Measurement-Prepared Quantum Critical States

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Acknowledgement

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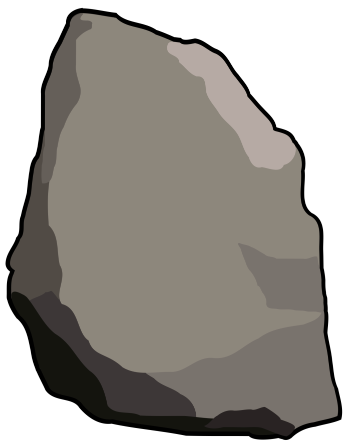
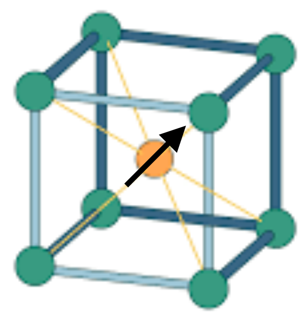
Reference: 2208.11699

Also see: 2208.11136 by G. Zhu, et al.

Scope

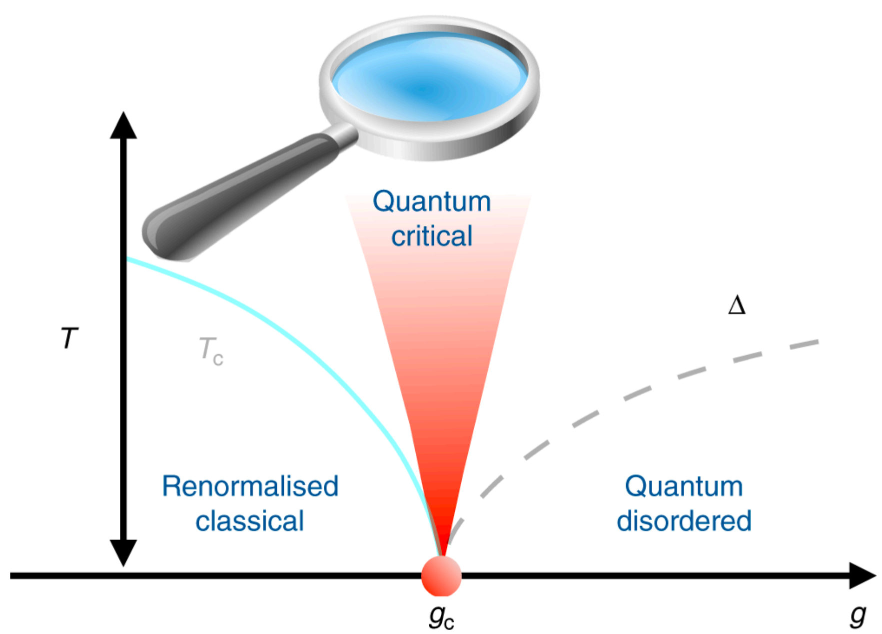
- Condensed matter usually studies equilibrium many-body systems:





$$H(g) = \sum_i H_i(g)$$

$$\mathcal{L}[\phi, \partial\phi]$$

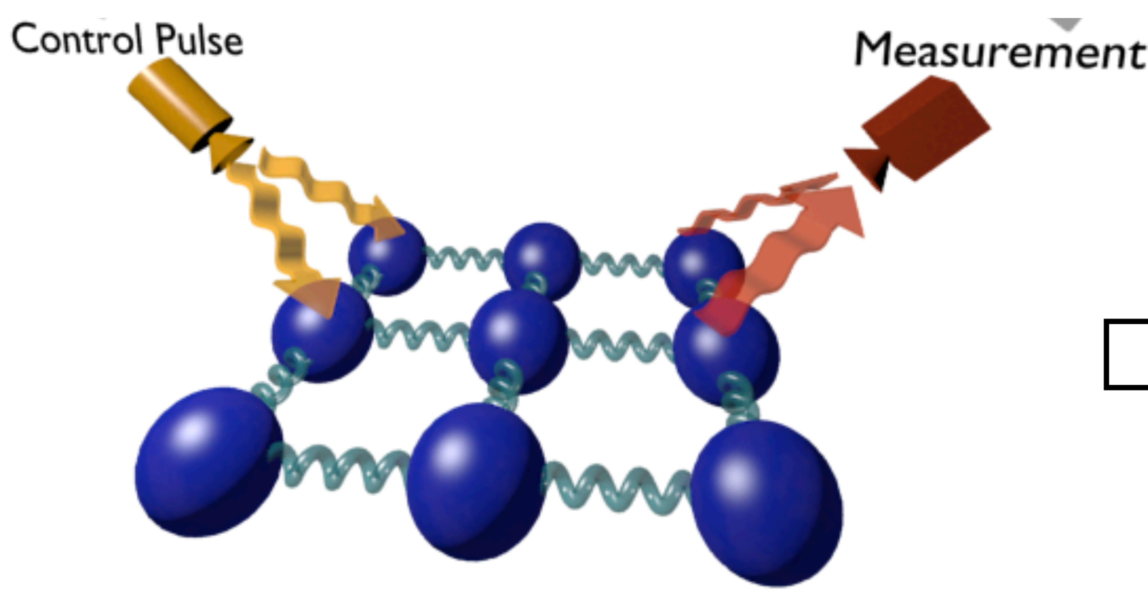


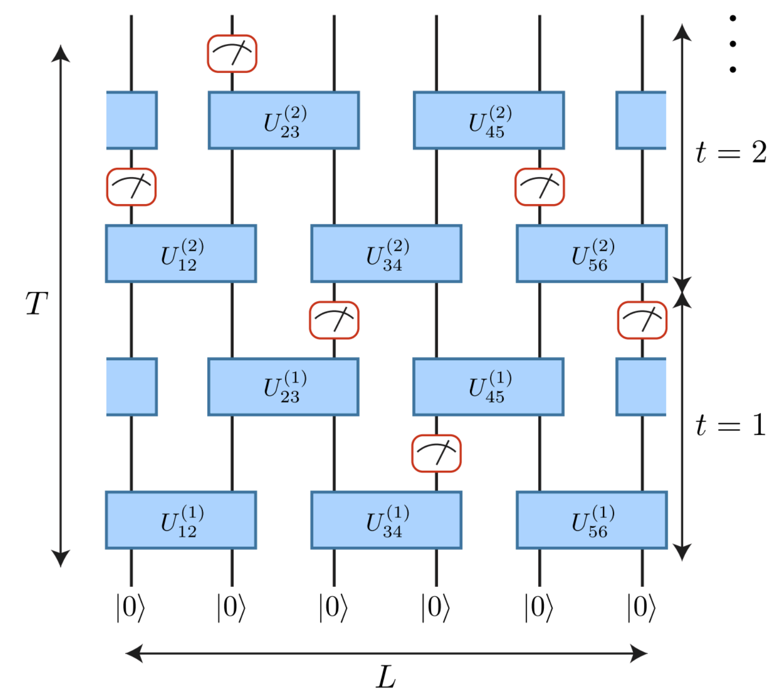
- IR phases & phase transitions emerging from microscopic lattice models

Book: Sachdev (2011)

Scope

- Quantum simulators — programmable lattice quantum systems





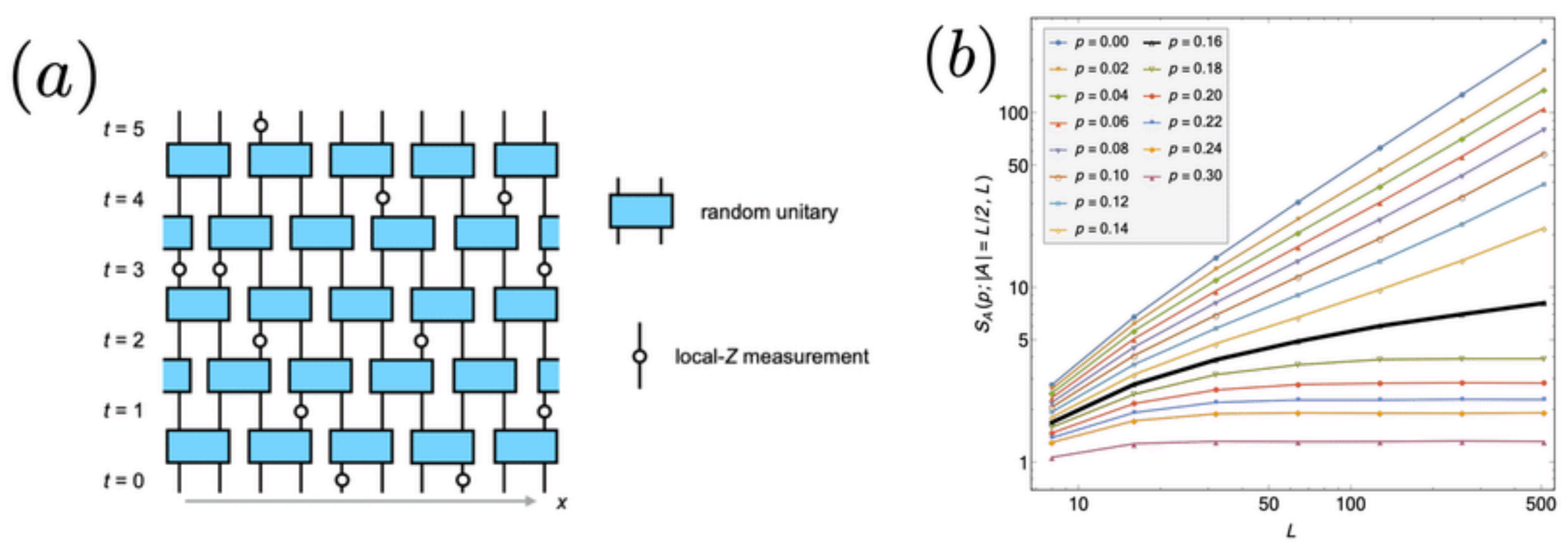
Platforms: cold atoms, trapped ions, Rydberg atoms, superconducting qubits...

Unitary evolutions + measurements

A review: E. Altman et al. (2021)

What to Do with Quantum Simulators?

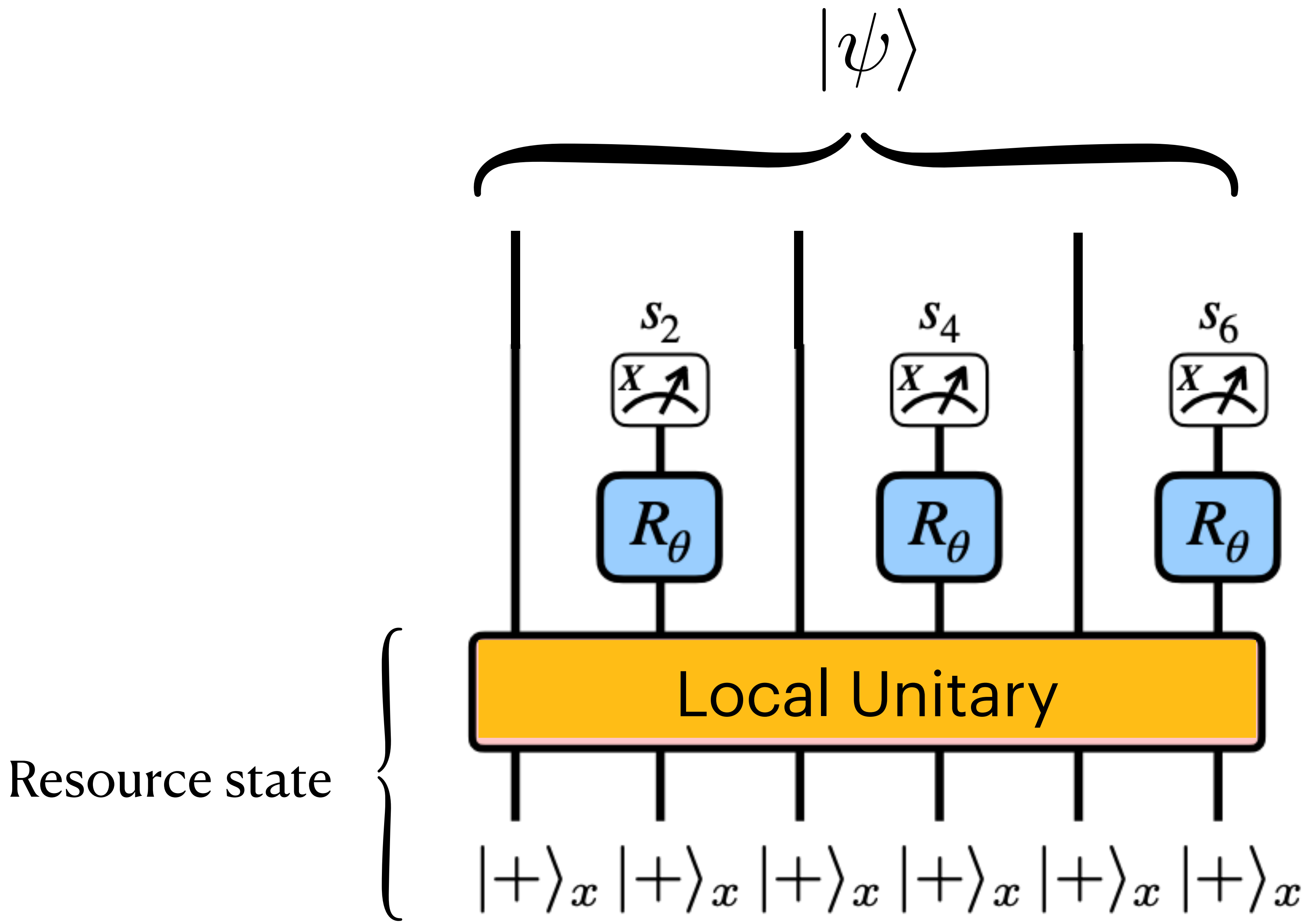
- Simulating quantum hamiltonians
  - e.g. Bose Hubbard model, Fermi Hubbard model, frustrated spin models…
- Quantum dynamics with unitaries and measurements



For a review: M. Fisher, V. Khemani, A. Nahum and S. Vijay (2022)

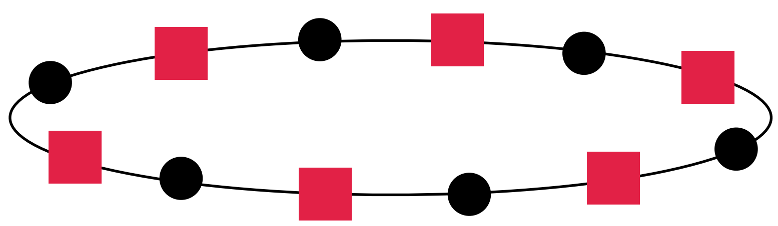
- Preparing interesting quantum states

State Preparation



Resource States

- Resource state: 1d Cluster state (Z2 x Z2 SPT)



2N spins, periodic boundary condition

$$H = -\sum_i Z_{i-1} X_i Z_{i+1}$$

$$Z_{i-1} X_i Z_{i+1} |\psi\rangle = |\psi\rangle$$

- Constraints (**symmetries**)

$$\Pi_{n=1}^{N} X_{2n} = 1$$

$$\Pi_{n=1}^{N} X_{2n-1} = 1$$

- This state can be generated from a product state by a finite depth unitary.

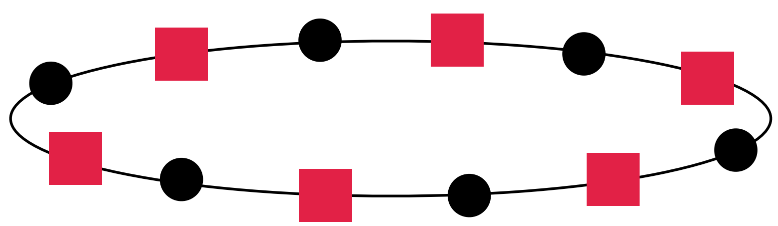
$$|\psi\rangle = \prod_n CZ_{n,n+1} |+\rangle^{\otimes 2N}$$

Preparing 1d Symmetry Breaking States

- Greenberger–Horne–Zeilinger (GHZ) state in 1 spatial dimension

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow \dots \uparrow\rangle + |\downarrow\downarrow \dots \downarrow\rangle)$$

- Starting point: 1d Cluster state



2N spins, periodic boundary condition

Constraints

$$Z_{i-1}X_iZ_{i+1} = 1$$

$$\Pi_{n=1}^{N} X_{2n-1} = 1$$

$$\Pi_{n=1}^{N} X_{2n} = 1$$

- Measurement X on **even** sites and **post-select +1 outcomes**

$$\mathcal{P}_X = \prod_{n=1}^{N} \frac{1+X_{2n}}{2} \Rightarrow \begin{array}{c} Z_{2n-1}Z_{2n+1} = 1 \\ \Pi_{n=1}^{N} X_{2n-1} = 1 \end{array} \Rightarrow \mathcal{P}_X|\psi\rangle \sim |GHZ\rangle$$

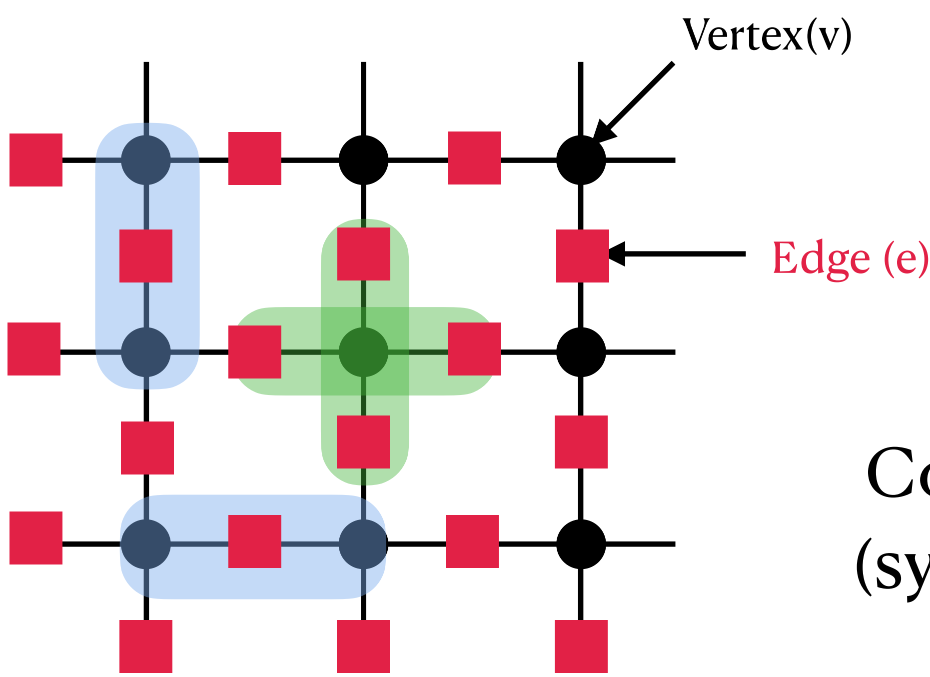
$$\langle Z_iZ_j\rangle_{|\mathcal{P}_X\psi\rangle} = \frac{\langle \mathcal{P}_X\psi|Z_iZ_j|\mathcal{P}_X\psi\rangle}{\langle \mathcal{P}_X\psi|\mathcal{P}_X\psi\rangle} = 1$$

H. J. Briegel and R. Raussendorf (2001)

Preparing 2d Symmetry Breaking States

2d GHZ state

- Cluster state on Lieb lattice



$$H = -\begin{matrix} & Z & \\ Z & X & Z \\ & Z & \end{matrix} - ZXZ - \begin{matrix} Z \\ X \\ Z \end{matrix}$$

Constraints (symmetries)

$\mathbb{Z}_2^{(0)}$ 0-form: $g = \prod_v X_v$

$\mathbb{Z}_2^{(1)}$ 1-form: $h_\gamma = \prod_{e \in \gamma} \boldsymbol{X}_e$

- Measure X on the edges, post-select the +1 outcome

$$\mathcal{P}_X = \prod_e \frac{1 + X_e}{2} \Rightarrow \boxed{ZZ} = \boxed{\begin{matrix} Z \\ Z \end{matrix}} = 1 \Rightarrow \mathcal{P}_X |\psi\rangle \sim |GHZ\rangle$$

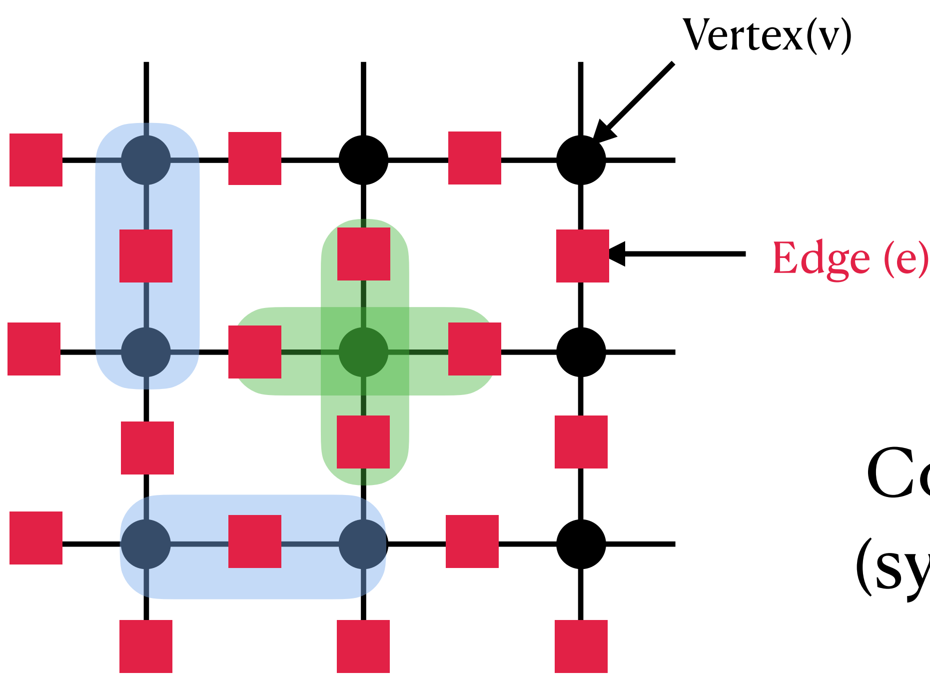
Also $\prod_v X_v = 1$

$$\langle Z_i Z_j \rangle_{|\mathcal{P}_X \psi\rangle} = 1$$

Preparing 2d Long-Range Entangled States

2d toric code state

- Cluster state on Lieb lattice



$$H = -\begin{array}{ccc} & Z & \\ Z & X & Z \\ & Z & \end{array} - ZXZ - \begin{array}{c} Z \\ X \\ Z \end{array}$$

Constraints (symmetries)

$\mathbb{Z}_2^{(0)}$ 0-form: $g = \prod_v X_v$

$\mathbb{Z}_2^{(1)}$ 1-form: $h_\gamma = \prod_{e \in \gamma} \boldsymbol{X}_e$

- Measure X on vertices, post-select the +1 outcome

$$\mathcal{P}_X = \prod_v \frac{1 + X_v}{2} \quad \Rightarrow \quad \begin{array}{ccc} & Z & \\ Z & + & Z \\ & Z & \end{array} = 1 \;\; \& \;\; \begin{array}{ccc} & X & \\ X & & X \\ & X & \end{array} = 1$$

Toric code ground state Wave function

$$\mathcal{P}|\psi\rangle \sim |T.C.\rangle$$

R. Raussendorf, S. Bravyi, and J. Harrington. (2005).

N. Tantivasadakarn, R. Thorngren, A. Vishwanath, and R. Verresen (2021)

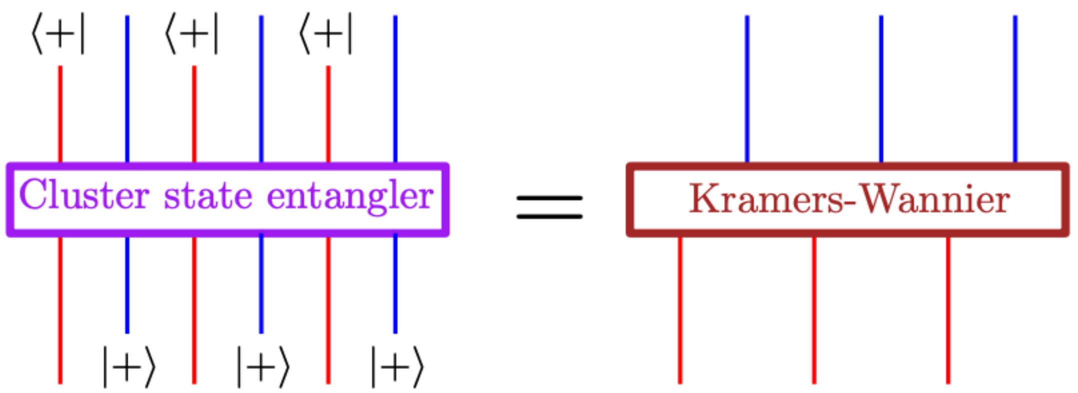
More on State Preparation

Some remarks

- **Unitaries** + **measurements** can prepare states that are **not reachable with only unitaries.**

- The cluster state is equivalent to a product state evolved by a shallow unitary evolution. Combined with the measurement, the whole circuit effectively implements a **Kramers-Wannier duality (gauging).**

$$|\psi\rangle = \prod_n CZ_{n,n+1} |+\rangle^{\otimes 2N}$$



- Recently, many **non-abelian** topological order states are also shown to be constructed within the unitary+measurement scheme.

N. Tantivasadakarn, R. Thorngren, A. Vishwanath, and R. Verresen (2021)

N. Tantivasadakarn, A. Vishwanath, and R. Verresen (2021)(2022)

S. Bravy, I. Kim, A. Kliesch, and R. Koenig (2022)

TC Lu, L. Lessa, I. Kim, and T. Hseih (2022)

Outline

- Motivations - Prepare Exotic States in Quantum Simulators
- Measurement-prepared Quantum Critical States (with Post-selections)
- No Post-selection — Randomness — 2d Random bond Ising model — Nishimori Line

A Trivial Limit of the Measurements

Symmetric product state

- Consider the 1d cluster state, now measure all spins on the **even sites** along the **Z** direction and post-select +1 outcomes.

$$\left.\begin{array}{ll} \text{From} & Z_{i-1}X_iZ_{i+1} = 1 \\ \text{\& measurement outcome} & Z_{2n} = 1 \end{array}\right\} \Rightarrow X_{2n-1} = 1$$

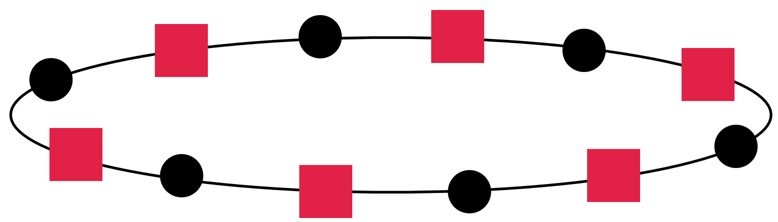
- The resulting state is a **symmetric product state**

$$\mathcal{P}_Z|\psi\rangle \sim |++++...\rangle \qquad \langle Z_i Z_j\rangle_{|\mathcal{P}_Z\psi\rangle} = 0$$

- Q: what if we measure along an intermediate angle? Will there be a transition?

Decorated Domain Wall States

A closer look to the 1d cluster state



2N spins, periodic boundary condition

$$\begin{cases} Z_{i-1}X_iZ_{i+1} = 1 \\ \Pi_{n=1}^{N} X_{2n-1} = 1 \\ \Pi_{n=1}^{N} X_{2n} = 1 \end{cases}$$

$$|\psi\rangle \sim |\uparrow + \uparrow - \downarrow - \uparrow + \uparrow \ldots\rangle + |\downarrow + \downarrow - \uparrow - \downarrow + \downarrow \ldots\rangle$$
$$+ |\uparrow - \downarrow + \downarrow - \uparrow + \uparrow \ldots\rangle + |\downarrow - \uparrow + \uparrow - \downarrow + \downarrow \ldots\rangle$$
$$+ \ldots$$

- Measuring **X** operators on the even sites —> **fixing** a unique domain wall configuration — the **zero temperature** limit of a classical Ising model.

$$\langle Z_i Z_j \rangle_{|\mathcal{P}_X \psi\rangle} = 1$$

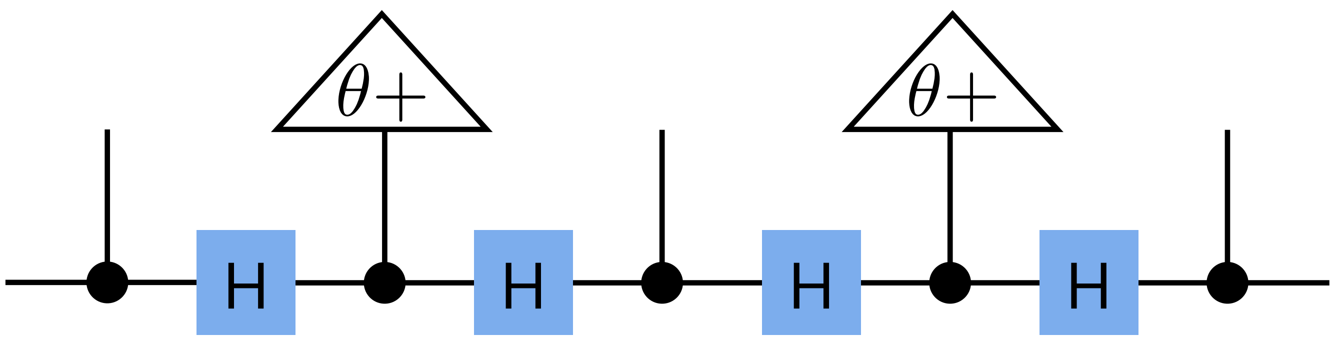
- Measuring **Z** operators on the even sites —> **fluctuating** all configurations of domain walls — the **high-temperature** limit.

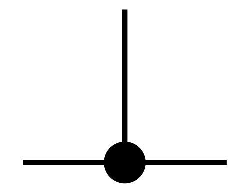
$$\langle Z_i Z_j \rangle_{|\mathcal{P}_Z \psi\rangle} = 0$$

Mapping to Stat. Mech. Model

Measurement angle as effective temperature

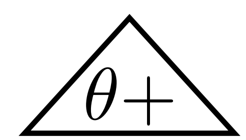
- A tensor network rep. of the cluster state





$= \delta_{ijk}$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(X + Z)$$



$=$ Projection to the +1 state

$$X\cos\theta + Z\sin\theta$$

- The resulting wavefunction

$$\mathcal{P}_\theta|\psi\rangle \sim \otimes_{i=1}^{N} \frac{1}{\sqrt{2}} \left(\sqrt{1+\cos\theta}\,|Z_{2i-1}Z_{2i+1} = +1\rangle + \frac{\sin\theta}{\sqrt{1+\cos\theta}}|Z_{2i-1}Z_{2i+1} = -1\rangle \right)$$

$$\sim e^{-\beta/2\sum_{i=1}^{N} Z_{2i-1}Z_{2i+1}} \otimes_{i=1}^{N} |+\rangle_{2i-1}$$

Where $\tanh\beta = \cos\theta$

- The amplitude of the wave function is mapped to the **Boltzmann weight of a classical Ising model.**

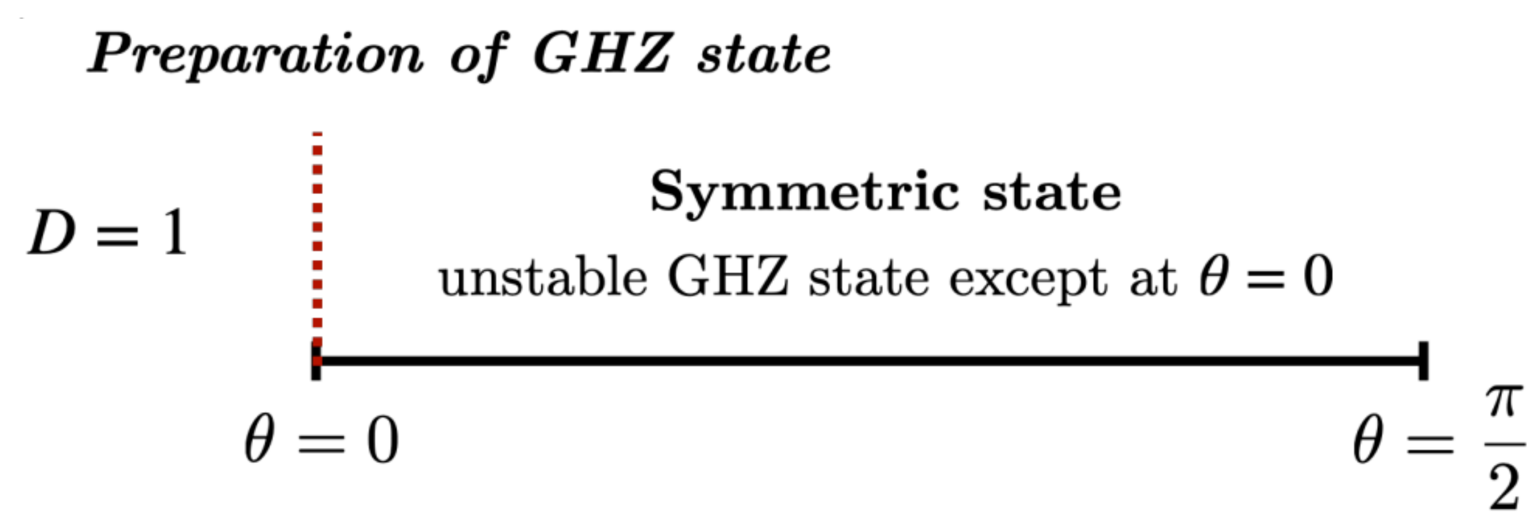
Mapping to Stat. Mech. Model

Can we get a phase transition?

- Turns out that the 1d case, at any small θ, **long-range order is destroyed** as at any finite temperature 1d Ising model is thermally disordered.

$$\langle Z_i Z_j \rangle_{\mathcal{P}_\theta |\psi\rangle} \sim e^{-|i-j|/\xi} \qquad \text{With} \quad \xi = |\ln\cos\theta|^{-1}$$

- Preparations of **1d GHZ states are not robust against measurement angle error.**



- Situations will be different in higher dimensions.

2 Dimensions

- Measuring the **vertices** — **2d classical Ising gauge theory** at finite temperature

$$\mathcal{P}_\theta|\psi\rangle \sim e^{-\frac{\beta}{2}H} \otimes_e |+\rangle_e \quad \text{With} \quad H = -\sum_v \prod_{e\in v} Z_e$$

  - **No finite temperature transition**
  - 1-form symmetry restored for $0 < \theta < \pi/2$

$$\langle \prod_{e\in\partial S} Z_e \rangle_{\mathcal{P}_\theta|\psi\rangle} \sim (\cos\theta)^{|S|}$$

- Measuring the **edges** — **2d classical Ising model** at finite temperature

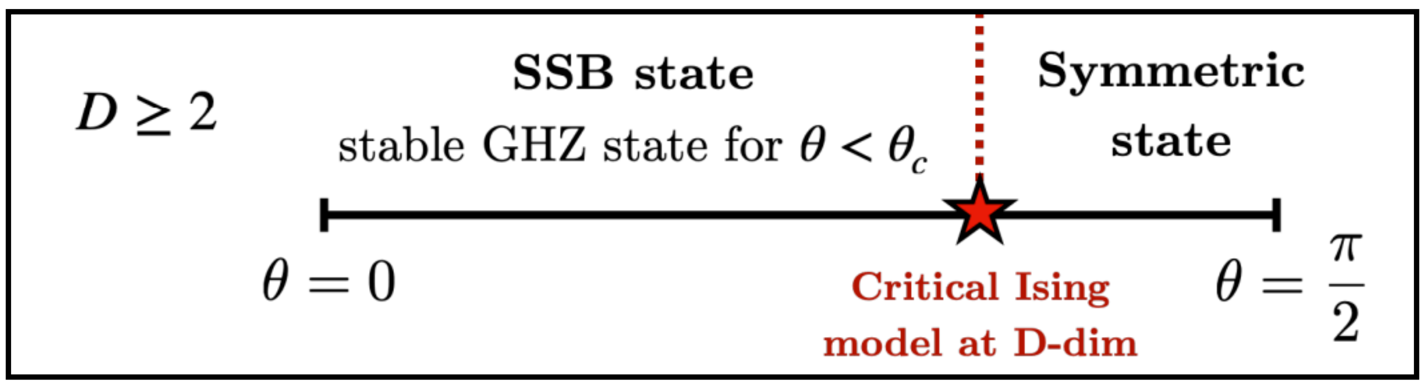
$$\mathcal{P}_\theta|\psi\rangle \sim e^{-\frac{\beta}{2}H} \otimes_v |+\rangle_v \quad \text{With} \quad H = -\sum_{\langle i,j\rangle} Z_i Z_j$$

  - Symmetry breaking state is **stable up to a finite temperature**
  - We get a **transition** in the behavior of the projected wave function

Properties of the Critical States

Spatial correlation functions

$$\mathcal{P}_\theta|\psi\rangle \sim e^{-\frac{\beta}{2}H} \otimes_v |+\rangle_v \quad \text{With} \quad H = -\sum_{\langle i,j \rangle} Z_i Z_j$$



- For D=2 $\qquad \beta_c = \sqrt{2} - 1,$ i.e. $\theta_c \sim 65°$
- At the critical measurement angle, the **spatial correlation function** maps to the correlation function of D-dimensional **Ising CFT.**

$$\langle Z_i Z_j \rangle_{\mathcal{P}_{\theta_c}|\psi\rangle} = \frac{\sum_{\{Z\}} Z_i Z_j e^{-\beta_c \sum_{\langle a,b \rangle} Z_a Z_b}}{\sum_{\{Z\}} e^{-\beta_c \sum_{\langle a,b \rangle} Z_a Z_b}} = \langle Z_i Z_j \rangle_{D-Ising}$$

E. Ardonne, P. Fendley, and E. Fradkin (2004)

Properties of the Critical States

Parent hamiltonians

- There is a general way to construct **a parent hamiltonian** (certain quantum dimer models) such that **the critical wave function is the ground state.**

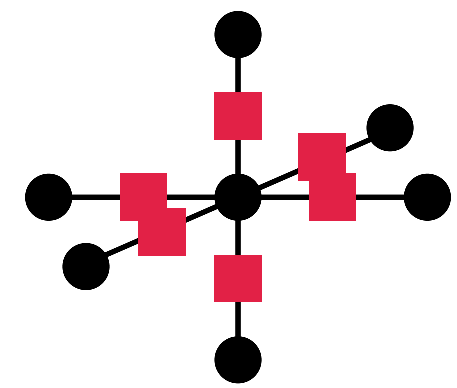
C L Henley (2004)

S. V. Isakov, P. Fendley, A. W. W. Ludwig, S. Trebst, and M. Troyer (2011)

- The quantum hamiltonian so constructed will have a dynamical exponent which is exactly equal to the dynamical exponent of the **classical relaxation dynamics** of the classical Ising model. $z=2.1667(5)$.

3 Dimensional Generalizations

- 3d cluster state with $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(2)}$ symmetry
  - Measure vertices — 3d classical **2-form** Ising gauge theory — no transition

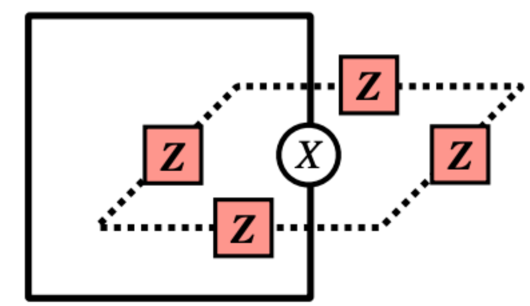


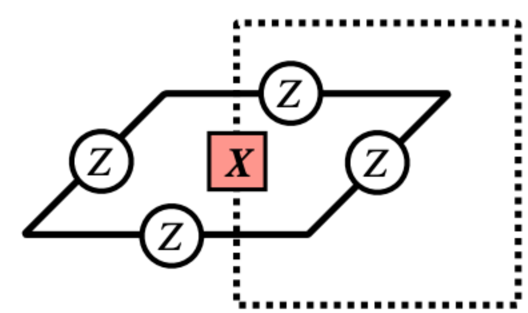
  - Measure edges — 3d classical Ising model — transition at $\theta_c \approx 78°$

- 3d cluster state with $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$ symmetry

$$\mathcal{P}_\theta |\psi\rangle \sim e^{-\frac{\beta}{2} H} \otimes_e |+\rangle_e$$

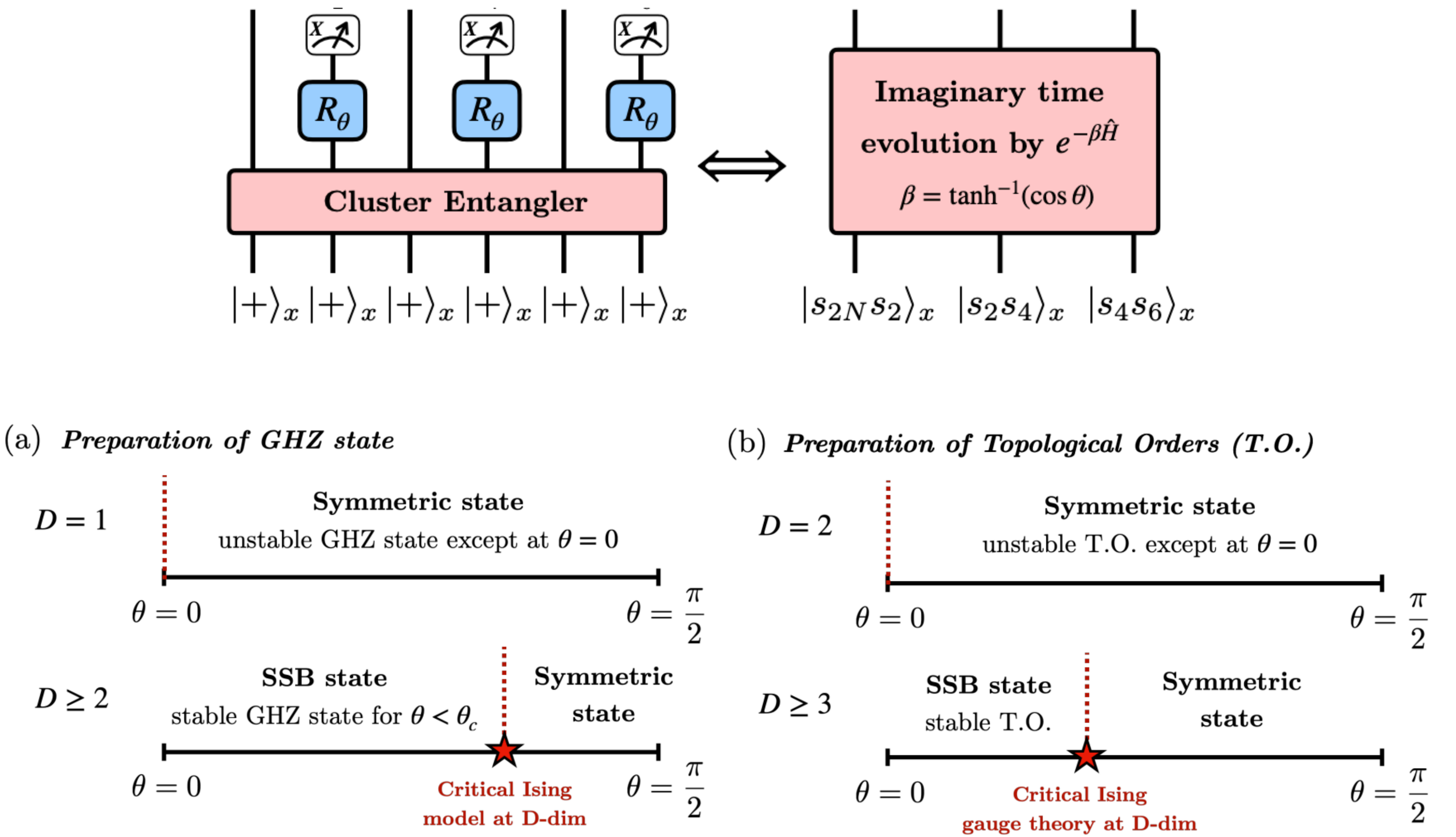
$$H = -\sum_f \prod_{e \in f} Z_e - \sum_v \prod_{e \in v} X_e$$





  - Thermal transition of 3d Ising gauge theory at $\theta_c \approx 50°$

Phase Diagrams



Outline

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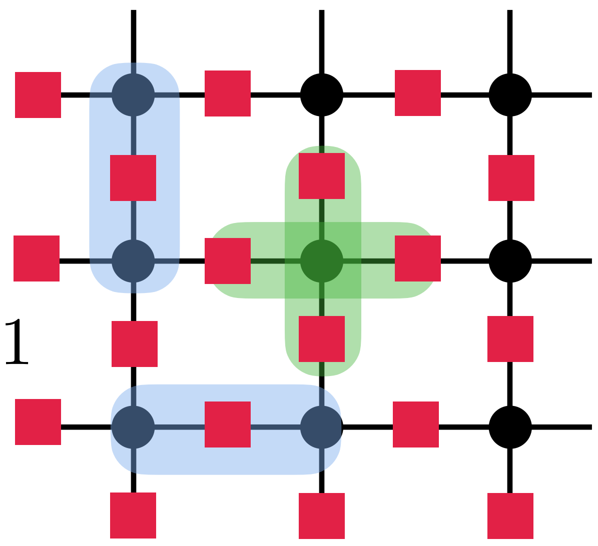
Random Outcomes

- Let us focus on the 2d case measuring along an angle θ



$s_e = \pm 1$

- In general, the outcome can be +1 or -1.

$$X_e \cos\theta + Z_e \sin\theta = s_e = \pm 1$$

- After many rounds, we will get a density matrix.

$$|\psi\rangle\langle\psi| \to \rho_\theta \equiv \sum_{\boldsymbol{s}} \mathcal{P}_{\boldsymbol{s}} |\psi\rangle\langle\psi| \mathcal{P}_{\boldsymbol{s}} = \sum_{\boldsymbol{s}} P_s(\boldsymbol{s}) \frac{|\mathcal{P}_{\boldsymbol{s}}\psi\rangle\langle\mathcal{P}_{\boldsymbol{s}}\psi|}{\langle\mathcal{P}_{\boldsymbol{s}}\psi|\mathcal{P}_{\boldsymbol{s}}\psi\rangle}$$

- Q: Are there structures in this density matrix? How can we extract them?

Random Outcomes

- Looking at the state with a specific set of measurement outcomes.

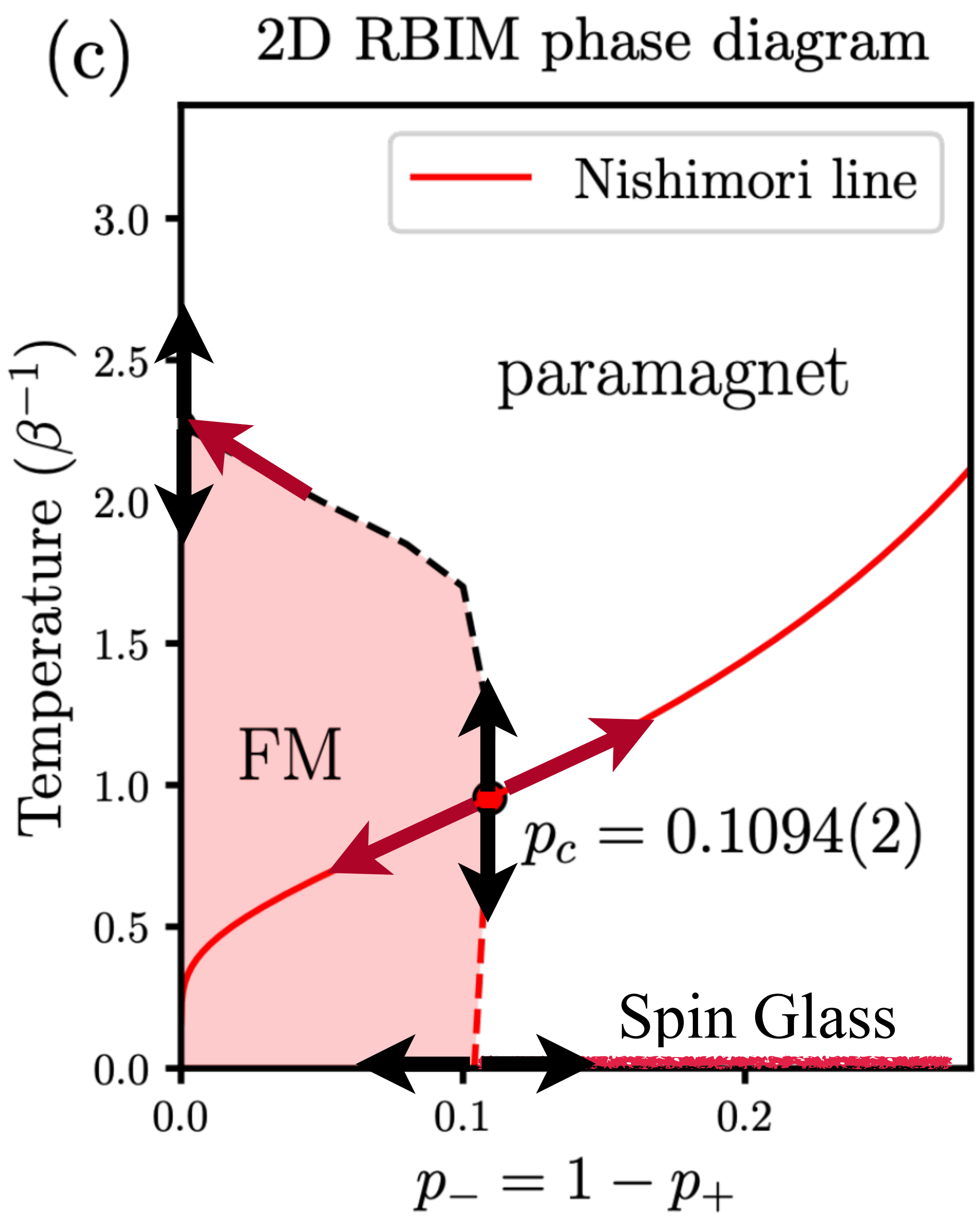
- Mapping to the classical stat. mech. model is still valid with modified coupling.

$$\mathcal{P}_{\mathbf{s}}|\psi\rangle \sim \otimes_{i=1}^{N} \frac{1}{\sqrt{2}} \left(\sqrt{1 + s_e \cos\theta} |(ZZ)_e = +1\rangle + \frac{s_e \sin\theta}{\sqrt{1 + s_e \cos\theta}} |(ZZ)_e = -1\rangle \right)$$

$$\sim e^{-\beta/2 \sum_e s_e (ZZ)_e} \otimes_v |X_v = \prod_{e \in v} s_e\rangle_v \qquad \text{With} \qquad \tanh\beta = \cos\theta$$

- Looks like a 2d classical random bond Ising model at finite temperature

2d Random Bond Ising Model



$$H = -\sum_{\langle i,j \rangle} s_{ij}\sigma_i\sigma_j$$

- Different bonds are **independent** of each other.
- P- is the probability of having antiferromagnetic coupling for a given bond.
- The Nishimori line is a special manifold in the phase diagram, where an **exact** solution of the free energy is known.

H. Nishimori (1981) (1986)

- Q: which part of the phase diagram is related to our measurement model?

Correlated Randomness

- The random outcomes are actually **correlated**.
- Consider the limit of $\theta = 0$, i.e. measure the X operators. Each outcome is random. But the **product of X along a loop is a stabilizer** of the original state and we have the following constraint.

$$\prod_{e \in \gamma} X_e = 1 \quad \Rightarrow \quad \prod_{e \in \gamma} s_e = 1$$

  - This means that the random bond Ising model we obtained at $\theta = 0$ is **frustration-free**.

$$H = -\sum_{\langle i,j \rangle} s_{ij} Z_i Z_j = -\sum_{\langle i,j \rangle} \tilde{s}_{ij} \tilde{Z}_i \tilde{Z}_j$$

$$t_i = \pm 1 \qquad \tilde{Z}_i = t_i Z_i \qquad \tilde{s}_{ij} = s_{ij} t_i t_j = 1$$

  - The Mattis model — by **gauge transformations** it can be brought into a **ferromagnetic** Ising model

For Generic Angles

- At a generic angle, we can calculate the **expectation value** of the loop of measurement outcome.

$$\mathbb{E}[\prod_{e\in\gamma_{\text{loop}}} s_e] = \sum_{\boldsymbol{s}} P_s(\boldsymbol{s}) \prod_{e\in\gamma_{\text{loop}}} s_e = (\cos\theta)^{|\gamma_{\text{loop}}|}$$

- The measurement angle also controls **the level of frustration.**

- Q: can we relate this correlated randomness to the random bond Ising model?

  - Key: The probability distribution is invariant under **gauge transformation**. i.e. two configurations **{s}** and **{s'}** have the same probability if they are related by gauge transformation.

  - Under gauge transformation, one can actually map the correlated distribution to the independent distribution.

Relating to 2d RBIM

- Gauge invariance

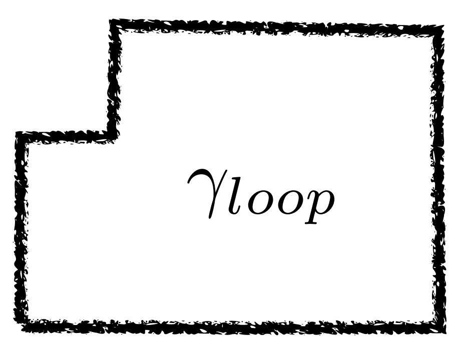
$$P_s(\boldsymbol{s}) = \langle \mathcal{P}_{\boldsymbol{s}}\psi | \mathcal{P}_{\boldsymbol{s}}\psi \rangle = \frac{1}{2^N Z_0} \sum_{\sigma=\pm 1} e^{\beta \sum_{\langle ij \rangle} s_{ij}\sigma_i\sigma_j}$$

$$= \frac{1}{2^N Z_0} \sum_{t\sigma=\pm 1} e^{-\beta \sum_{\langle i,j \rangle} s_{ij} t_i t_j \sigma_i\sigma_j} \quad = P_s(\mathbf{s}') \quad \text{With} \quad s'_{ij} = s_{ij} t_i t_j$$

- The correlated distribution is equivalent to **a gauge symmetrized random bond Ising model**.

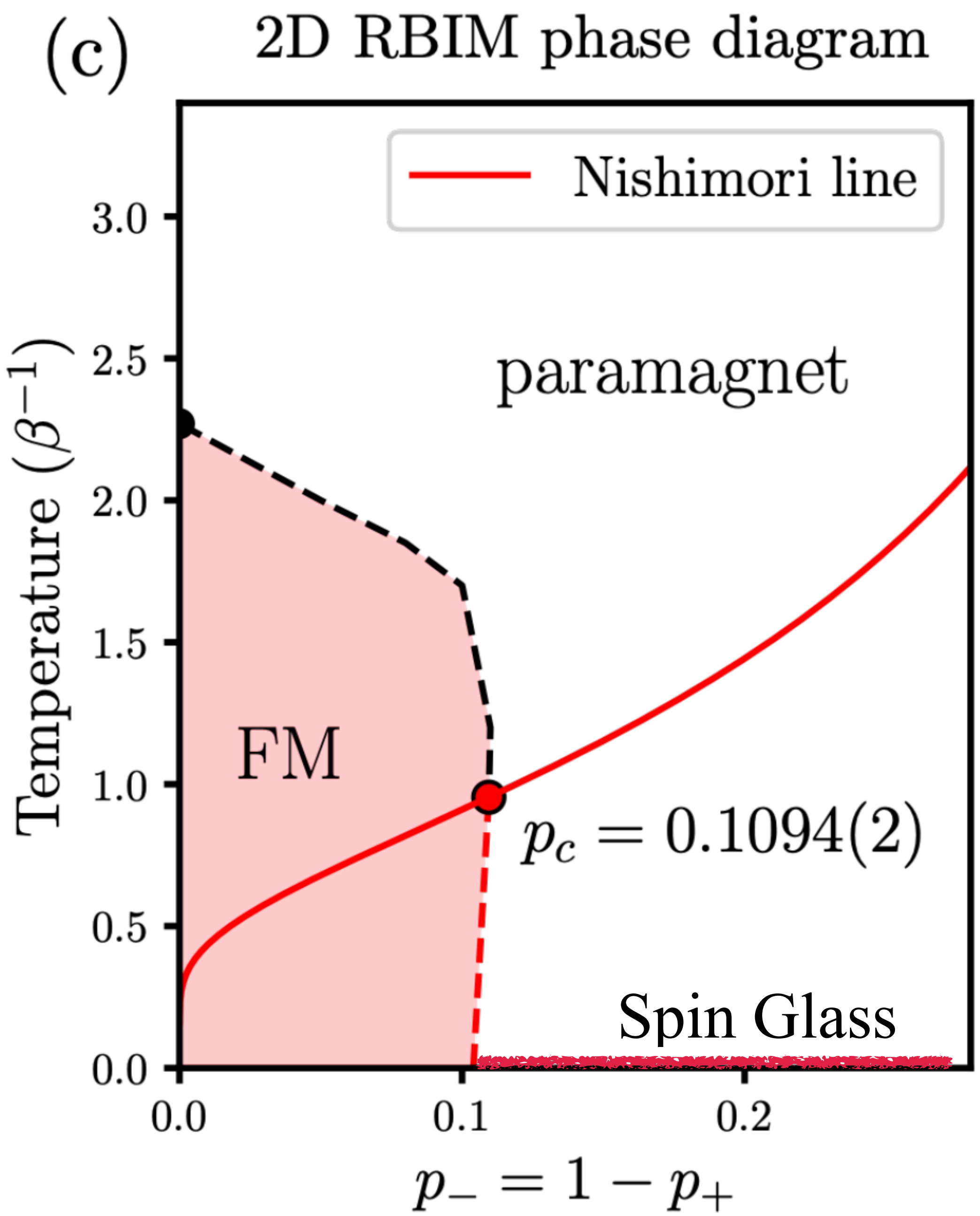
$$\frac{1}{2^N} \sum_{t_i=\pm 1} \prod_{\langle ij \rangle} p^{\mathrm{RBIM}}(t_i s_{ij} t_j) \quad = P_s(\boldsymbol{s}) \qquad \text{where} \qquad p^{\mathrm{RBIM}}(s) = \frac{1 + s\cos\theta}{2}$$

- **Gauge invariant quantities** — expectations of loops — will be the same for the two distributions.



$$\mathbb{E}[\prod_{e\in\gamma_{\text{loop}}} s_e] = \ (\cos\theta)^{|\gamma_{\text{loop}}|}$$

On the Nishimori Line



$$(p_-, \beta) = \left(\frac{1-\cos\theta}{2}, \tanh^{-1}(\cos\theta)\right)$$

$$\Downarrow$$

$$e^{2\beta} = p_+/p_-$$

- Tuning the measurement angle turns out to be exactly moving along the Nishimori line.

Decoding the Hidden Order

- Consider the ferromagnetic **susceptibility** $\langle\chi(\boldsymbol{s})\rangle_\beta \equiv \frac{1}{N}\sum_{i,j}\left[\frac{1}{Z_\beta[\boldsymbol{s}]}\sum_{\boldsymbol{\sigma}}\sigma_i\sigma_j e^{\beta\sum_{\langle ij\rangle}\sigma_i s_{ij}\sigma_j}\right]$

We know
$$\overline{\langle\chi(\boldsymbol{s})\rangle_\beta}^{\mathrm{RBIM}} \equiv \sum_{\boldsymbol{s}} P_s^{\mathrm{RBIM}}(\boldsymbol{s})\,\langle\chi(\boldsymbol{s})\rangle_\beta$$
$$\propto \begin{cases} N & \text{(in FM phase)} \\ \text{const} & \text{(in PM phase)} \end{cases}$$

- Notice this quantity is **not gauge invariant**. If we calculate this with the correlated distribution, it will give **zero** identically.
- But we can tease out this hidden order by doing some **post-editing**.
- In the case of $\theta = 0$ (Mattis model) we already know how to do this:

$$H = -\sum_{\langle i,j\rangle} s_{ij}Z_iZ_j = -\sum_{\langle i,j\rangle}\tilde{s}_{ij}\tilde{Z}_i\tilde{Z}_j$$

$$\langle Z_iZ_j\rangle_{FM} = t_it_j\langle Z_iZ_j\rangle_{Mattis}$$

$$t_i = \pm 1 \qquad \tilde{Z}_i = t_iZ_i \qquad \tilde{s}_{ij} = s_{ij}t_it_j = 1$$

Decoding the Hidden Order

An algorithm to decode the ferromagnetic order

- Do a round of measurements on the bonds with outcome **{s}**, then measure all the vertex spins **{σ}**.
- The gauge-invariant information is the **distribution of frustrated plaquettes** — **flux configurations**. **A stochastic sampling** of **{s'}** conditioned on the **same flux configuration** under a decoder's probability distribution. In our case, the decoder's probability is the 2d RBIM.
- Find the gauge transformation **{t}** from **{s}** to **{s'}**.

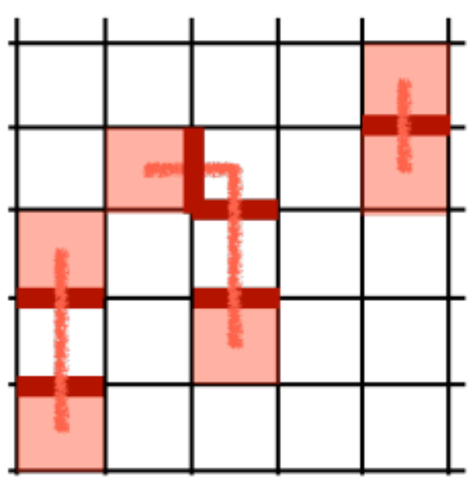
$$C_{ij}^{\text{decode}}(\boldsymbol{s},\boldsymbol{\sigma}) \equiv \frac{1}{|T|}\sum_{\boldsymbol{t}^n\in T} t_i^n(\boldsymbol{s})t_j^n(\boldsymbol{s})\sigma_i\sigma_j .$$

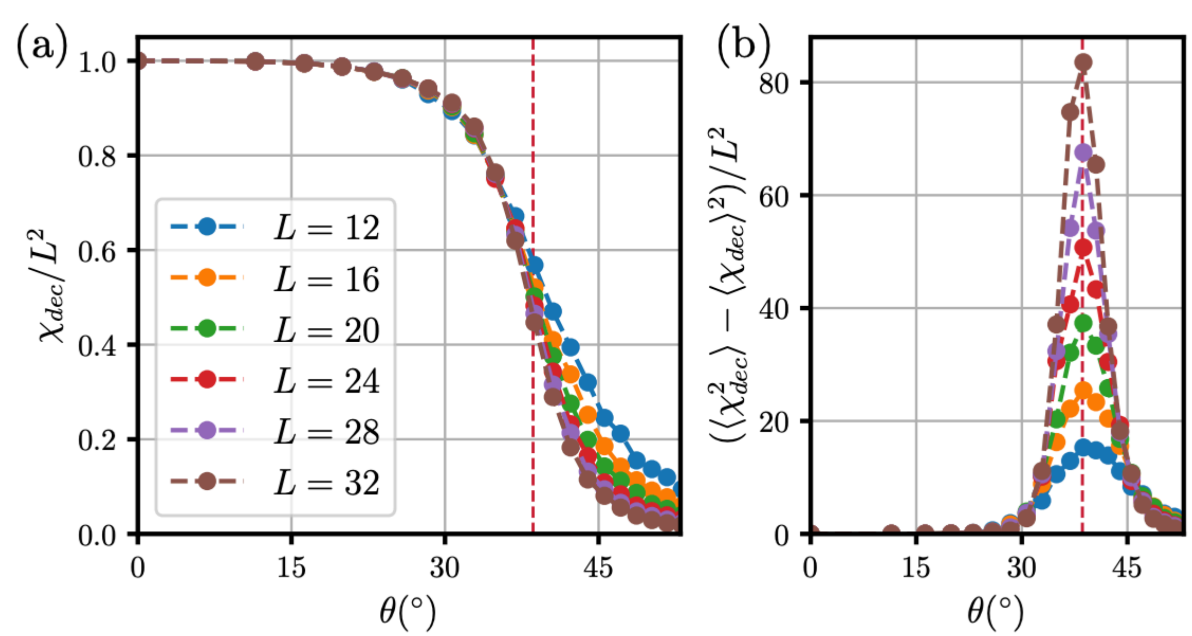
- Repeat and average the correlation function above.
- One can prove the result is the same as the susceptibility in 2d RBIM

Decoding the Hidden Order

A fast decoder

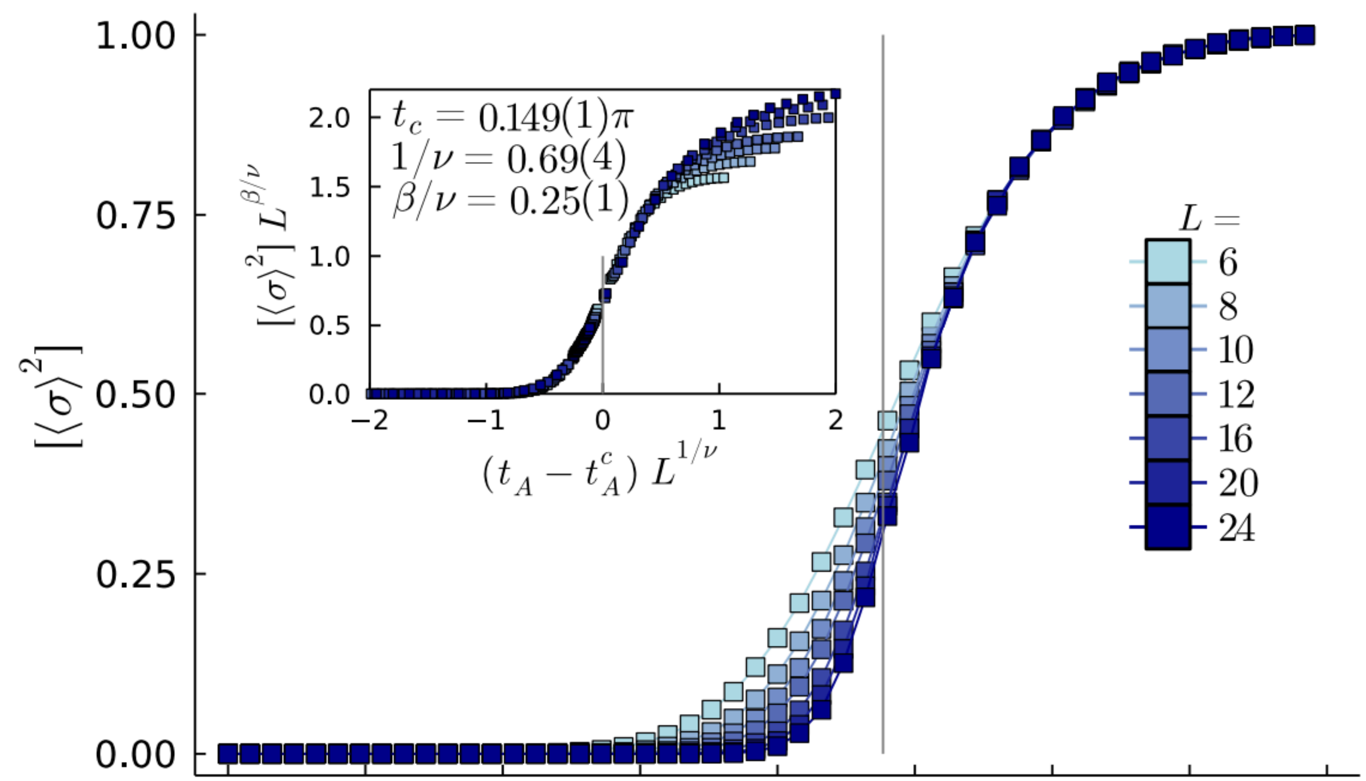
- To just know whether there will be a ferromagnetic tendency, we don't need stochastic sampling. We only need to transform the configuration {s} to the **minimal number of antiferromagnetic bonds that give the same flux configurations**. This will amplify the ferromagnetic ordering.





Edwards-Anderson Order Parameter

$$\mathbb{E}[\langle Z\rangle^2] = \sum_{\{s\}} P_s(\{s\})\langle Z\rangle^2$$



$\theta = \pi/2$ $\theta = 0$

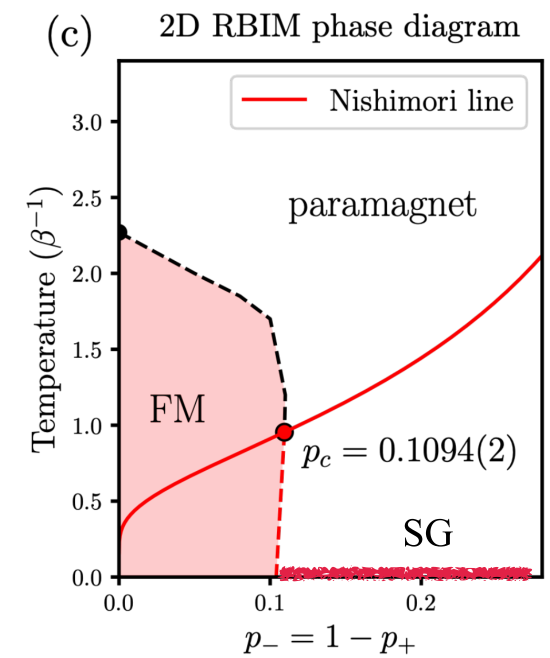
*in the presence of symmetry-breaking field

See: 2208.11136 by G. Zhu, et al.

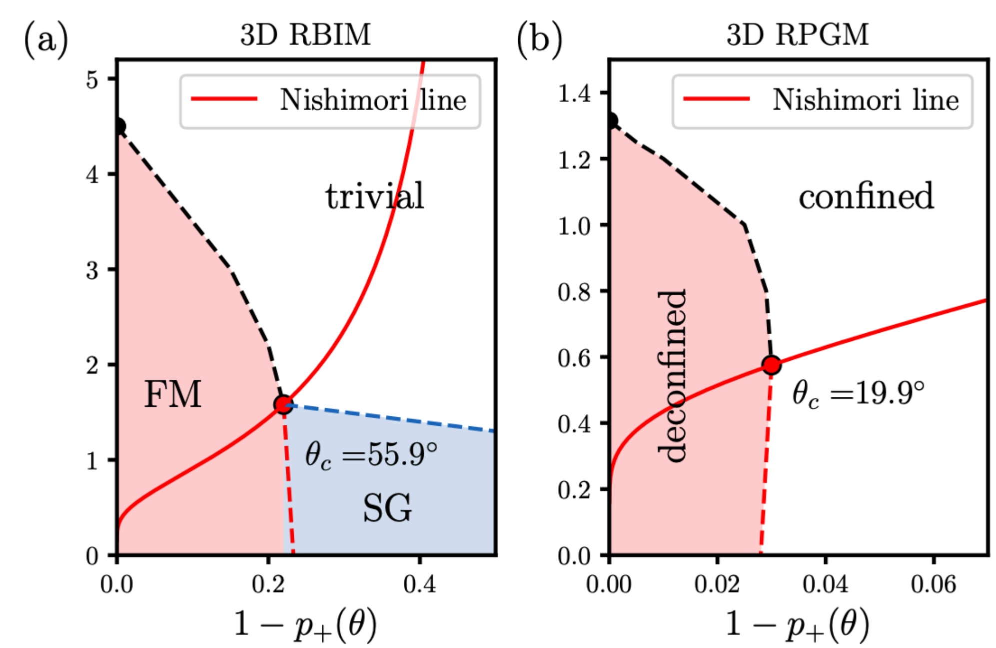
Discussion

- Random outcome + some post-selections

  Q: can we explore other parts of the phase diagram with this?



- Generalization to 3-dimensions.



- How to generalize this scheme to continuous symmetry? Can we get O(n) critical point?