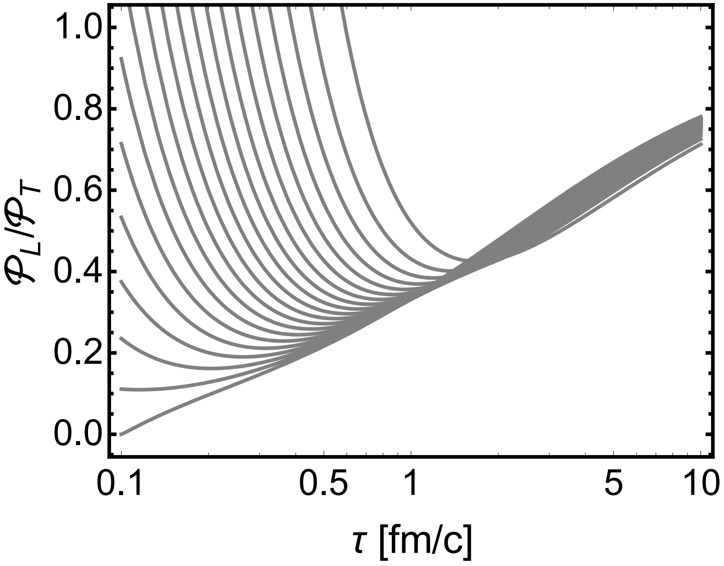
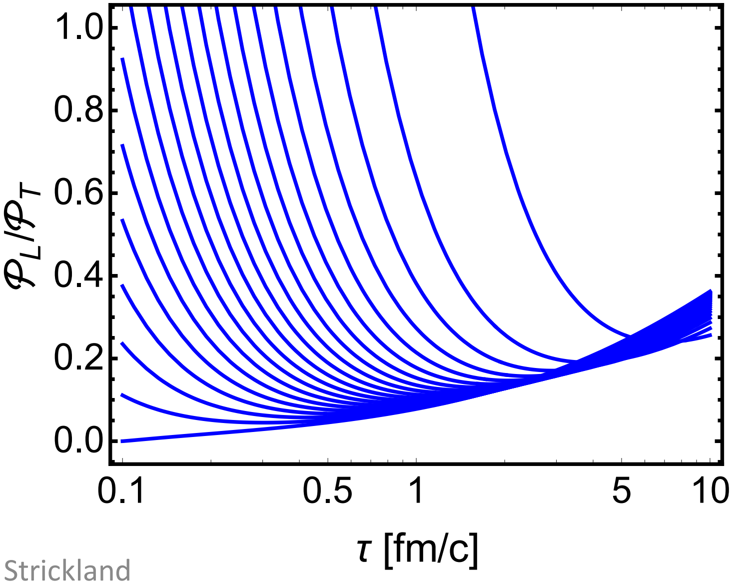
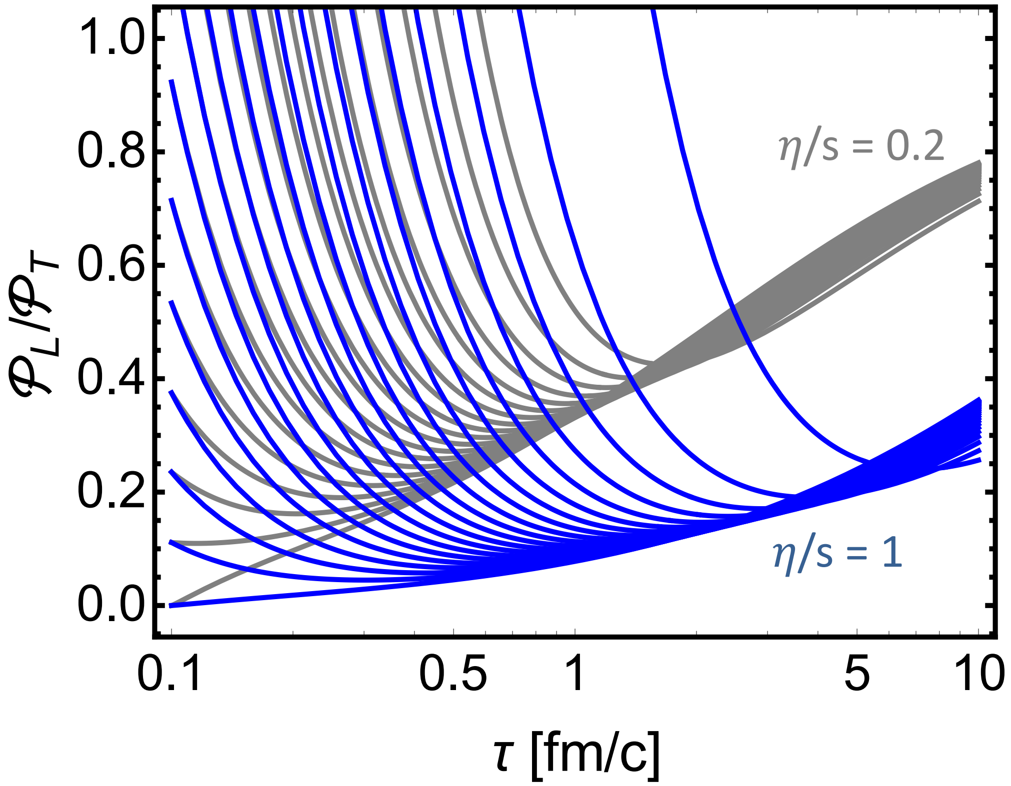
Non-equilibrium attractors in hydro and kinetic theory

Assuming η/s = 0.2 and T = 500 MeV



Assuming η/s = 1 and T = 500 MeV

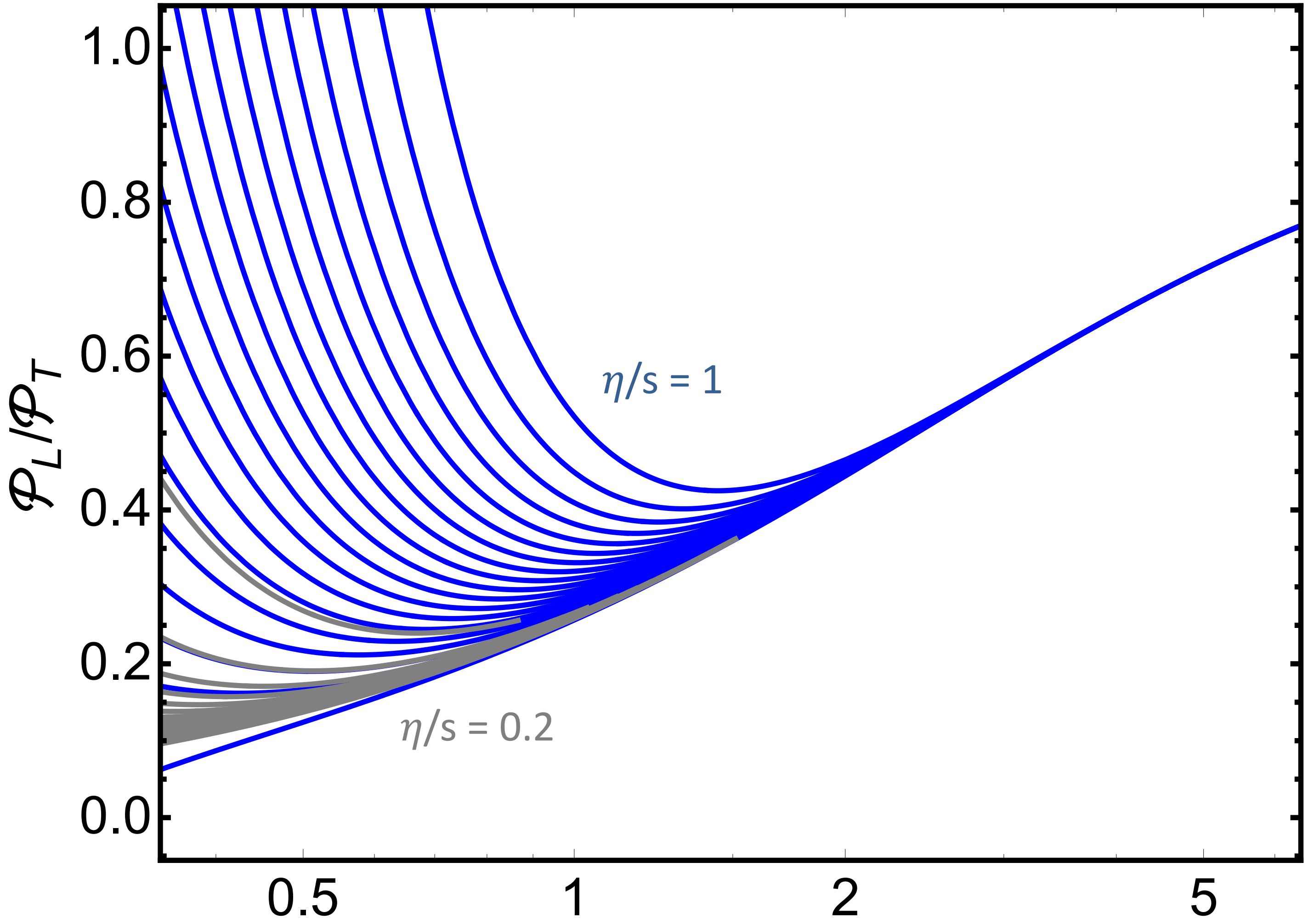




- Solve equations for different initial conditions and different values of the shear viscosity (gray vs blue)

- Hints of universal behavior at late times visible (similar levels of momentum anisotropy)

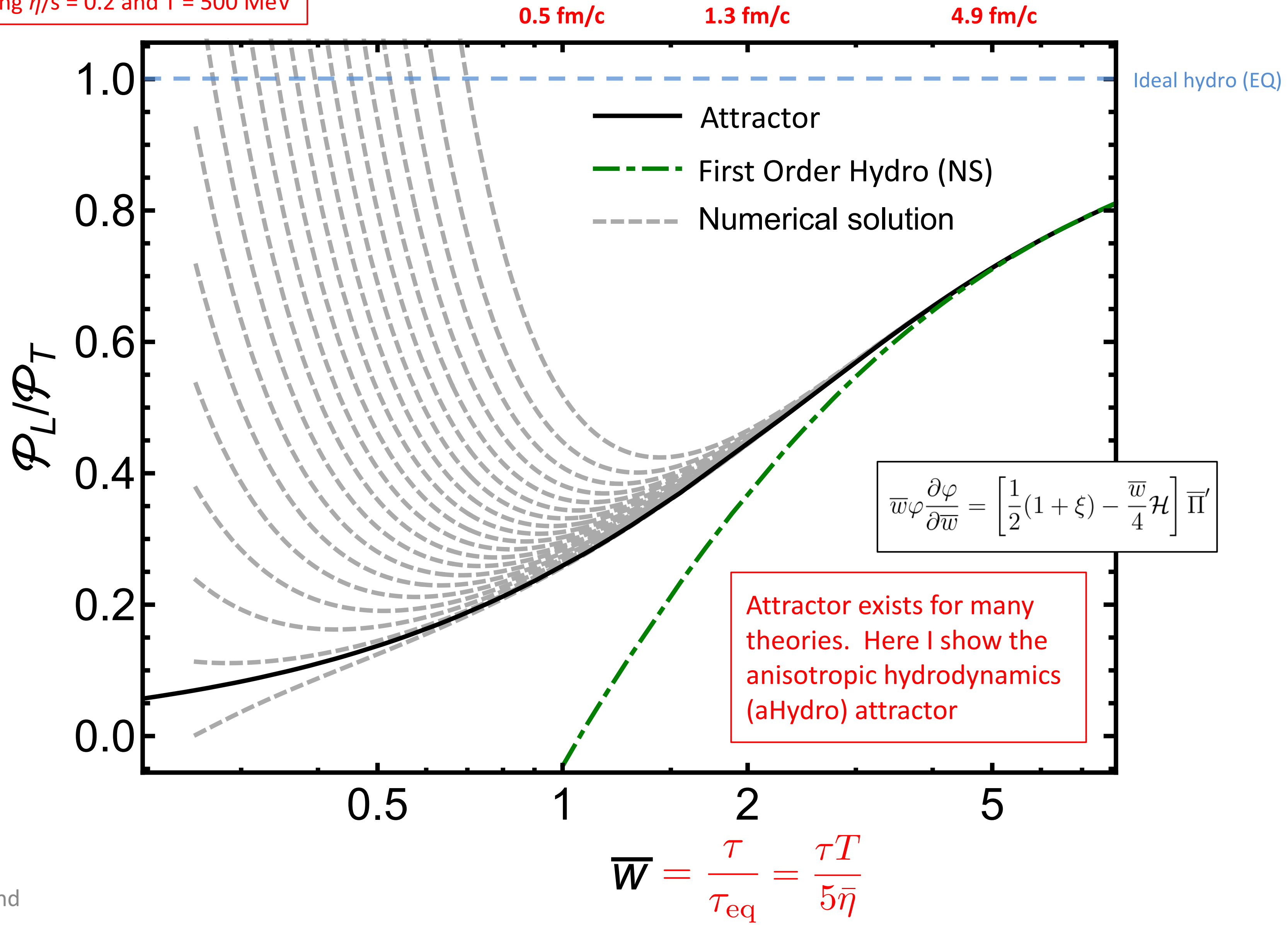
Evidence for an attractor



$$\overline{w} = \frac{\tau}{\tau_{\rm eq}(\tau)} = \frac{\tau T(\tau)}{5\bar{\eta}(\tau)}$$

The attractor concept

Assuming η/s = 0.2 and T = 500 MeV



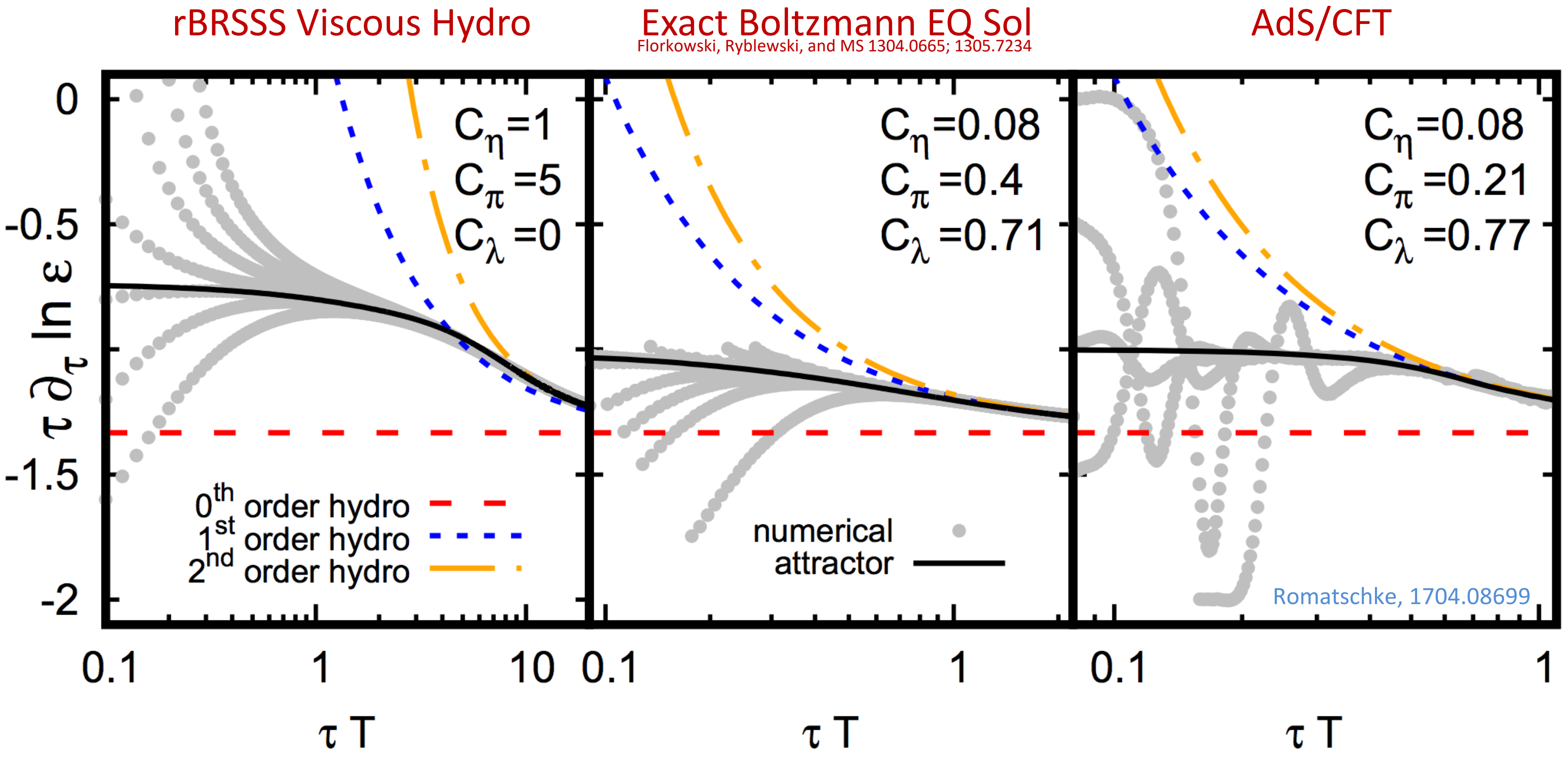
$$\overline{w}\varphi\frac{\partial\varphi}{\partial\overline{w}} = \left[\frac{1}{2}(1+\xi) - \frac{\overline{w}}{4}\mathcal{H}\right]\overline{\Pi}'$$

Attractor exists for many theories. Here I show the anisotropic hydrodynamics (aHydro) attractor

Universality and non-universality

- Universality → All microscopic theories are equally good starting points; pick the one that's easiest to deal with
- Non-universality → We have to pick which microscopic theory we think best describes the system's dynamics
- I would argue that some sort of pQCD/kinetic theory inspired microscopic theory makes the most sense at early times (particles + CGC) and in the dilute regions (hadronic transport) which are precisely where "non-universal" physics pops up.
- And, since kinetic-theory based models also share the "universal properties" of hydrodynamics at later times, they will also work well in this region.

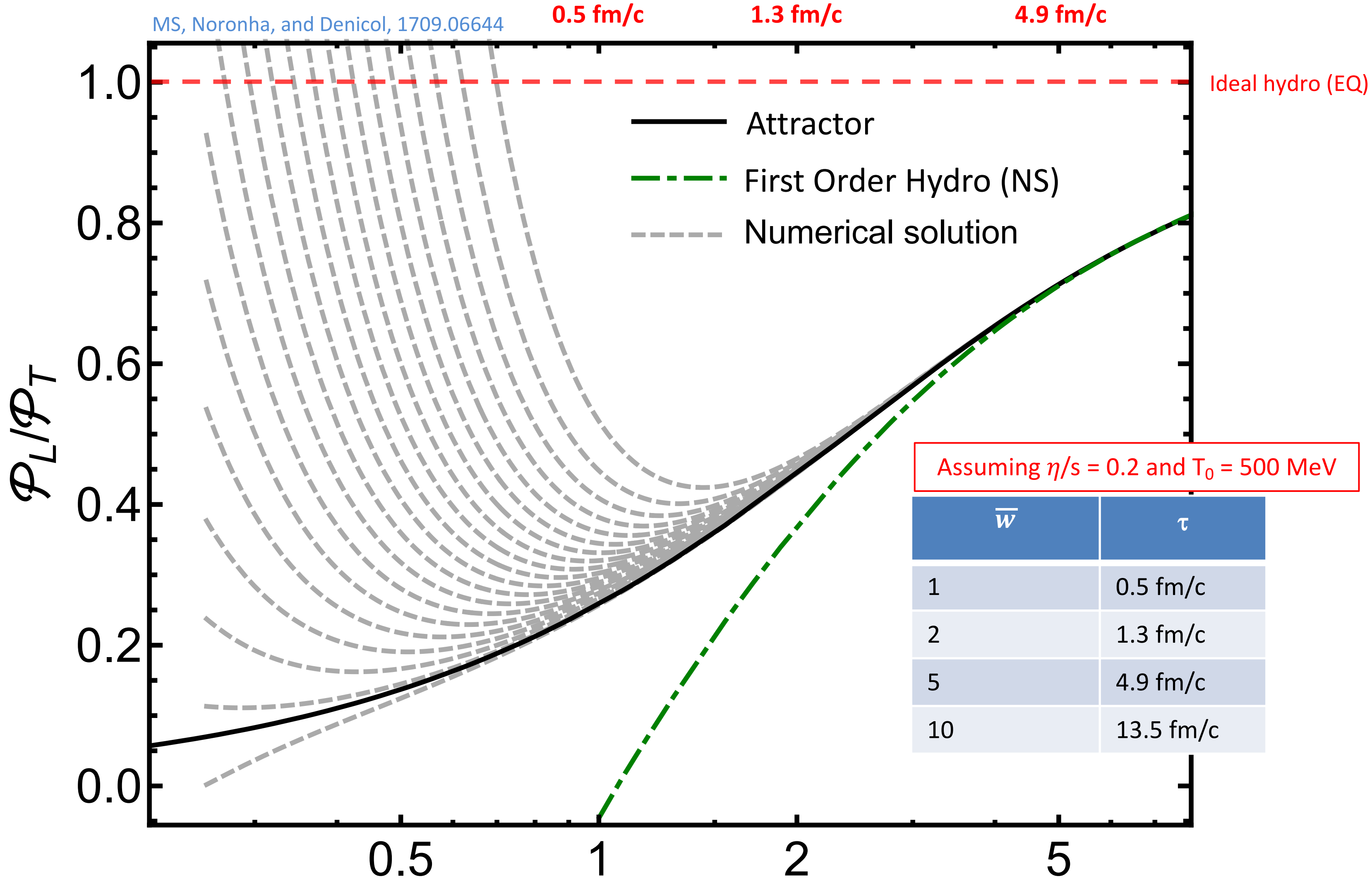
Attractor exists in many theories



Romatschke, 1704.08699; see also Keegan et al, 1512.05347

The attractor concept

MS, Noronha, and Denicol, 1709.06644

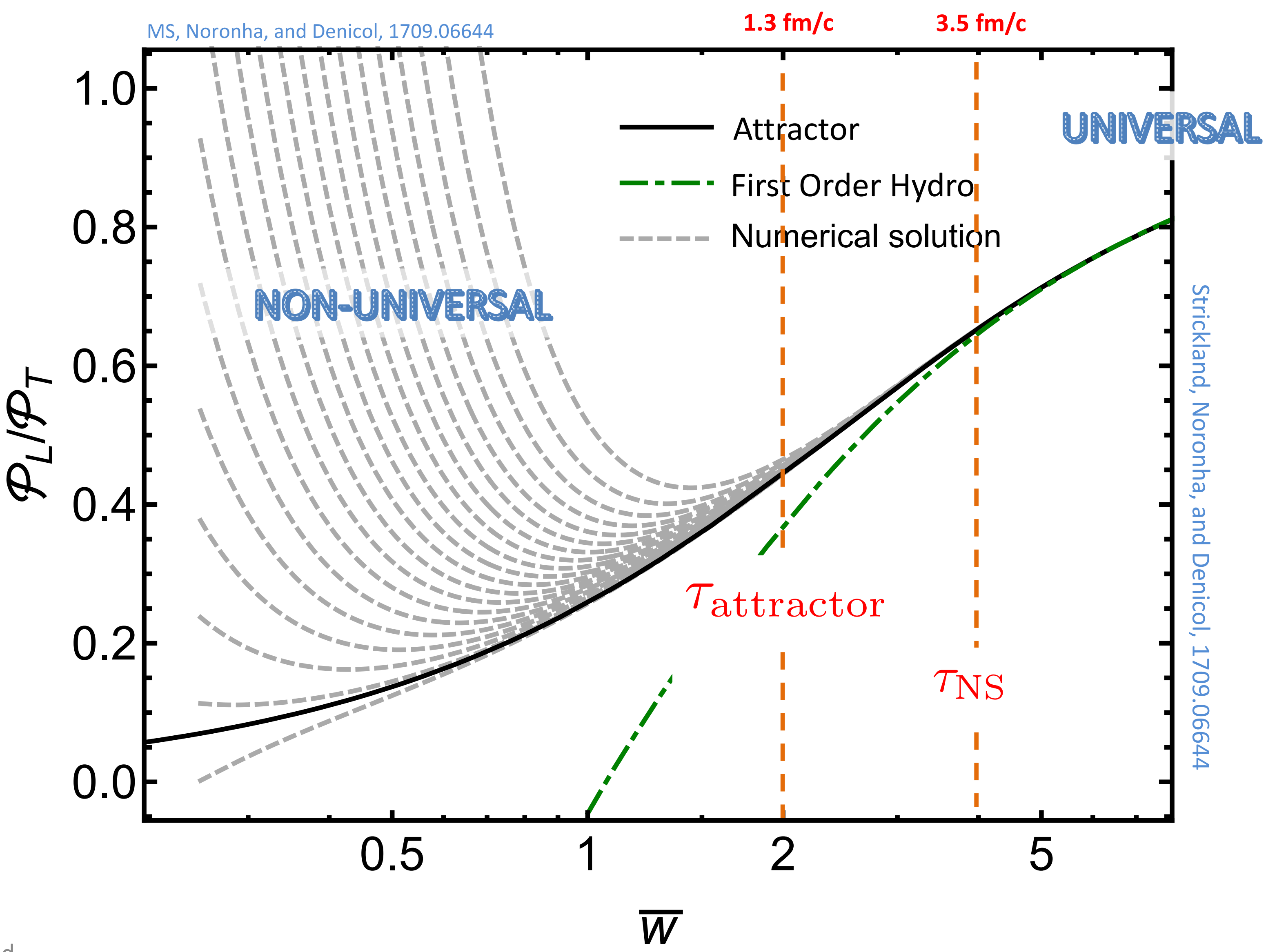


Assuming $\eta/s = 0.2$ and $T_0 = 500$ MeV

$\overline{w}$	τ
1	0.5 fm/c
2	1.3 fm/c
5	4.9 fm/c
10	13.5 fm/c

$$\overline{w} = \frac{\tau}{\tau_{\text{eq}}} = \frac{\tau T}{5\bar{\eta}} = \text{Inverse Knudsen \#}$$

The attractor concept



How does one obtain the attractor?

- Let's look at hydrodynamics-like theories for simplicity (e.g. MIS, DNMR, aHydro, etc.)
- Start with the 0+1 d energy conservation equation

$$\tau\dot{\epsilon} = -\frac{4}{3}\epsilon + \Pi \qquad \Pi = \Pi^{\varsigma}{}_{\varsigma}$$

- Change variables to

$$\boxed{w = \tau T} \qquad \boxed{\varphi(w) \equiv \tau\frac{\dot{w}}{w} = 1 + \frac{\tau}{4}\partial_\tau \log\epsilon}$$

$$\boxed{w\varphi\frac{\partial\varphi}{\partial w} = -\frac{8}{3} + \frac{20}{3}\varphi - 4\varphi^2 + \frac{\tau}{4}\frac{\dot{\Pi}}{\epsilon}}$$

How does one obtain the attractor?

- Need the evolution equation for the viscous correction.
- To linear order in the shear correction (e.g. MIS, DNMR) one has

$$\dot{\Pi} = \frac{4\eta}{3\tau\tau_\pi} - \beta_{\pi\pi}\frac{\Pi}{\tau} - \frac{\Pi}{\tau_\pi}$$

For DNMR in RTA $\beta_{\pi\pi} = \frac{38}{21}$

- Plugging this into the energy-momentum conservation equation gives

$$\boxed{\overline{w}\varphi\varphi' + 4\varphi^2 + \left[\overline{w} + \left(\beta_{\pi\pi} - \frac{20}{3}\right)\right]\varphi - \frac{4c_{\eta/\pi}}{9} - \frac{2}{3}(\beta_{\pi\pi} - 4) - \frac{2\overline{w}}{3} = 0}$$

$$\overline{w} \equiv \frac{w}{c_\pi} = \frac{\tau T}{5\overline{\eta}} \qquad c_{\eta/\pi} \equiv \frac{c_\eta}{c_\pi} = \frac{1}{5}$$

How does one solve for the attractor?

$$\boxed{\overline{w}\varphi\varphi' + 4\varphi^2 + \left[\overline{w} + \left(\beta_{\pi\pi} - \frac{20}{3}\right)\right]\varphi - \frac{4c_{\eta/\pi}}{9} - \frac{2}{3}(\beta_{\pi\pi} - 4) - \frac{2\overline{w}}{3} = 0}$$

$$\overline{w} \equiv \frac{w}{c_\pi} = \frac{\tau T}{5\overline{\eta}} \qquad\qquad c_{\eta/\pi} \equiv \frac{c_\eta}{c_\pi} = \frac{1}{5}$$

- First try to approximate using “slow-roll” approx $(\varphi' = 0)$
- From this, we can read off the boundary condition as $w \to 0$

$$\lim_{\overline{w}\to 0} \varphi(\overline{w}) = \frac{1}{24}\left(-3\beta_{\pi\pi} + \sqrt{64c_{\eta/\pi} + (3\beta_{\pi\pi} - 4)^2} + 20\right)$$

- Then numerically solve the ODE at the top of the slide

Beyond hydrodynamics?

- Can the concept of a non-equilibrium attractor be extended beyond the 14 degrees of freedom described using the energy-momentum tensor, number density, and diffusion current?
- In kinetic theory we describe things in terms of a one-particle distribution function **f(x,p)** and the energy-momentum tensor is obtained from low-order moments:

$$T^{\mu\nu} = \int dP \, p^\mu p^\nu f(x,p) \qquad\qquad \int dP \equiv \int \frac{d^3p}{(2\pi)^2 E}$$

- What about more general moments of f? Particularly ones that are sensitive to higher momenta?

Beyond hydrodynamics?

- For a conformal system we consider

$$\mathcal{M}^{nm}[f] \equiv \int dP \, (p \cdot u)^n \, (p \cdot z)^{2m} \, f(x,p)$$

- This encompasses the moments necessary to construct the energy momentum tensor, e.g. below, and more

$$\varepsilon = \mathcal{M}^{20} = \int dP \, (p \cdot u)^2 \, f(\tau, w, p_T) = T^{00}_{\mathrm{LRF}}$$

$$P_L = \mathcal{M}^{01} = \int dP \, (p \cdot z)^2 \, f(\tau, w, p_T) = T^{zz}_{\mathrm{LRF}}$$

Use 0+1d RTA Exact Solution

- Simple model: Boost-invariant transversally homogeneous Boltzmann equation in relaxation time approximation (RTA).
- Many results in this model, so we can compare with the literature.

Boltzmann EQ $p^\mu \partial_\mu f(x,p) = C[f(x,p)]$

RTA $C[f] = \frac{p_\mu u^\mu}{\tau_{\rm eq}} \left[f_{\rm eq}\Big(p_\mu u^\mu, T(x)\Big) - f(x,p) \right]$

Massless Particles
W. Florkowski, R. Ryblewski, and MS,
1304.0665 and 1305.7234

Massive Particles
W. Florkowski, E. Maksymiuk,
R. Ryblewski, and MS, 1402.7348

Solution for the energy density (massless particle case)

$$T^4(\tau) = D(\tau,\tau_0) T_0^4 \frac{\mathcal{H}\big(\frac{\alpha_0 \tau_0}{\tau}\big)}{\mathcal{H}(\alpha_0)} + \int_{\tau_0}^{\tau} \frac{d\tau'}{2\tau_{\rm eq}(\tau')} D(\tau,\tau')\, T^4(\tau') \mathcal{H}\left(\frac{\tau'}{\tau}\right)$$

Time-dependent relaxation time $\tau_{\rm eq}(\tau) = \frac{5\bar{\eta}}{T(\tau)}$

Damping Function $D(\tau_2,\tau_1) = \exp\left[- \int_{\tau_1}^{\tau_2} d\tau \, \tau_{\rm eq}^{-1}(\tau) \right]$

0+1d RTA Exact Solution

$$T^4(\tau) = D(\tau,\tau_0)\,T_0^4\,\frac{\mathcal{H}\left(\frac{\alpha_0\tau_0}{\tau}\right)}{\mathcal{H}(\alpha_0)} + \int_{\tau_0}^{\tau} \frac{d\tau'}{2\tau_{\rm eq}(\tau')}\, D(\tau,\tau')\, T^4(\tau')\,\mathcal{H}\!\left(\frac{\tau'}{\tau}\right)$$

Once this integral equation is solved (by numerical iteration), we can construct the full one-particle distribution function f(τ,p) OR we can compute general moments:

$$f(\tau,w,p_T) = D(\tau,\tau_0)\,f_0(w,p_T) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')}\, D(\tau,\tau')\, f_{\rm eq}(\tau',w,p_T)$$

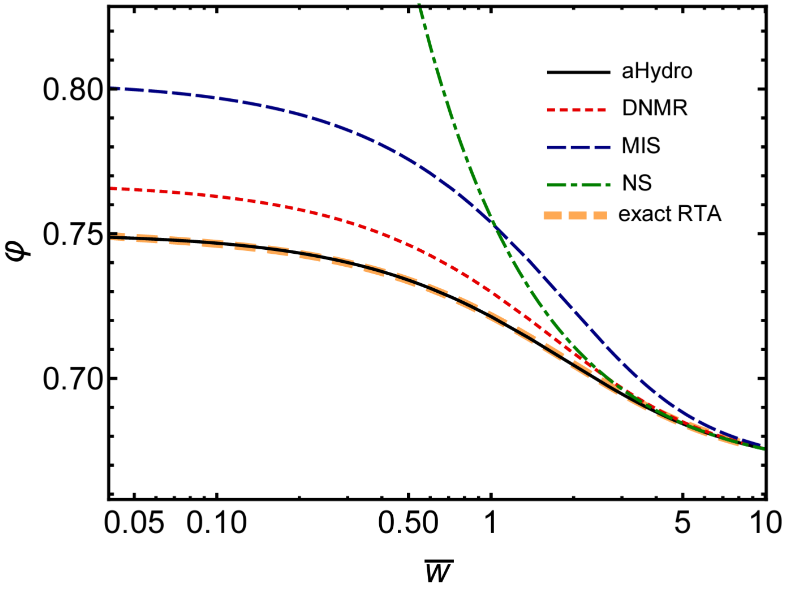
MS, 1809.01200

$$\mathcal{M}^{nm}(\tau) = \frac{\Gamma(n+2m+2)}{(2\pi)^2}\Bigg[D(\tau,\tau_0)\,2^{(n+2m+2)/4}\,T_0^{n+2m+2}\,\frac{\mathcal{H}^{nm}\left(\frac{\alpha_0\tau_0}{\tau}\right)}{\left[\mathcal{H}^{20}(\alpha_0)\right]^{(n+2m+2)/4}} + \int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{\rm eq}(\tau')}\,D(\tau,\tau')\,T^{n+2m+2}(\tau')\,\mathcal{H}^{nm}\!\left(\frac{\tau'}{\tau}\right)\Bigg],$$

$$\mathcal{H}^{nm}(y) = \tfrac{2y^{2m+1}}{2m+1}\,{}_2F_1\!\left(\tfrac{1}{2}+m, \tfrac{1-n}{2}; \tfrac{3}{2}+m; 1-y^2\right).$$

0+1d RTA Exact Solution

MS, G. Denicol, and J. Noronha, 1709.06644



Assuming η/s = 0.2 and T_0 = 500 MeV

$\overline{w}$	τ
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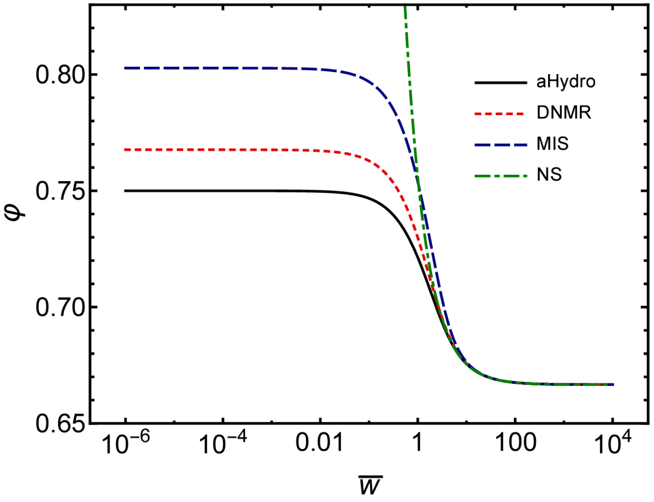
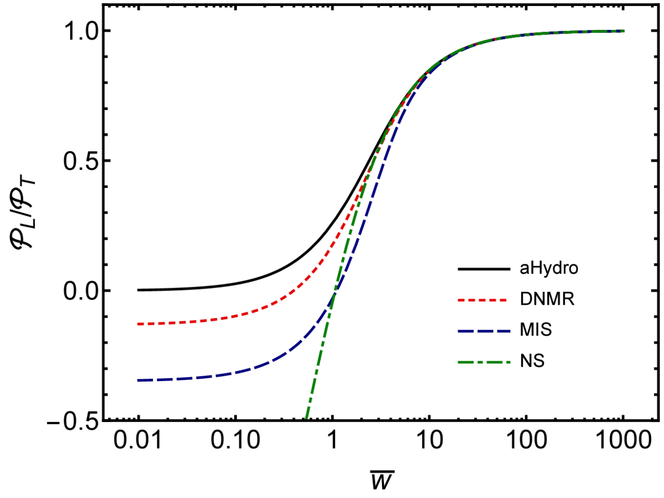
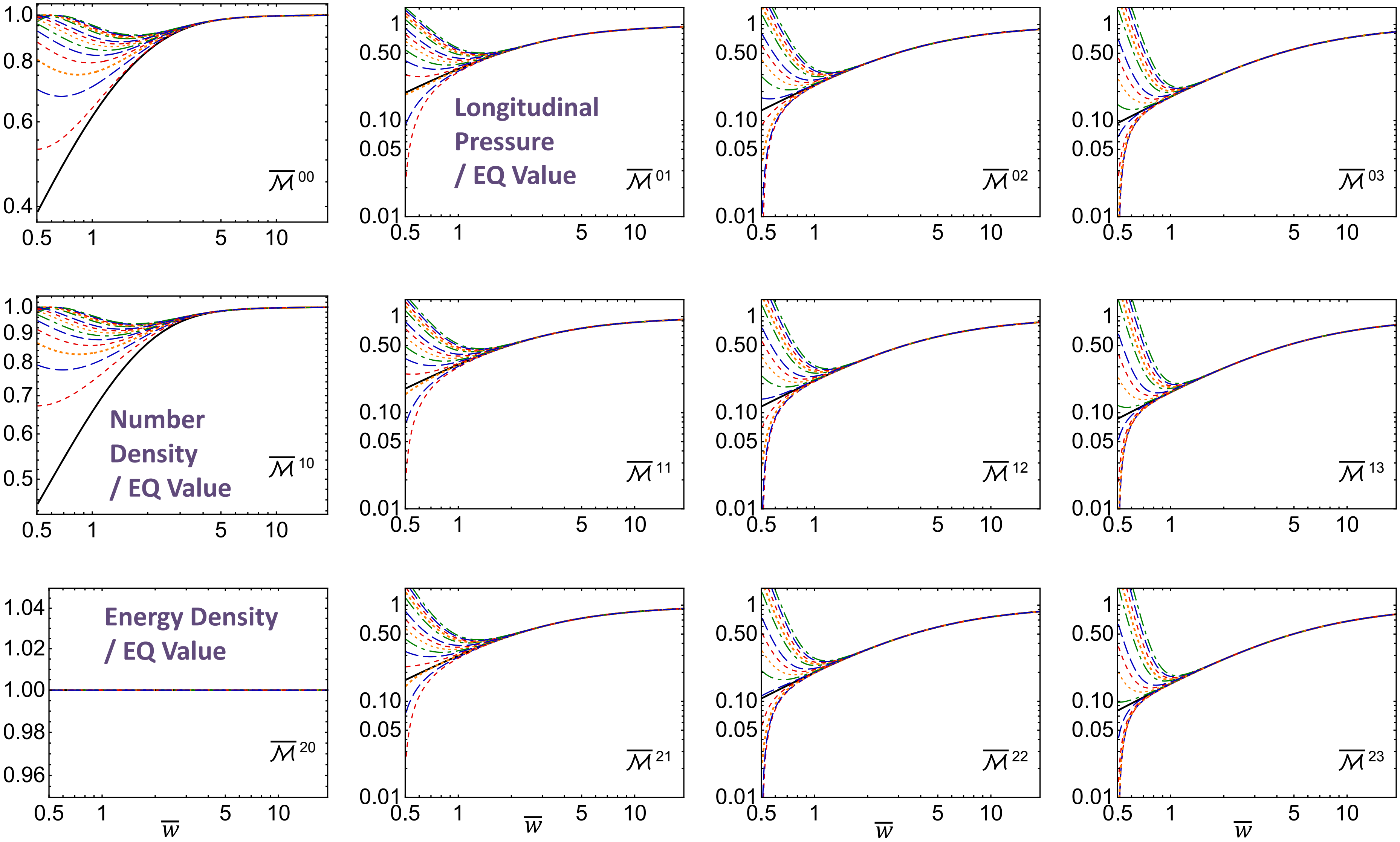




Behavior of higher order moments

MS, 1809.01200



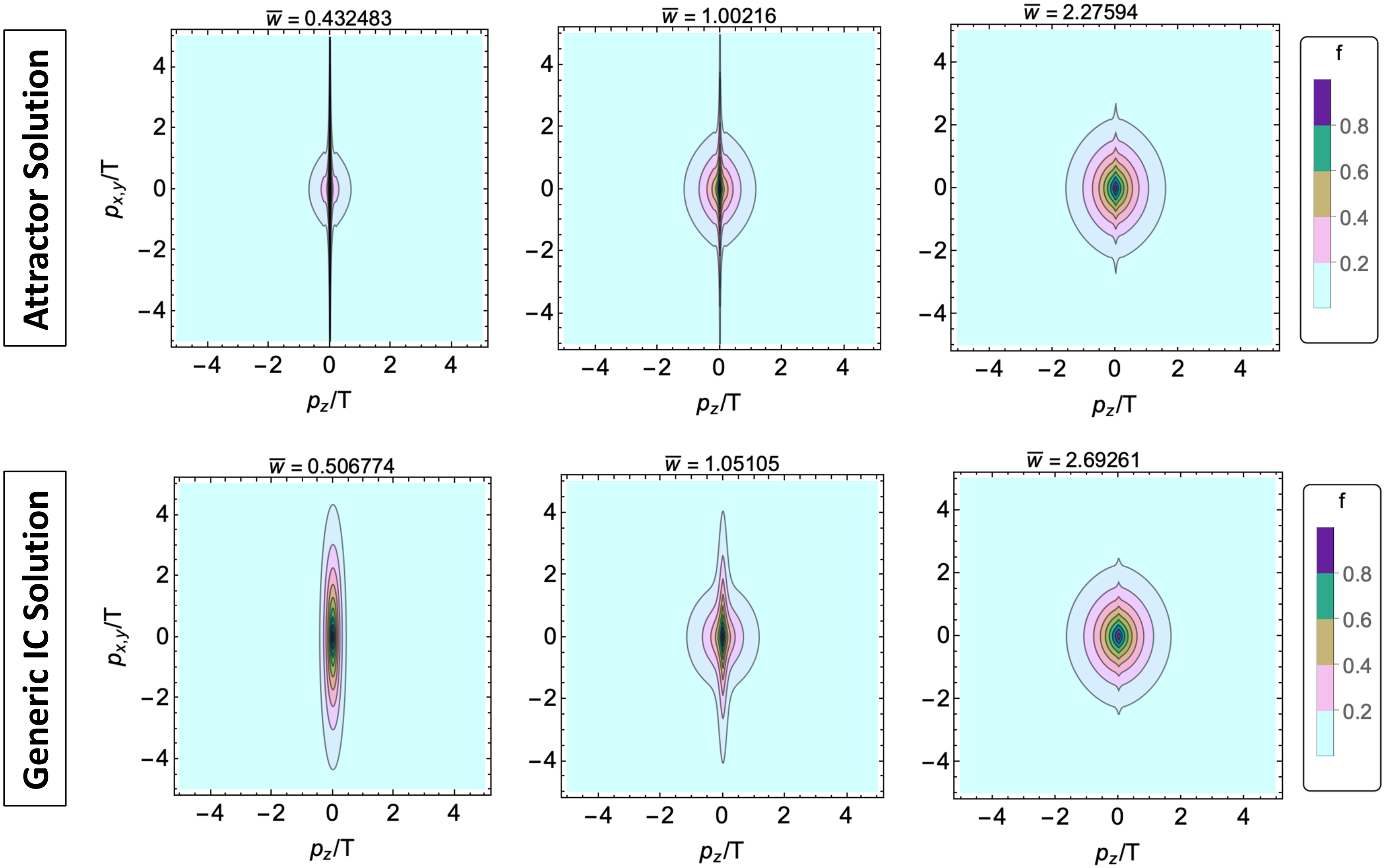
Black Line = Attractor Solution

Dashed colored lines = scan of initial conditions

The attractor for the distribution function itself

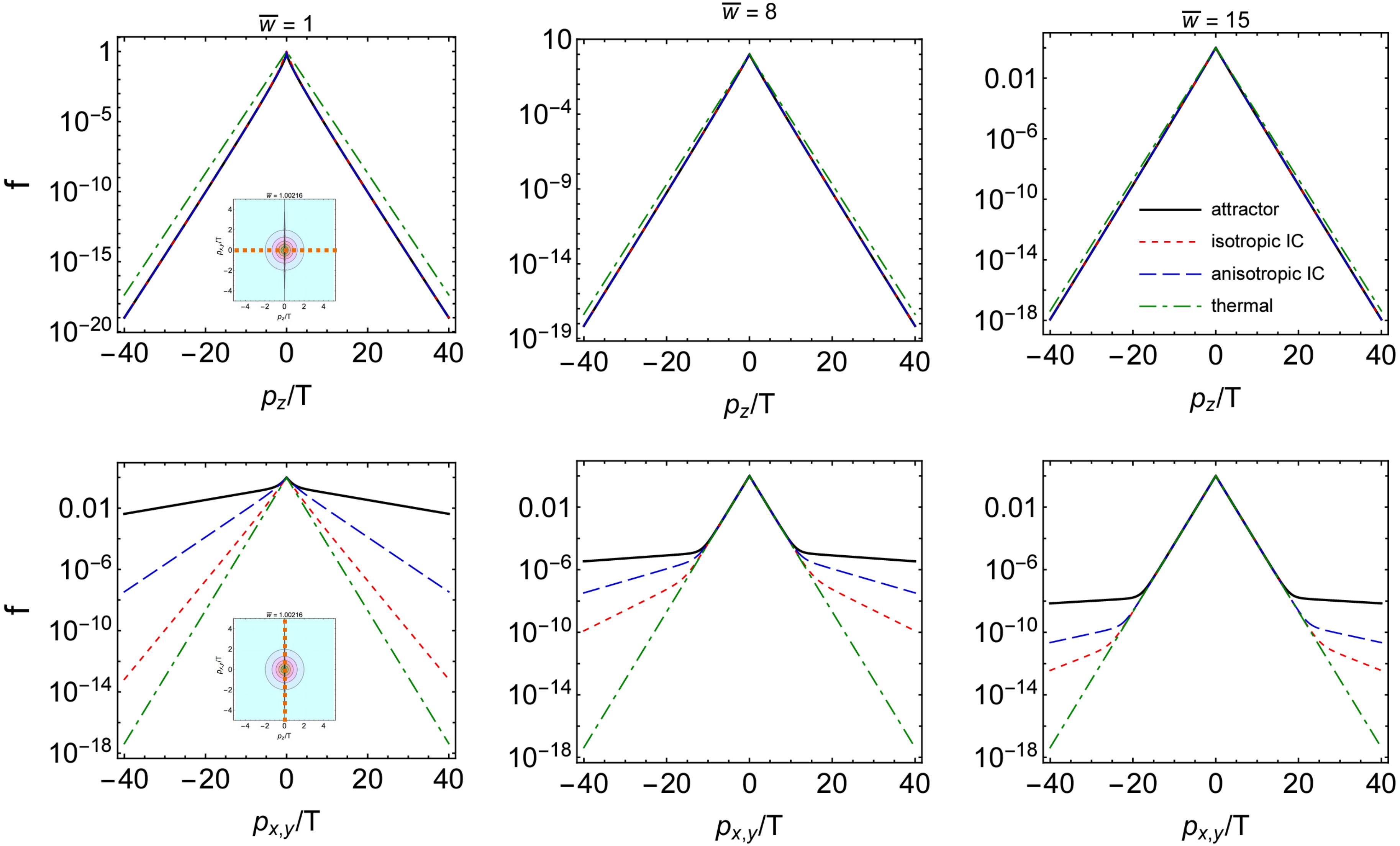
Attractor distribution function

MS, 1809.01200



Attractor distribution function

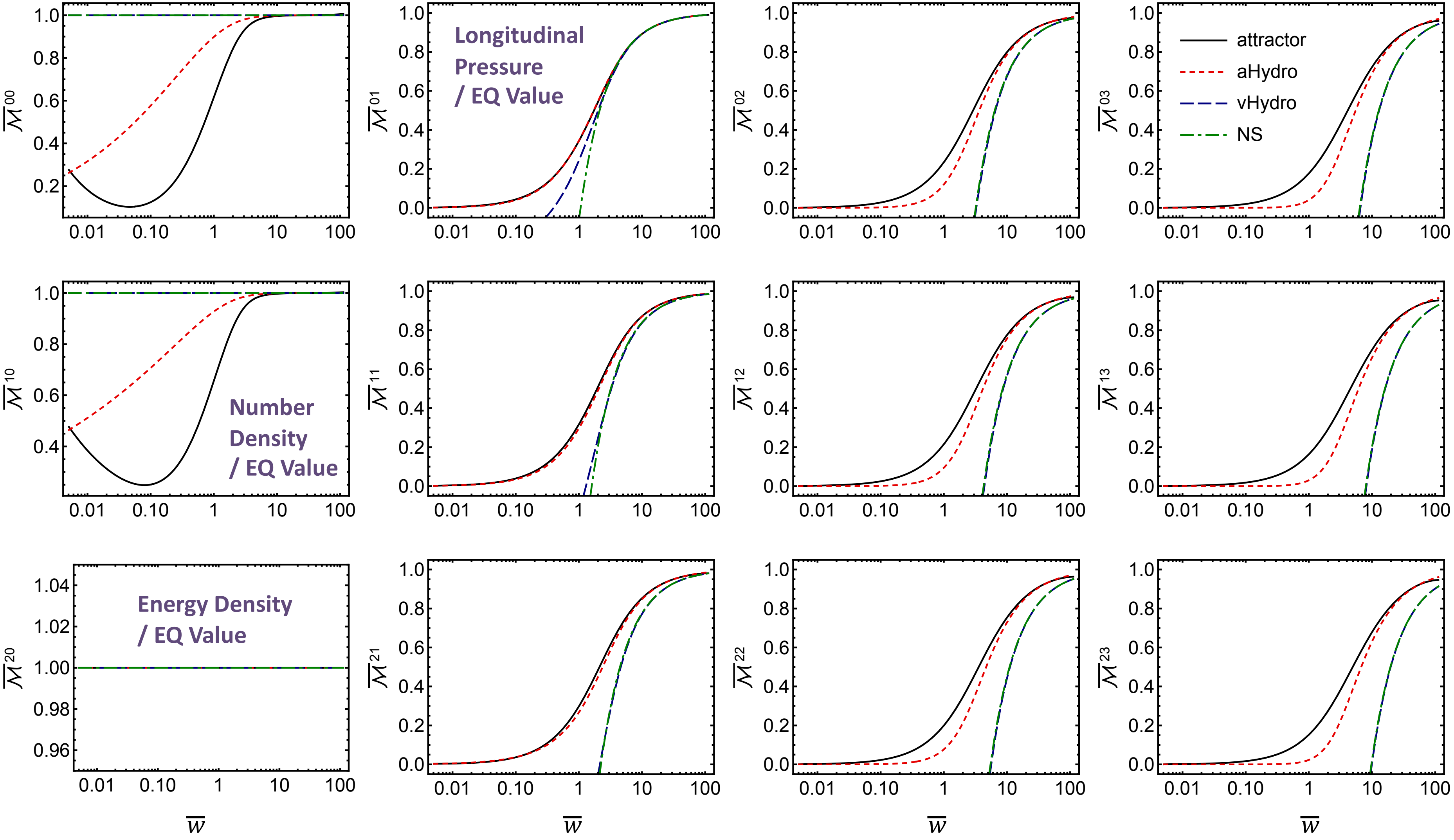
MS, 1809.01200



Comparison of exact attractor for moments with different hydrodynamics approximations

Hydrodynamic comparisons

MS, 1809.01200

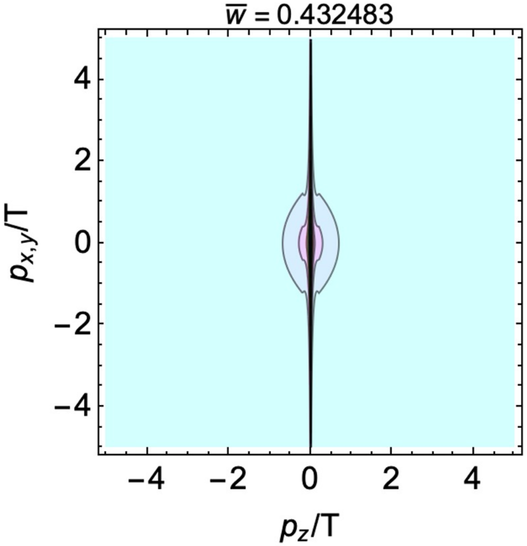


Longitudinal Pressure / EQ Value

Number Density / EQ Value

Energy Density / EQ Value

Can we do better than this?



$$f(\tau, w, p_T) = D(\tau, \tau_0) f_0(w, p_T) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')} D(\tau, \tau') \, f_{\rm eq}(\tau', w, p_T)$$

An improved aHydro ansatz

H. Alalawi and MS, 2006.13834

$$\boxed{f(\mathbf{p},\tau) = f_0(\xi_{\rm FS},\Lambda_0)D(\tau,\tau_0) + f_{\rm RS}(\xi,\Lambda)[1-D(\tau,\tau_0)]}$$

$$\xi_{\rm FS} = (1+\xi_0)\frac{\tau^2}{\tau_0^2} - 1 \qquad D(\tau_2,\tau_1) = \exp\left[-\int_{\tau_1}^{\tau_2} d\tau\, \tau_{\rm eq}^{-1}(\tau)\right] \qquad f_{\rm RS}(\xi,\Lambda) = f_{\rm eq}(\sqrt{\mathbf{p}^2+\xi p_z^2}/\Lambda)$$

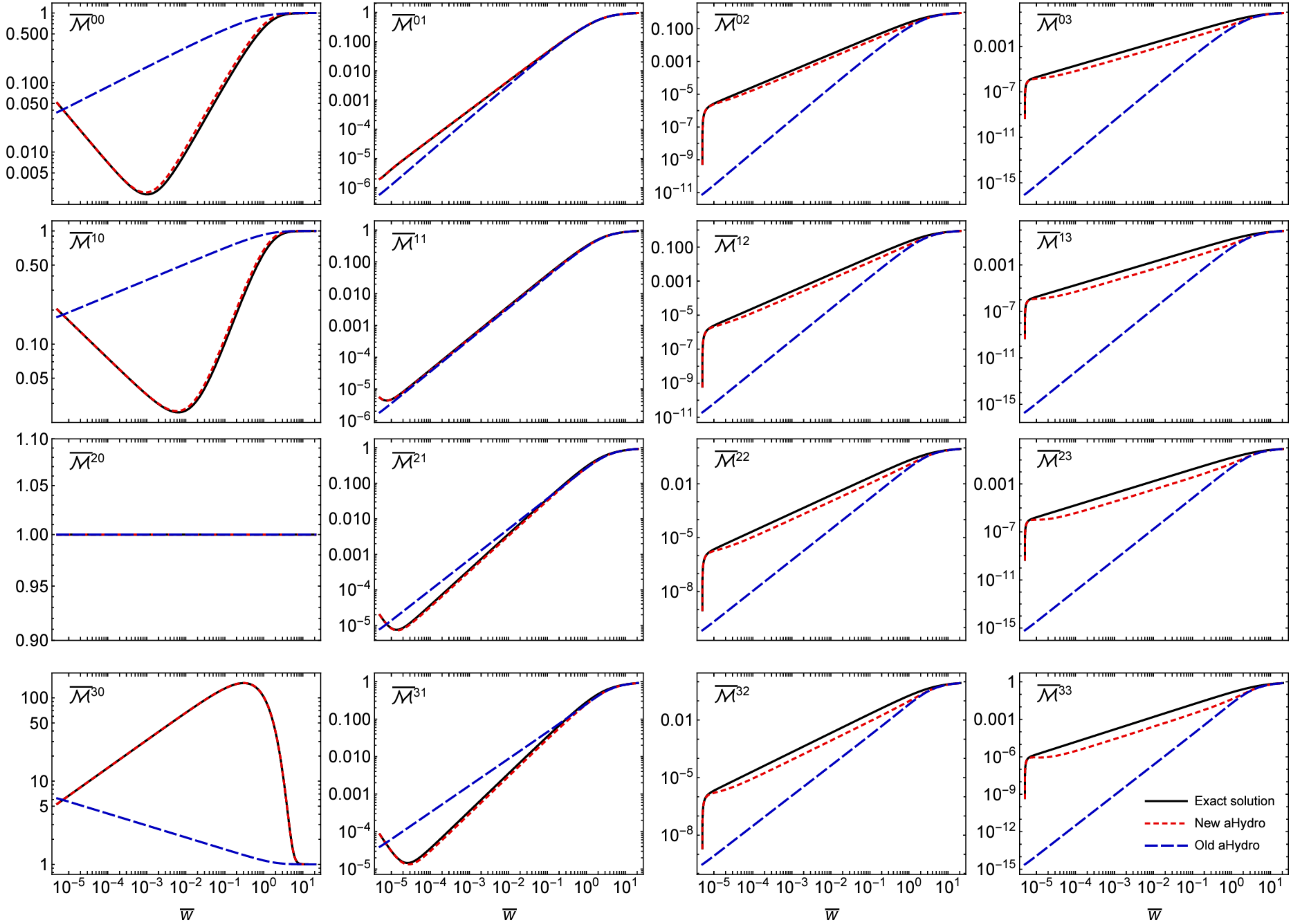
First moment of Boltzmann equation

$$\boxed{\begin{aligned} &[1-D(\tau,\tau_0)]\left[\hat\Lambda^4\mathcal{R}'(\xi)\dot\xi + 4\hat\Lambda^3\mathcal{R}(\xi)\dot{\hat\Lambda} + \frac{\mathcal{R}(\xi)\hat\Lambda^4}{\tau}\left(1+\frac{1}{3}\frac{\mathcal{R}_L(\xi)}{\mathcal{R}(\xi)}\right)\right] + \\ &\quad D(\tau,\tau_0)\left[\mathcal{R}'(\xi_{\rm FS})\dot\xi_{\rm FS} - \left(\frac{1}{\tau_{\rm eq}}-\frac{1}{\tau}\right)\mathcal{R}(\xi_{\rm FS}) + \frac{1}{3\tau}\mathcal{R}_L(\xi_{\rm FS}) + \frac{\hat\Lambda^4\mathcal{R}(\xi)}{\tau_{\rm eq}}\right] = 0\end{aligned}}$$

Second moment of Boltzmann equation

$$\boxed{[1-D(\tau,\tau_0)]\left(\frac{1}{1+\xi}\dot\xi - \frac{2}{\tau}\right) + \frac{\xi\sqrt{1+\xi}}{\tau_{\rm eq}}\frac{\hat T^5}{\hat\Lambda^5} = 0}$$

H. Alalawi and MS, 2006.13834



What happens if we break the conformal symmetry?

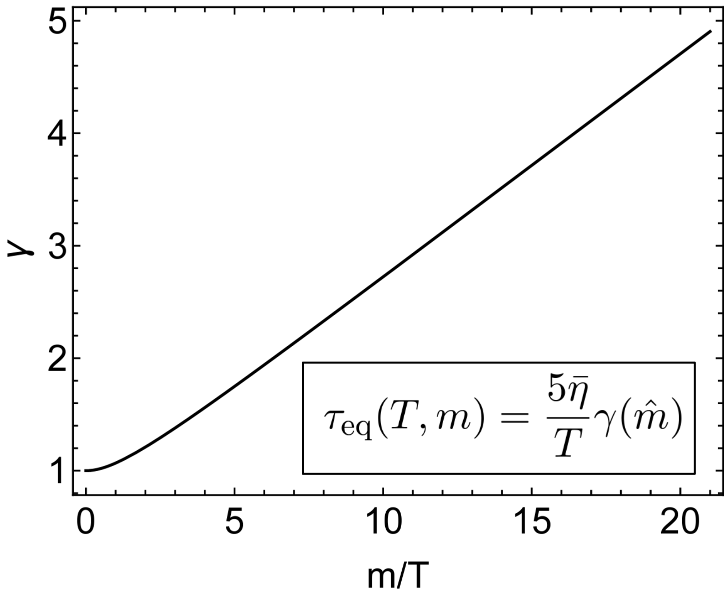
Non-conformal attractor?

H. Alalawi and MS, 2211.06363

W. Florkowski, E. Maksymiuk, R. Ryblewski, and MS, 1402.7348 (constant relaxation time)

Consider a massive gas with a mass- and temperature-dependent relaxation time.

$$\boxed{\begin{aligned} &2T^4(\tau)\,\hat{m}^2\left[3K_2\left(\frac{m}{T(\tau)}\right)+\hat{m}K_1\left(\frac{m}{T(\tau)}\right)\right] \\ &\qquad = D(\tau,\tau_0)\Lambda_0^4\tilde{H}^{20}\left(\frac{\tau_0}{\tau\sqrt{1+\xi_0}},\frac{m}{\Lambda_0}\right)+\int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{\rm eq}(\tau')}\,D(\tau,\tau')\,T^4(\tau')\tilde{H}^{20}\left(\frac{\tau'}{\tau},\frac{m}{T(\tau')}\right)\end{aligned}}$$



$$\tau_{\rm eq}(T,m)=\frac{5\bar{\eta}}{T}\gamma(\hat{m})$$

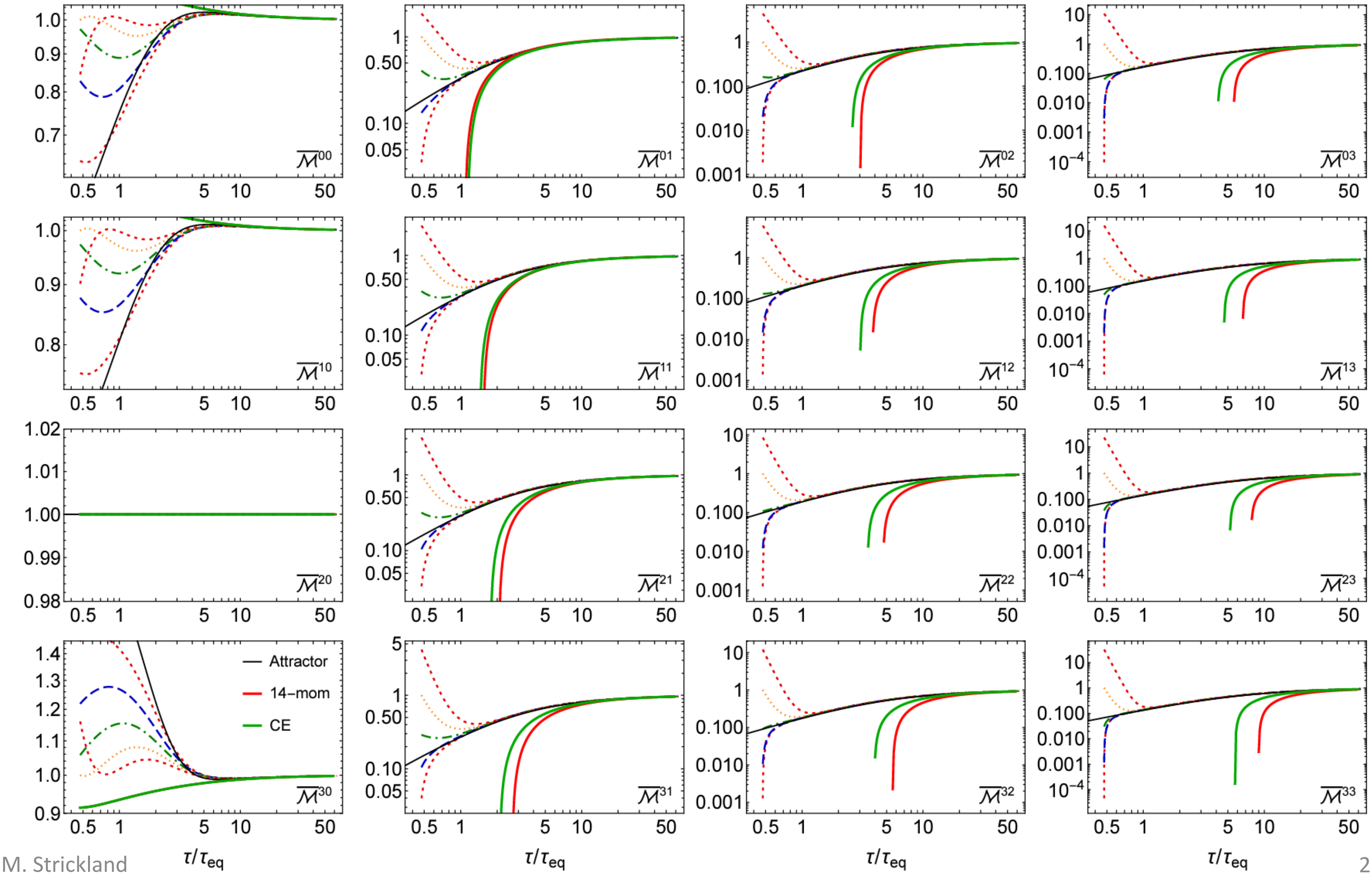
$$\begin{aligned}\mathcal{M}^{nl} &= \frac{D(\tau,\tau_0)\Lambda_0^{n+2l+2}}{(2\pi)^2}\tilde{H}^{nl}\left(\frac{\tau_0}{\tau\sqrt{1+\xi_0}},\frac{m}{\Lambda_0}\right) \\ &\qquad +\frac{1}{(2\pi)^2}\int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{\rm eq}(\tau')}\,D(\tau,\tau')\,T^{n+2l+2}(\tau')\,\tilde{H}^{nl}\left(\frac{\tau'}{\tau},\frac{m}{T(\tau')}\right)\end{aligned}$$

$$\tilde{H}^{nl}(y,z)=\int_0^\infty du\,u^{n+2l+1}e^{-\sqrt{u^2+z^2}}\,H^{nl}\left(y,\frac{z}{u}\right),$$

$$H^{nl}(y,x)=\frac{2\,y^{2l+1}(1+x^2)^{\frac{n-1}{2}}}{2l+1}\,{}_2F_1\left(l+\frac{1}{2},\frac{1-n}{2};l+\frac{3}{2};\frac{1-y^2}{1+x^2}\right),$$

Non-conformal attractor?

H. Alalawi and MS, 2211.06363



Non-conformal attractor?

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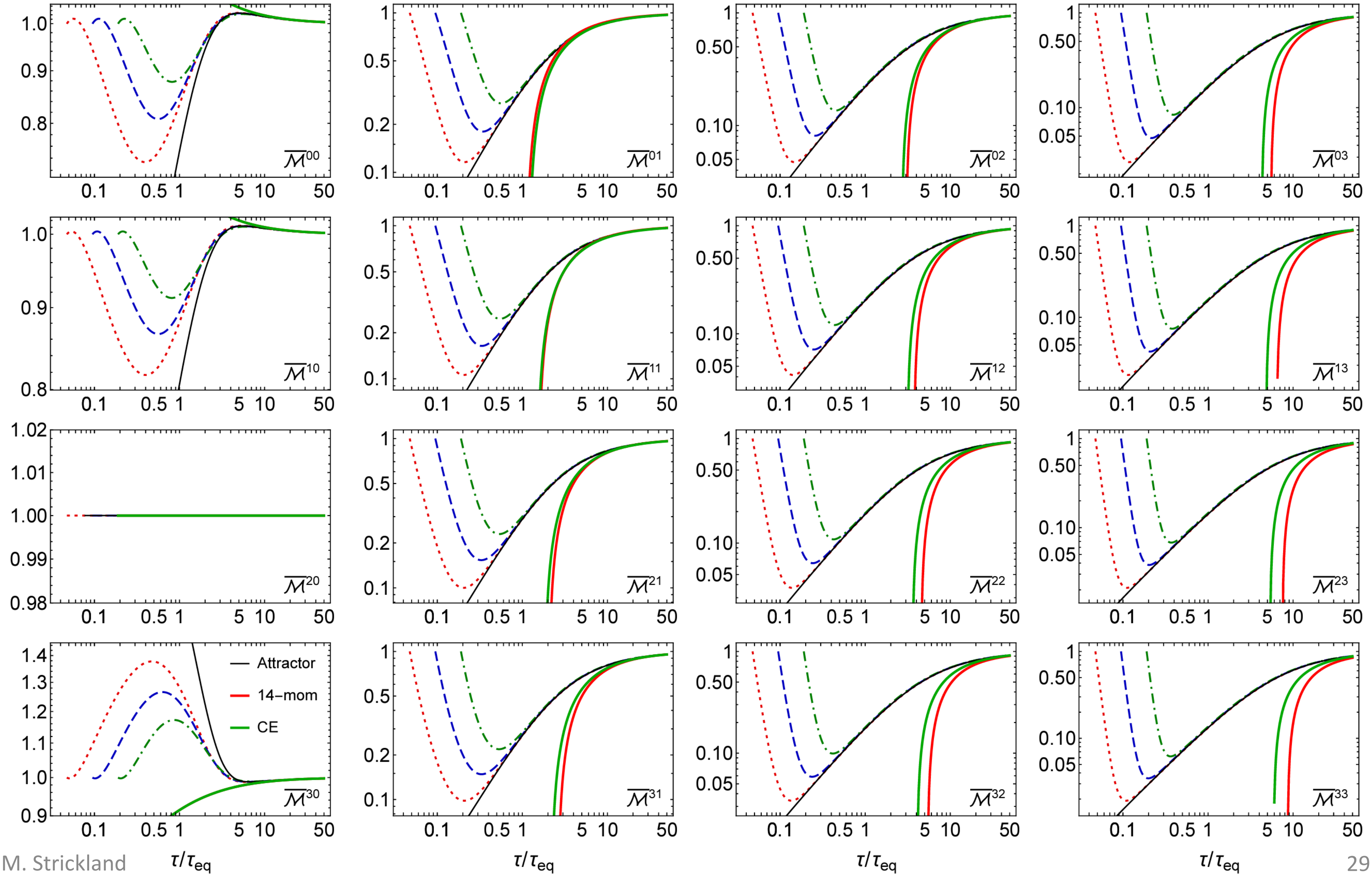
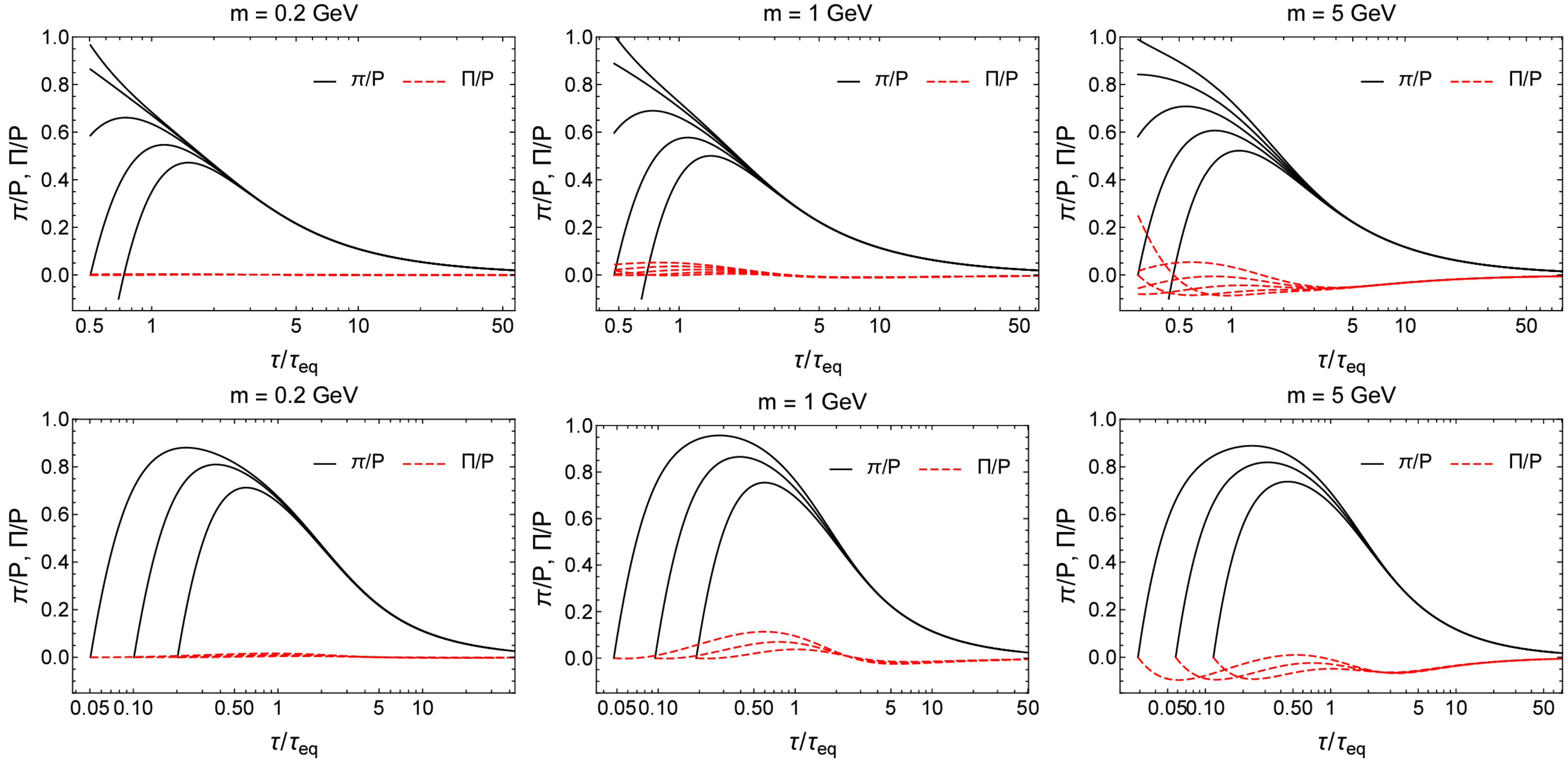


Non-conformal attractor?

H. Alalawi and MS, 2211.06363

$$\tilde{\Pi} \equiv \frac{\Pi}{P} = -\frac{m^2\left(\mathcal{M}^{00} - \mathcal{M}^{00}_{\rm eq}\right)}{\mathcal{M}^{20}_{\rm eq} - m^2 \mathcal{M}^{00}_{\rm eq}}$$

$$\tilde{\pi} \equiv \frac{\pi}{P} = 1 - \overline{\mathcal{M}}^{01} + \tilde{\Pi}$$



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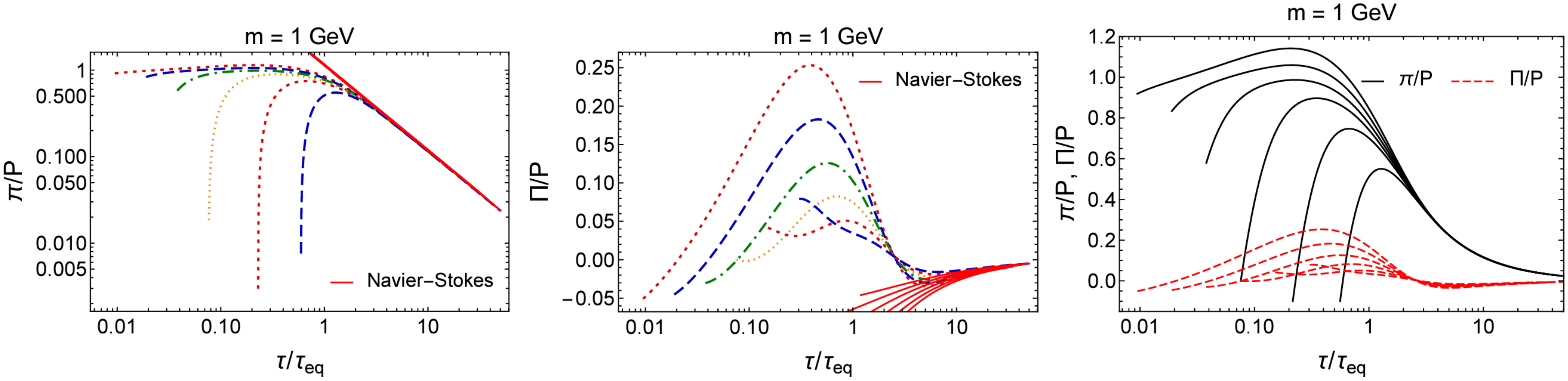


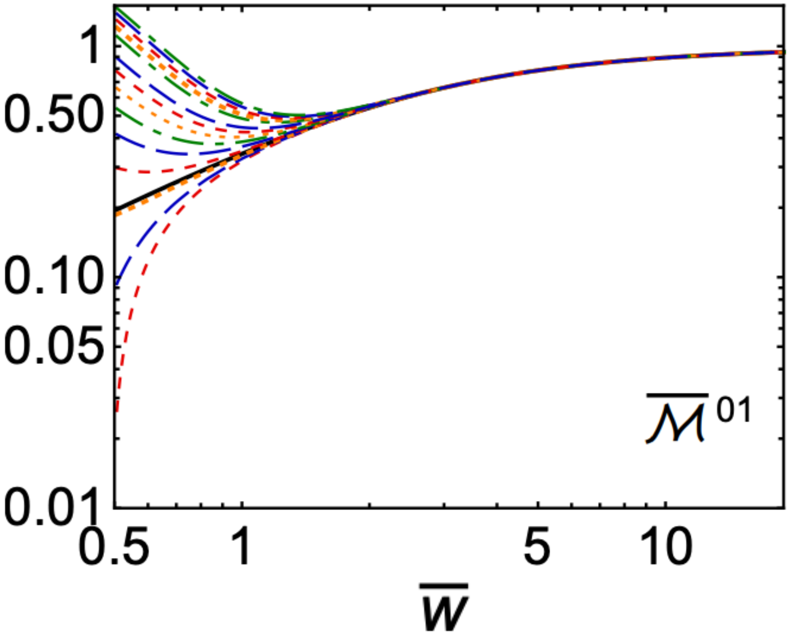
- Above plot shows varying initialization time, initial anisotropy, and initial fugacity.
- **Shear and bulk don't have early-time attractors. Bulk is much worse than shear; however, longitudinal pressure (M^{01}) does possess an attractor and so do all moments with m > 0!**
- The violation of universality occurs at both early and late times in the case of the bulk viscous pressure!
- However, there remains a semi-universality at late times that you can only see broken in you zoom in a lot.

Evidence for a QCD EKT attractor

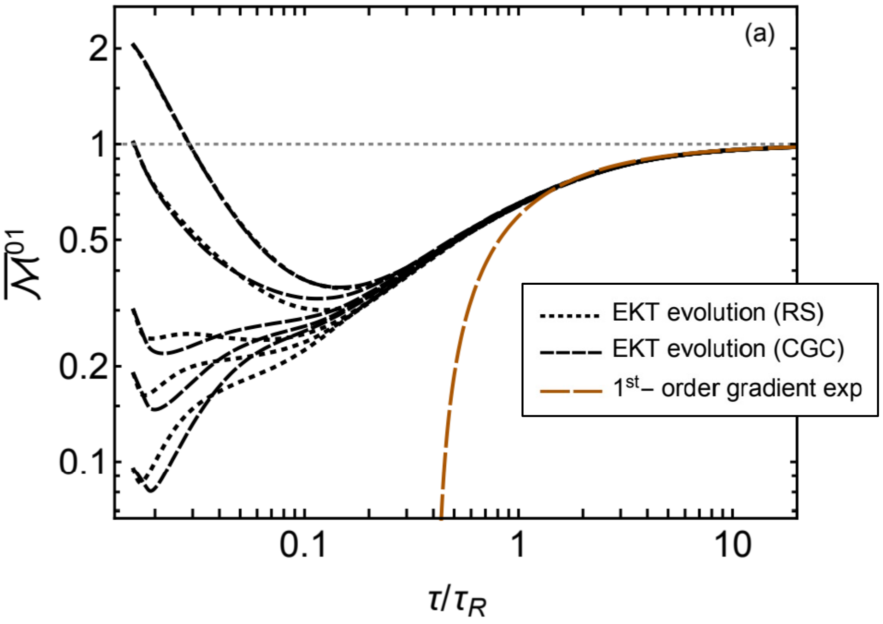
D. Almaalol, A. Kurkela, and MS, PRL 125, 122302(2020)

- Numerical implementation of pure glue AMY effective kinetic theory (EKT)
- Includes **elastic gluon scattering** and **inelastic gluon splitting** with LPM suppression and detailed balance.
- We use the “pure glue” EKT code of Kurkela and Zhu PRL 115, 182301 (2015)
- 250 x 2000 x 1 grid in momentum space (n_p x n_θ x n_ϕ)

RTA



EKT QCD



MS, JHEP2018, 128; 1809.01200

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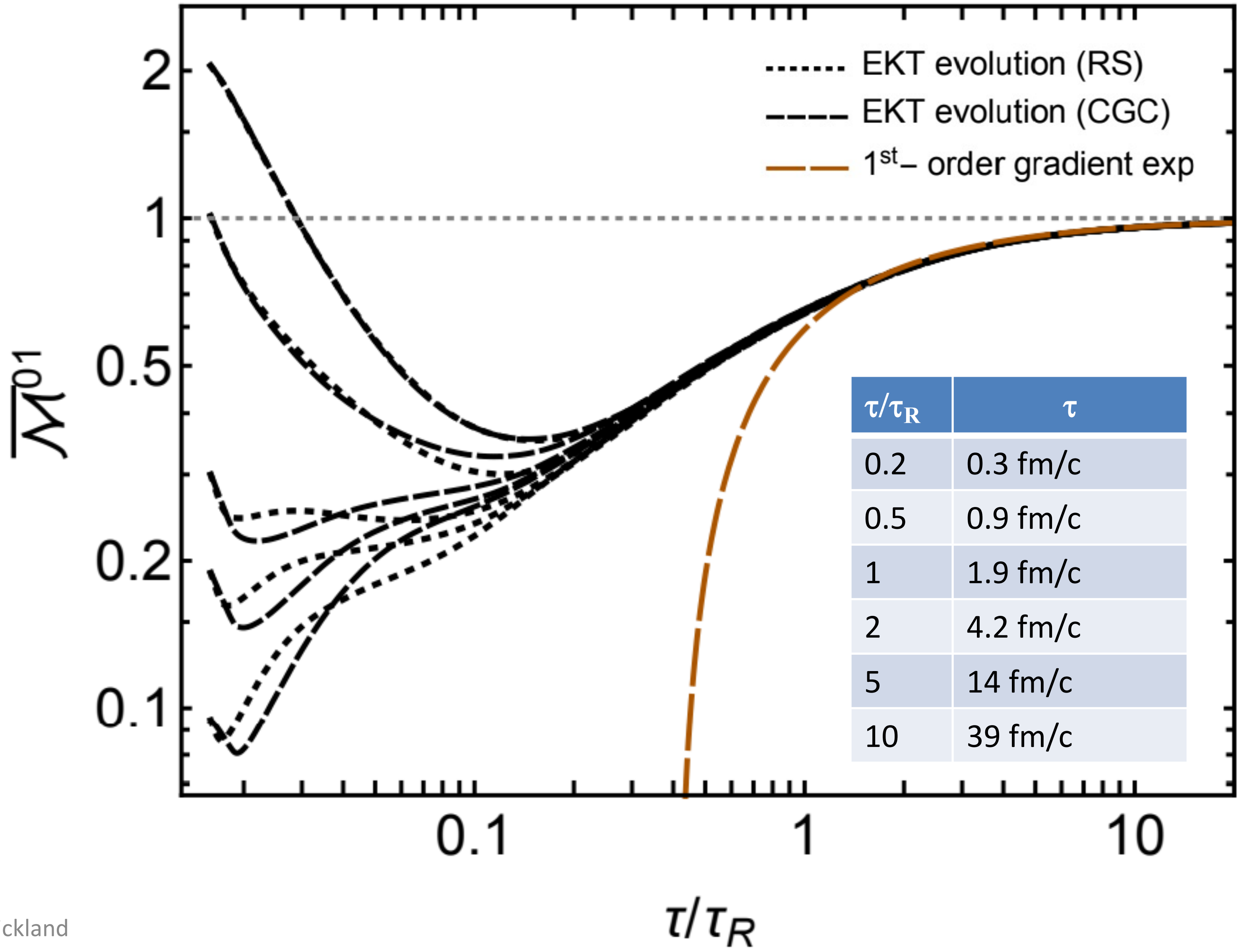
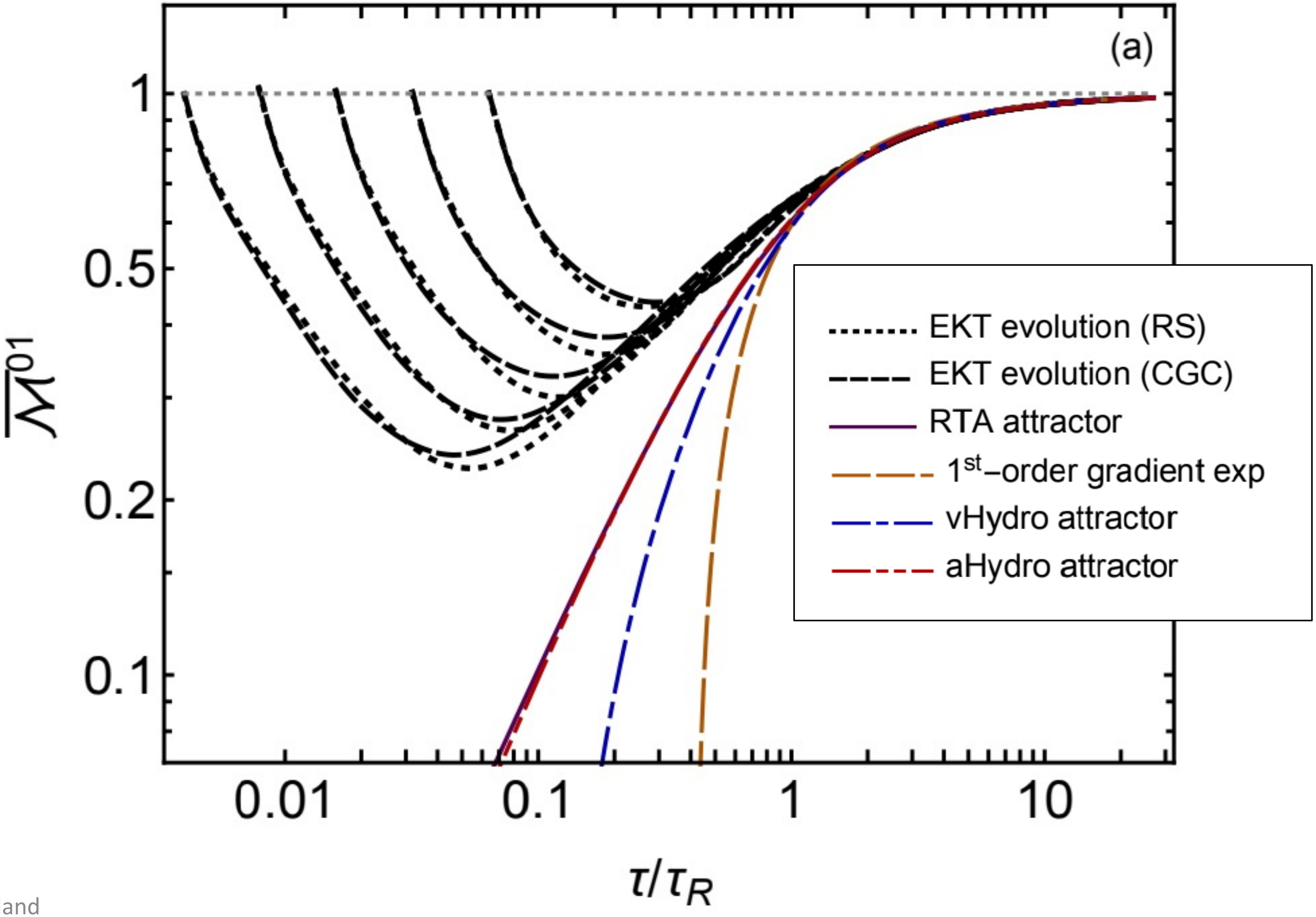


τ/τ_R	τ
0.2	0.3 fm/c
0.5	0.9 fm/c
1	1.9 fm/c
2	4.2 fm/c
5	14 fm/c
10	39 fm/c

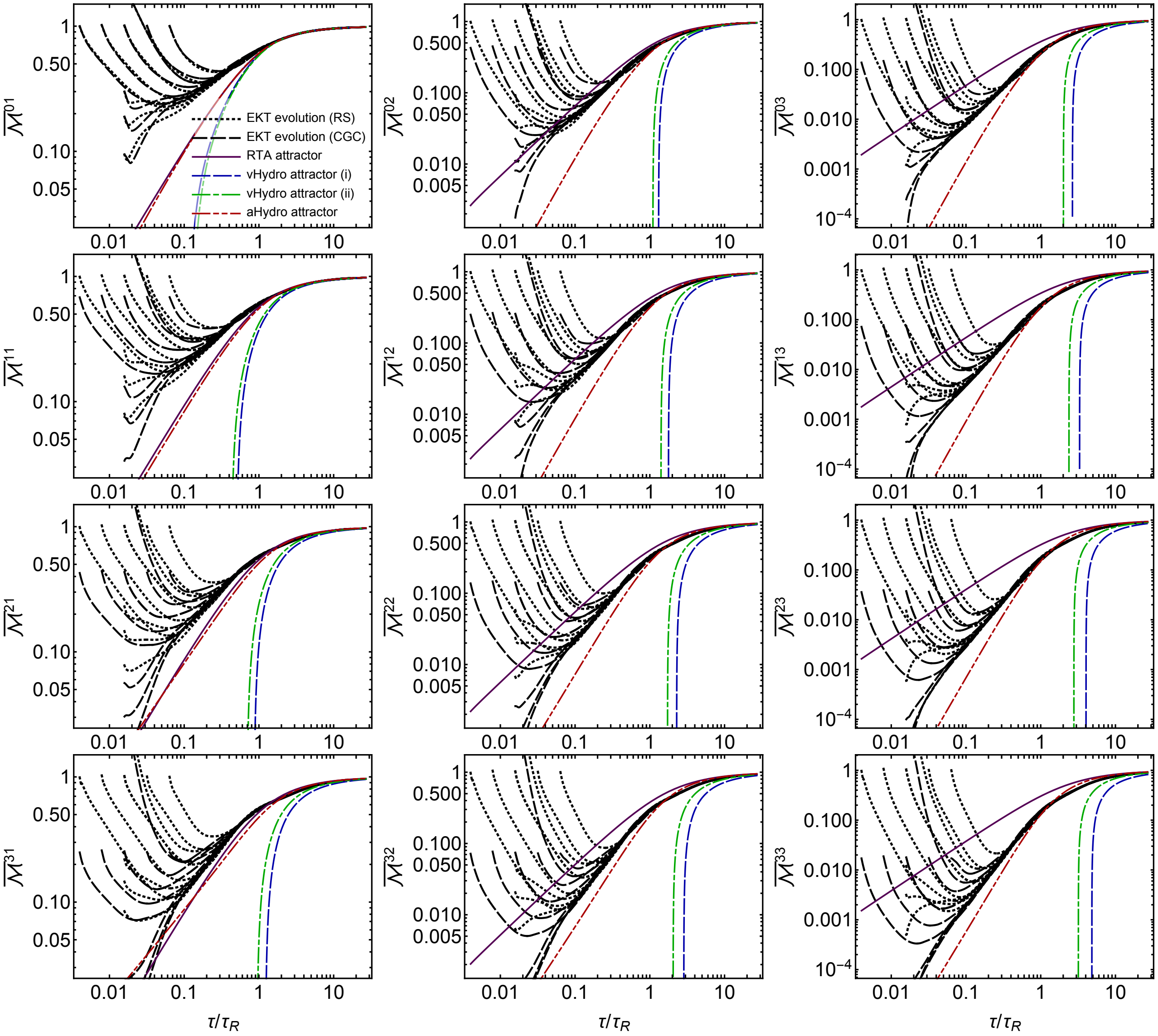
Is there an early-time attractor?

D. Almaalol, A. Kurkela, and MS, PRL 125, 122302(2020)



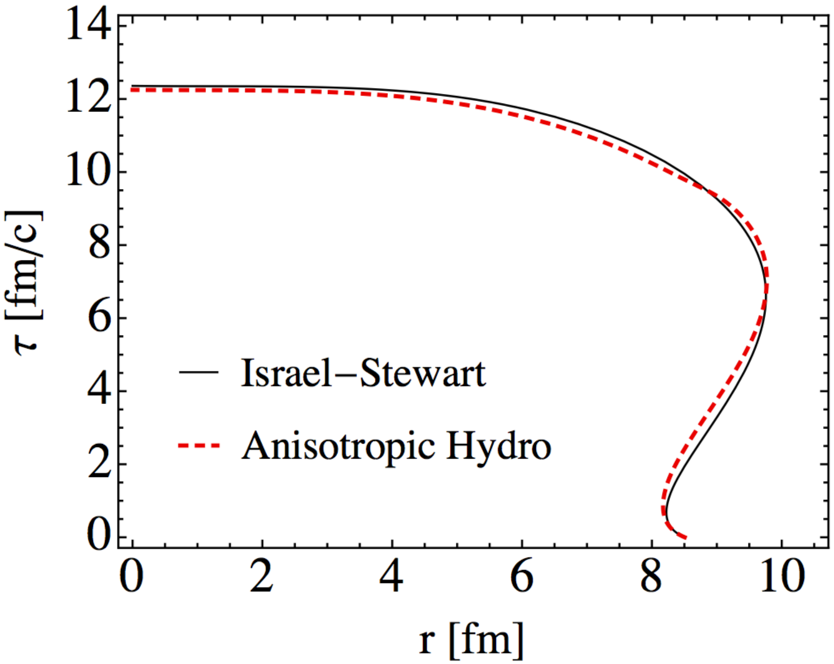
Varying initial anisotropy and t_0

D. Almaalol, A. Kurkela, and MS, PRL 125, 122302(2020)



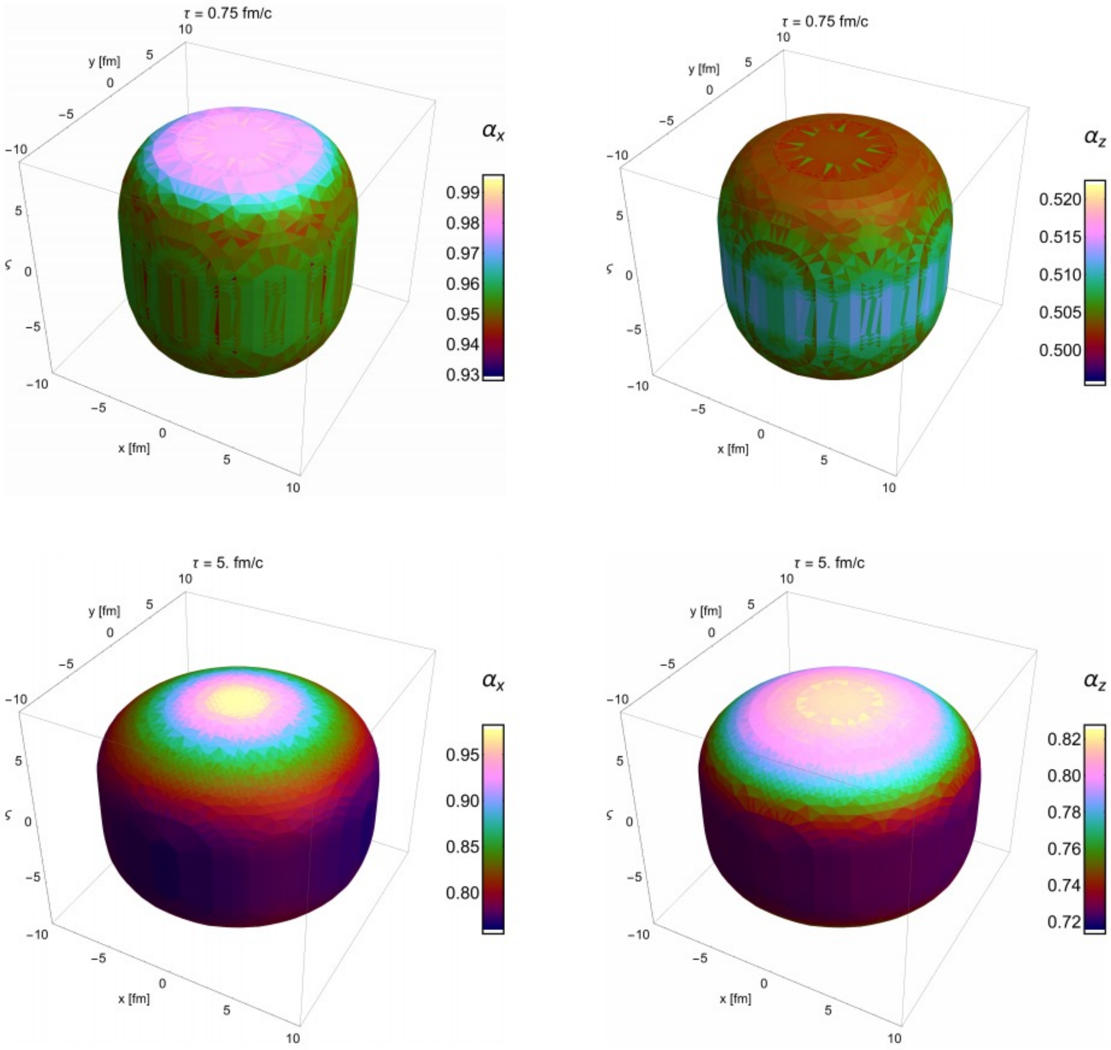
- Attractor seen in all moments and is the same for both types of initial conditions.
- For low order moments EKT QCD is closer to EQ than RTA and hydro predictions.
- For high order moments the opposite is true.
- **Hydrodynamization is only the tip of the iceberg → Pseudothermalization**

Freeze-out?



τ [fm/c]
14
12
10
8
6
4
2
0
0
2
4
6
8
10
r [fm]
Israel–Stewart
Anisotropic Hydro

Testing different hydro freeze-out methods



i. vHyro f = f_{eq} + δf.
Standard quadratic ansatz

$$\frac{\delta f_{(i)}}{f_{\rm eq}(1+f_{\rm eq})} = \frac{3\overline{\Pi}}{16T^2}(p^2 - 3p_z^2)$$

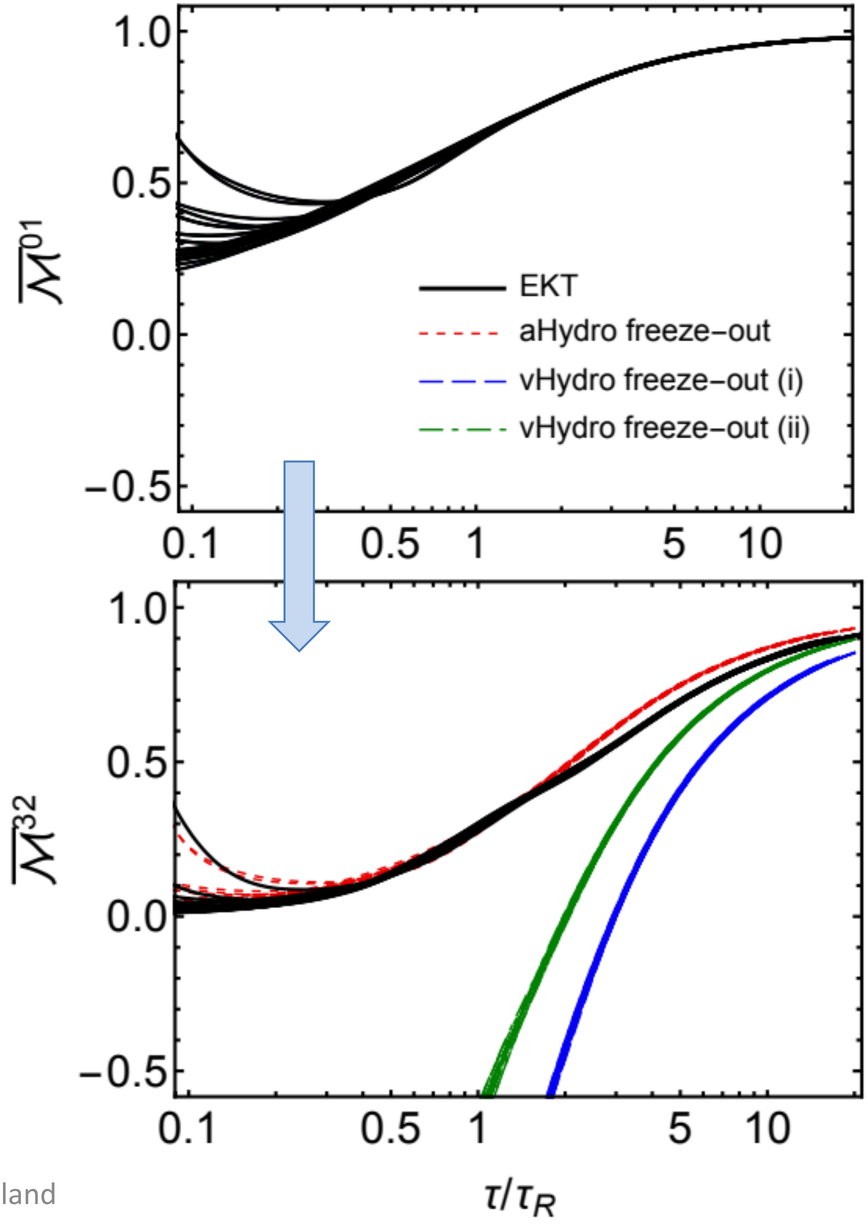
ii. vHyro f = f_{eq} + δf.
LPM-improved ansatz

$$\frac{\delta f_{(ii)}}{f_{\rm eq}(1+f_{\rm eq})} = \frac{16\overline{\Pi}}{21\sqrt{\pi}\,T^{3/2}}\left(p^{3/2} - \frac{3p_z^2}{\sqrt{p}}\right)$$

iii. aHydro ansatz

$$f_{\rm aniso}^{LRF} = f_{\rm iso}\left(\frac{\sqrt{\mathbf{p}^2 + \xi(\mathbf{x},\tau)p_z^2}}{\Lambda(\mathbf{x},\tau)}\right)$$

Testing different hydro freeze-out methods



i. vHyro f = f_{eq} + δf.
Standard quadratic ansatz

$$\frac{\delta f_{(i)}}{f_{\rm eq}(1+f_{\rm eq})} = \frac{3\overline{\Pi}}{16T^2}(p^2-3p_z^2)$$

ii. vHyro f = f_{eq} + δf.
LPM-improved ansatz

$$\frac{\delta f_{(ii)}}{f_{\rm eq}(1+f_{\rm eq})} = \frac{16\overline{\Pi}}{21\sqrt{\pi}\,T^{3/2}}\left(p^{3/2}-\frac{3p_z^2}{\sqrt{p}}\right)$$

iii. aHydro ansatz

$$f^{LRF}_{\rm aniso} = f_{\rm iso}\left(\frac{\sqrt{\mathbf{p}^2+\xi(\mathbf{x},\tau)p_z^2}}{\Lambda(\mathbf{x},\tau)}\right)$$

Anisotropic Hydrodynamics

Generalized aHydro formalism

In generalized aHydro, one assumes that the distribution function is of the form

$$f(x,p) = f_{\rm eq}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu}(x) p^\nu}}{\lambda(x)}, \frac{\mu(x)}{\lambda(x)}\right) + \delta\tilde{f}(x,p)$$

$$\Xi^{\mu\nu} = \underbrace{u^\mu u^\nu}_{\text{LRF four velocity}} + \underbrace{\xi^{\mu\nu}}_{\text{Traceless symmetric anisotropy tensor}} - \underbrace{\Delta^{\mu\nu}}_{\text{Transverse projector}} \underbrace{\Phi}_{\text{“Bulk”}}$$

$$u^\mu u_\mu = 1$$
$$\xi^\mu{}_\mu = 0$$
$$\Delta^\mu{}_\mu = 3$$
$$u_\mu \xi^{\mu\nu} = u_\mu \Delta^{\mu\nu} = 0$$

- 3 degrees of freedom in u^μ
- 5 degrees of freedom in $\xi^{\mu\nu}$
- 1 degree of freedom in Φ
- 1 degree of freedom in λ
- 1 degree of freedom in μ

→ 11 DOFs

See e.g.

- M. Martinez, R. Ryblewski, and MS, 1204.1473
- L. Tinti and W. Florkowski, 1312.6614
- M. Nopoush, R. Ryblewski, and MS, 1405.1355

Equations of Motion

- The EOM are obtained from moments of the Boltzmann equation in the relaxation time approximation (RTA) including temperature -dependent quasiparticle mass

$$p^\mu \partial_\mu f + \frac{1}{2}\partial_i m^2 \partial^i_{(p)} f = -\mathcal{C}[f] \qquad \mathcal{C}[f] = \frac{p^\mu u_\mu}{\tau_{\rm eq}}(f - f_{\rm eq})$$

- It is relatively straightforward to use other collisional kernels (see paper with D. Almaalol arXiv:1801.10173)

- 1 equation from the 0th moment [number (non-)conservation]
- 4 equations from the 1st moment [energy-momentum conservation]
- 6 equations from the 2nd moment [dissipative dynamics – **shear + bulk + ∞**]
- We must also specify the relation between the equilibrium (isotropic) pressure and energy density (EoS).

$$\begin{aligned} D_u n + n\theta_u &= \frac{1}{\tau_{\rm eq}}(n_{\rm eq} - n) \\ \partial_\mu T^{\mu\nu} &= 0 \\ \partial_\mu \mathcal{I}^{\mu\nu\lambda} &= \frac{1}{\tau_{\rm eq}}(u_\mu \mathcal{I}^{\mu\nu\lambda}_{\rm eq} - u_\mu \mathcal{I}^{\mu\nu\lambda}) \end{aligned}$$

$$\mathcal{I}^{\mu\nu\lambda} \equiv \int dP\, p^\mu p^\nu p^\lambda f(x,p)\,.$$

Implementing the equation of state

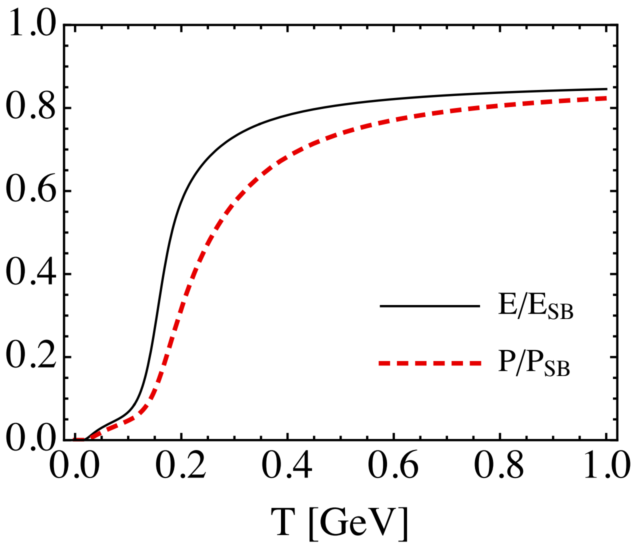
M. Alqahtani, M. Nopoush, and MS, 1509.02913; 1605.02101
M. Alqahtani, M. Nopoush, R. Ryblewski, and MS, 1703.05808; 1705.10191

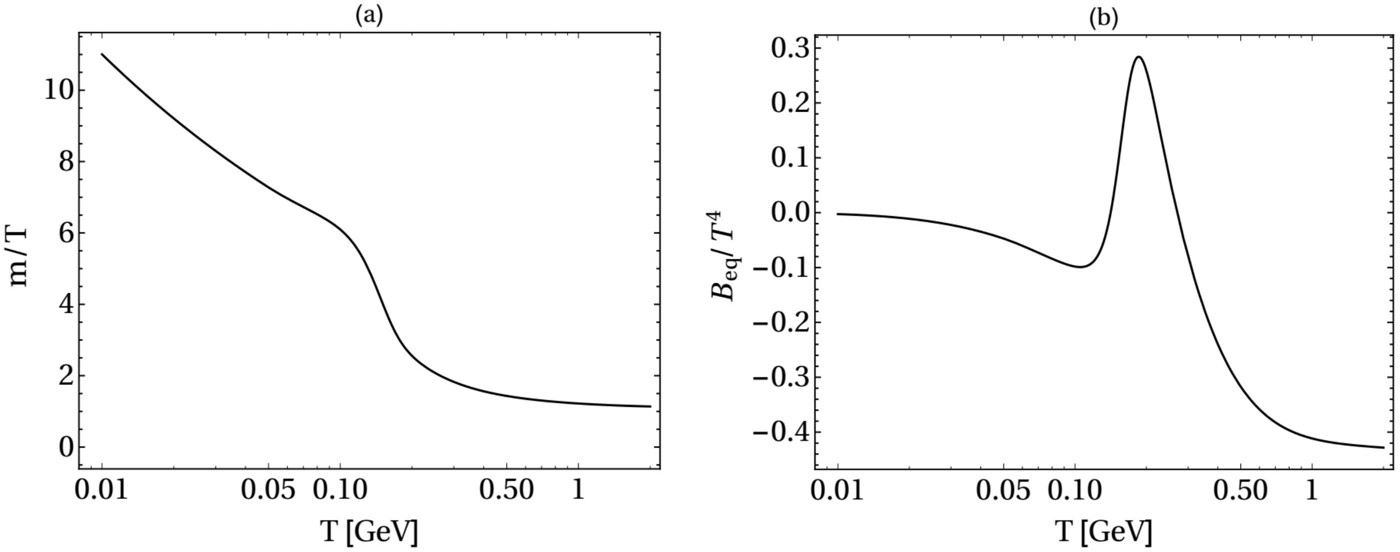
Quasiparticle Method

$$T^{\mu\nu} = T^{\mu\nu}_{\text{kinetic}} + B g^{\mu\nu}$$

$$p^\mu \partial_\mu f + \frac{1}{2}\partial_i m^2 \partial^i_{(p)} f = -\mathcal{C}[f]$$

$$\partial_\mu B = -\frac{1}{2}\partial_\mu m^2 \int dP\, f(x,p)$$





3+1d aHydro Equations of Motion

- Assuming an ellipsoidal form for the anisotropy tensor (ignoring off-diagonal components for now), one has seven degrees of freedom ξ_x, ξ_y, ξ_z, u_x, u_y, u_z, and λ which are all fields of space and time.
- Ignore $\delta\tilde{f}$ for now

$$\begin{aligned}
D_u\mathcal{E} + \mathcal{E}\theta_u + \mathcal{P}_x u_\mu D_x X^\mu + \mathcal{P}_y u_\mu D_y Y^\mu + \mathcal{P}_z u_\mu D_z Z^\mu &= 0\,, \\
D_x\mathcal{P}_x + \mathcal{P}_x\theta_x - \mathcal{E} X_\mu D_u u^\mu - \mathcal{P}_y X_\mu D_y Y^\mu - \mathcal{P}_z X_\mu D_z Z^\mu &= 0\,, \\
D_y\mathcal{P}_y + \mathcal{P}_y\theta_y - \mathcal{E} Y_\mu D_u u^\mu - \mathcal{P}_x Y_\mu D_x X^\mu - \mathcal{P}_z Y_\mu D_z Z^\mu &= 0\,, \\
D_z\mathcal{P}_z + \mathcal{P}_z\theta_z - \mathcal{E} Z_\mu D_u u^\mu - \mathcal{P}_x Z_\mu D_x X^\mu - \mathcal{P}_y Z_\mu D_y Y^\mu &= 0\,.
\end{aligned}$$

First Moment

$$\mathcal{I}^{\mu\nu\lambda} \equiv \int dP\, p^\mu p^\nu p^\lambda f(x,p)\,.$$

$$\begin{aligned}
\mathcal{I}_i &= \alpha\,\alpha_i^2\,\mathcal{I}_{\rm eq}(\lambda,m)\,, \\
\mathcal{I}_{\rm eq}(\lambda,m) &= 4\pi\tilde{N}\lambda^5\hat{m}^3 K_3(\hat{m})\,,
\end{aligned}$$

$$\begin{aligned}
D_u\mathcal{I}_x + \mathcal{I}_x(\theta_u + 2u_\mu D_x X^\mu) &= \frac{1}{\tau_{\rm eq}}(\mathcal{I}_{\rm eq} - \mathcal{I}_x)\,, \\
D_u\mathcal{I}_y + \mathcal{I}_y(\theta_u + 2u_\mu D_y Y^\mu) &= \frac{1}{\tau_{\rm eq}}(\mathcal{I}_{\rm eq} - \mathcal{I}_y)\,, \\
D_u\mathcal{I}_z + \mathcal{I}_z(\theta_u + 2u_\mu D_z Z^\mu) &= \frac{1}{\tau_{\rm eq}}(\mathcal{I}_{\rm eq} - \mathcal{I}_z)\,.
\end{aligned}$$

Second Moment

M. Alqahtani, M. Nopoush, and MS, 1509.02913; 1605.02101
M. Alqahtani, M. Nopoush, R. Ryblewski, and MS, 1703.05808; 1705.10191

Anisotropic Cooper-Frye Freezeout

M. Alqahtani, M. Nopoush, and MS, 1605.02101
M. Alqahtani, M. Nopoush, R. Ryblewski, and MS, 1703.05808; 1705.10191

- Use same generalized-RS form for "anisotropic freeze-out" at LO
- Form includes both shear and bulk corrections to the distribution function
- Use energy density (scalar) to determine the freeze-out hyper-surface Σ → e.g. $T_{\rm eff,FO}$ = 130 MeV

$$f(x,p) = f_{\rm iso}\left(\frac{1}{\lambda}\sqrt{p_\mu \Xi^{\mu\nu} p_\nu}\right)$$

$$\Xi^{\mu\nu} = \underbrace{u^\mu u^\nu}_{\text{isotropic}} + \underbrace{\xi^{\mu\nu}}_{\text{anisotropy tensor}} - \underbrace{\Phi\Delta^{\mu\nu}}_{\text{bulk correction}}$$

$$\xi^{\mu\nu}_{\rm LRF} \equiv {\rm diag}(0,\xi_x,\xi_y,\xi_z)$$

$$\xi^\mu{}_\mu = 0 \qquad u_\mu \xi^\mu{}_\nu = 0$$

$$\left(p^0\frac{dN}{dp^3}\right)_i = \frac{\mathcal{N}_i}{(2\pi)^3}\int f_i(x,p)\,p^\mu d\Sigma_\mu\,,$$

NOTE: Usual 2nd-order viscous hydro form

$$f(p,x) = f_{\rm eq}\left[1 + (1 - a f_{\rm eq})\frac{p_\mu p_\nu \Pi^{\mu\nu}}{2(\epsilon+P)T^2}\right]$$

$$f_{\rm eq} = 1/[\exp(p\cdot u/T) + a] \qquad a = \text{-}1,\ +1,\ \text{or}\ 0$$

- This form suffers from the problem that the distribution function can be negative in some regions of phase space → unphysical
- **Problem becomes worse when including the bulk viscous correction**.

Anisotropic Cooper-Frye Freezeout

M. Alqahtani, M. Nopoush, and MS, 1605.02101
M. Alqahtani, M. Nopoush, R. Ryblewski, and MS, 1703.05808; 1705.10191

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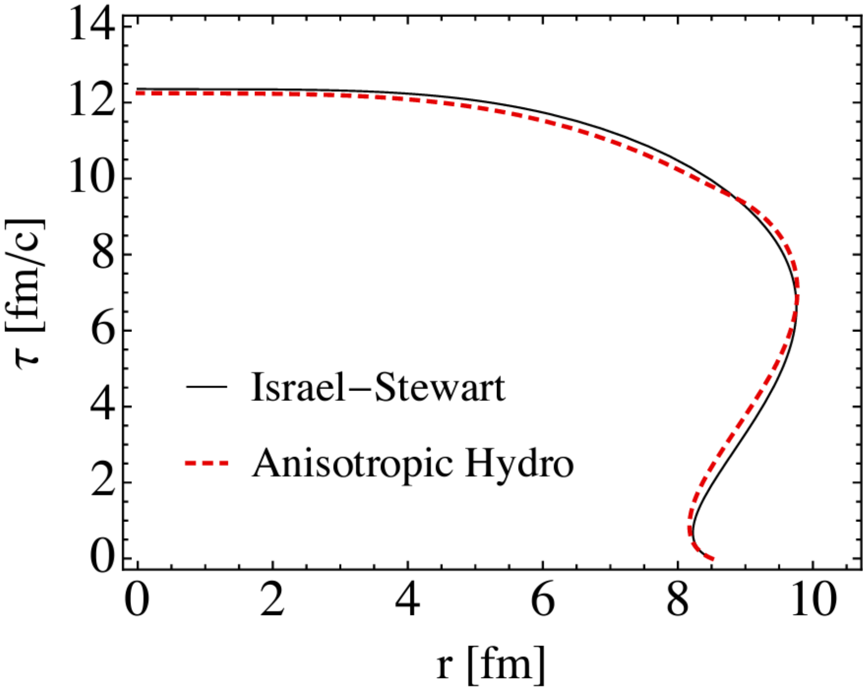

Anisotropic Cooper-Frye Freezeout

M. Alqahtani, M. Nopoush, and MS, 1605.02101
M. Alqahtani, M. Nopoush, R. Ryblewski, and MS, 1703.05808; 1705.10191

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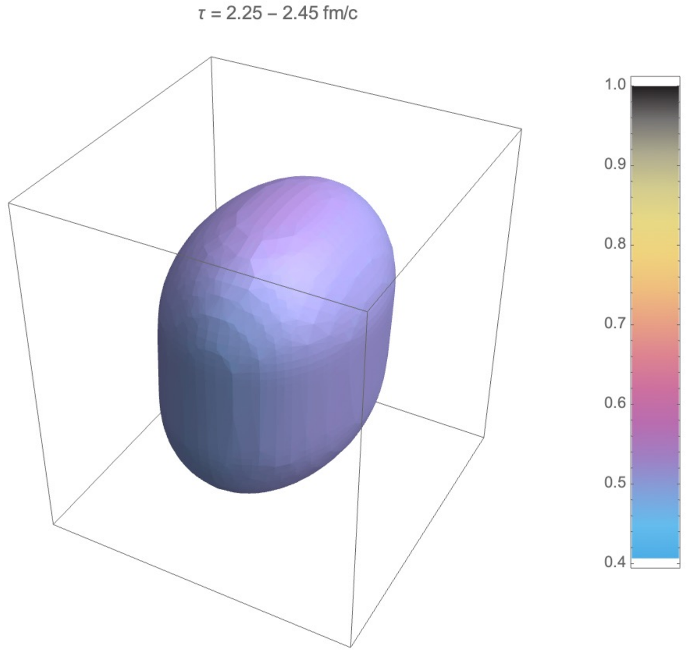
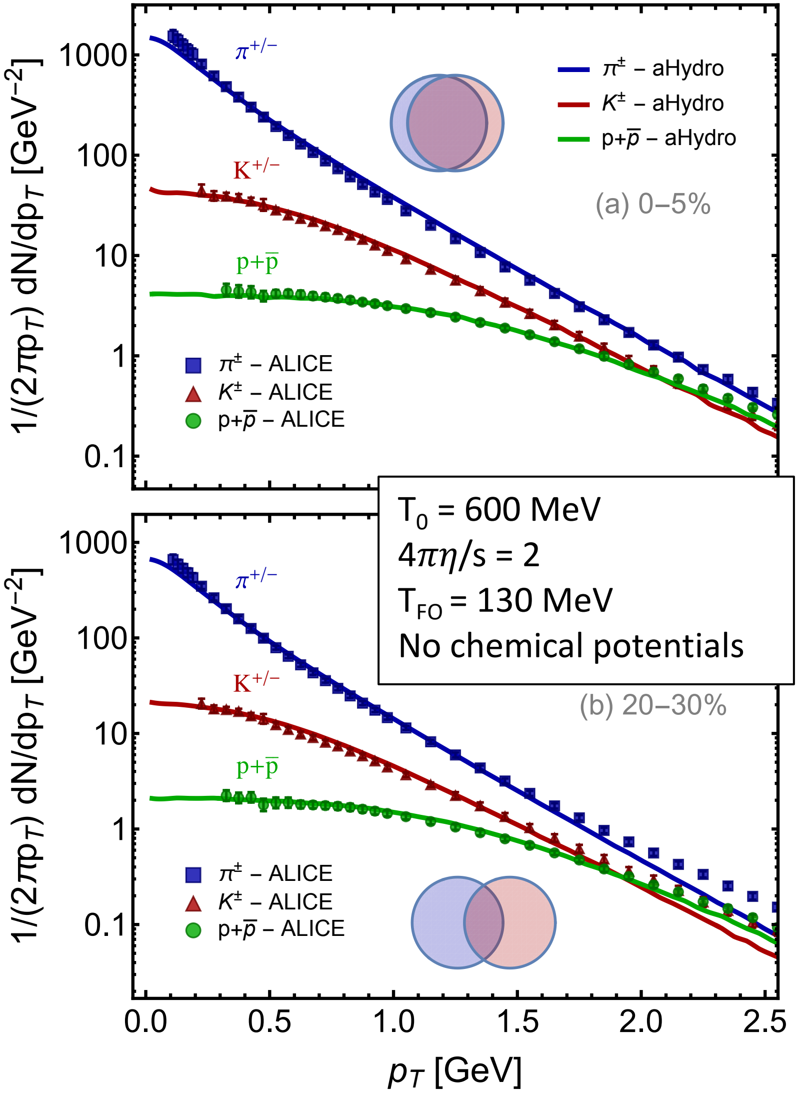


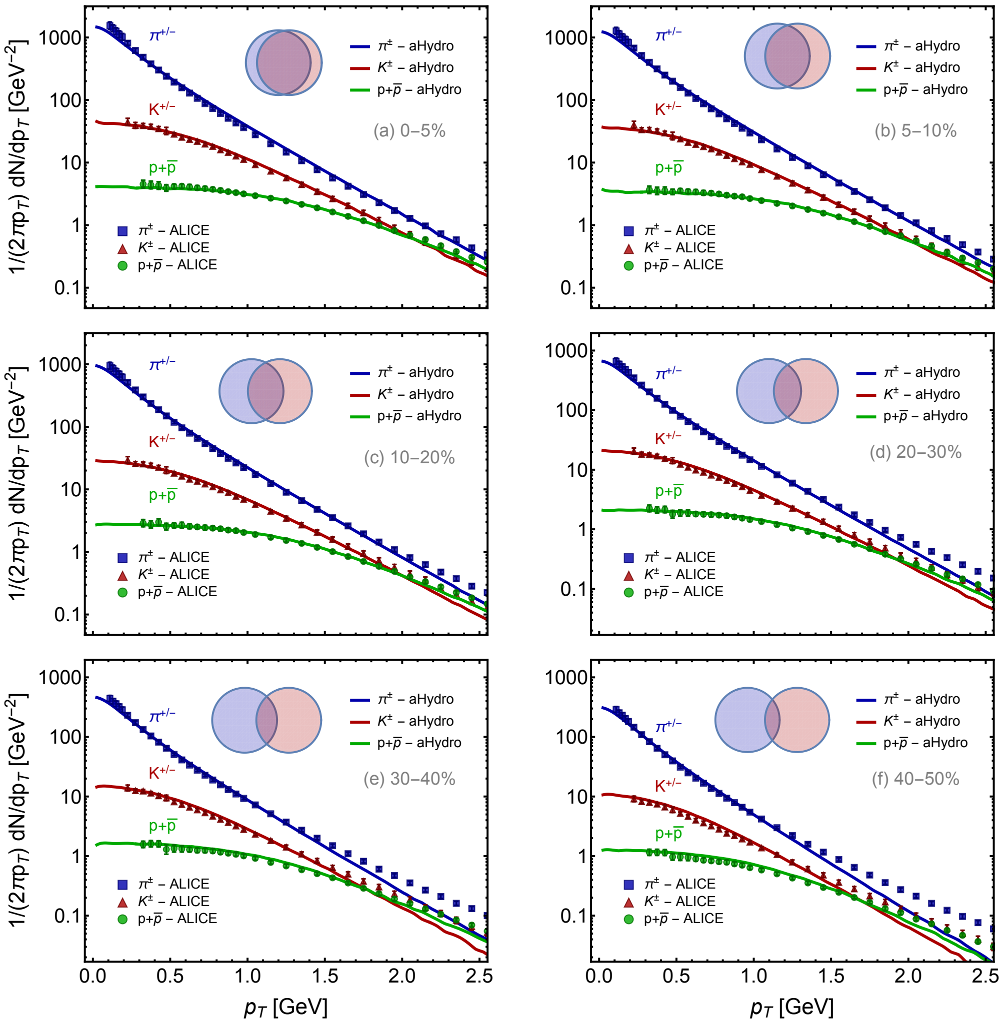
LHC 2.76 TeV

Identified particle spectra @ 2.76 TeV

Alqahtani, Nopoush, Ryblewski, MS, 1703.05808 (PRL); 1705.10191



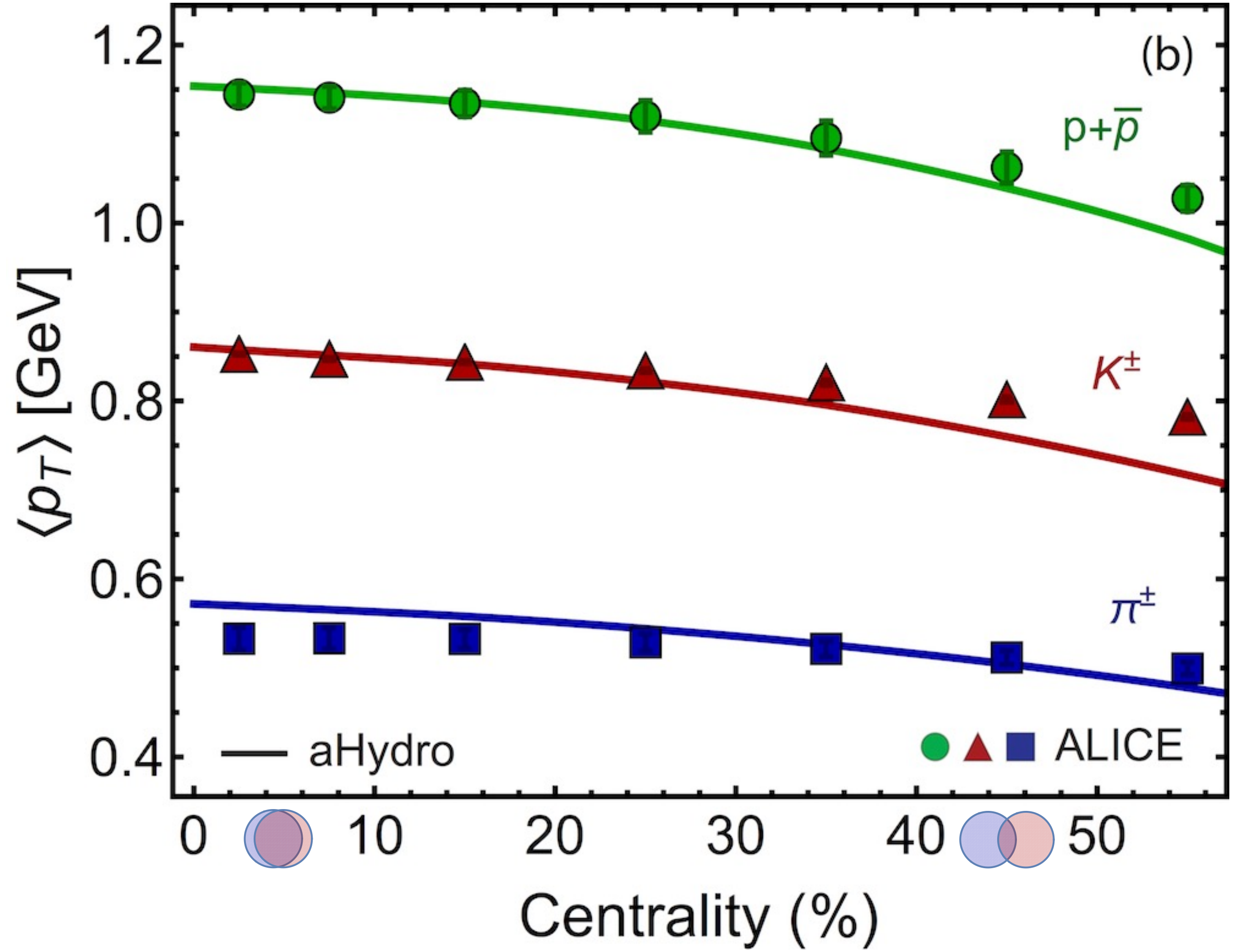
T$_0$ = 600 MeV
$4\pi\eta/s = 2$
T$_{FO}$ = 130 MeV
No chemical potentials



Data are from the ALICE collaboration data for **Pb-Pb collisions @ 2.76 TeV/nucleon**

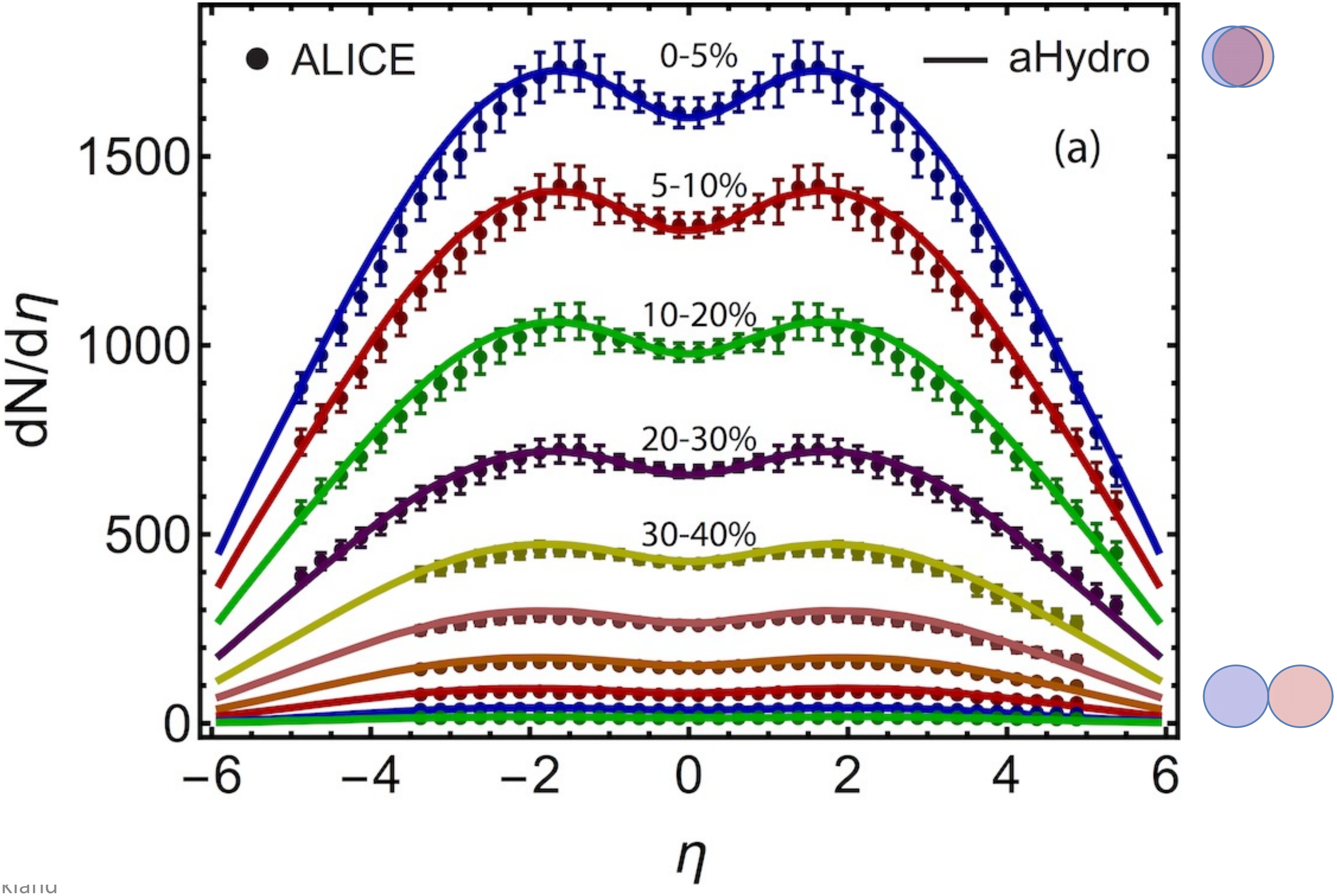
Identified particle average p_T @ 2.76 TeV

Alqahtani, Nopoush, Ryblewski, MS, 1703.05808; 1705.10191

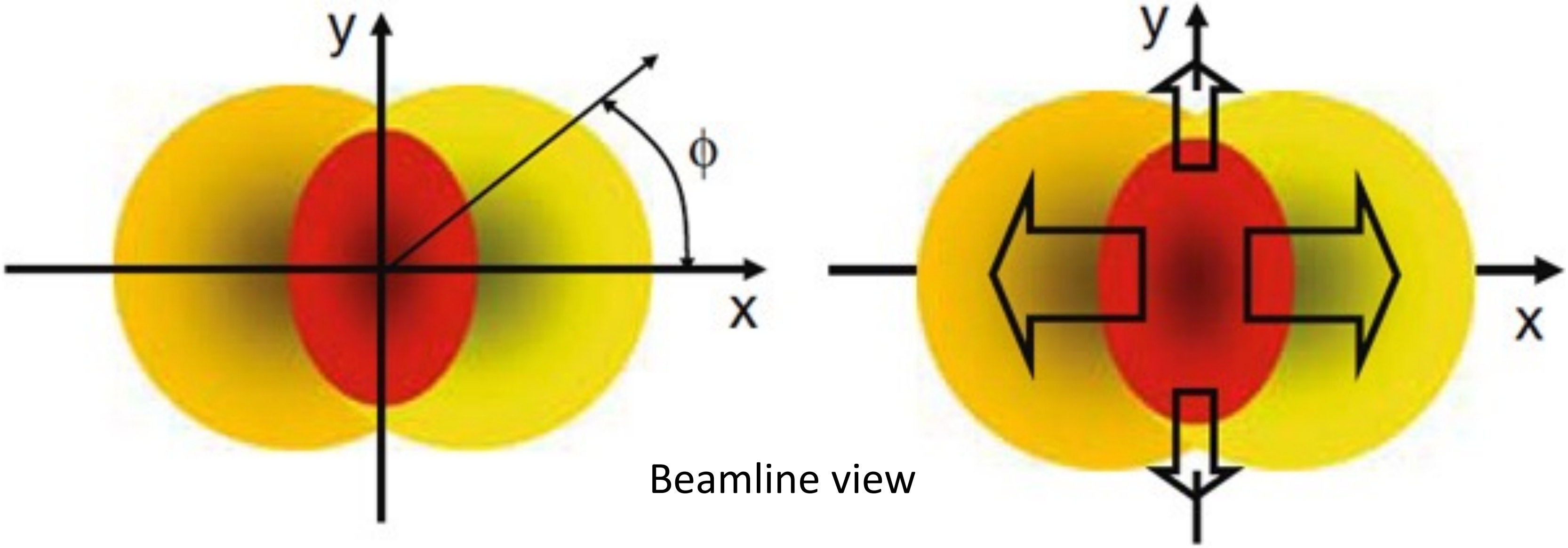


Charged particle multiplicity @ 2.76 TeV

Alqahtani, Nopoush, Ryblewski, MS, 1703.05808 (PRL); 1705.10191



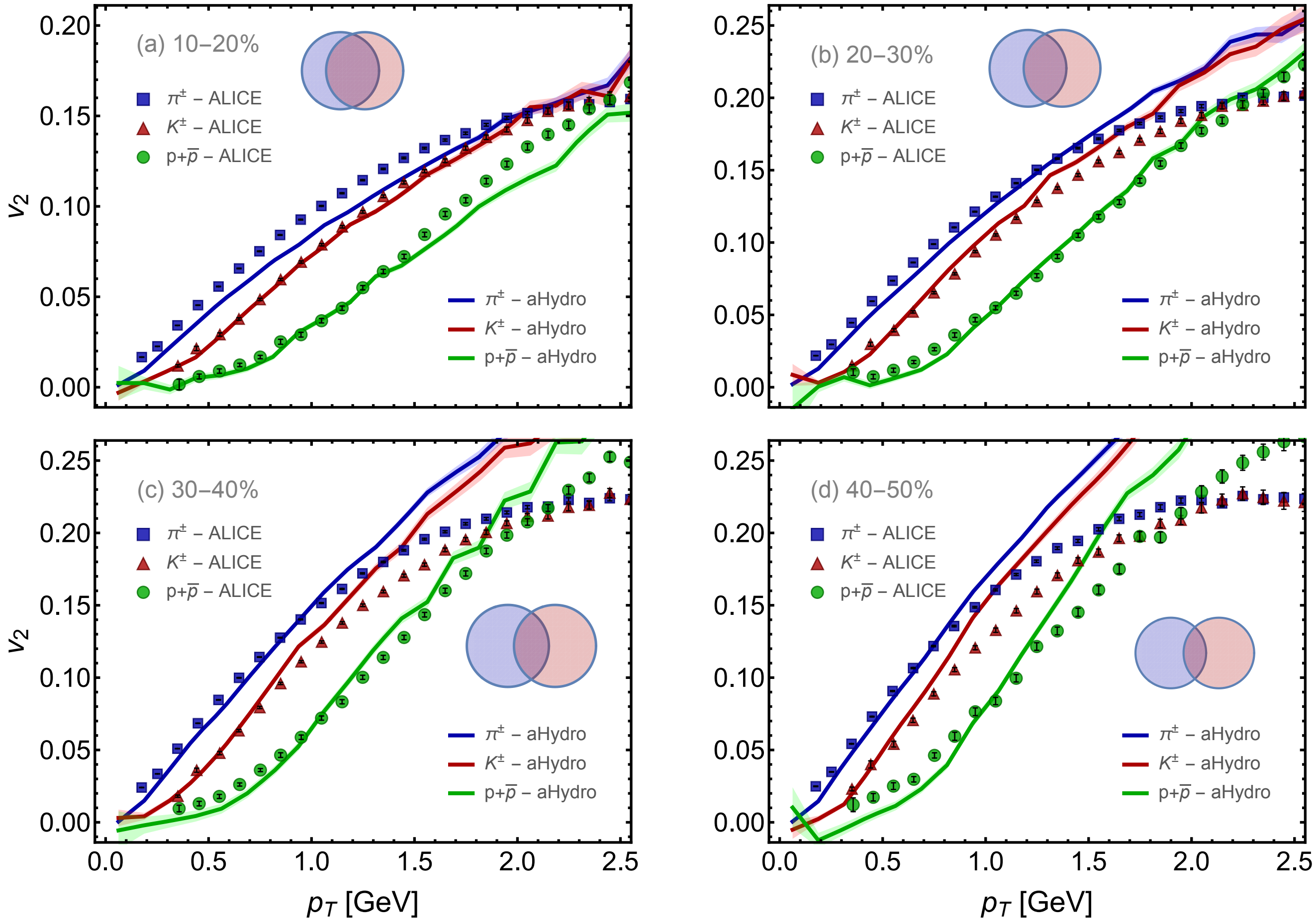
Elliptic flow



Geometry of overlap region creates anisotropic pressure gradients which result in “anisotropic flow” of plasma constituents.

Elliptic flow @ 2.76 TeV

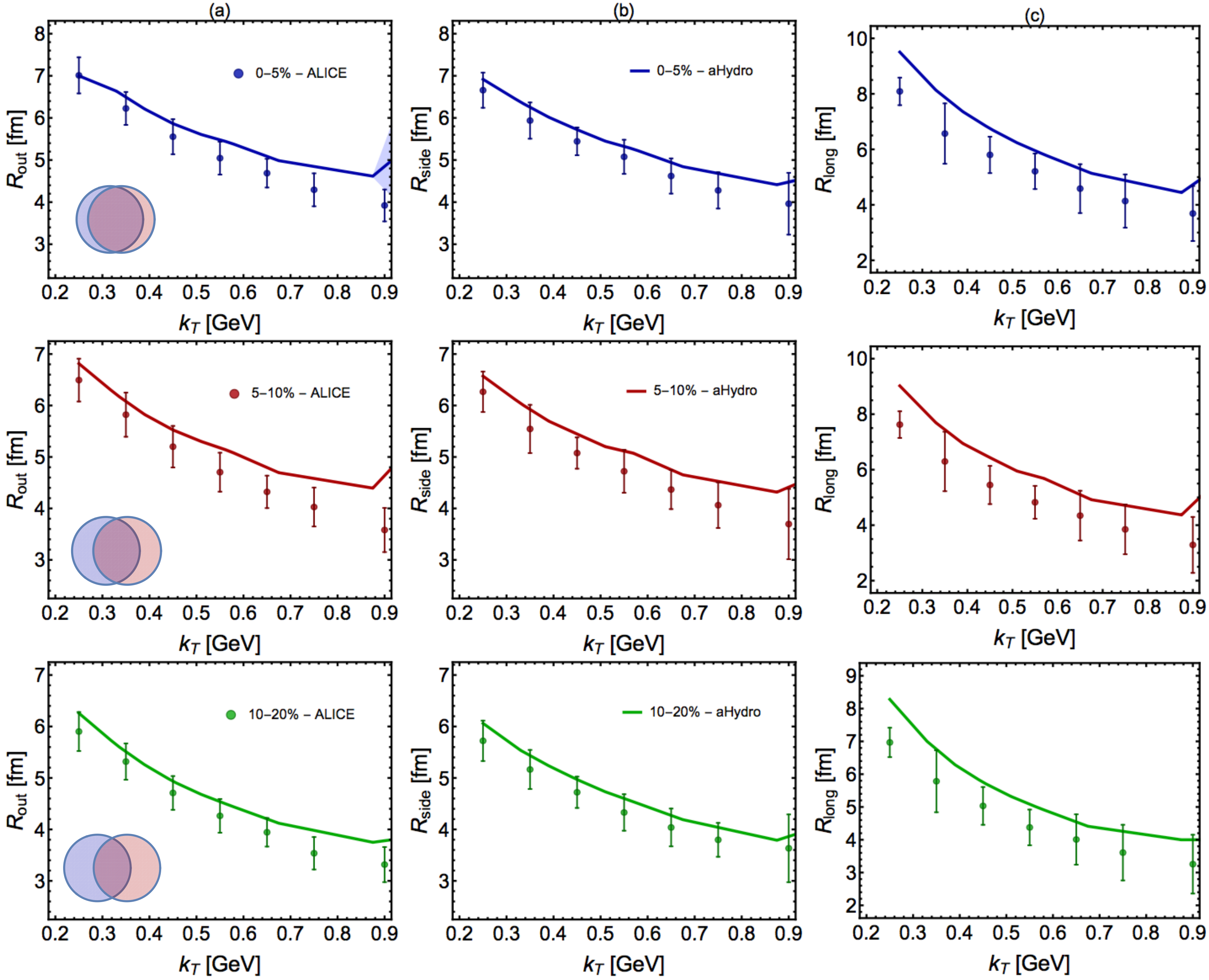
- Quite good description of identified particle elliptic flow as well
- Central collisions → need to include fluctuating init. Conditions!



Alqahtani, Nopoush, Ryblewski, MS, 1703.05808 (PRL); 1705.10191

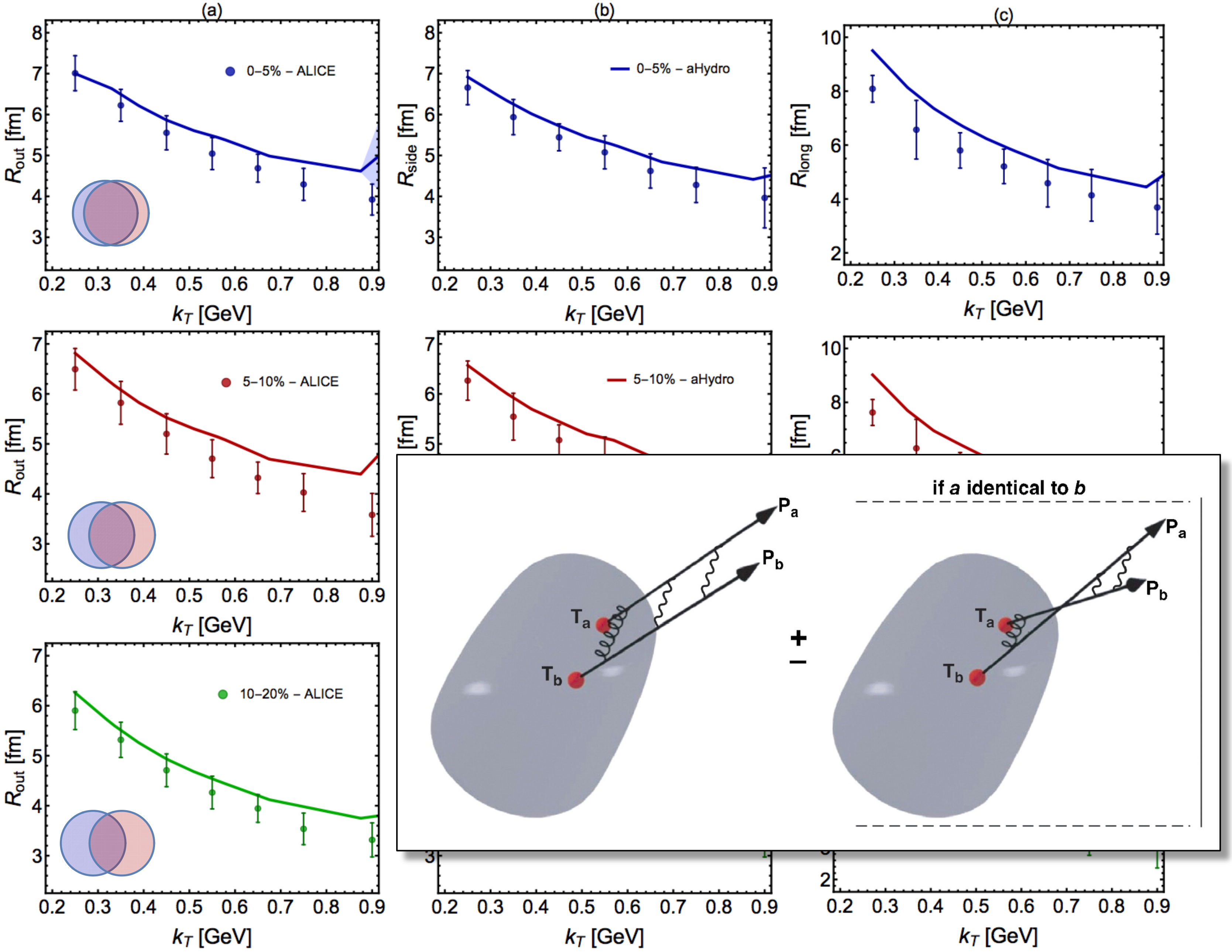
Pionic HBT Radii @ 2.76 TeV

Alqahtani, Nopoush, Ryblewski, MS, 1705.10191



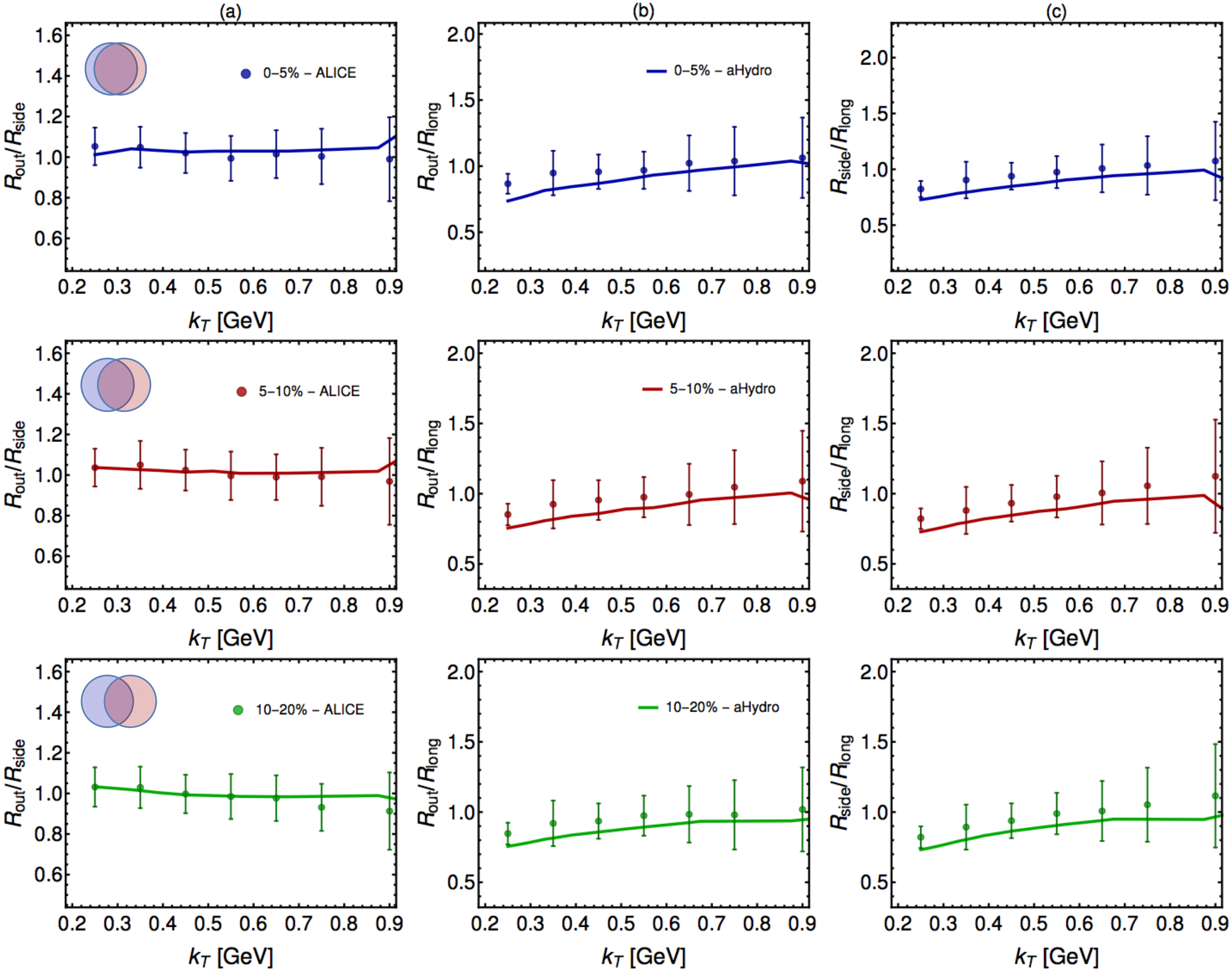
Pionic HBT Radii @ 2.76 TeV

Alqahtani, Nopoush, Ryblewski, MS, 1705.10191



Pionic HBT Radii Ratios @ 2.76 TeV

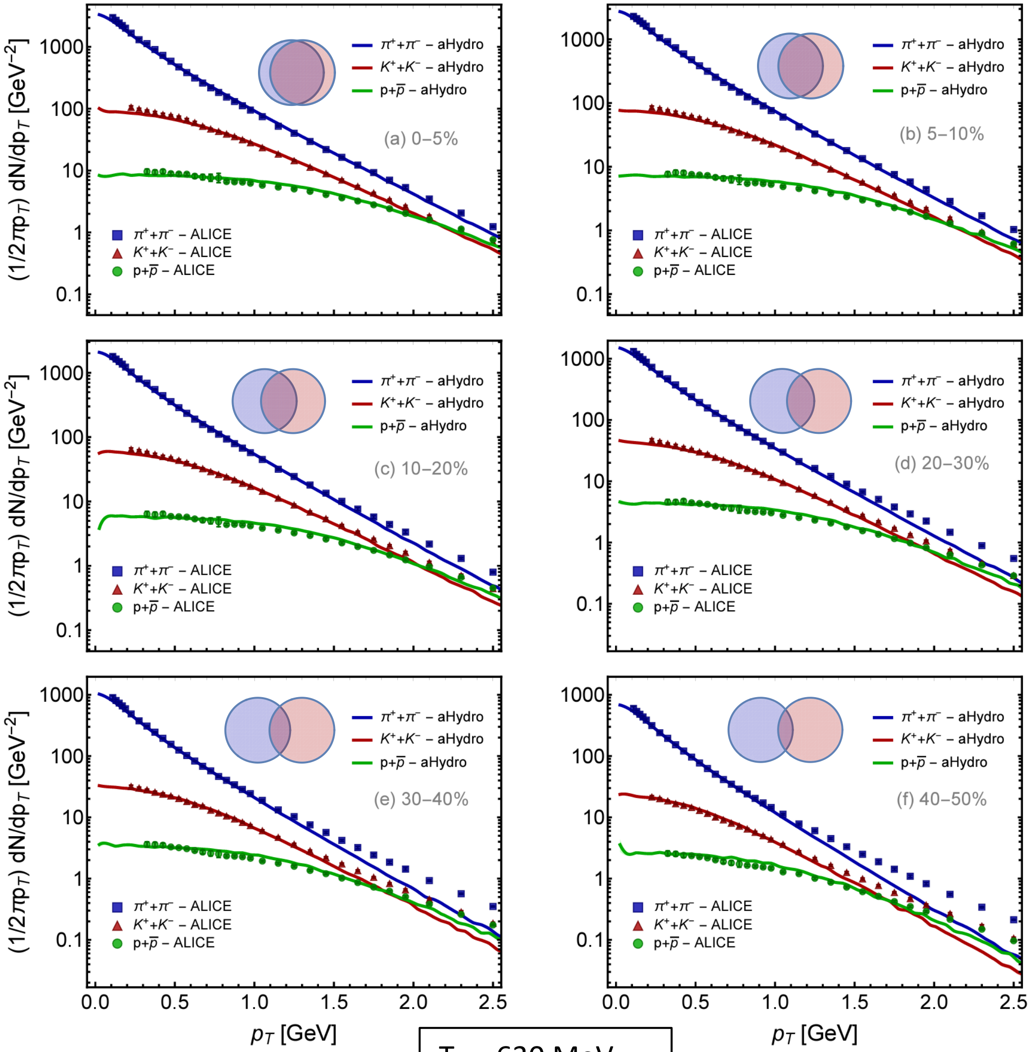
Alqahtani, Nopoush, Ryblewski, MS, 1705.10191

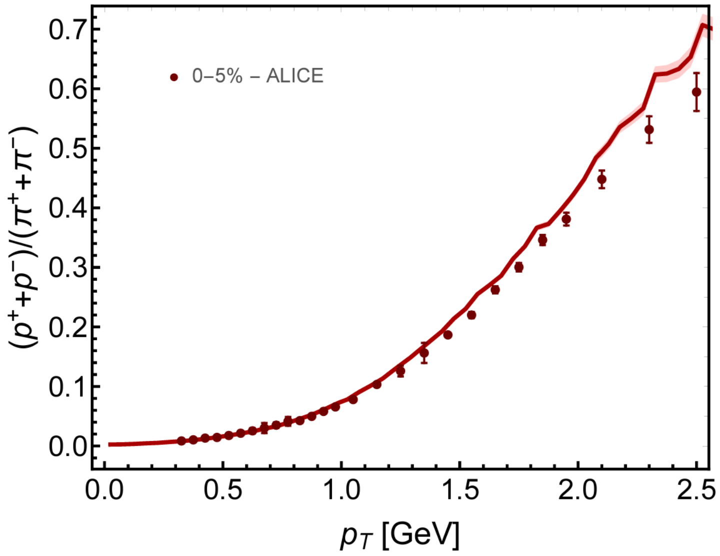


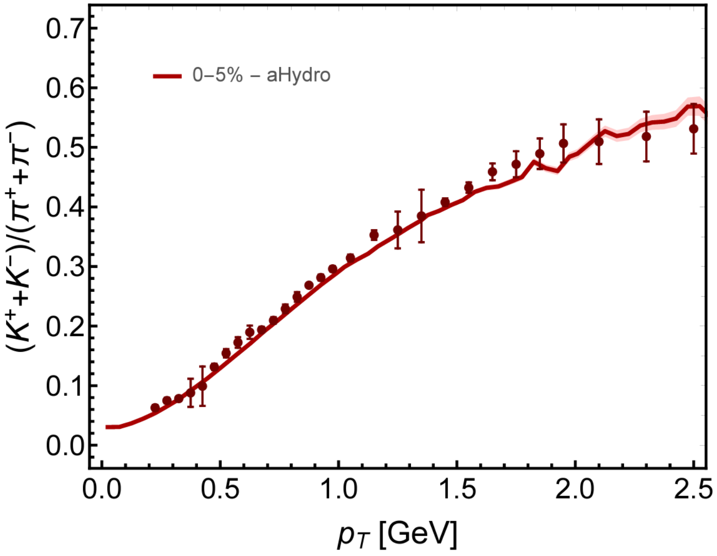
LHC 5 TeV

Identified particle spectra @ 5 TeV

Alqahtani and MS, 2008.07657





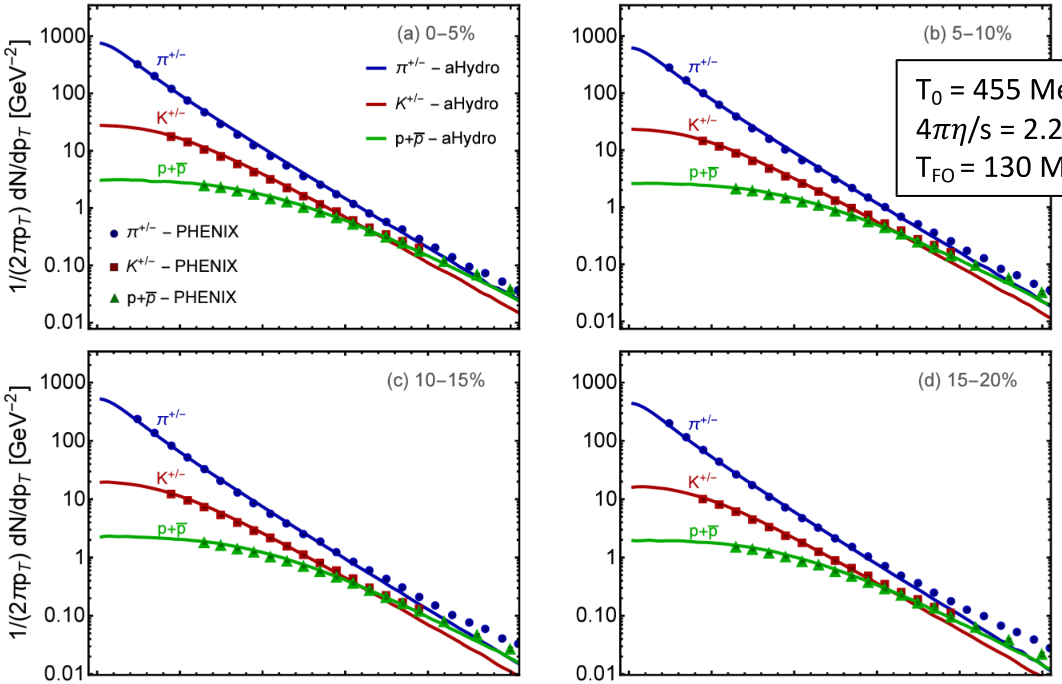


T_0 = 630 MeV
$4\pi\eta/s$ = 2
T_{FO} = 130 MeV

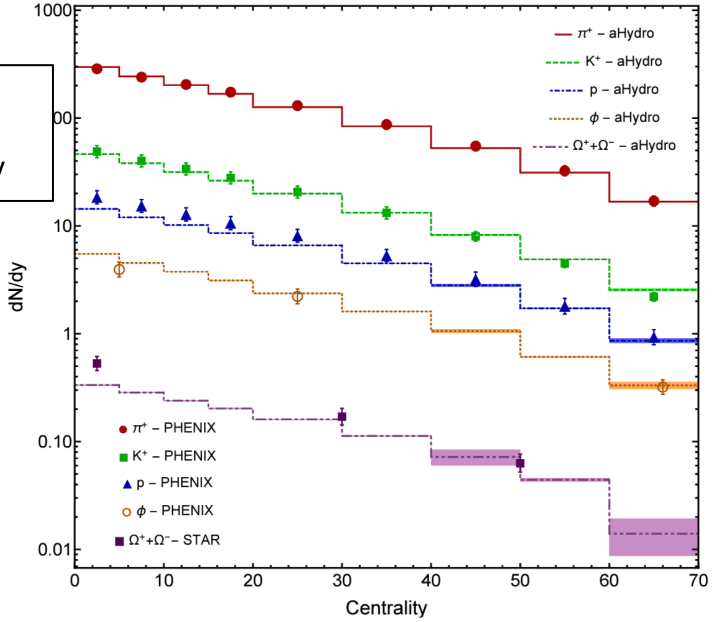
RHIC 200 GeV

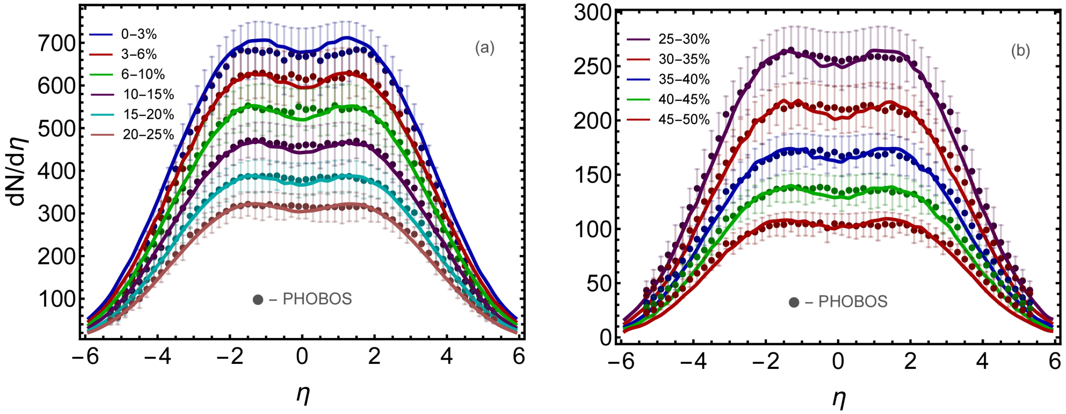
Identified particle spectra @ RHIC

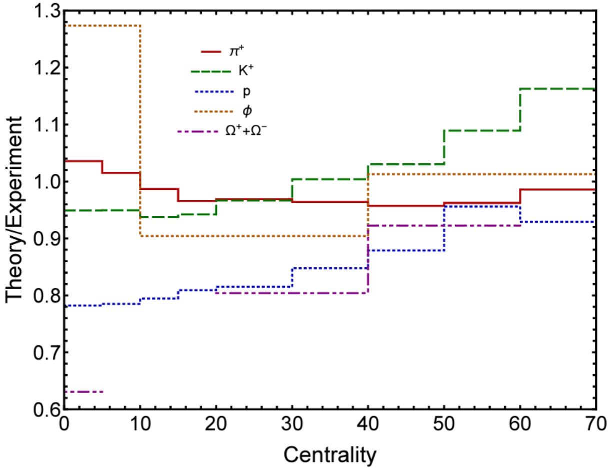
Almaalol, Alqahtani, MS, 1807.04337



T$_0$ = 455 MeV
$4\pi\eta/s$ = 2.25
T$_{FO}$ = 130 MeV







Kaonic HBT Radii @ RHIC

Alqahtani, MS, 2209.10894