, -s+1, \cdots, s$ spin-$s$ rep of $SO(3)$

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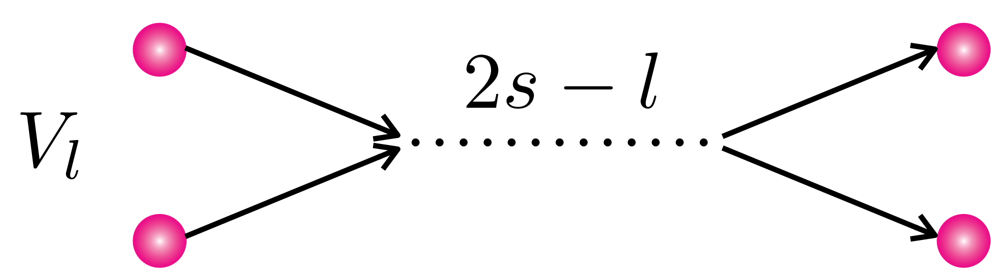


# Model and phase diagram

$$H_{int} = -\int d\Omega_a d\Omega_b\, U(\Omega_a, \Omega_b) n^z(\theta_a, \varphi_a) n^z(\theta_b, \varphi_b) - h \int d\Omega\, n^x(\theta, \varphi),$$

$$n^\alpha(\theta, \varphi) = (\hat{\psi}^\dagger_\uparrow(\theta, \varphi),\, \hat{\psi}^\dagger_\downarrow(\theta, \varphi))\, \sigma^\alpha \begin{pmatrix} \hat{\psi}_\uparrow(\theta, \varphi) \\ \hat{\psi}_\downarrow(\theta, \varphi), \end{pmatrix}$$

$$U(\Omega_a, \Omega_b) = g_1\, \delta(\Omega_{ab}) + g_2\, \nabla^2 \delta(\Omega_{ab})$$

LLL projection

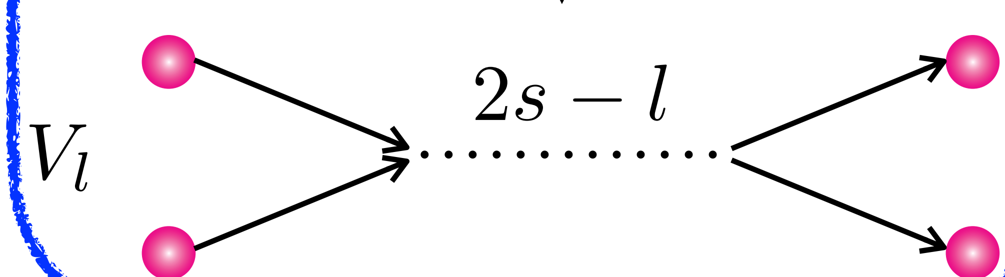
$$\int d\Omega\, n^x(\theta, \varphi)$$

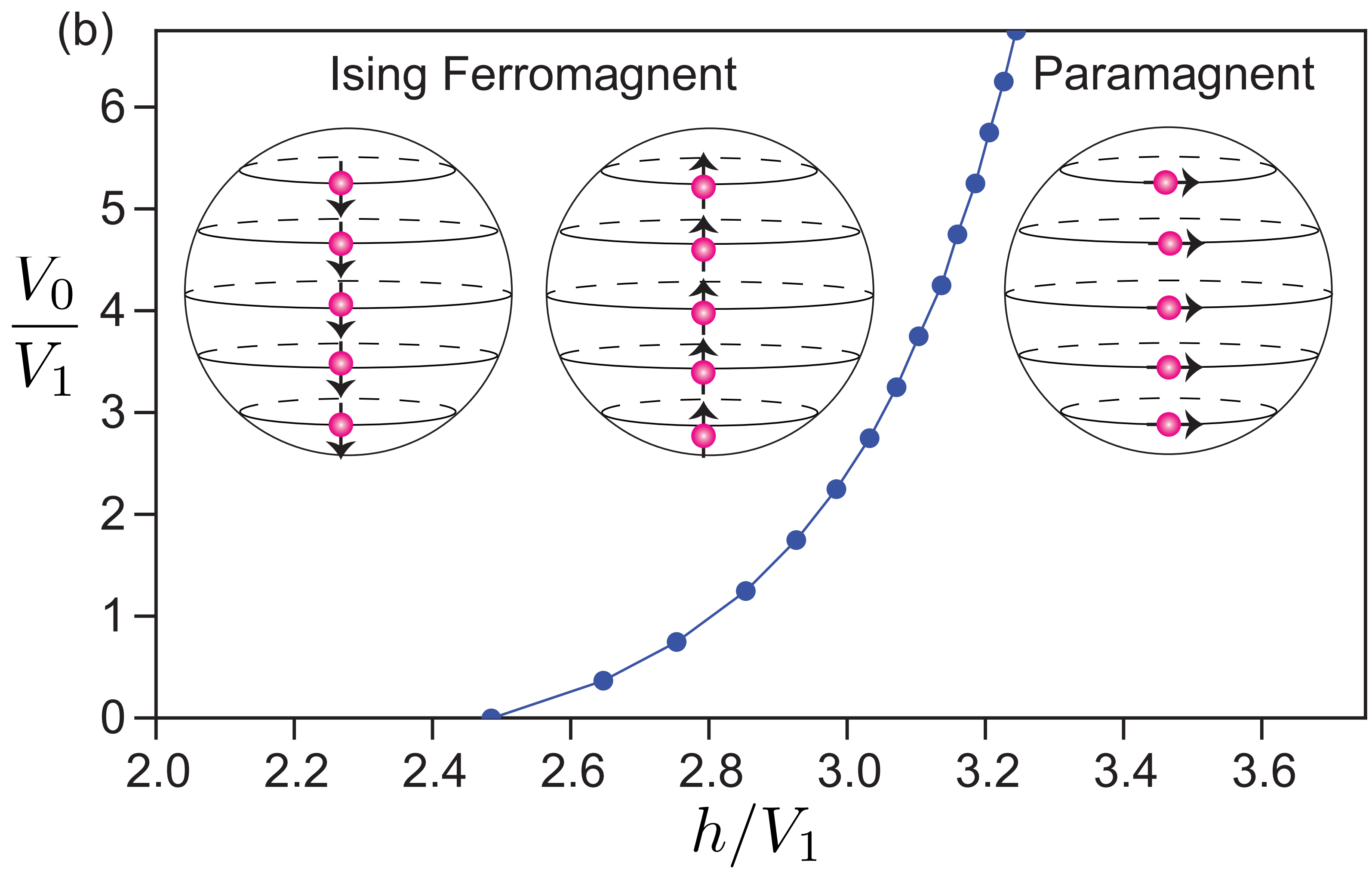
$$\downarrow$$

$$\sum_{m=-s}^{s} \mathbf{c}^\dagger_m \sigma^x \mathbf{c}_m$$

$$\int d\Omega_a d\Omega_b\, U(\Omega_a, \Omega_b) n^z_a n^z_b$$

$$\downarrow$$




Similar model on fuzzy torus has been studied.

Ippoliti, Mong, Assaad, Zaletel (2018)

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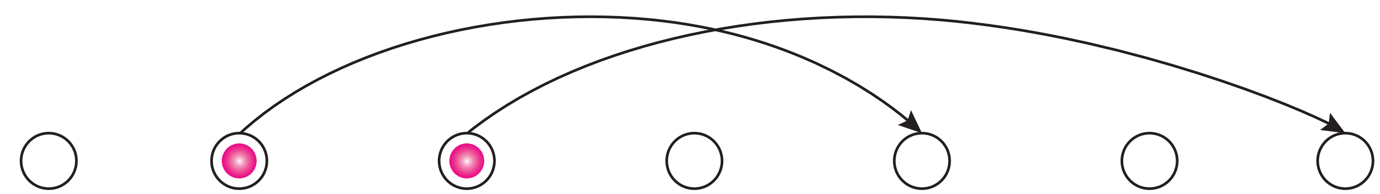
# Symmetries and order parameter

$2s+1$-site "fermionic chain"



Fermions are at half filling: $N = 2s+1$

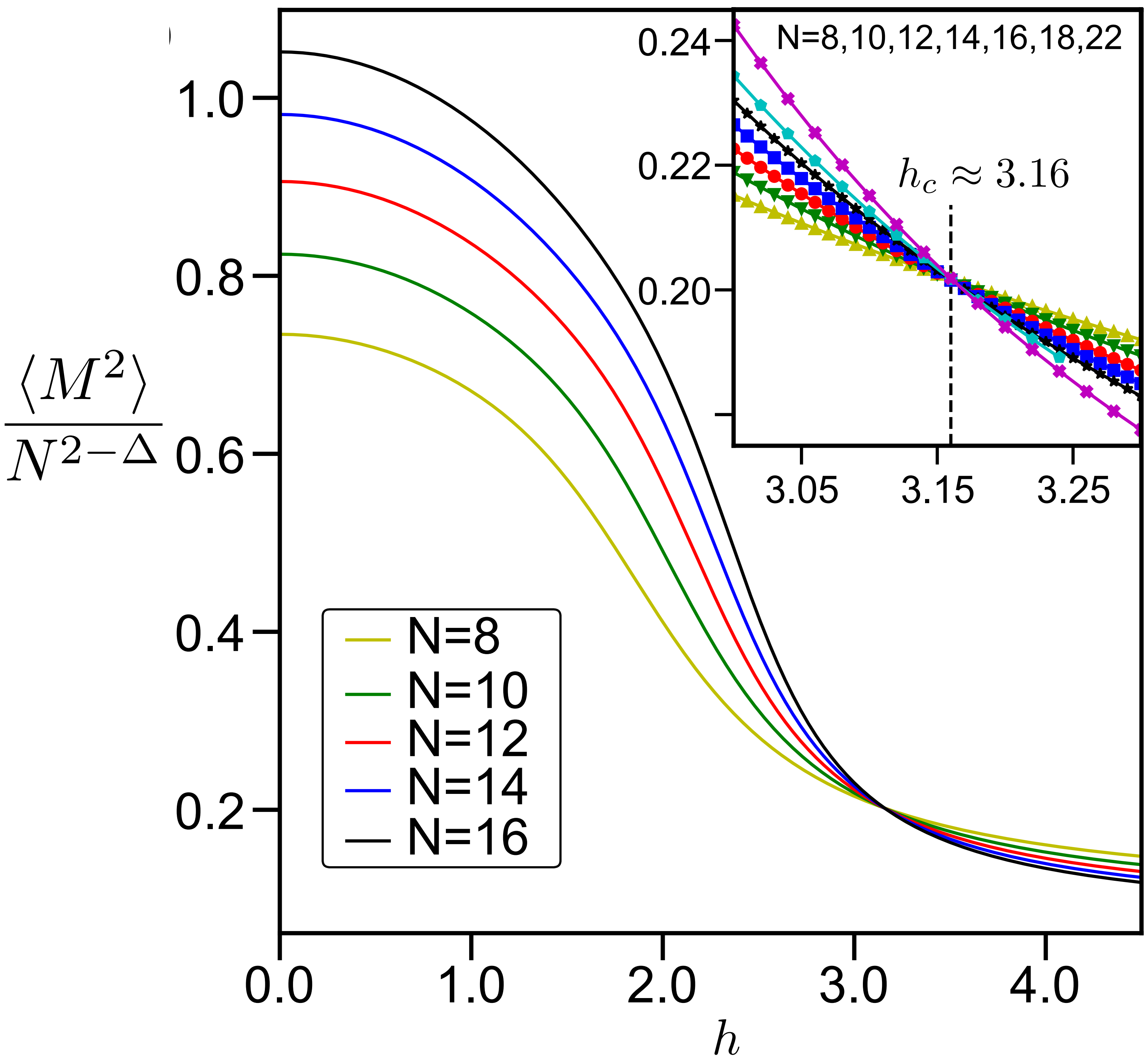
$$\mathbf{c}_m^\dagger = (c_{m,\uparrow}^\dagger, c_{m,\downarrow}^\dagger)$$
$$m = -s, -s+1, \cdots, s$$

$N = 2s+1$ is the space volume, so $N \sim L_x \times L_x$

| UV model | IR Ising transition |
|---|---|
| $\mathbf{c}_m \to \sigma^x \mathbf{c}_m$ | Z2 Ising symmetry |
| SO(3): $\mathbf{c}_{m=-s,\cdots,s}$ spin-s irrep | SO(3) Lorentz rotation |
| Particle-hole symmetry $\mathbf{c}_m \to i\sigma^y \mathbf{c}_m^\dagger$, $i \to -i$ | Space-time parity symmetry |

Ising order parameter: $M = \sum_{m=-s}^{s} \mathbf{c}_m^\dagger \frac{\sigma^z}{2} \mathbf{c}_m$

# Finite size scaling: order parameter




$$M = \sum_{m=-s}^{s} \mathbf{c}_m^\dagger \frac{\sigma^z}{2} \mathbf{c}_m$$

Ordered phase:

$$\langle M^2 \rangle \sim L_x^4 = N^2$$

Phase transition:

$$\langle M^2 \rangle \sim L_x^{4-2\Delta} = N^{2-\Delta}$$

$$\Delta \approx 0.518148$$

Disorder phase:

$$\langle M^2 \rangle \sim O(N^0)$$

# State-operator correspondence

$$\delta E_n = E_n - E_0 = \frac{v}{R}\Delta_n$$

A. We use exact diagonalization to obtain low lying eigenstates of our model.

B. We rescale the energy gaps by calibrating the energy momentum tensor.

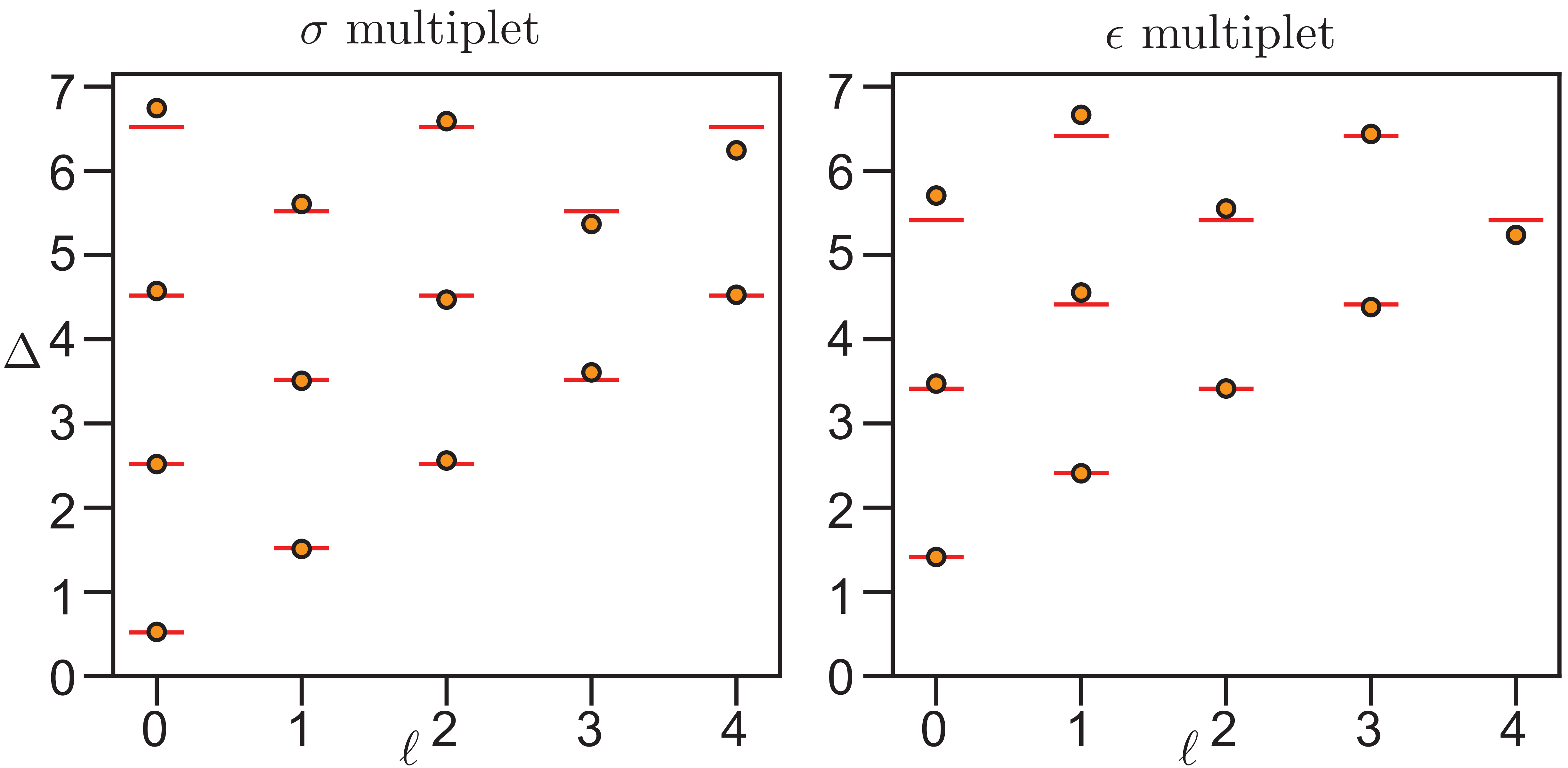
C. For each quantum number, we enumerate scaling operators and compare them with rescaled energy gaps.

Z2 odd, parity even, L=0: $\sigma,\ \Box\sigma,\ \Box^2\sigma,\ \sigma',\ \partial_{\mu_1}\partial_{\mu_2}\sigma_{\mu_1\mu_2},\ \cdots$

Rescaled gaps of N=16 spins: 0.524, 2.517, 4.572 5.303, 6.523

# State-operator correspondence

descendents: $\partial_{\mu_1} \cdots \partial_{\mu_j} \Box^n O, \quad n, j \geq 0$

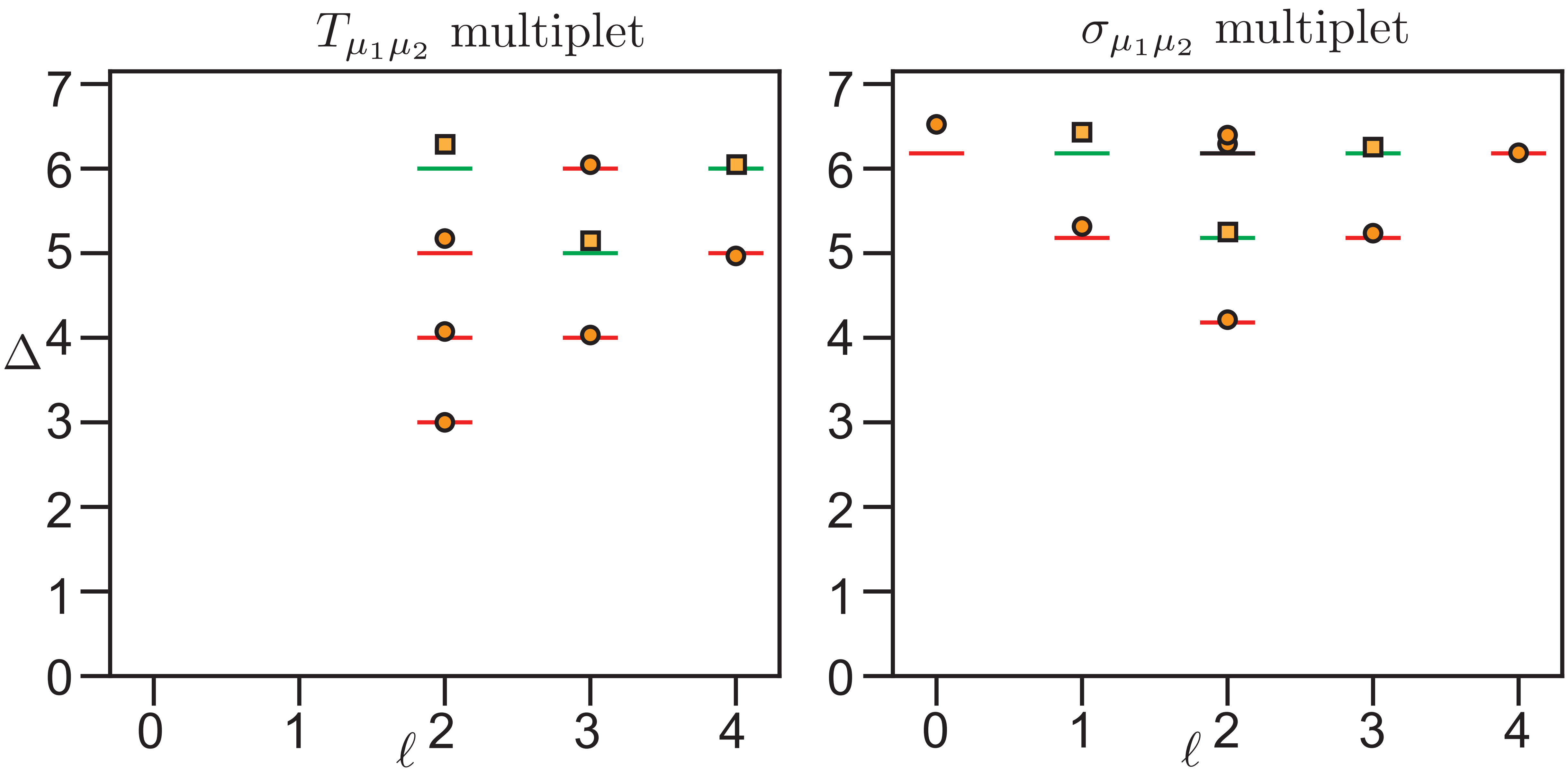


# State-operator correspondence

Conformal multiplet of spinning operator:

$$\partial_{\nu_1}\cdots\partial_{\nu_j}\partial_{\mu_1}\cdots\partial_{\mu_s}\Box^n O_{\mu_1\cdots\mu_l} \qquad \varepsilon_{\mu_l\rho\tau}\partial_\rho\partial_{\nu_1}\cdots\partial_{\nu_j}\partial_{\mu_1}\cdots\partial_{\mu_s}\Box^n O_{\mu_1\cdots\mu_l}$$

1. Energy momentum tensor is conserved.
2. Parity odd descendant.



# State-operator correspondence

- We identified 15 primary operators, the numerical errors for all primaries are within 1.6%.
- We looked at 70 lowest lying states with L<5, all of them match bootstrap results with small errors, namely 60 of them have errors smaller than 3%, 10 of them have errors 3%~5%.

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Bootstrap data from Simmons-Duffin, 2017

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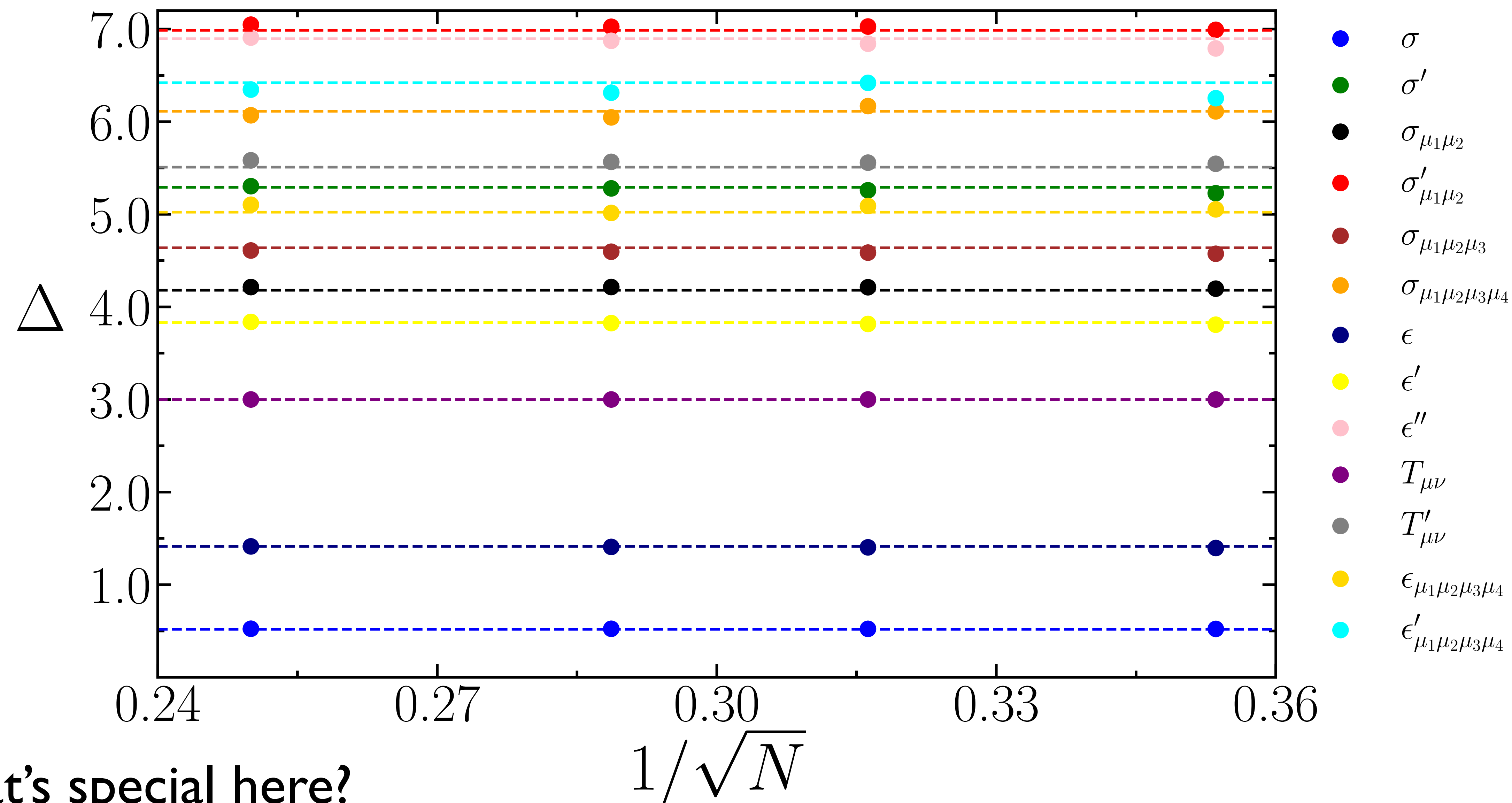
# Why does it work so well?!

Even N=8 spins work!

| Operator | Quantum number | CB data | $N=16$ | $N=8$ |
|---|---|---|---|---|
| $\sigma$ | $L=0$ | $0.5181489(10)$ | $0.52428857$ | $0.5223286$ |
| $\partial_\mu\sigma$ | $L=1$ | $1.5181489(10)$ | $1.50941793$ | $1.50472673$ |
| $\Box\sigma$ | $L=0$ | $2.5181489(10)$ | $2.51722181$ | $2.47849119$ |
| $\partial_{\mu_1}\partial_{\mu_2}\sigma$ | $L=2$ | $2.5181489(10)$ | $2.55937503$ | $2.54111144$ |
| $\Box\partial_\mu\sigma$ | $L=1$ | $3.5181489(10)$ | $3.50635346$ | $3.42826396$ |
| $\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\sigma$ | $L=3$ | $3.5181489(10)$ | $3.6059226$ | $3.40372029$ |
| $\Box\partial_{\mu_1}\partial_{\mu_2}\sigma$ | $L=2$ | $4.5181489(10)$ | $4.47002281$ | $4.2798376$ |
| $\Box^2\sigma$ | $L=0$ | $4.5181489(10)$ | $4.57231367$ | $4.59472215$ |
| $\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\partial_{\mu_4}\sigma$ | $L=4$ | $4.5181489(10)$ | $4.52727499$ | |

# Why does it work so well?!

Even N=8 spins work!




What's special here?

- 3D Ising CFT is simple?
- Sphere? (Forbidding lots of irrelevant operators: flows to IR faster.)
- Fuzzy sphere? (Thermodynamic limit equals to continuum limit)

# Opportunity to tackle 3D CFTs numerically

A zoo of 3D CFTs can be regularized using lowest Landau level:

- O(2), O(3) Wilson-Fisher
- Non-linear sigma model with a WZW term:
    1. QCD3 (fermions coupled to SU(2) gauge field)
    2. QED3?
    3. Stiefel liquid?
- Gross-Neveu-Yukawa?

| Amenable to various numerical methods | System size | # of eigenstates for a given quantum number |
|---|---|---|
| Exact diagonalization | $N = 16 \sim 20$ | $50 \sim 100$ |
| DMRG | $N = 30 \sim 50$ | $2 \sim 10$ |
| Determinant Monte Carlo | $N = 50 \sim 100$ | $1$ |

# LLL projection and fuzzy sphere

For the relation between LLL and fuzzy sphere see e.g. K. Hasebe

$$H = \frac{1}{2Mr^2}\Lambda_\mu^2 = \frac{1}{2Mr^2}(\partial_\mu + iA_\mu)^2$$

$$L_\mu = \Lambda_\mu + s\frac{x_\mu}{r} \qquad\qquad [L_\mu, L_\nu] = i\varepsilon_{\mu\nu\rho}L_\rho$$

Projecting to LLL, covariant angular momentum gets quenched: $L_\mu \sim s\frac{x_\mu}{r}$

Lowest Landau level orbitals <—> States on the fuzzy sphere

$$m = -s, -s+1, \cdots, s$$

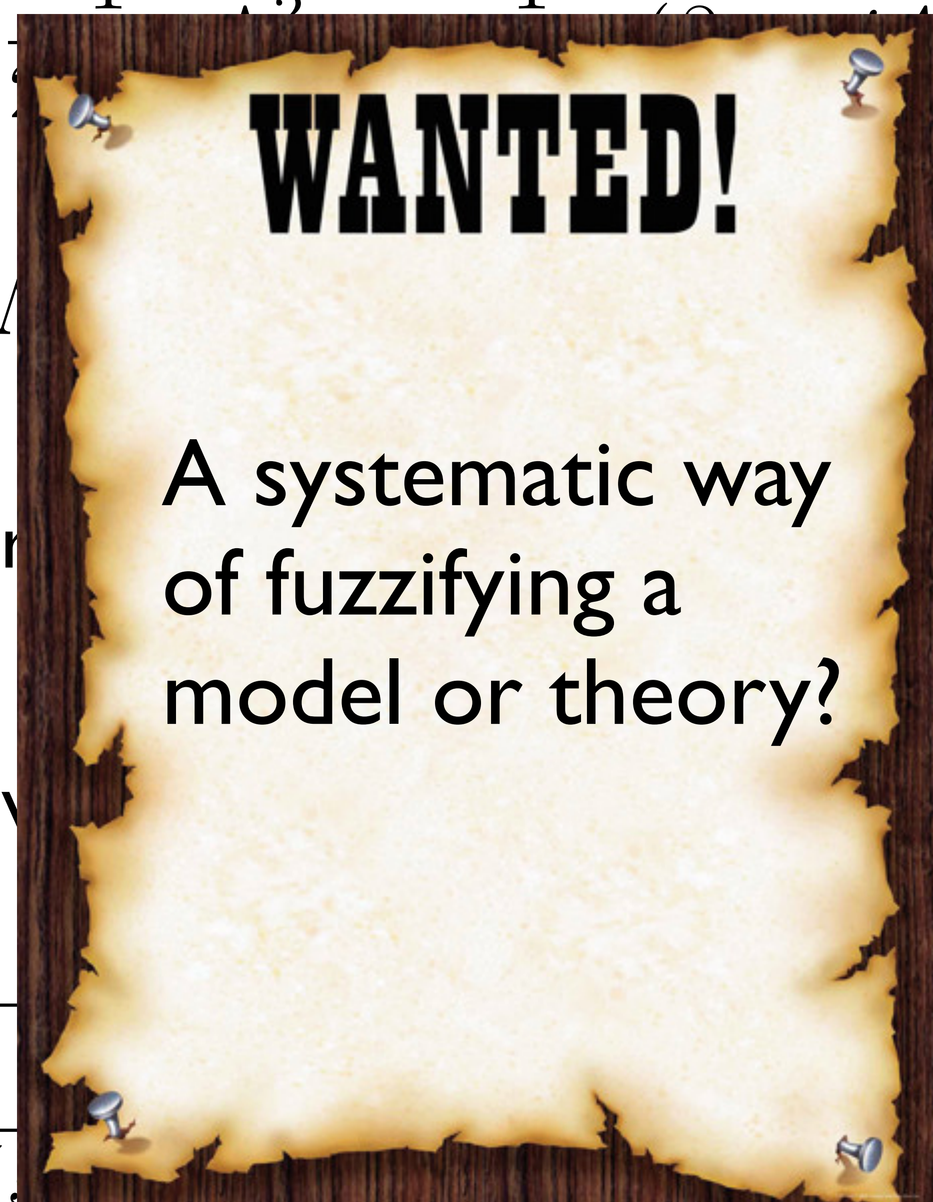
$$\Phi_{m,s}(\theta,\varphi) = \sqrt{\frac{(2s+1)!}{4\pi(s+m)!(s-m)!}}e^{im\varphi}\cos^{s+m}\left(\frac{\theta}{2}\right)\sin^{s-m}\left(\frac{\theta}{2}\right)$$

# LLL projection and fuzzy sphere

For the relation between LLL and fuzzy sphere see e.g. K. Hasebe

$$H = \frac{1}{\cdots} \quad \cdots \quad \frac{1}{\cdots} \quad \cdots {}_{\mu})^2$$

$$L_\mu = \cdots \quad \cdots i\varepsilon_{\mu\nu\rho}L_\rho$$




Projecting to LLL, covar … quenched: $L_\mu \sim s\frac{x_\mu}{r}$

Lowest Landau lev … he fuzzy sphere

$$-s, -s+1, \cdots, s$$

$$\Phi_{m,s}(\theta, \varphi) = \sqrt{\frac{\cdots}{4\pi(\cdots}} \cdots \left(\frac{\theta}{2}\right) \sin^{s-m}\left(\frac{\theta}{2}\right)$$

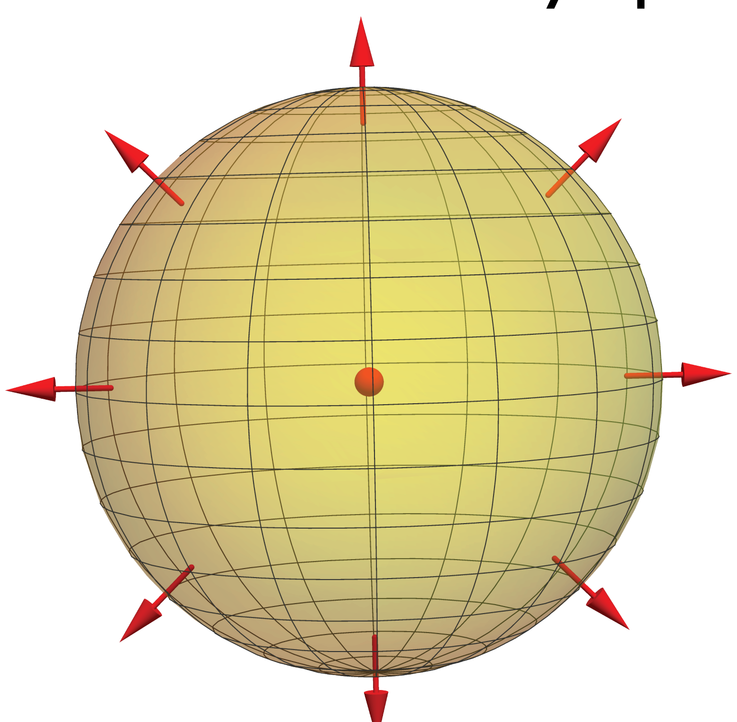
# NOT a non-commutative field theory

- Non-commutative field theory is a field theory defined on a fuzzy manifold, and it suffers the issue called UV-IR mixing.

Do we have a similar UV-IR mixing here? No!

The fuzziness comes from the monopole.

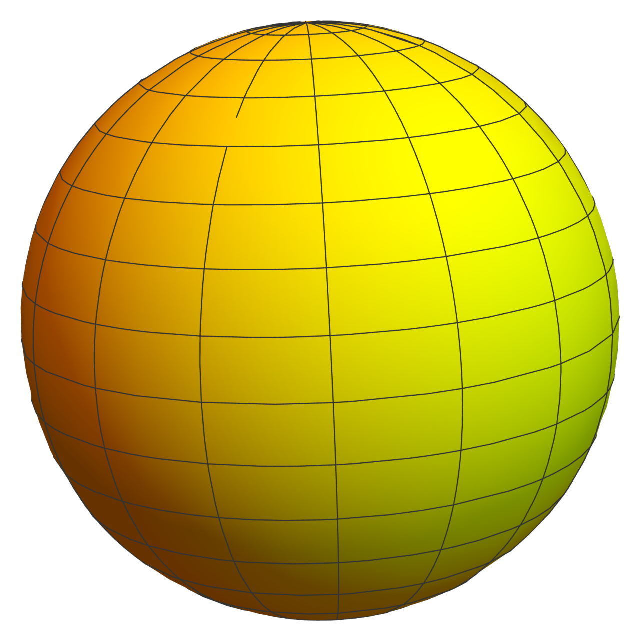
Electrons see a fuzzy sphere.



Ising spin:

$$\sigma^z = c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow$$

Ising spins see a normal sphere.



# Summary

Thank you!

- We simulated 3D Ising transition on the sphere S^2xR, and for the first time we observed almost perfect state-operator correspondence on an incredible small system size.

- We have identified 13 parity even primary operators, all of them agree with bootstrap results with a small error (within 1.6%).

- We find two parity odd primary operators which were unknown.

- Future study will be devoted to explore other transitions, and a systematic framework of fuzzifying model or theory is needed.

Let's explore the fuzzy world!