Conformal field theory approach to quantum loop models

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“Bootstrapping Nature”, GGI, Italy

“Fully packed quantum loop model on the triangular lattice: Hidden vison plaquette phase and cubic phase transitions”, Zheng Yan, Xiaoxue Ran, Yan-Cheng Wang, Rhine Samajdar, JR, Subir Sachdev, Yang Qi, Zi Yang Meng, arxiv: 2205.04472
“Scalar CFTs from structural phase transitions”, JR, to appear
“On the critical exponents η_* of the Cubic$_*$ conformal field theory”, JR, Ning Su, to appear

From lattice to field theory

Instead of talking about quantum dimer models, we first discuss structural phase transitions.

When temperature is changed, the crystal structural may change, which can be understood as spontaneous symmetry breaking .

Crystal can be understood as three dimensional lattices. The subgroup of E(3) which leaves the lattice structural unchanged is called the “space group”. As temperature change, spontaneous breaking happens

$G\to H$

Both G and H are space groups. The space group is infinite dimensional, in particular, it contains a subgroup which is the lattice translational group $\mathcal{T}$.

In quantum field theory, the type of symmetry we deal with is usually a direct product of the euclidean group with certain flavor symmetry group.

There is clearly some “mysterious” steps that one need to follow to get a field theory description of this phase transition. In particular, some structural phase transition are know to be in the 3D Ising universality class, one may ask where is the Z_2 symmetry of the Ising model coming from.

From lattice to field theory

The answer lies in the representation theory of the space group.

Let us define a lattice to be

$$\vec{R} = \sum_{i=1}^{3} n_i \vec{a}_i,$$

One can then define reciprocal lattice vectors

$$b_1^\mu = \frac{2\pi}{\Omega} \epsilon_{\mu\nu\rho} a_2^\nu a_3^\rho, \quad b_2^\mu = \frac{2\pi}{\Omega} \epsilon_{\mu\nu\rho} a_3^\nu a_1^\rho, \quad b_3^\mu = \frac{2\pi}{\Omega} \epsilon_{\mu\nu\rho} a_1^\nu a_2^\rho.$$

They satisfies the following relation

$$\vec{a}_i \cdot \vec{b}_j = 2\pi \delta_{ij},$$

The we have a lattice in the momentum space.

$$\vec{K} = \sum_i n_i \vec{b}_i,$$

From lattice to field theory

Bloch’s theorem tells us that the eigenfunctions of the translations group $\mathcal{T}$ can be written as

$$\rho_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} u(\vec{r}),$$

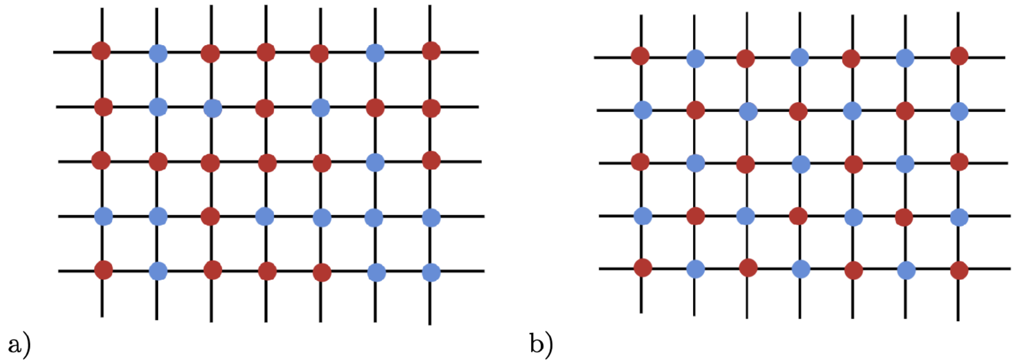
with $u(\vec{r}+\vec{a}_i) = u(\vec{r}), \quad \text{for} \quad i = 1, 2, 3.$

Notice $\vec{k}$ and $\vec{k}+\vec{b}_i$ are in the same irrep of the space group.

This introduce the Brillouin zone, which is a torus.

From lattice to field theory

Let us consider a “hypothetical” two dimensional structural phase transition



At high temperature, the density of the red atoms are

$$\rho(\vec{r}) = \frac{1}{2} u(\vec{r}). \text{ with } \quad u(\vec{r}) = \sum_{\vec{R}} \delta(\vec{r} - \vec{R})$$

At low temperature, the density of the red atoms are

$$\rho(\vec{r}) = u(\vec{r}) + e^{\mathrm{i}\frac{1}{2}(\vec{b}_1 + \vec{b}_2)\cdot\vec{r}} u(\vec{r}),$$

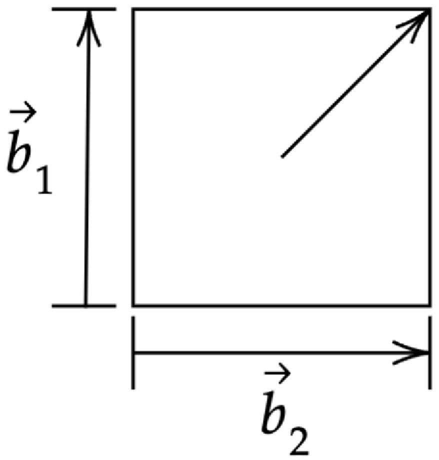
We can view the coefficient of the second term as the order parameter.

From lattice to field theory

The order parameter

$$\eta(\vec{r}) = \phi \times e^{\mathrm{i}\frac{1}{2}(\vec{b}_1+\vec{b}_2)\cdot\vec{r}} u(\vec{r}).$$

has a momentum at the cornor of the Brillouin zone.



How does the order parameter transform under the action of the space group?
Clearly, many of the space group elements acts trivially on the order parameter. The 90 degree rotations and reflections bring the momentum vector to a point which is equivalent to the original momentum vector.

Under lattice transition

$$\eta(\vec{r}+\vec{a}_1) = \phi e^{\mathrm{i}\frac{1}{2}(\vec{b}_1+\vec{b}_2)\cdot\vec{a}_1} u(\vec{r}) = -\eta(\vec{r}),$$

so that

$$\phi \to -\phi.$$

To summary, scalar fields in the Landau theory forms non-faithful representations of the space group.

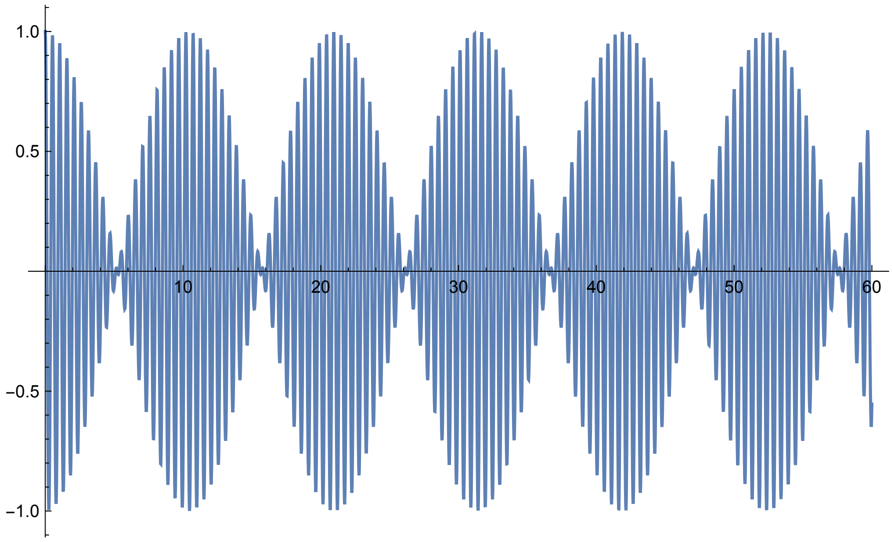
From lattice to field theory

So that the Landau theory is

$$F = a\phi^2 + \lambda\phi^4 + \cdots .$$

This is the mean field theory approximation of the $\lambda\phi^4$ theory. The order of the phase transition depends on the sign of λ, which depends on the UV details of the crystal. In other words, this is a phenomenological model.

What about fluctuations? Fluctuations are spatial modulations of the order parameter. We allow ϕ to have spacial dependence ϕ(x), however, this spatial modulation will have a length scale much large that the lattice scale.



Taking into account the fluctuation we get the action

$$S = \int dx^3 \frac{1}{2} \left(\vec{\nabla}\phi(\vec{x})\right)^2 + a\phi(x)^2 + \lambda\phi(x)^4 + \ldots .$$

which is the familiar field theory.

From lattice to field theory

1, group-subgroup relations,

2, Landau condition: The effective action should not contain cubic terms,

3, Lifshitz condition: The effective action should not contains one derivative terms such as $\phi^i(x)\,\partial^j\,\phi^j(x)$,

4, stability under RG.

...

The Landau and Lifshitz conditions can be written as group theoretical conditions on the irreps of the space group.

Remarkably, in late 70s and early 80s, people were able to classify all the possible Landau effective action that can come from a structural phase transition. They also analyzed the corresponding $\lambda\phi^4$ theory perturbatively in 4-ϵ expansion.

[Landau, Lifshitz, Michel, Brezin, Toledano, Toledano, Kovalev, Kim, Hatch, Stokes, ...]

crystal universalities

There are only 6 CFT that can be realized in commensurate structural phase transitions.

No.	Name	Images
1	Ising	A2a
2	XY	B4a, B6b, B8a, B12a, B12b, B24a
3	N=3 Cubic	C24a, C24c, C48a
4	XY^2	D32e, D64a, D64b, D64d, D72b, D128a, D144a
5	N=4 Cubic	D192a, D192c, D384a
6	XY^3	E96k, E192j, E768b, E768c, E1536a

[“Scalar CFTs from structural phase transitions”, JR, to appear] (no originality claimed)

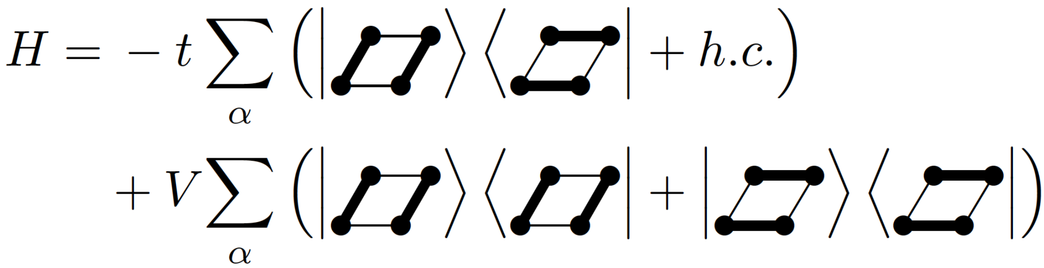
This is heavily based on “Isotropy Subgroups of the 230 Crystallographic Space Groups” by Stokes and Hatch.

These CFTs are probably lower hanging fruits on the bootstrap (and the Monte Carlo?) tree (as compared to gauge theories).

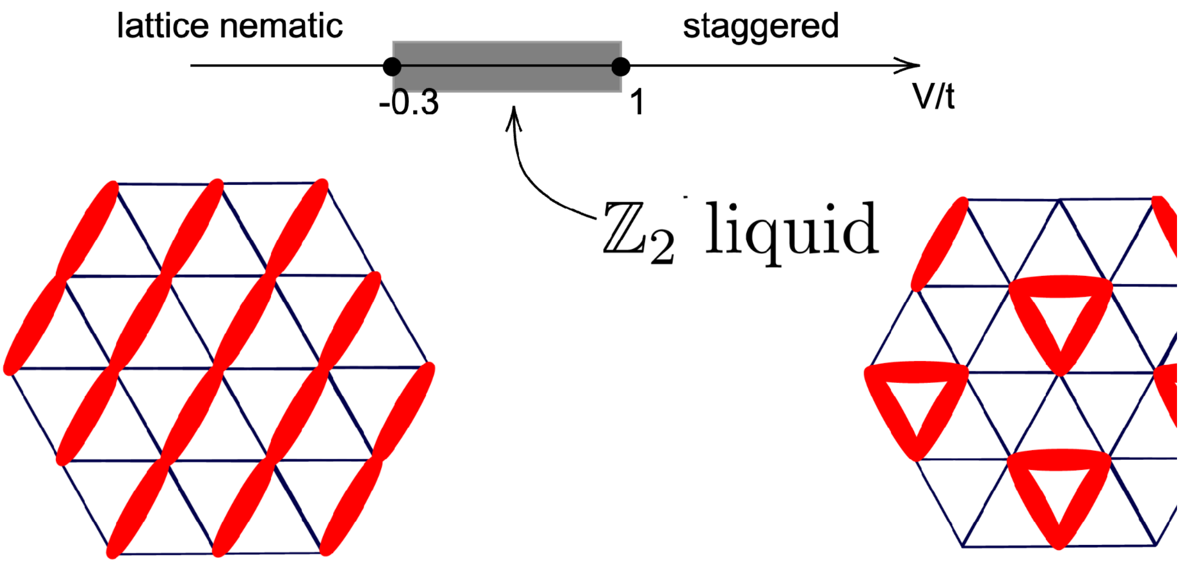
We will now discuss the N=3 Cubic model. See the talk by Martin Hasenbusch and also [M. Hasenbusch and E. Vicari, Phys. Rev. B 84, 125136]. Early bootstrap attempts can be found in [JR and Su 1712.00985, Stergiou 1801.07127, Kousvos and Stergiou 1810.10015, 1911.00522].

quantum dimer models

The quantum dimer models are given by



The phase diagram are believed to be

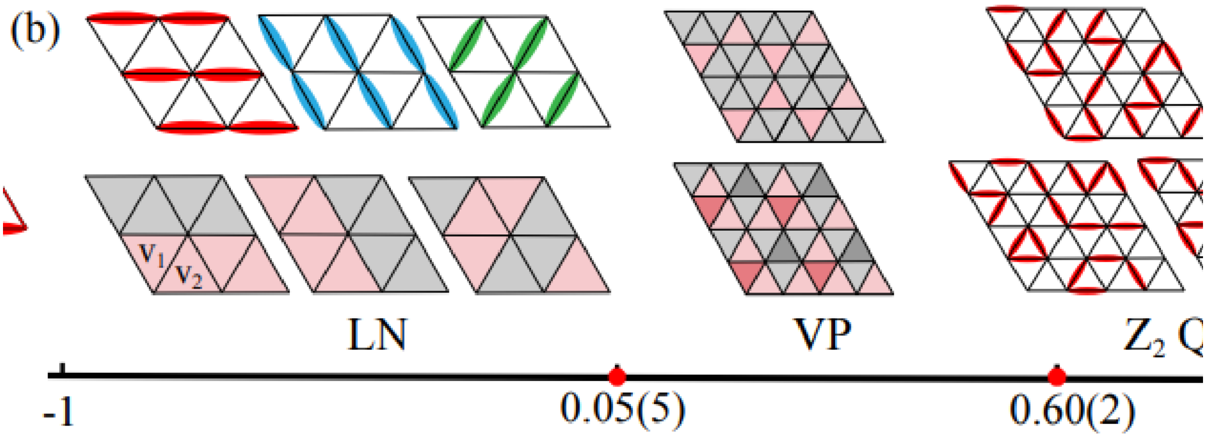


[K. Roychowdhury, S. Bhattacharjee, and F. Pollmann, Phys. Rev. B 92, 075141 (2015)]

The results are based on DMRG or exact diagonalization. Quantum Monte Carlo studies are made possible due to the development of new algorithms by Zheng Yan.

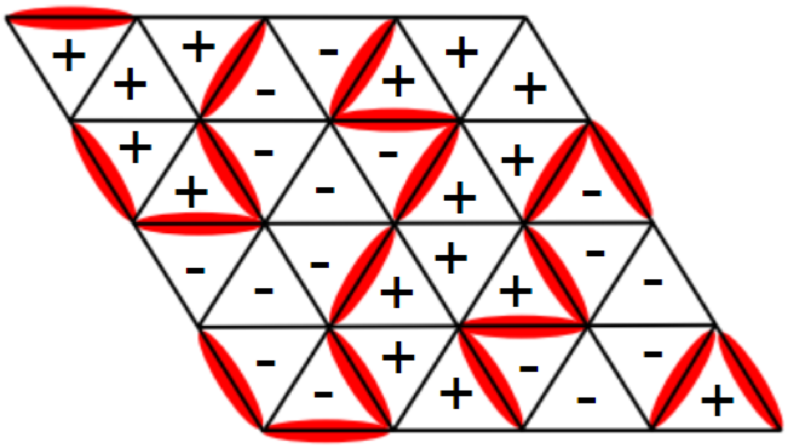
The new phase diagram

Our result suggests that there is more structure in the phase diagram, in particular, there is a new phase which we call Vison-Plaquette phase.



Vison

From the dimer configuration, one can easily define a vison configuration. The vison change sign when crossing the a dimer line



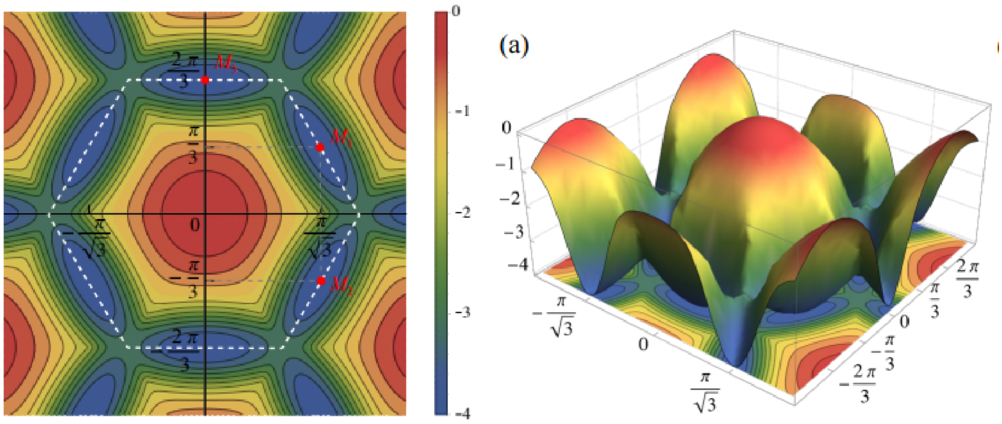
The Vison effective action

The vison is described by transverse-field Ising model on the dual honeycomb lattice

$$H_{dual} = J_1 \sum_{\langle IJ \rangle} v_I^z v_J^z + J_2 \sum_{\langle\langle IJ \rangle\rangle} v_I^z v_J^z + J_3 \sum_{\langle\langle\langle IJ \rangle\rangle\rangle} v_I^z v$$

At the Γ=0 limit, the quadratic Hamiltonian can be diagonalized.

If we set $J_1 = J_2 = J_3 = 1$, we get the spectrum to be



so that the eigen modes are

$$\phi_j = \sum_{\mathbf{r}} (v_{1,\mathbf{r}}, v_{2,\mathbf{r}}) \cdot \mathbf{u}_j e^{i\mathbf{M}_j \cdot \mathbf{r}}, \quad j = 1, 2, 3,$$

The details of the location of the minimum depends on the choice of J_1, J_2, J_3. We do not know exactly how J_1, J_2, J_3 depends on the coupling t and V. We conjecture from the above the modes at the M points are the critical modes.

[D. Blankschtein, M. Ma, A. N. Berker, G. S. Grest, and C. M. Soukoulis, Phys. Rev. B 29, 5250 (1984), D. Blankschtein, M. Ma, and A. N. Berker, Phys. Rev. B 30, 1362 (1984).]

The Vison effection action

We now need to understand how the space group acts on the vison modes

$$T_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I$$

$$R_6 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

These matrices generates the Cubic subgroup of O(3).

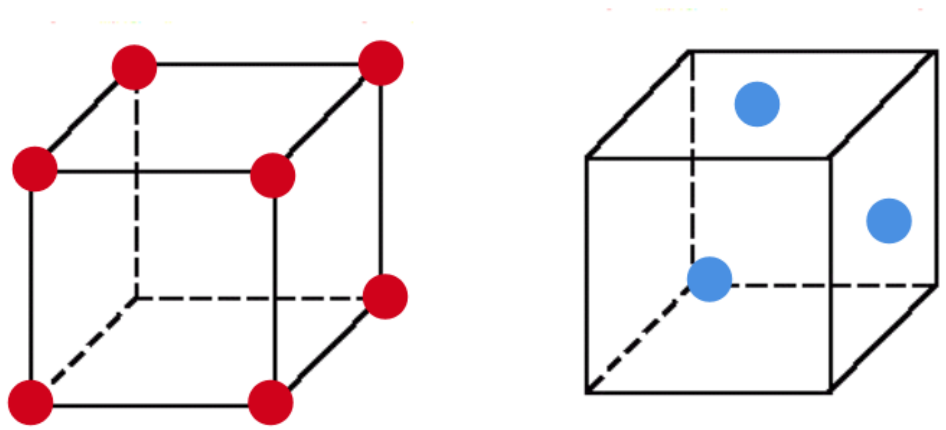
we can write down the effective action (coupled to a Z2 gauge theory)

$$L = \sum_{i=1}^{3}(\partial_\mu \phi_i)^2 + r\sum_{i=1}^{3}\phi_i^2 + \mu(\sum_{i=1}^{3}\phi_i\phi_i)^2 + \nu_4\sum_{i=1}^{3}(\phi_i$$

$$+ \mu_6(\sum_{i=1}^{3}\phi_i^2)^3 + \nu_6(\phi_1\phi_2\phi_3)^2 + \nu_6'(\sum_{i=1}^{3}\phi_i^2)(\sum_{i=1}^{3}\phi_i^4),$$

The Cubic model, phase structure

The mean field theory result tells us



The are two phases, the corner cubic ($v_4 > 0$)

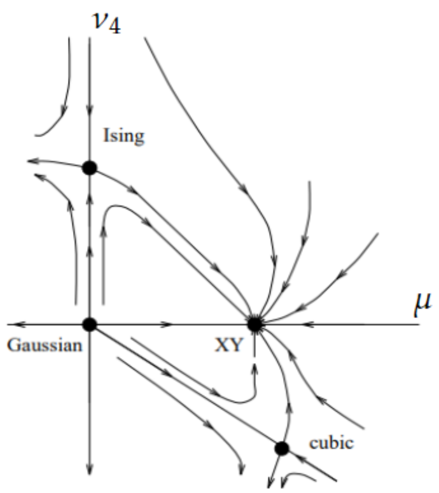
$$\langle \phi_1 \rangle = \pm v, \quad \langle \phi_2 \rangle = \pm v, \quad \langle \phi_3 \rangle = \pm v.$$

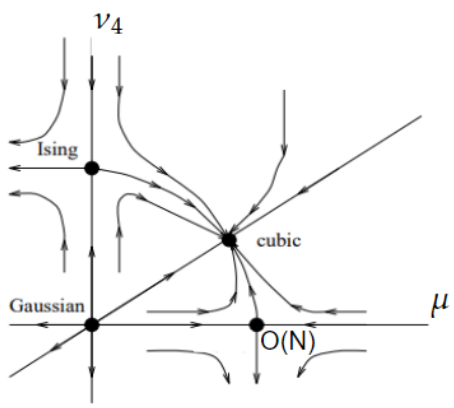
and the face cubic ($v_4 < 0$)

$$\langle \phi_1 \rangle = \pm v', \quad \langle \phi_2 \rangle = \langle \phi_3 \rangle = 0,$$

The Cubic model, perturbative RG and bootstrap

The RG structure of the Cubic Lagrangian depends on the size of the Cubic group.
$N < N_c$ vs $N > N_c$

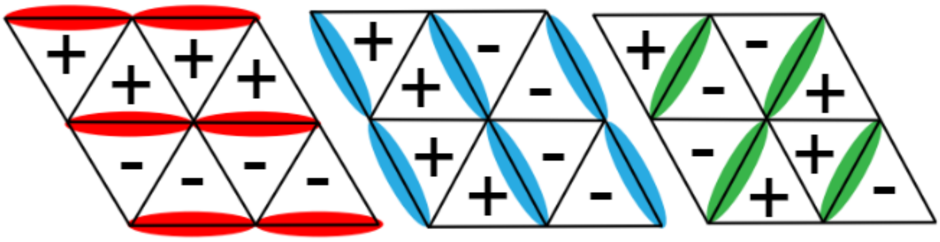




[A. Aharony, Physical Review B 8 (1973) 4270]
Recent conformal bootstrap studied confirmed the second picture, by analysing the scaling dimension of the symmetric traceless rank-4 tensor, which is the operator that drives the RG flow from the O(3) to Cubic. [S. M. Chester, W. Landry, J. Liu, D. Poland, D. Simmons-Duffin, N. Su, and A. Vichi, Phys. Rev. D 104, 105013 (2021).]

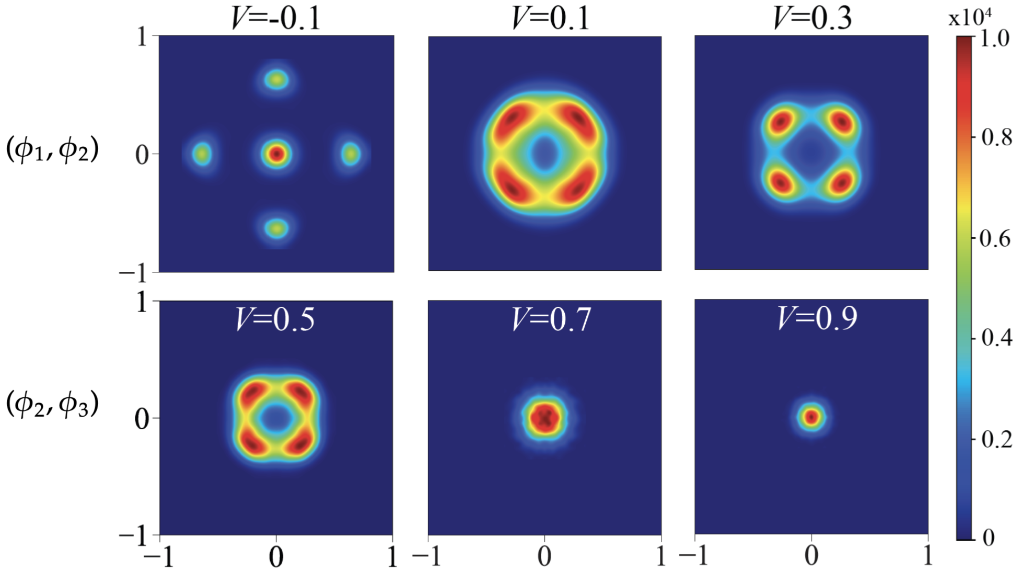
More interestingly, notice that (v4>0), this means that the transition from disordered to face cubic can not be second order. The VP phase must be there.

phases of the quantum dimer model and the $\lambda\phi^4$ action



Notice in terms of dimers, there are only three the lattice nematic phase (the face cubic phase). Vison phase related by gauge transformation corresponds to the same phase of dimers.

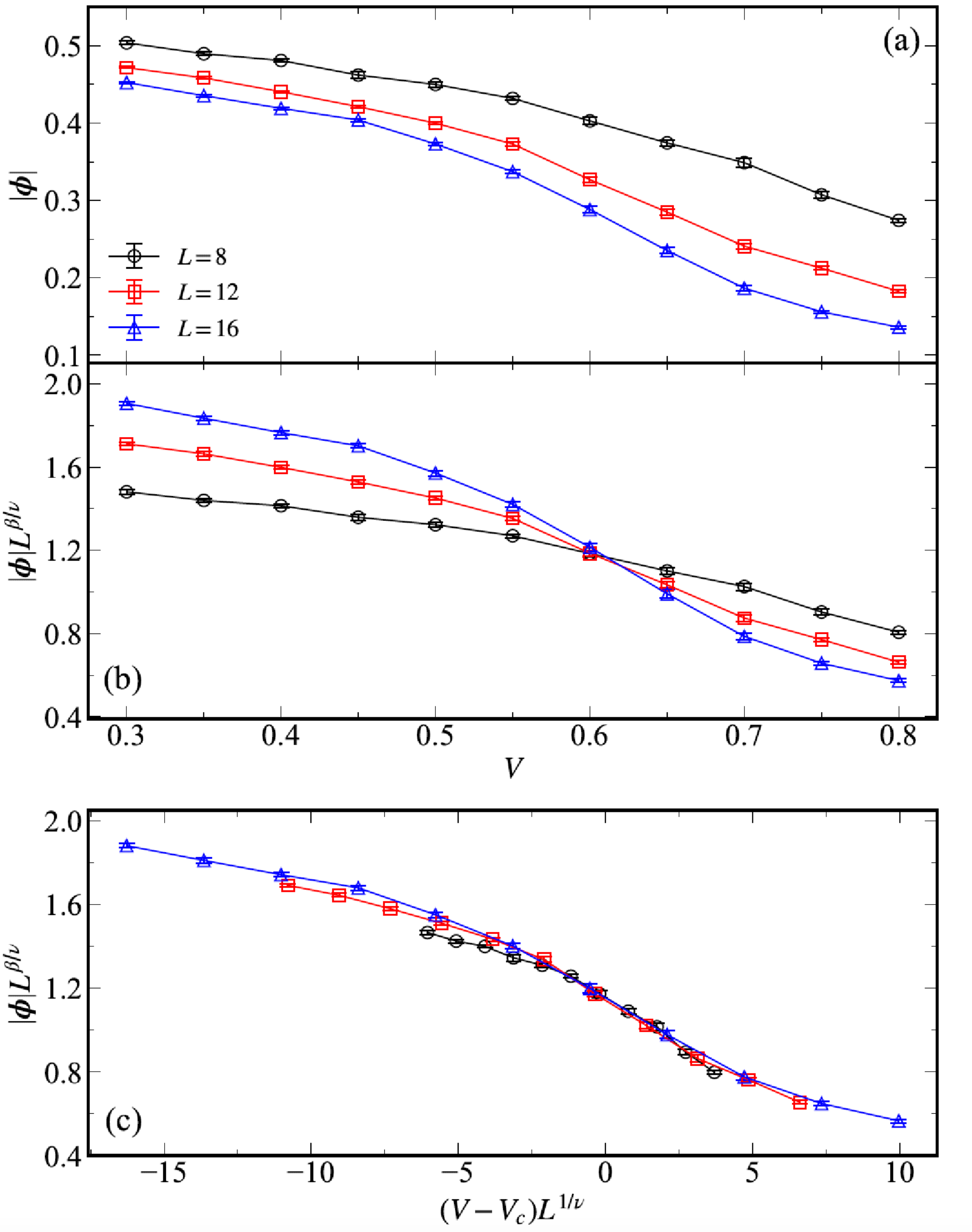
What we need to do is to compare the vison configuration with the above three configurations, and plot it in the histogram.



Clearly both face cubic and cornor cubic phases are visible.
It is important that the cubic fixed point has small and positive v_4. As we tune V/t, the lattice operator mix ϕ^2 and $\phi_1{}^4 + \phi_2{}^4 + \phi_3{}^4$, causing the Cubic anisotropy term to change sign.

critical behavior-- data collapsing

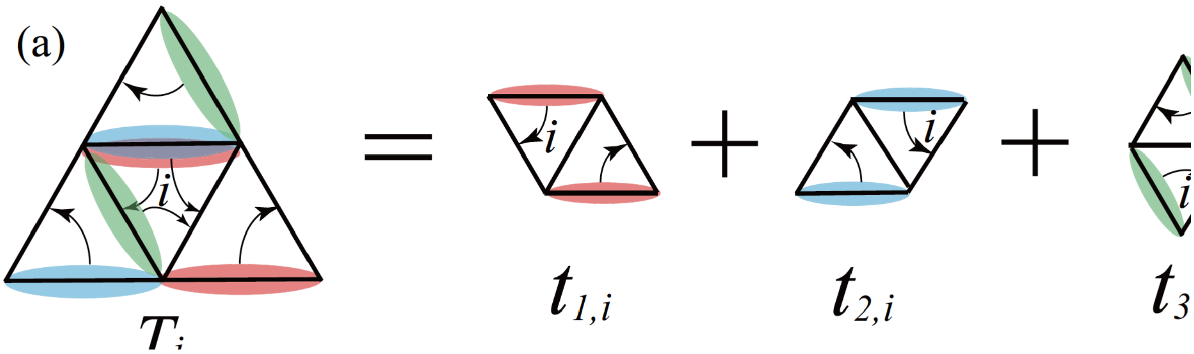
Near the critical point, the vison order parameter shows scaling behavior



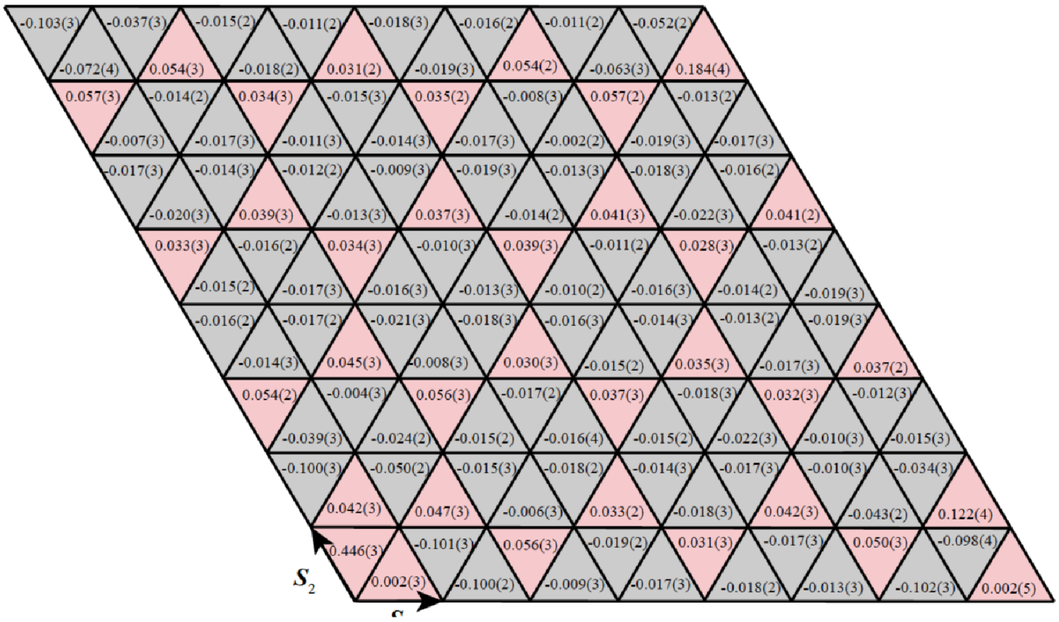
We used the O(3) critical exponents to rescale the data.

The order parameter

To make it more explicit, we define a local operator to be



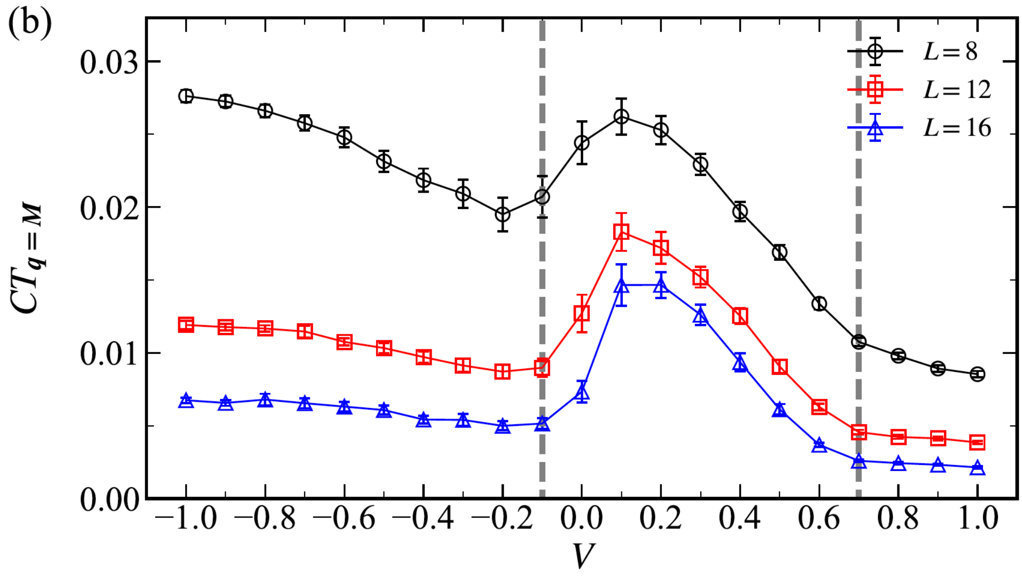
Then the real space correlation of the T_i operator is



The operator preserves R_6rotation symmetry. Notice $R_6 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. The operator should be

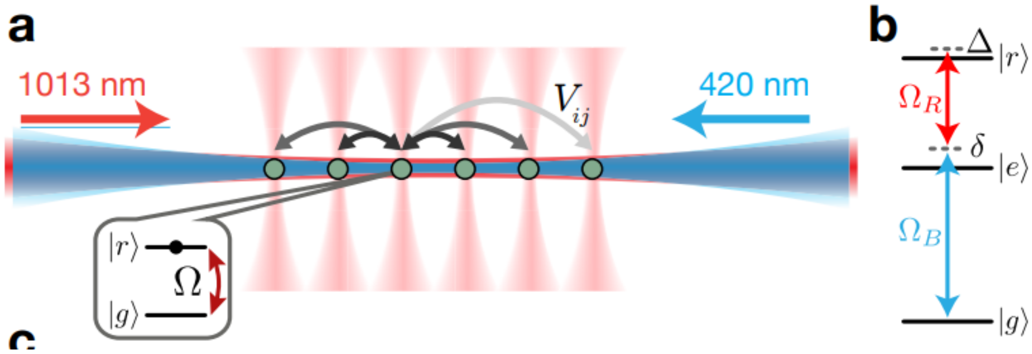
$O = \phi_1\, \phi_2 + \phi_2\, \phi_3 - \phi_3\, \phi_1$.

Do a fourier transformation, and sum over the M points, we get



This agrees with the expectation. Notice for in the face cubic phase, only one of the $\langle \phi_i \rangle$ gets expectation value, the quadratic operator has no expectation value. The corresponding critical exponents $\eta_* = 1.42$, which is in the symmetric trace less tensor "T" representation of O(3).

Rydberg atoms



H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletić, and M. D. Lukin, Nature 551, 579 (2017)

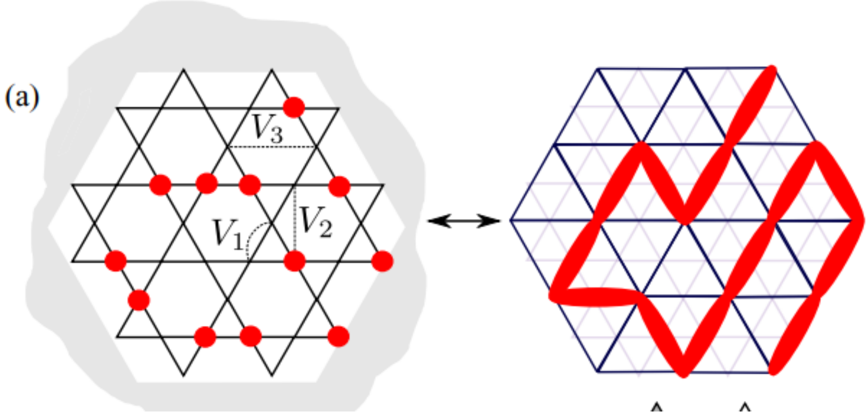
hard-core boson

The Rydberg atoms corresponds to a two dimensional version of the so called Fendley-Sengupta-Sachdev model

$$H=\frac{\Omega}{2}\sum_{i}(b_i+b_i^{\dagger})-\delta\sum_{i}n_i+\frac{1}{2}\sum_{i,j}V(|\boldsymbol{i}-\boldsymbol{j}|)n_in_j.$$

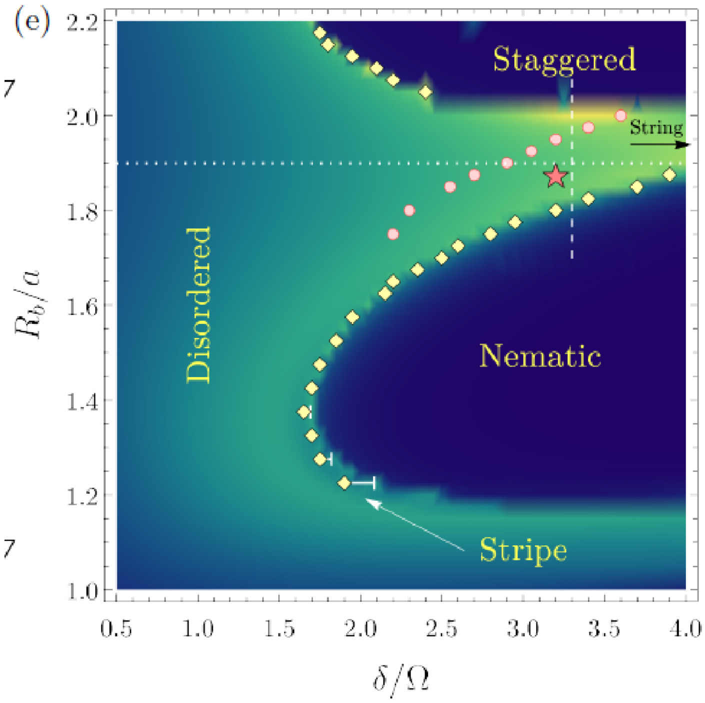
where b are hard core boson.

Hardcore boson on Kagome lattice can be mapped to dimer models on tri-angular lattice (K. Roychowdhury, S. Bhattacharjee, and F. Pollmann, Phys. Rev. B 92, 075141 (2015).)



hard-core bosons

There is a DMRG studies in R. Samajdar, Wen Wei Ho, H. Pichler, M. D. Lukin, and S. Sachdev, PNAS 118, e2015785118 (2021)



DMRG study are based on two dimensional generalization of the so called matrix product states. The region of most interest is denoted by the star.

There is a slightly different set up (Rydberg atom on ruby lattice, while dimers on kagome lattice) that has been proposed in
R. Verresen, M. D. Lukin, and A. Vishwanath, Phys. Rev. X 11, 031005 (2021).

and realized experimentally in

G. Semeghini, H. Levine, A. Keesling, S. Ebadi, T. T.Wang, D. Bluvstein, R. Verresen, H. Pichler, M. Kalinowski, R. Samajdar, A. Omran, S. Sachdev, A. Vishwanath, M. Greiner, V. Vuletić, and M. D. Lukin, Science 374, 1242 (2021)

conformal perturbation from O(3)

Interestingly, the Cubic CFT appearing in the quantum dimer model is infact the Z2 gauged version, which is Cubic*.

This means the true order parameter are boson bi-linears.

Perturb the O(3) CFT by the Cubic anisotropy term, which is in the T4 representation of O(3).

$S_{\text{cubic}} = S_{O(3)} + \int \text{dx}^3\, T^4$

The perturbation parameter is $\delta = 3 - \Delta_{T^4} \approx 0.012$.

For an arbitrary operator O, its anomalous dimension depends on the OPE coefficients [Zohar Komargodski, David Simmons-Duffin 1603.04444, Connor Behan, Leonardo Rastelli, Slava Rychkov, Bernardo Zan, 1703.05325]

$\Delta_O{}^{\text{IR}} = \Delta_O{}^{\text{UV}} + \frac{\lambda_{OOT^4}}{\lambda_{T^4 T^4 T^4}} S_{d-1}\, \delta$

The η and ν, on the other hand, receive corrections in order δ^2.

The T2 opeartor of O(3) on the other hand, get correction at order δ^1.

$5 \to 2+3$,

2: $(\phi_1)^2 + (\phi_2)^2 - 2\,(\phi_3)^2$ and $(\phi_2)^2 + (\phi_3)^2 - 2\,(\phi_1)^2$
3: $\phi_1\,\phi_2$, $\phi_2\,\phi_3$, and $\phi_3\,\phi_1$

Take in account the group theory factors, we get ($\Delta_t = 1.2096$)

$\Delta_2 \approx 1.2200$
$\Delta_3 \approx 1.2091$

[“On the critical exponents η_* of the Cubic$_*$ conformal field theory”, JR, Ning Su, to appear]

Will this be visible in M. Hasenbusch’s upcoming work?

future directions

1 Fully packed loop model on square lattice [2209.10728], the Lifshitz fixed point

2 new phases in one dimer per site model, that is the O(4)* CFT? (work in progress)

3 magnetic phase transitions? Incommensurate phase transition?

4 classify similar 2+1D CFTs using (projective) representation of wallpaper groups?

...


Thank you!