# Gapless SPTs

Ryan Thorngren, October 2022, GGI

w/ Ruben Verresen, Nick G. Jones, Saran Prembabu, Frank Pollmann, Ashvin Vishwanath

1905.06969 Symmetry-enriched quantum criticality
2008.06638 Intrinsically gapless SPTs
2208.12258 Boundary DQCP in an SPT-SPT' transition

SPTs are a large but well understood family of topological *gapped* phases, which have edge modes protected by 't Hooft anomalies.

We are interested in the stability of these edge modes when coupling to gapless modes, which are always present.

We will find new kinds of stable boundary fixed points of CFTs in this problem, an "anomalous Kondo problem".

We are also interested in how SPT edge modes change anomalies at bulk transitions between distinct SPT phases.

There is even SPT-like physics which can *only* occur in a gapless phase: *(intrinsically) gapless SPT phases.*

1. Review of (gapped) SPTs

2. Gapped SPTs driven to a symmetry-breaking critical point

3. SPT-SPT transitions

4. Intrinsically gapless SPTs

$\mathscr{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathscr{H}_x$ with "on-site" $G$ action $U(g) = \bigotimes_{x\in\Lambda} U_x(g)$

An **SPT *state*** is a $G$-singlet in $|\mathrm{SPT}\rangle \in \mathscr{H}_\Lambda$ of the form

$$\boxed{|\mathrm{SPT}\rangle = e^{ih}|\mathrm{trivial}\rangle}$$

where $h$ is local and $|\mathrm{trivial}\rangle = \bigotimes_x |x\rangle$, $|x\rangle$ a singlet

$|\mathrm{SPT}\rangle$ is automatically a gapped ground state:

Let $H_0$ be the parent Hamiltonian of $|\mathrm{trivial}\rangle$, then $H_{\mathrm{SPT}} = e^{ih}H_0e^{-ih}$

The set of gapped Hamiltonians with a given ground state is convex, so states and gapped Hamiltonians are used interchangeably.

$$|\mathrm{SPT}\rangle = e^{ih}|\mathrm{trivial}\rangle$$

$|\mathrm{SPT}\rangle$, hence $e^{ih}$ is *G*-symmetric, but *h* may not be

Can classify *G*-SPT states mod evolution by symmetric *h*

These form a group $\Omega_G^D$ under tensor product (stacking)

Strategy for classification: construction *topological invariants*, ie. homomorphisms (under stacking)

$$\Omega_G^D \to A$$

where *A* is some other group.

Construct enough of these to capture $\Omega_G^D$ and then understand their relations.

Symmetry fractionalization:

$$|\mathrm{SPT}\rangle = e^{ih}|\mathrm{trivial}\rangle$$

$$\prod_{i<j<k} U_j(g)|\mathrm{SPT}\rangle = V_L(g)V_R(g)|\mathrm{SPT}\rangle$$

$V_L V_R$ is a linear rep of $G$, but $V_{L,R}$ is only *projective*

This gives a topological invariant of the SPT:

$$e^{i\omega(g_1,g_2)} = V_L(g_1)V_L(g_2)V_L(g_1g_2)^\dagger$$

$V_L$ only defined up to phase factors, pass to group cohomology

$$[\omega] \in H^2(G, U(1))$$

This projective representation will appear at the boundary of the SPT, as a form of anomaly in-flow.

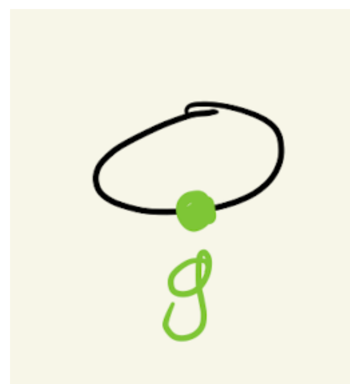
Can prove in spin systems this invariant is *complete* [Kapustin-Sopenko-B. Yang]

Expect SPTs to flow to topological gauge theories in the IR.

The 2-cocycle appears as the Dijkgraaf-Witten action

$$Z(\Sigma, A) = e^{i \int_\Sigma \omega(A)}$$

One state in each twisted sector, $V_L \prod U$





Partition functions on decorated surfaces generated by the pair of pants

Expect this TQFT to be a complete invariant in general, but it's unknown how to construct it from $|\mathrm{SPT}\rangle$.

For gapless SPT we will need to combine this with CFT, similar to discrete torsion in the old days of orbifolds, but our symmetries are global.

The simplest SPT, the cluster state

$$H_0 = -\sum X_i \quad H_{\mathrm{SPT}} = -\sum Z_{i-1}X_iZ_{i+1} \quad h = \frac{\pi}{4}\sum_i (-1)^i Z_i Z_{i+1}$$

Unitary symmetry $\mathbb{Z}_2^2$: $\prod X_{2i(+1)}$, time reversal $T = K$ (complex conj)

"Stabilizer state" $Z_{i-1}X_iZ_{i+1}|\mathrm{SPT}\rangle = |\mathrm{SPT}\rangle$

Acting on $|\mathrm{SPT}\rangle$ with boundary at i = 0, one missing stabilizer

$$X_0X_2X_4\cdots = X_0(Z_1Z_3)(Z_3Z_5)\cdots = X_0Z_1$$

$$X_1X_3X_5\cdots = Z_0,$$

Projective representation => 2x edge degeneracy

Also protected by TRS $\tilde{T} = K\prod X_i$, $\tilde{T}_L = KY_0Z_1$, $\tilde{T}_L^2 = -1$ Kramers doublet

Partition function, $Z(\Sigma, A, B) = (-1)^{\int_\Sigma A\cup B + w_1 \cup A}$

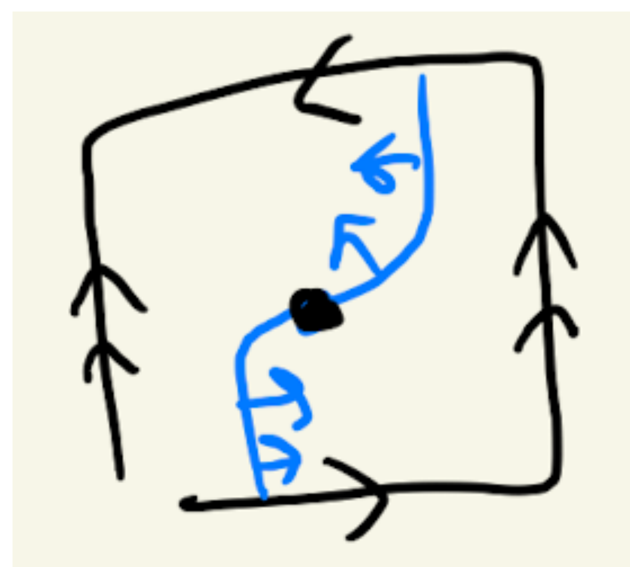
$T^2$ partition function

$$\mathrm{Tr}\ U(g) e^{-\beta H_h} =$$

$$= -1$$

$K^2$ partition function

$$\mathrm{Tr}\ R e^{-\beta H_h} =$$



$$= -1$$

Invariants correspond to $U_1 U_2 U_1^{-1} U_2^{-1}$ and $UTUT^{-1}$ on boundary.

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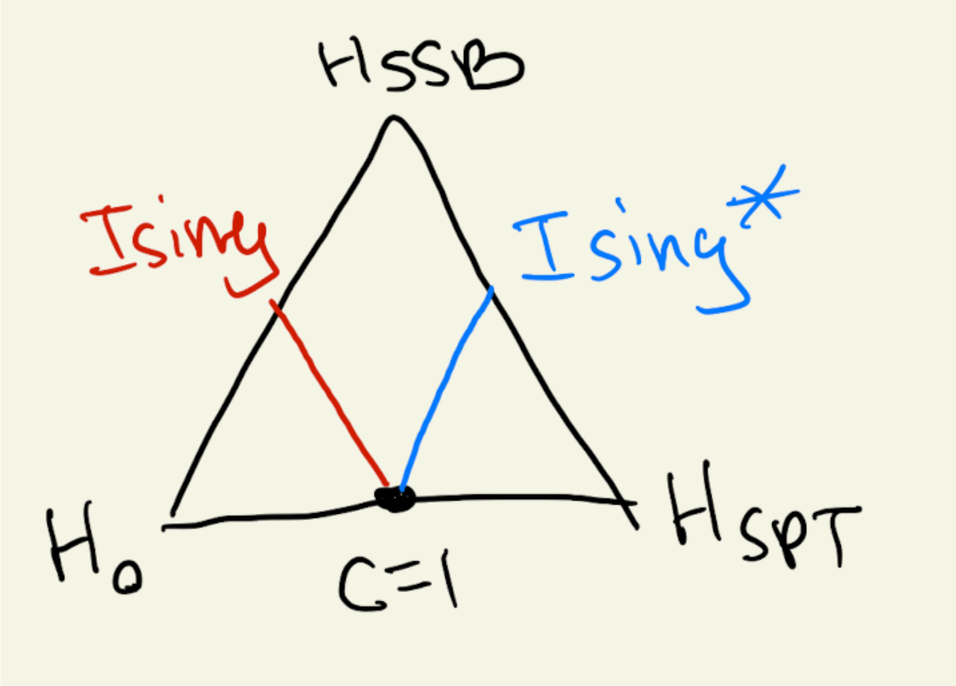
Ising-Cluster model, $T = K, U = \prod X_i$

$$H_0 = -\sum X_i \quad H_{\mathrm{SPT}} = -\sum Z_{i-1}X_iZ_{i+1} \quad H_{\mathrm{SSB}} = \sum -Z_iZ_{i+1}$$

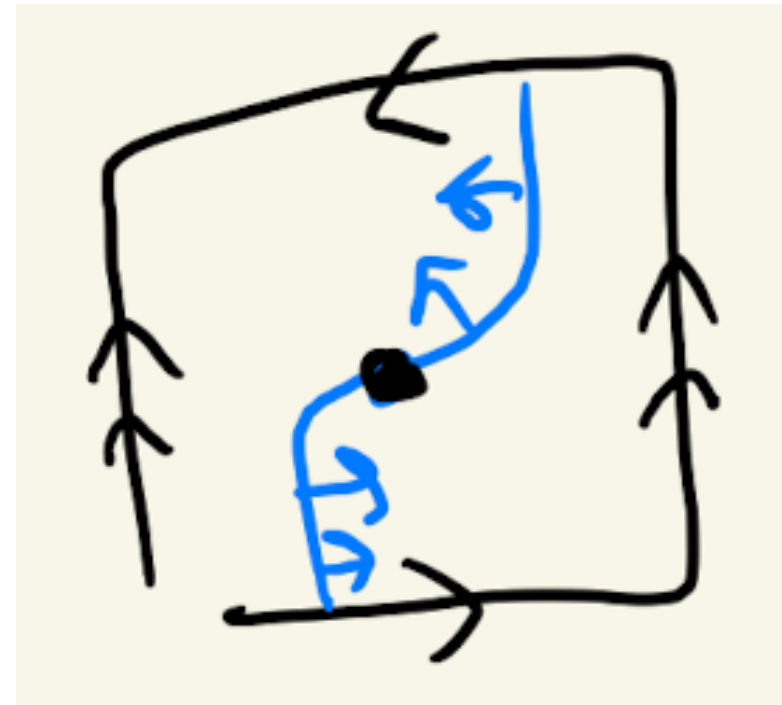



The two Ising CFTs have all the same local correlation functions, since they are related by the entangler $e^{ih}$ (symmetry on central axis)

However, we will show they have different BCFT: the SPT edge degeneracy survives at the transition, a new stable fixed point.

Ising and Ising* distinguished by twisted partition functions on $K^2$

$$\mathrm{Tr}\ R e^{-\beta H_h} =$$

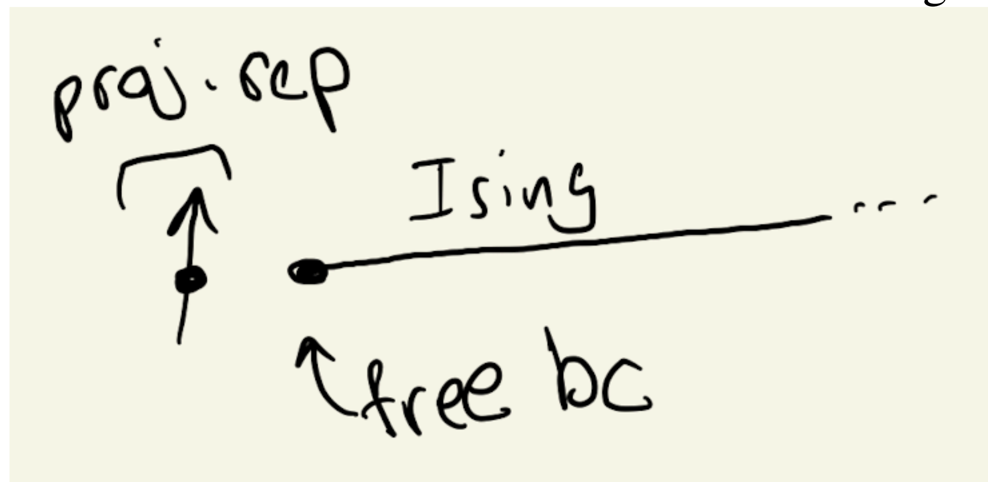


$$Z_{\text{Ising}}(K^2, U, q) = \chi_\sigma(q^2)$$

$$Z_{\text{Ising}^*}(K^2, U, q) = -\chi_\sigma(q^2)$$

The primary disorder operator comes from the charged string operator $Z_0 Y_1 \prod_{i>1} X_i$. The undecorated T-even version is a descendant!

Can study Ising* as an "anomalous Kondo problem", $H_{\text{Ising}}$+ decoupled spin




Symmetry action modified on the "impurity" site $\tilde{T} = Y_L(\prod_i X_i)K$

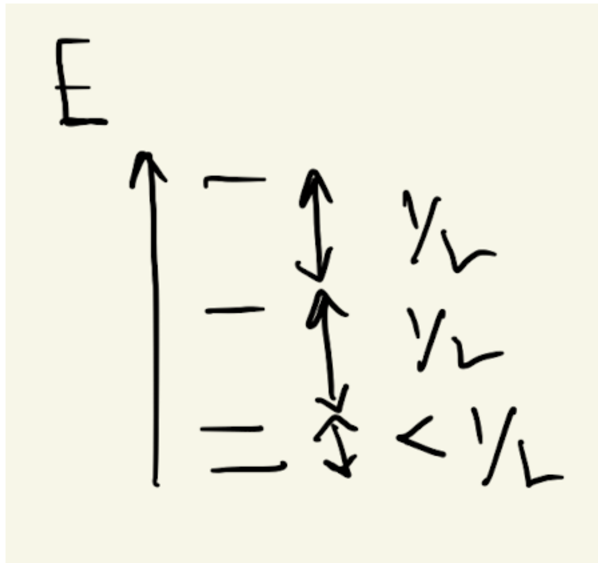
Start in 2x free bc. $Z_L\sigma(0)$ is a relevant symmetric perturbation, flows to $|\uparrow\rangle + |\downarrow\rangle$, $H_{\text{eff}} = -Z_LZ_0 + H_{\text{Ising}}$

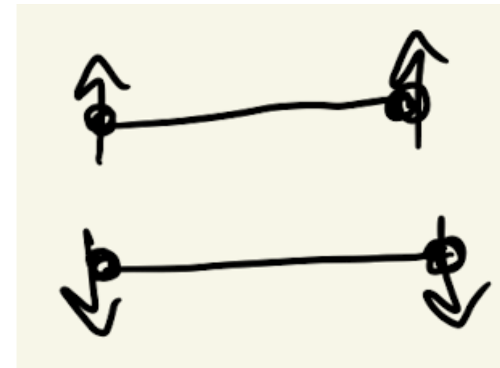
In Ising, there is a relevant bcc operator $X_L \sim \mu(0)$ which induces a flow to $|\text{free}\rangle$, which is the unique stable boundary.

But because of the funny symmetry action, this perturbation is not allowed in Ising*, so *this bc is stable*.

This bcc operator comes from fusion of the bulk disorder operator $\mu$, which is charged because of the SPT twist.

For Ising* on an interval, we find a *2-fold* degeneracy in CFT limit



The bcc operator $\mu$ was disallowed, splitting given by first allowed (irrelevant) operator

If $\mu$ was charged under a unitary symmetry, the splitting must be exponential (and the symmetry must be gapped)

all other bcc operators are descendants of $\mu$ and have the same charge.

TRS is more interesting because descendants can have different charges

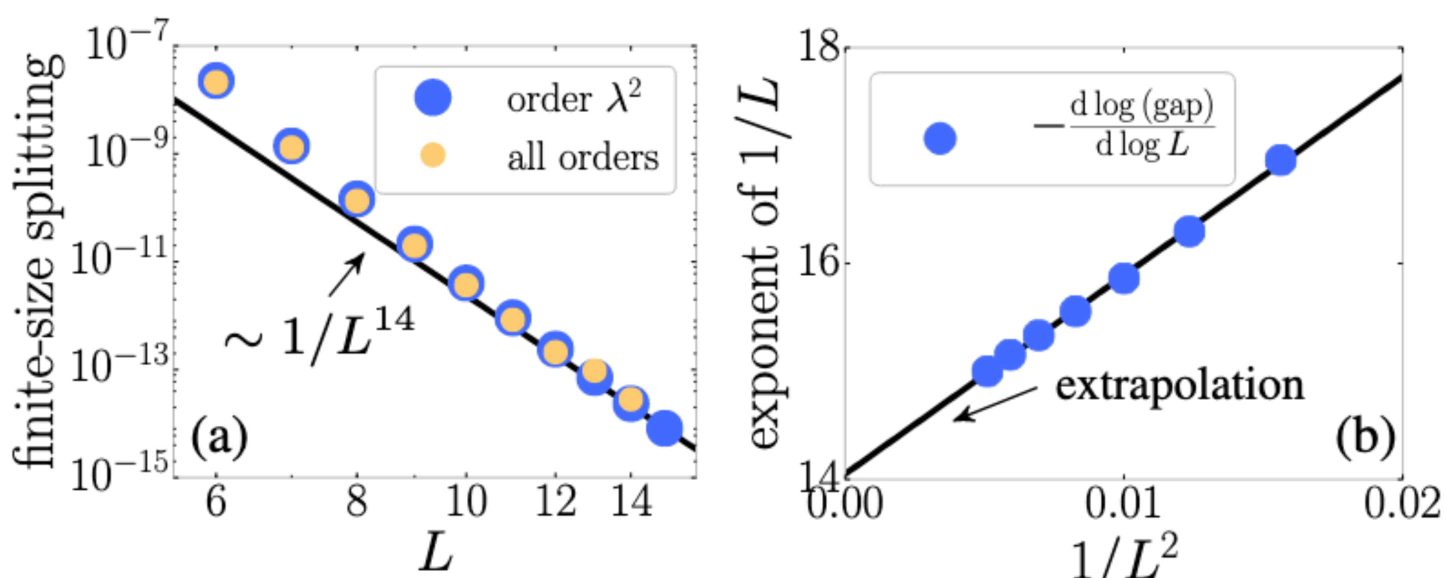
The first descendant is $T$-even but can be rotated away

$$H + g\partial_t \mu(0) = e^{i\lambda\mu(0)} H e^{-i\lambda\mu(0)} + O(\lambda^2)$$

...as can the next few descendants

$$q^{-\frac{23}{48}} \chi_{1,2}(q) = \big(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 4q^7 + \cdots\big).$$

First contribution from 7th descendant, of dimension 1/2+7



Mermin-Wagner theorem implies no discrete SSB in 0+1D (domain walls favored by entropic log vs constant energy)

In (gapped) SPTs, the degeneracy is not SSB. Recall we have zero modes

$$X_0X_2X_4\cdots = X_0(Z_1Z_3)(Z_3Z_5)\cdots = X_0Z_1$$

$$X_1X_3X_5\cdots = Z_0,$$

It's tempting to consider these as SSB order parameters.

There are ground states where $\langle Z_0 \rangle \neq 0$ but also where $\langle Z_0 \rangle = 0$, such as eigenstates of $X_0Z_1$, and these are on equal footing: quantum degeneracy at the edge.

In the CFT we spoiled $X_0Z_1$ in the first step, there is really only $Z_0$. In all local ground states, $\langle Z_0 \rangle \neq 0$, so this degeneracy is true SSB.

The SPT "twist" $e^{ih}$ does not modify local correlation functions, but it does modify twisted sectors and the stability of boundary CFTs.

We saw a degeneracy in the finite-size spectrum with a surprising power law.

We can prove these twisted CFTs have different boundary states from their untwisted partner (a free bc is not allowed), but we can't prove there is an edge degeneracy.

Next we'll see an example where the edge degeneracy briefly disappears as a boundary critical point.

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Ising-Cluster model, $T = K, U = \prod X_i$

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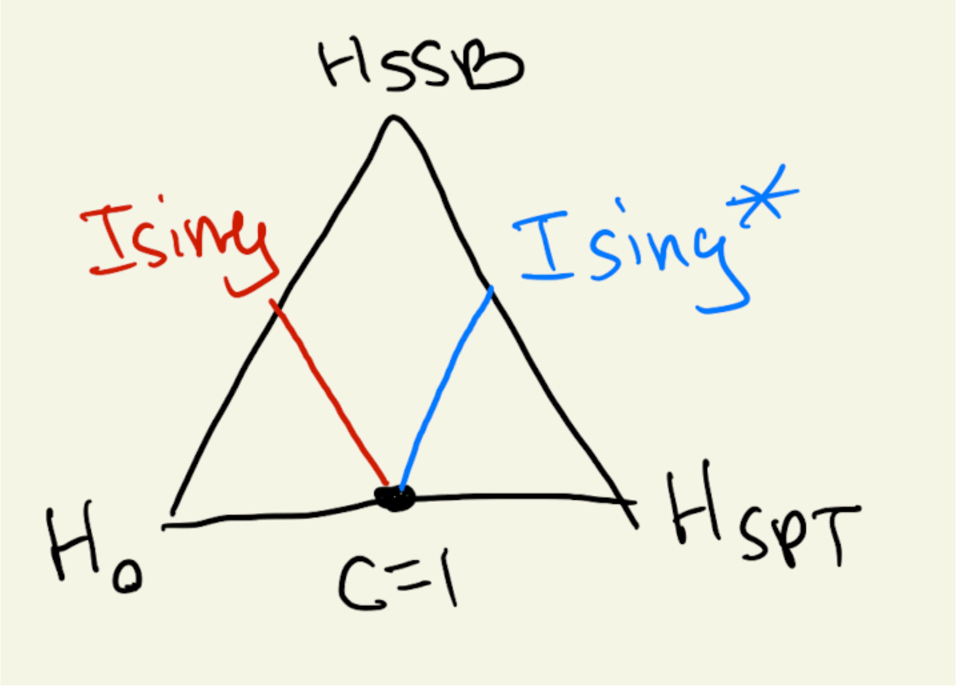



The SPT-trivial transition $H_0 + H_{\mathrm{SPT}}$ is a c=1 boson.

$e^{ih}$ acts is a (anomalous) symmetry of this point - no boson* CFT! The twisted torus partition function vanishes due to the anomaly.

The edge mode disappears at a boundary KT transition.

$\mathbb{Z}_3^2$ Cluster models, $H^2(\mathbb{Z}_3^2) = \mathbb{Z}_3$

qutrits with symmetry $\prod_i X_{2i(+1)}$

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}$$

$$H_0 = -\sum X_i + X_i^\dagger + hc$$

This transition is described by a c=8/5 CFT ~ Potts^2 orbifold

$$H_{\mathrm{SPT}} = -\sum Z_{2i-1} X_{2i} Z_{2i+1}^\dagger + Z_{2i} X_{2i+1}^\dagger Z_{2i+2}^\dagger$$

The three transitions are related by entanglers $e^{ih}$ and all have the same spectrum of local operators.

$$H_{\mathrm{SPT}'} = -\sum Z_{2i-1} X_{2i}^\dagger Z_{2i+1}^\dagger + Z_{2i} X_{2i+1} Z_{2i+2}^\dagger$$

SPT-SPT' is distinguished from the other two by the torus partition function, twist spectrum is now degenerate, two lowest operators are charged for SPT-SPT'

The triv-SPT transitions have only nondegen stable bcs

The SPT-SPT' transition has two SSB edge phases, and a "continuous" transition between them, reminiscent of DQCP.

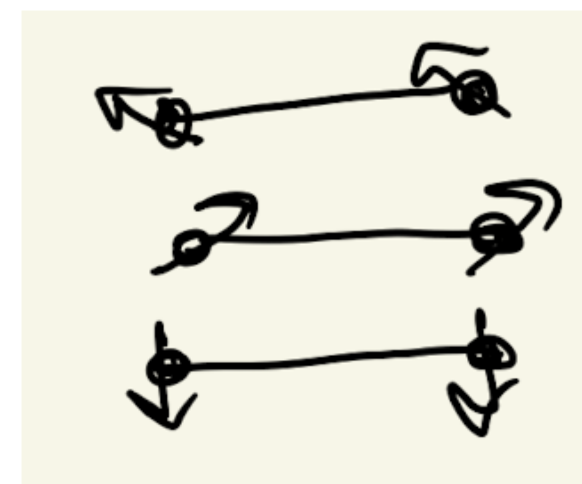
$$H_{\mathrm{SPT}} = -\sum Z_{2i-1}X_{2i}Z^\dagger_{2i+1} + Z_{2i}X^\dagger_{2i+1}Z^\dagger_{2i+2}$$

Study open chain [1,2N+1]:

$$H_{\mathrm{SPT'}} = -\sum Z_{2i-1}X^\dagger_{2i}Z^\dagger_{2i+1} + Z_{2i}X_{2i+1}Z^\dagger_{2i+2}$$

$$H = H_{\mathrm{SPT}} + H_{\mathrm{SPT'}} - b_L X_1 - b_R X_{2N+1}$$

For b=0 there are zero mode operators $Z_{1,2N+1}$ commuting with the Hamiltonian and not with the symmetry => exact 3 fold degeneracy (with perfect end-to-end correlation)



Think of these as $\mathbb{Z}_3^{\mathrm{odd}}$-SSB order params

For infinite b, we can project $X_1 = X_{2N+1} = 1$

We find this shortens the chain to [2,2N], and now $\mathbb{Z}_3^{\mathrm{even}}$ is spontaneously broken with order param $Z_2$. Can also arrange each edge to break each symmetry if $b_L \neq b_R$.

We will analyze the stability of these boundary phases and study the phase diagram connecting them. We will need some new tricks.

$$H = H_{\mathrm{SPT}} + H_{\mathrm{SPT}'} - b_L X_1 - b_R X_{2N+1}$$

Equivalent to a single Potts chain on a ring with two defects, where we sum over flux sectors through the ring

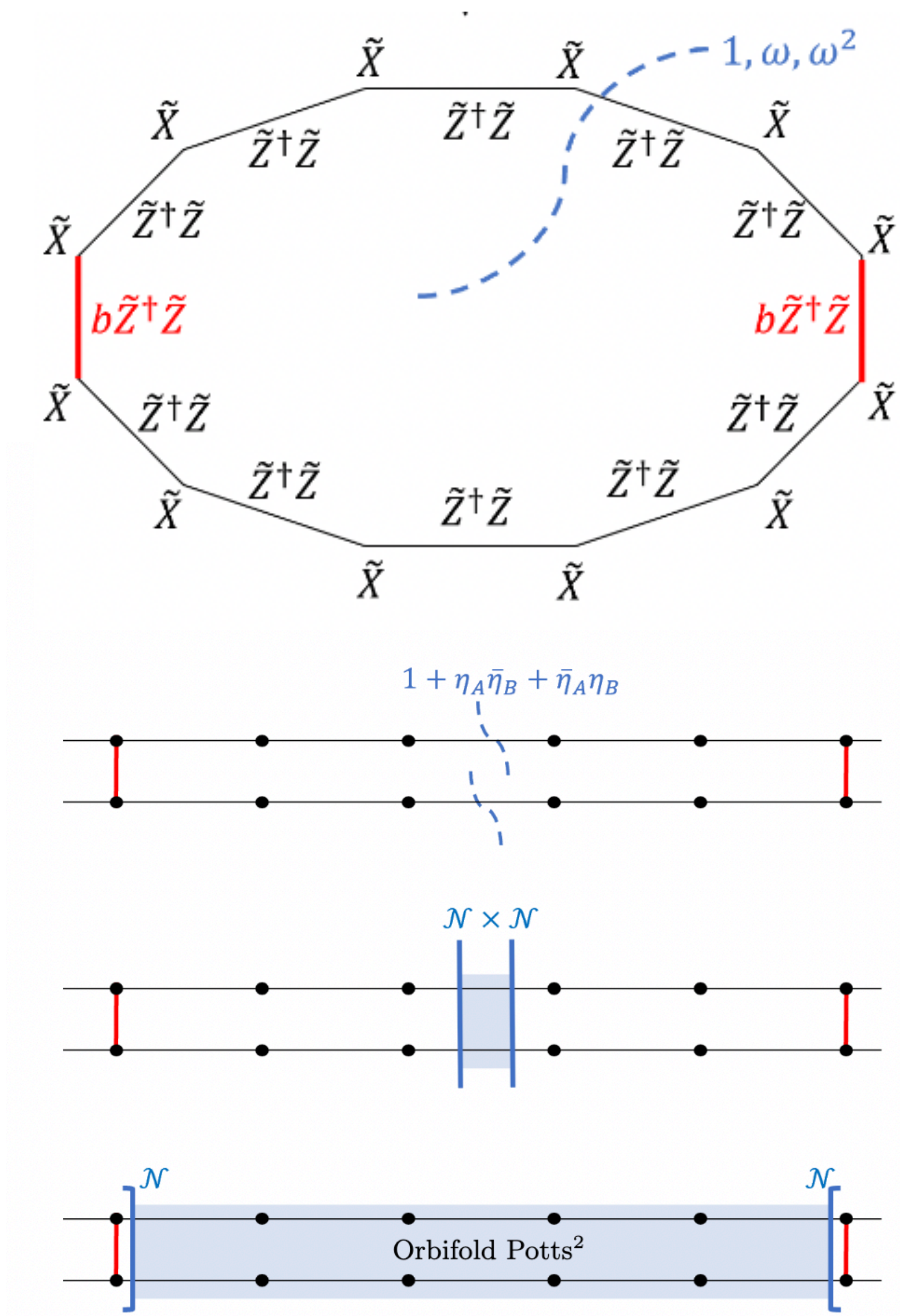
The symmetries are now the $\mathbb{Z}_3$ Potts symmetry and the magnetic flux symmetry

b=0 is a cut chain with free boundaries: magnetic symmetry is spontaneously broken, 3x degen

b=infty gives the (A,A)+(B,B)+(C,C) defects, also 3x degen

b=1 is a trivial defect - **no degeneracy**

KW duality acts at b=1, exchanging two nearby phases (only emergent duality)



$$H = H_{\text{SPT}} + H_{\text{SPT}'} - b_L X_1 - b_R X_{2N+1}$$

Let's study the stability of the (free,free) defect at b = 0, the other is similar

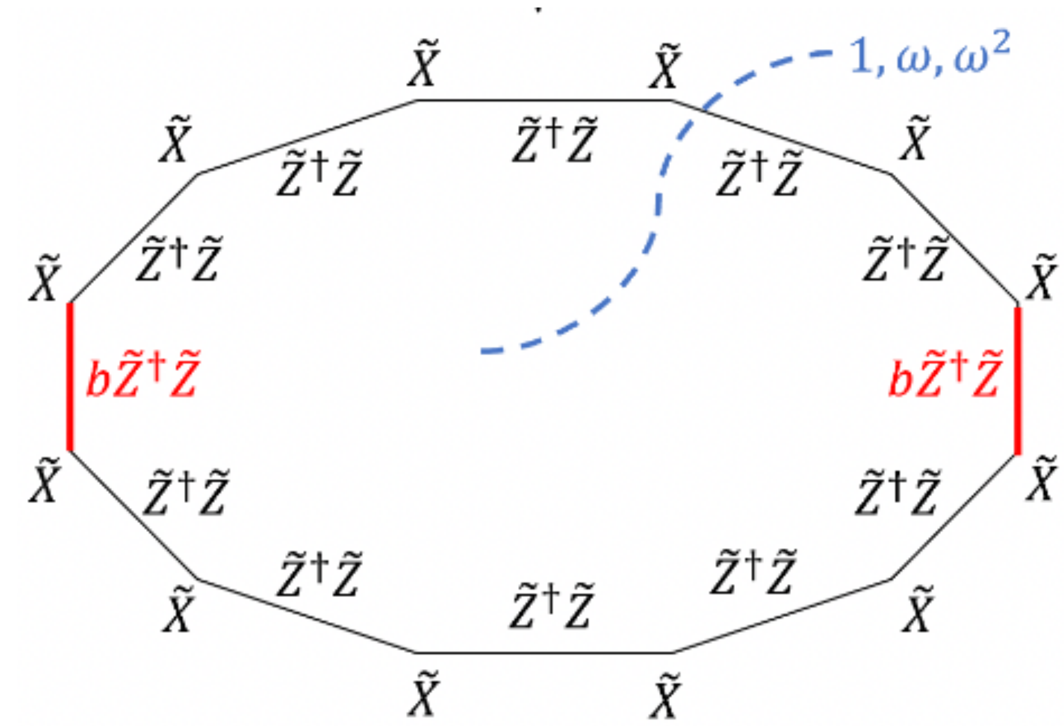
The most relevant symmetry-allowed perturbation is $b\tilde{Z}^\dagger\tilde{Z}$ itself, which we recognize as the double order parameter $\sigma_1(0)\sigma_2(0)$,

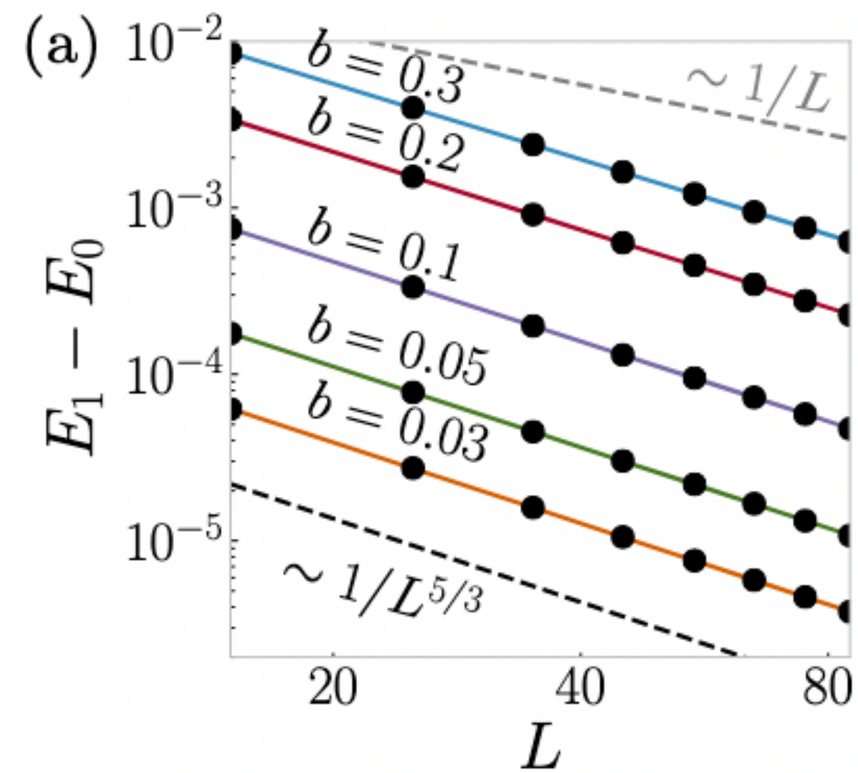
This has dimension 2*2/3 => **irrelevant**

The finite-size splitting is given at second order by $g^2/L^{2\Delta-1} \sim 1/L^{5/3}$

Observed this splitting in numerics

Implies both that this is a stable boundary phase and the 3x degeneracy persists beyond the fine-tuned point



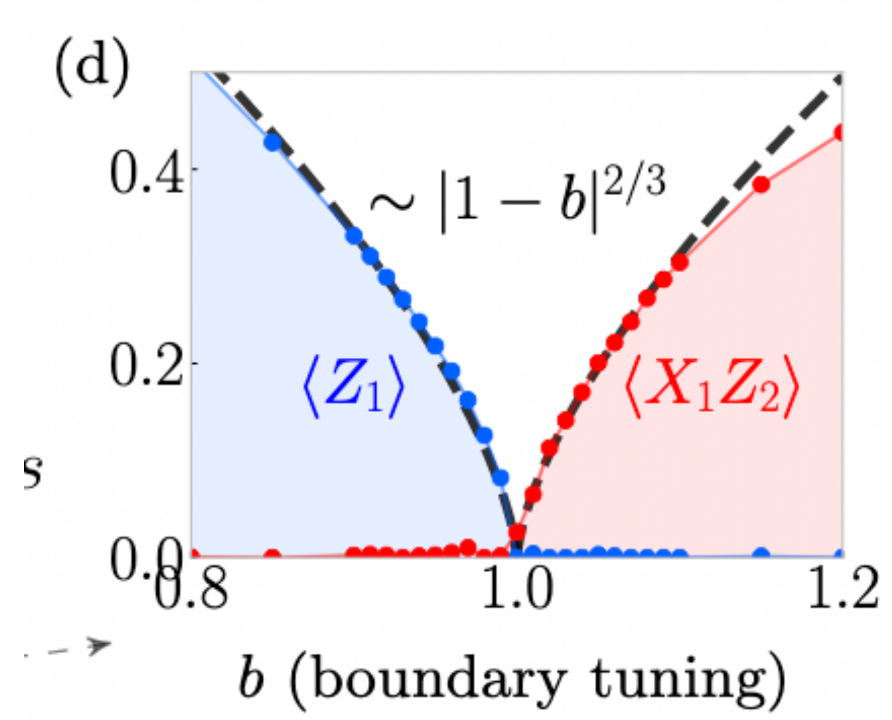
Now we study the boundary order parameters as we approach the transition.

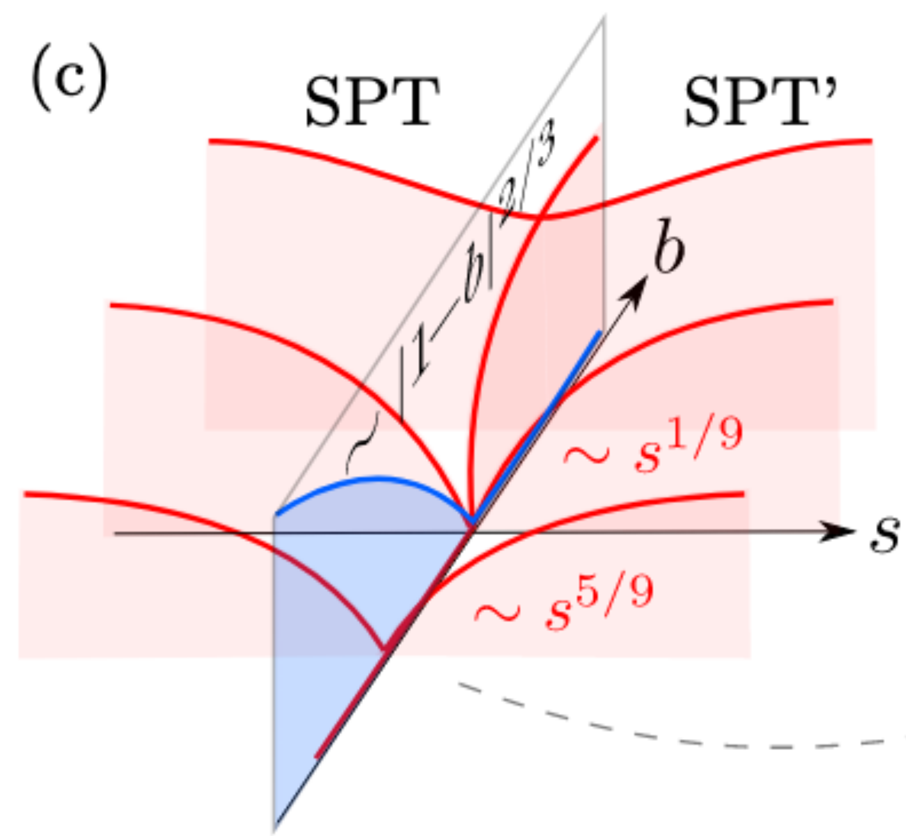
As anticipated, there is no SSB at the b=1 point - it's a continuous transition.

Can identify the order parameter of the b=infty (or b=0) defect with the Potts order/disorder parameter. The observed 2/3 scaling is $\Delta_\sigma/(1-\Delta_\epsilon)$.

In a sense, the fate of the edge modes is that they delocalize: the translation symmetry in the ring at b=1 means nothing is localized at the boundary anymore.

In the nearby SPT phases, both order parameters are nonzero, but each vanishes on half the transition line.



At a transition between two non-trivial SPTs, found an interesting boundary phase diagram with two ordered phases and a "DQCP" between them.

Can compare to a twist of the same theory, like $H_0 + H_{\mathrm{SPT}}$ or $H_0 + H_{\mathrm{SPT}'}$, but there is no boundary degeneracy in this case.

$$H_{\mathrm{SPT}} = -\sum Z_{2i-1}X_{2i}Z^{\dagger}_{2i+1} + Z_{2i}X^{\dagger}_{2i+1}Z^{\dagger}_{2i+2} + X_i + X_i^{\dagger}$$

Is there a bulk-boundary correspondence in these systems, or do we have to do this analysis on a case-by-case basis?

Are all stable boundaries symmetry breaking?

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So far all the topological ingredients have come from gapped SPTs.

It is possible for gapless systems to realize topology not possible in a gapped theory.

We study on-site symmetries, which have vanishing 't Hooft anomaly.

It is possible for global anomalies to emerge in the IR, so long as they satisfy anomaly-matching

$$\rho : G_{UV} \to G_{IR}$$

$$\rho^* \omega = d\alpha$$

A simple case is if the IR symmetry is a quotient

$$G_{IR} = G_{UV}/G_{gap}$$

$$\rho^*\omega = d\alpha$$

$$Z(X, A) = Z_{\text{CFT}}(X, A_{IR})e^{i\int \alpha(A_{IR}, A_{gap})}$$

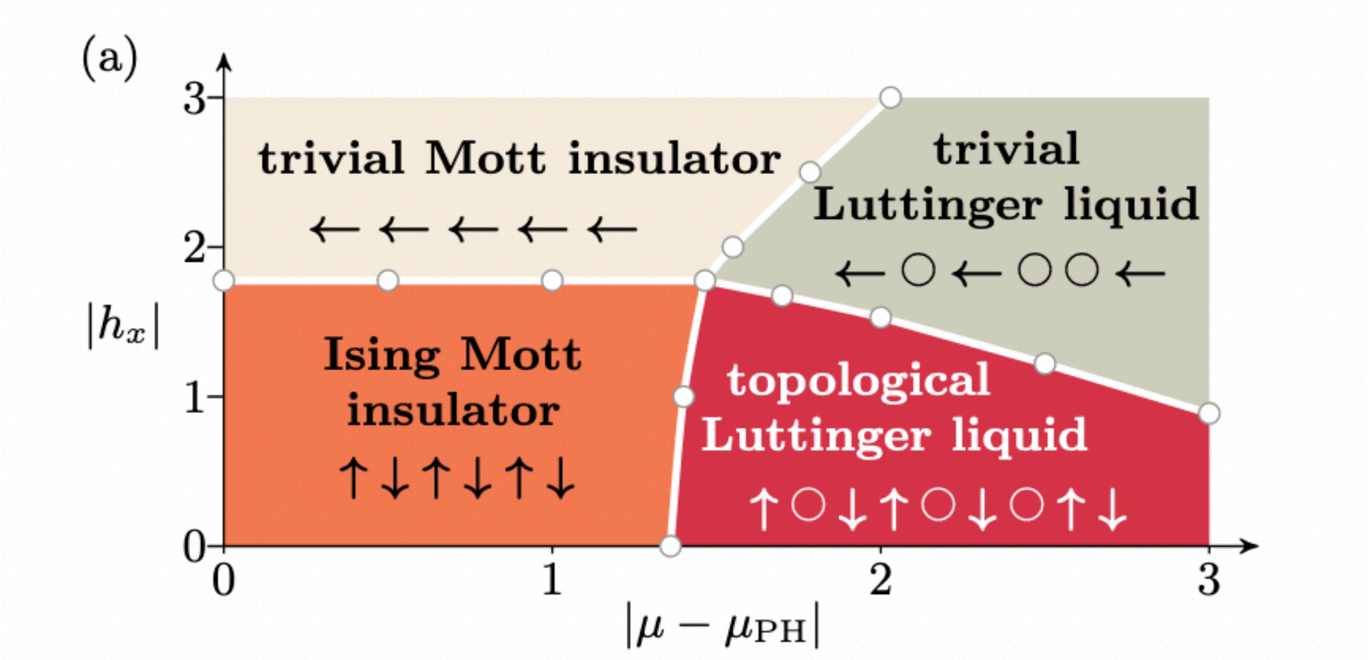
$Z_{CFT}$ does not need to be gauge invariant under $A_{IR}$ gauge transformations, so long as we have the counterterm $\alpha$

We can consider the counterterm as coming from integrating out gapped fundamental $G_{UV}$ charges. All gapless modes are only charged under $G_{IR}$ (this is where we need gaplessness).

The simplest solution is in 1+1d, $G_{UV} = \mathbb{Z}_4$, $G_{IR} = \mathbb{Z}_2$, $\omega = \frac{1}{4}A_{IR}dA_{IR}$, $\alpha = \frac{1}{2}A_{IR}A_{gap}$, $dA_{gap} = \frac{1}{2}dA_{IR}$.

t-J model, "Hubbard-Ising" model, hole-doped Ising model

$$H = \sum_i -t(c^\dagger_{j+1,s}c_{j,s} + hc) + Un_{j,\uparrow}n_{j,\downarrow} - \mu n_{j,s} + J_z S^z_j S^z_{j+1} + h_x S^x_j$$




We study symmetry $G_{UV} = \mathbb{Z}_4$,

$R_x = \prod_i e^{\frac{i\pi}{2}S^x_i}$, $R_x^2 = (-1)^F$.

Crucially, the fermions are gapped, so $G_{gap} = \mathbb{Z}_2^F$.

The holes destroy long-range order in AFM, but the AFM pattern is preserved in the "squeezed space". Long range string order parameter:

$$S^x_i \prod_{j>i} (-1)^{ic^\dagger_{j,s}c_{j,s}}$$

This string order parameter already sees the intrinsically gapless SPT here.

It's a $(-1)^F$-twisted sector operator which is charged under $G_{IR} = \mathbb{Z}_2$, reflects the counterterm $\frac{1}{2}A_{gap}A_{IR}$ (if we imagine the torus partition function)

We can describe the model in bosonization by two compact scalars $(\varphi_s, \theta_s)$, $s = \uparrow, \downarrow$,

$$\psi^\dagger_{s,\pm} = U_s e^{\pm i\varphi_s/2 + i\theta_s} \qquad R_x : \begin{cases} \varphi_s \mapsto \varphi_{-s} \\ \theta_\uparrow \mapsto \theta_\downarrow + \pi/2 \\ \theta_\downarrow \mapsto \theta_\uparrow - \pi/2 \end{cases} \qquad \begin{cases} U_\uparrow \mapsto U_\downarrow \\ U_\downarrow \mapsto -U_\uparrow \end{cases}$$

The Ising interaction is $S^z S^z \sim \cos(\varphi_\uparrow - \varphi_\downarrow)$, which pins $\Phi_1 = \varphi_\uparrow - \varphi_\downarrow$.

The remaining gapless degrees of freedom are Cooper pairs (which preserve the AFM pattern)

$$\psi^\dagger_{\uparrow,+}\psi^\dagger_{\downarrow,-} + hc \sim \cos(\theta_\uparrow + \theta_\downarrow - \varphi_\uparrow/2 + \varphi_\downarrow/2)$$

This motivates the definition $\Phi_2 = \varphi_\downarrow$, $\Theta_2 = \theta_\uparrow + \theta_\downarrow - \varphi_\uparrow/2 + \varphi_\downarrow/2$. In these variables, we have

$$R_x : \begin{cases} \Phi_2 \mapsto \Phi_2 + \langle \Phi_1 \rangle \\ \Theta_2 \mapsto \Theta_2 + \langle \Phi_1 \rangle, \end{cases}$$

**Anomalous $\mathbb{Z}_2$ symmetry depending on the sign of $S^z S^z$!**

The lightest twist sector operator is $c_{i,s} \prod_{j>i} R^x_j$, which has "fractional" $G_{IR}$ charge. Of course, the charge is not really fractional from the UV pov.

Any gapless theory with a global anomaly can occur as an intrinsically gapless SPT.

These models have edge modes where either the symmetry is broken or the gapped dofs become gapless again.

R. Ma, L. Zou, and C. Wang (2110.08280) studied a realization of the usual 2+1d DQCP at a phase transition between a QSH state and verified this prediction in a large N calculation. The edge has an "impossible" mixed SU(2) x U(1) anomaly.

R. Wen and A. Potter (2208.09001) recently devised many new examples and gave an explanation of the counterterm $\alpha$ in terms of an anomalous SPT pump.

A mechanism that doesn't require a gapped charge sector is still lacking, although we have some preliminary results in this direction.

We would like to better understand igSPT transitions as well.

There's a world of quantum phases and phase transitions beyond SSB.

SPTs are building blocks of many interesting systems.

We need to get better at studying extended operators and boundaries if we want to push this to higher dimensions.

2+1d SPT-SSB transitions seem especially promising. Room for the bootstrap?

Bulk-boundary correspondence? These theories differ in their twisted bulk sectors, do they have the same BCFTs but with different stability conditions? How do we compute this?

Are the stable boundaries always SSB?