Gravitational scattering at high energies

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Motivation

Selected for a Viewpoint in *Physics*
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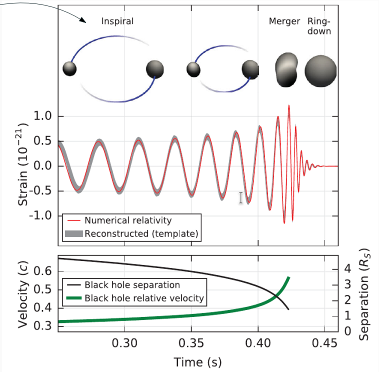
Observation of Gravitational Waves from a Binary Black Hole Merger

1602.03837

B. P. Abbott *et al.*[*]
(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 21 January 2016; published 11 February 2016)

We will focus on the inspiral part



Broad brush picture

Black holes/compact objects can be approximated as point-like particles when the separation b_J is large with respect to their size R

For a spinless object, we start from a minimally coupled scalar

$$S = -\int d^D x \sqrt{|g|}\, \frac{1}{2} \left[\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + m^2 \phi^2 + \ldots\right]$$

higher derivative terms ($\delta S \sim \int \frac{c^X}{m} \phi^2 W_X^2$) encode tidal effects ...

It is instructive to use an amplitude-based approach (starting from S) to derive quantitative information for the inspiral phase

The idea is to study the scattering process (rather than a bound system) and extract the effective gravitational "potential" of the EoB approach

Buonanno Damour 1998, ..., Damour 2016, ...

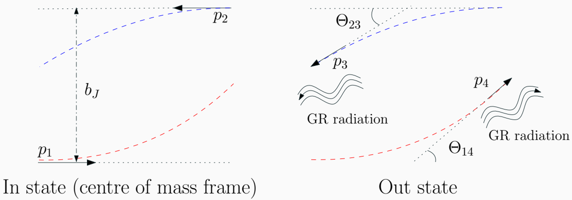
It allows to recycle techniques developed for particle physics and led to new results/ideas in classical GR

The aim

Consider two "elementary" objects interacting gravitationally. What is the final state if they scatter with an initial relative Lorentz factor $\sigma = -\frac{p_1 p_2}{m_1 m_2}$ and impact parameter b_J?

A rich (not fully solved) problem even in the classical limit

We expect the following qualitative picture



The aim: give a quantitative description of the final state when $\frac{R}{b_J} \ll 1$

How do the classical observables (deflection angles, radiation spectrum) behave in the UR limit $\sigma \gg 1$ at large b_J?

Based on:

2210.12118, 2204.02378, 2203.11915, 2101.05772: the eikonal operator

2008.12743, 2104.03256: detailed 3PM discussion (including integrals)

Physics Reports (in progress): the gravitational eikonal

in collaboration with: P. Di Vecchia, C. Heissenberg, G. Veneziano

The eikonal approach

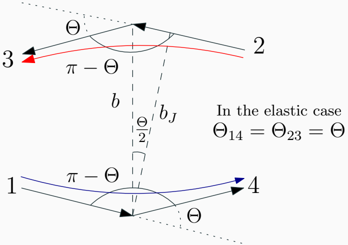
1. We treat the BH's/shockwaves as objects with known couplings to gravity (massless fields in general)
2. Use perturbative amplitudes to describe the large-distance scattering, take the classical limit and extract classical observables.

The second step can be tackled technically in several ways. I will focus on the eikonal approach. Two main features:

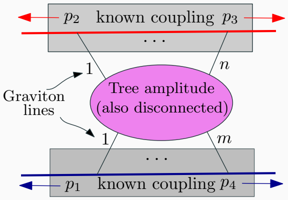
- It is a general approach applicable to all perturbative gravitational theories (GR, supergravity, string theory) and different types of objects (Schwarzschild, Kerr, shockwaves, strings . . .)
- Classical physics is obtained by resumming an infinite set of contributions which leads to exponentiation

The eikonal phase I (elastic case)

Calculate the $2 \to 2$ scattering amplitude $\mathcal{A}(E, q^2)$ focusing on the non-analytic terms as $q \to 0$ ($q \sim \hbar/b$ is the typical momentum carried by a single graviton exchanged between m_1 and m_2). In pictures



A spacetime picture of the scattering



Diagrammatic picture

Key classical quantities:

The centre-of-mass energy $E^2 = s = -(p_1 + p_2)^2 = (m_1^2 + m_2^2 + 2m_1 m_2 \sigma)$.
The angular momentum $J = p\, b_J$, $p = |\vec{p}_i|$, $Ep = m_1 m_2 \sqrt{\sigma^2 - 1}$
The momentum transferred $Q^\mu = p_1^\mu + p_4^\mu$, $Q = 2p \sin\left(\frac{\Theta}{2}\right)$

The eikonal phase II (elastic case)

It is convenient to go to impact parameter space

$$\tilde{\mathcal{A}}(s,b) = \int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q}\, \frac{\mathcal{A}(s,q^2)}{4Ep}$$

The semiclassical limit requires that the long range part of $\widetilde{\mathcal{A}}$ takes the form

$$\boxed{1 + i\widetilde{\mathcal{A}}(s,b) = \Big(1 + 2i\Delta(s,b)\Big)\, e^{i2\delta(s,b)}}$$

where δ is $\mathcal{O}(\hbar^{-1})$ and Δ encodes the quantum terms $\mathcal{O}(\hbar^m)$ with $m \geq 0$

$\delta = \delta_0 + \delta_1 + \ldots$, $\Delta = \Delta_1 + \ldots$, with $\delta_k, \Delta_k \sim \mathcal{O}(G^{k+1})$ (PM expansion)

Expanding formally in G the expression above we get

$$= \frac{16\pi G}{q^2} m_1^2 m_2^2 \left(2\sigma^2 - \frac{2}{D-2}\right) + \ldots \;\Rightarrow\; \tilde{\mathcal{A}}_0 = 2\delta_0 = \frac{G m_1 m_2 (\pi b^2)^{\epsilon} (2\sigma^2 - \frac{1}{1-\epsilon})\Gamma(-\epsilon)}{\sqrt{\sigma^2 - 1}}$$

$$+ \ldots \quad \Rightarrow \quad (\tilde{\mathcal{A}}_1)_{\frac{1}{\hbar^2}} = \frac{i}{2}(2\delta_0)^2\,, \quad (\tilde{\mathcal{A}}_1)_{\frac{1}{\hbar}} = 2\delta_1 \xrightarrow{D\to 4} \frac{3\pi G^2 m_1 m_2 (m_1 + m_2)(5\sigma^2 - 1)}{4b\sqrt{\sigma^2 - 1}}$$

The eikonal phase III (elastic case)

The relation between b and Q follows from a stationary phase argument

$$\mathcal{S}(s,Q^2) = 4Ep\int d^{D-2}b\, e^{-\frac{i}{\hbar}bQ+2i\delta(s,b)} \quad \Rightarrow \quad Q^\mu = \hbar\frac{\partial\, 2\delta}{\partial b_\mu}$$

$$\text{At leading order} \quad Q^{D=4}_{1PM} = \left|\hbar\frac{\partial 2\delta_0}{\partial b}\right| = \frac{2Gm_1m_2\left(2\sigma^2-1\right)}{b\sqrt{\sigma^2-1}}$$

An interesting limit is: $\sigma \gg 1$ with $R_E^{D-3} \equiv GE \sim G\sqrt{2m_1m_2\sigma} < b^{D-3}$

In this ultrarelativistic regime the 2PM angle has a finite limit. For $D=4$

$$\frac{Q}{p} \simeq \Theta_{\rm 2PM} \xrightarrow{UR} \frac{4R_E}{b} + \mathcal{O}\left(\frac{R_E^3}{b^3}\right)$$

It agrees with the scattering of two Aichelburg and Sexl shockwaves

't Hooft; Fabbrichesi, Pettorino, Veneziano, Vilkovisky

Is this a general feature (i.e. holding at all PM orders)?

Novelties at 3PM

From the 2-loop amplitude we extract 3PM data Bern, Cheung, Roiban, Shen, Solon, Zeng; Herrmann, Parra-Martinez, Ruf, Zeng; DHRV; Bjerrum-Bohr, Damgaard, Plante, Vanhove; Brandhuber, Chen, Travaglini, Wen

In the eikonal approach a first result is δ_2 2008.12743 and 2104.03256

$$
\begin{aligned}
2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\Bigg\{ & \frac{(2\sigma^2-1)^2(8-5\sigma^2)}{6(\sigma^2-1)^2} - \frac{\sigma(14\sigma^2+25)}{3\sqrt{\sigma^2-1}} \\
& + \frac{s(12\sigma^4-10\sigma^2+1)}{2m_1 m_2(\sigma^2-1)^{\frac{3}{2}}} + \cosh^{-1}\sigma \left[\frac{\sigma(2\sigma^2-1)^2(2\sigma^2-3)}{2(\sigma^2-1)^{\frac{5}{2}}} + \frac{-4\sigma^4+12\sigma^2+3}{\sigma^2-1}\right]\Bigg\} \\
& + i\,\frac{2m_1^2 m_2^2 G^3}{\pi b^2}\frac{(2\sigma^2-1)^2}{(\sigma^2-1)^2}\Bigg\{ -\frac{1}{\epsilon}\left[\frac{8-5\sigma^2}{3} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}}\cosh^{-1}(\sigma)\right] \\
& + \left(\log(4(\sigma^2-1)) - 3\log(\pi b^2 e^{\gamma_E})\right)\left[\frac{8-5\sigma^2}{3} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}}\cosh^{-1}(\sigma)\right] \\
& + (\cosh^{-1}(\sigma))^2\left[\frac{\sigma\left(3-2\sigma^2\right)}{(\sigma^2-1)^{\frac{1}{2}}} - 2\frac{4\sigma^6-16\sigma^4+9\sigma^2+3}{(2\sigma^2-1)^2}\right] \\
& + \cosh^{-1}(\sigma)\left[\frac{\sigma\left(88\sigma^6-240\sigma^4+240\sigma^2-97\right)}{3(2\sigma^2-1)^2(\sigma^2-1)^{\frac{1}{2}}}\right] \\
& + \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}}\mathrm{Li}_2(1-z^2) + \frac{-140\sigma^6+220\sigma^4-127\sigma^2+56}{9(2\sigma^2-1)^2}\Bigg\}
\end{aligned}
$$

radiation reaction
(agrees with Damour's 2010.01641)

potential gravitons
Bern, Cheung, Roiban, Shen, Solon, Zeng 1901.04424

probe limit

A consequence of analyticity and crossing

$z = \sigma - \sqrt{\sigma^2 - 1}$

Energy crisis I

Let us first focus on the contribution from potential gravitons

- There is a log-divergent UR term
- In the opposite limit it matches all known PN data

We now include the RR contribution

- It yields only odd PN terms (so they don't spoil previous checks!)
- In the UR limit the log-divergent term cancel

The leading total UR contribution reads

$$2\delta_2 \xrightarrow{UR} \frac{4G^3 m_1^2 m_2^2}{b^2}\left[\sigma^2\left(-\frac{10}{3}-\frac{14}{3}+12\right)\right] = \frac{16G^3 m_1^2 m_2^2}{b^2}\sigma^2$$

Amati, Ciafaloni, Veneziano

A finite UR contribution to the 3PM angle $\Theta_{\mathrm{3PM}} \xrightarrow{UR} \frac{1}{12}\Theta_{\mathrm{1PM}}^3$

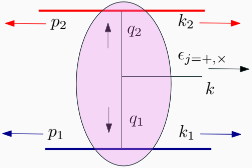
It's universal: a consequence of the "graviton dominance in the Ultrahigh-Energy Scattering" (checked in several supergravity theories)

2008.12743; Bern, Ita, Parra-Martinez, Ruf

Including radiation

The new ingredient is the classical $2 \to 3$ amplitude

$A_j(q_1, q_2, k) =$



Many gravitons are emitted classically during the scattering.

We expect that in A_j exponentiates as well and yields an eikonal operator including a coherent radiation. Schematically we have

$$e^{2i\hat{\delta}(b_1,b_2)} = e^{2i\tilde{\delta}(b)} e^{i\int_k \left[\tilde{\mathcal{A}}_j(b_1,b_2,k)a_j^\dagger(k) + \tilde{\mathcal{A}}_j^*(b_1,b_2,k)a_j(k)\right]}$$

$\tilde{\delta}$ is real, see below for its relation with the elastic eikonal

Eikonal operator

The in-state is described by the wavepackets Φ_i

wavepackets $\quad b_J = b_1 - b_2$ impact parameter

$$|\psi\rangle = \int_{-p_1}\int_{-p_2} \Phi_1(-p_1)\Phi_2(-p_2)e^{ib_1\cdot p_1 + ib_2\cdot p_2}|-p_1,-p_2,0\rangle$$

The out-state contains many gravitons (a coherent state)

$$S|\psi\rangle \simeq \int_{p_3}\int_{p_4} e^{-ib_1\cdot p_4 - ib_2\cdot p_3} \int \frac{d^DQ_1}{(2\pi)^D}\int \frac{d^DQ_2}{(2\pi)^D}\,\Phi_1(p_4-Q_1)\,\Phi_2(p_3-Q_2)$$

$$\times \int d^Dx_1 \int d^Dx_2\, e^{i(b_1-x_1)\cdot Q_1 + i(b_2-x_2)\cdot Q_2}\, e^{2i\hat{\delta}(x_1,x_2)}|p_3,p_4,0\rangle$$

$$p_1 + p_4 = Q_1$$

$$p_2 + p_3 = Q_2$$

$$e^{2i\hat{\delta}(x_1,x_2)} = \int \frac{d^D\tilde{Q}}{(2\pi)^D}\int d^D\tilde{x}\, e^{-i\tilde{Q}(\tilde{x}-x_1+x_2)+i2\tilde{\delta}(b)}\, e^{i\int_k\left[\tilde{A}_j(x_1,x_2,k)a_j^\dagger(k) + \tilde{A}_j^*(x_1,x_2,k)a_j(k)\right]}$$

$$2\tilde{\delta}(b_e) = \text{Re}\delta(\sigma_{12},b) + \text{Re}\delta(\sigma_{34},b)$$

$$\sigma_{12} = -\frac{p_1\cdot p_2}{m_1m_2}\,, \quad \sigma_{34} = -\frac{p_3\cdot p_4}{m_1m_2}$$

Derived from the $2\to 2$ amplitude

Derived from the $2\to 2+1gr$ amplitude

Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White

$\tilde{A}$ is the Fourier Transform of the $2\to 2+1gr$ classical amplitude

Goldberger, Ridgway; Luna, Nicholson, O'Connell, White; Mogull, Plefka, Steinhoff

Unitarity and classical constraints

Notice that Imδ is now absent and the eikonal operator has a chance to be (classically) unitary

When calculating the elastic transition $\langle\psi|S|\psi\rangle$, we use the BCH formula: this should produce at the stationary point Imδ (checked at 3PM)

For instance unitarity requires $\langle\psi|S^\dagger S|\psi\rangle \simeq \langle\psi|\psi\rangle = 1$. It implies

$$(x_i - b_i)_\mu = \frac{\partial 2\delta_s(b_e)}{\partial Q_i^\mu} - i\int_k \tilde{\mathcal{A}}^*(x_1,x_2,k)\frac{\overleftrightarrow{\partial}}{\partial Q_i^\mu}\tilde{\mathcal{A}}(x_1,x_2,k) \qquad \tilde{x}_\mu = (x_1 - x_2)_\mu + \frac{\partial 2\delta_s(b_e)}{\partial \tilde{Q}^\mu}$$

$$Q_{i\,\mu} = (-1)^{i+1}\tilde{Q}_\mu - i\int_k \tilde{\mathcal{A}}^*(x_1,x_2,k)\frac{\overleftrightarrow{\partial}}{\partial x_i^\mu}\tilde{\mathcal{A}}(x_1,x_2,k) \qquad \tilde{Q}_\mu = \frac{\partial 2\delta_s(b_e)}{\partial \tilde{x}^\mu}$$

These are stationary phase conditions for the integrals over Q_i, $\tilde{Q}$, x_i and $\tilde{x}$

Classical observables are derived from expectation values $\langle\psi|S^\dagger O S|\psi\rangle$
Evaluating scalar products at the stationary point provides the classical physics results (as usual)

Other observables at 3PM

Full inclusive radiative observables at 3PM: the total radiated energy, the impulses, the the mechanical, gravitational angular momenta

$$\mathbf{Q}_1^\alpha \simeq -\frac{G^3 m_1^2 m_2^2}{b^3}\,\check{u}_2^\alpha\,\mathcal{E}\,,\quad \mathbf{Q}_2^\alpha \simeq -\frac{G^3 m_1^2 m_2^2}{b^3}\,\check{u}_1^\alpha\,\mathcal{E}\,,\quad \mathbf{P}^\alpha \simeq \frac{G^3 m_1^2 m_2^2}{b^3}\,(\check{u}_1^\mu + \check{u}_2^\mu)\,\mathcal{E}$$

$$\Delta\mathbf{L}_1 \simeq \frac{G^3 m_1^2 m_2^2}{b^3}\left[\frac{\mathcal{E}_+ b^{[\alpha} u_1^{\beta]}}{\sigma-1} - \frac{1}{2}\,\mathcal{E}\, b^{[\alpha}\check{u}_2^{\beta]}\right]\,,\quad \mathbf{J}^{\alpha\beta} \simeq \frac{G^3 m_1^2 m_2^2}{b^3}\,\mathcal{F}\left(b^{[\alpha}\check{u}_1^{\beta]} - b^{[\alpha}\check{u}_2^{\beta]}\right)$$

$$\Delta\mathbf{L}_2 = \Delta\mathbf{L}_1(1\leftrightarrow 2, b^\alpha \leftrightarrow -b^\alpha)\,,\quad \check{u}_1^\mu = \frac{\sigma\, u_2^\mu - u_1^\mu}{\sigma^2-1}\,,\ \check{u}_2^\mu = \frac{\sigma\, u_1^\mu - u_2^\mu}{\sigma^2-1}\,,\ u_i^\mu = -\frac{p_i^\mu}{m_i} \qquad \mathcal{C}\sqrt{\sigma^2-1} = -\mathcal{E}_+ + \sigma\mathcal{E}_-$$

$$\frac{\mathcal{E}}{\pi} = f_1 + f_2 \log\frac{\sigma+1}{2} + f_3\frac{\sigma\cosh^{-1}\sigma}{2\sqrt{\sigma^2-1}} \qquad \frac{\mathcal{C}}{\pi} = g_1 + g_2\log\frac{\sigma+1}{2} + g_3\frac{\sigma\cosh^{-1}\sigma}{2\sqrt{\sigma^2-1}} \qquad \mathcal{F} = \mathcal{E}_+ - \frac{1}{2}\,\mathcal{E} = -\mathcal{E}_- + \frac{1}{2}\mathcal{E}$$

$$f_1 = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48(\sigma^2-1)^{\frac{3}{2}}}$$

In the centre of mass frame with $b_1 + b_2 = 0$

$$f_2 = -\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2-1}}\,,\quad f_3 = \frac{(2\sigma^2-3)(35\sigma^4 - 30\sigma^2 + 11)}{8(\sigma^2-1)^{\frac{3}{2}}}$$

$$g_1 = \frac{105\sigma^7 - 411\sigma^6 + 240\sigma^5 + 537\sigma^4 - 683\sigma^3 + 111\sigma^2 + 386\sigma - 237}{24(\sigma^2-1)^2}$$

Herrmann, Parra-Martinez,Ruf, Zeng
Manohar, Ridgway, Shen
DHRV: 2210.12118

$$g_2 = -\frac{35\sigma^5 - 90\sigma^4 - 70\sigma^3 + 16\sigma^2 + 155\sigma - 62}{4(\sigma^2-1)}\,,\quad g_3 = -\frac{(2\sigma^2-3)\left(35\sigma^5 - 60\sigma^4 - 70\sigma^3 + 72\sigma^2 + 19\sigma - 12\right)}{4\left(\sigma^2-1\right)^2}$$

Conservations laws checked explicitly. Subtleties for J, L from zero-frequency "gravitons" Damour; Veneziano, Vilkovisky; 2203.11915; Riva, Vernizzi, Wonng; Bini, Damour; ...

Many checks (and predictions) in the PN limit Bini, Geralico, Damour; ...

Energy crisis II: a UR threshold

A dangerous region $\omega^\star \sim \frac{\sqrt{\sigma}}{b}$

There is a cancellation between the leading terms in f_3, g_3 and $f_2\, g_2$! A consequence of the radiation being focused along the the BH trajectories: $\theta_{\rm gr} < 1/(\omega b)$, so at $\omega^\star$ we have $\theta_{\rm gr} < 1/\sqrt{\sigma}$

D'Eath; Colferai, Ciafaloni, Veneziano

However this is not sufficient to avoid another energy crisis in $E^{\rm rad}$

$$\frac{E^{\rm rad}}{E} \xrightarrow{UR} \frac{R_E^3}{b^3} \frac{(m_1+m_2)E}{m_1 m_2} \left[\frac{35\pi}{64}(1+2\ln 2)\right]$$

There is a threshold where the PM expansion breaks down: $\sqrt{\frac{m_j \sigma}{2m_i}}\Theta \gtrsim 1$ (it appears clearly in the soft energy spectrum)

D'Eath 1978 (DHRV 2204.02378)

After crossing D'Eath's threshold, $\mathbf{Q}$, ... should stop increasing

Gruzinov, Veneziano; Ciafaloni, Colferai, Coradeschi, Veneziano

Transverse and longitudinal 3PM impulses Q_i behave very differently in the UR regime

Results at 4PM and energy crisis III

At 3PM the transverse impulse (along b^μ) and the longitudinal one have different UR behaviours: the first has a finite limit, while the second one requires a resummation of the PM expansion

The full 4PM impulses Q_i have been recently derived

Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng; Dlapa, Kälin, Liu, Neef, Porto

At low velocities this analysis agrees with recent PN results Bini, Damour, Geralico

In the UR limit the pattern seen at 3PM seems to be broken

- the transverse components diverge as $Q_{4\mathrm{PM}} \sim \Theta_{1PM}^4 \frac{E}{\sqrt{m_1 m_2}}$
- the longitudinal part is log-divergent $\frac{E_{4\mathrm{PM}}^{\mathrm{rad}}}{E} \sim \Theta_{1PM}^4 \ln\left[\frac{E}{\sqrt{m_1 m_2}}\right]$

Two possibilities: we should always resum the PM-expansion before taking the UR limit or we have not disentangled radiative and conservative effects at 4PM and there exists a transverse part of the 4PM impulses with a finite UR limit

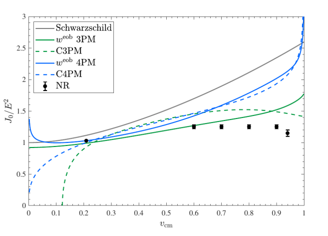
Connections to other approaches

The classical deflection angle in a spherically symmetric metric is

$$\Theta = -\pi + 2J \int_{r_*}^{\infty} \frac{dr}{r^2\sqrt{p^2 - \frac{J^2}{r^2} - V(r)}}\,, \quad V(r) = -\sum_{n=1}^{\infty} \frac{G^n}{r^n} f_n \quad \text{EOB approach}$$

In the $\frac{R}{b_J} \ll 1$ limit we can derive f_n from δ_{n-1} ...after that, use the lhs to calculate Θ at $b_J \simeq \mathcal{O}(R)$ Damour and Rettegno; see yesterday's discussion lead by Ceresole and Rettegno

An observable exploring a range velocities. From Damour Rettegno 2211.01399



One can exploit $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$ as a perturbative parameter (instead of G). A large literature: the self-force approach). A very recent comparison with 4PM data and self-force (with scalar interaction)

Barack, Bern, Herrmann, Long, Parra-Martinez, Roiban, Ruf, Shen, Solon, Teng, Zeng 2304.09200

Conclusion and outlook

We can use an amplitudes based approach to study gravitational binaries

It captures all aspects: conservative, radiation-reaction and real radiation

The approach is flexible and can be applied to different theories/objects and yields explicit, Lorentz invariant expressions.

The high-energy (UR) limit provides challenges and useful tests

I discussed a UR threshold and different "energy crisis". The study of the UR limit is likely to play a useful role in the future. Open questions:

- at 4PM we have an unexpected pattern: the UR divergence in Q is worse than in $E^{\rm rad}$ (true even in the $m_1 = m_2$ case)
- understand better how to distinguish between conservative and dissipative effects
- exploit the recent NLO waveforms Brandhuber, Brown, Chen, Angelis, Gowdy, Travaglini; Elkhidir, O'Connell, Sergola, Vazquez-Holm; Herderschee, Roiban, Teng; Georgoudis, Heissenbergb, Vazquez-Holm
- refine the links with the EoB approach

Extra slides

Soft radiation

Dress the elastic scattering with soft gravitons ($\omega < \omega^* \ll \frac{v}{b}$). The emission of such gravitons exponentiate in momentum space

Bloch-Nordsieck, Weinberg; Laddha, Saha, Sahoo, Sen; Addazi, Bianchi, Veneziano

Combining this with the eikonal exponentiation we have

$$e^{2i\hat{\delta}_{s.r.}} \simeq \exp\left(\frac{1}{\hbar}\int_k^{\omega^*}\sum_j \left[f_j(k)a_j^\dagger(k) - f_j^*(k)a_j(k)\right]\right) \times [1+2i\Delta(\sigma,b)]\, e^{i\,\mathrm{Reg}(2\delta)}$$

S-matrix with soft gravitons

$$f_j(k) = \varepsilon_j^{*\mu\nu}(k)\sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n\cdot k}$$

The f_j's act on δ: $Q^\mu = p_1^\mu + p_4^\mu = \frac{\partial \mathrm{Reg}(2\delta)}{\partial b_\mu} = \frac{-b^\mu}{b} 2p\sin\frac{\Theta}{2} = -(p_2^\mu + p_3^\mu)$

$\mathrm{Reg}(2\delta)$ is the ϵ^0 part of the elastic eikonal. The IR-singular part of the elastic process follows from normal ordering (via the BCH formula) $\hat{\delta}$

$$\langle\Psi_0|e^{2i\hat{\delta}_{s.r.}}|\Psi_0\rangle \simeq \exp\left[\underbrace{-\left(\frac{(\omega^*)^{-2\epsilon}}{-2\epsilon}\right)\frac{G}{\pi}\sum_{n,m} m_n m_m\left(\sigma_{nm}^2 - \frac{1}{2}\right)F_{nm}}_{\stackrel{?}{=}\,\mathrm{Im}(2\delta^{\mathrm{sing}})}\right] e^{i\,\mathrm{Reg}(2\delta)}$$

with

$$a_i|\Psi_0\rangle = 0 \qquad \sigma_{nm} = -\eta_i\eta_j\frac{p_n p_m}{m_n m_m} \qquad F_{nm} = \int_k^{\omega^*}\frac{m_n m_m}{(p_m k)(p_m k)} = \left(\frac{(\omega^*)^{-2\epsilon}}{-2\epsilon}\right)\frac{\eta_n\eta_m \mathrm{arccosh}\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}}$$

$\eta_n = +1$ ($\eta_n = -1$) if n is a final (initial) state in the elastic process

The soft spectrum

The standard PM approach is a Taylor expansion in $\frac{Q^2}{2m_i^2}$ (small momentum exchange in comparison to the BH masses)

$$\sigma_{jj} = F_{jj} = 1\,, \quad \sigma_{14} = 1 + \frac{Q^2}{2m_1^2}\,, \quad \sigma_{23} = 1 + \frac{Q^2}{2m_2^2}$$

$$\sigma_{12} = \sigma_{34} = \sigma\,, \quad \sigma_{13} = \sigma_{24} = \sigma - \frac{Q^2}{2m_1 m_2}\,,$$

$$\left[\mathrm{Im}(2\delta^{\mathrm{sing}})\right] \overset{\checkmark}{=} \left(-\tfrac{1}{2\epsilon}\right)\frac{G}{\pi}\,Q^2_{1\mathrm{PM}}\left[\frac{8-5\sigma^2}{3(\sigma^2-1)} + \frac{(2\sigma^2-3)\sigma}{(\sigma^2-1)^{3/2}}\cosh^{-1}\sigma\right] + \ldots$$

Classical observables are derived as expected: $\langle\psi|e^{-2i\hat{\delta}}\,O\,e^{2i\hat{\delta}}|\psi\rangle$, where $|\psi\rangle$ is the initial (2-particle) state

For instance $\frac{dE^{\mathrm{rad}}_{\mathrm{soft}}}{d\omega} = \langle\psi|e^{-2i\hat{\delta}_{s.r.}}\left[\sum_i \int \frac{d\Omega_2}{2\omega(2\pi)^3}\left(\omega a_i^\dagger a_i\right)\right]e^{2i\hat{\delta}_{s.r.}}|\psi\rangle$

Then, for the soft radiation, we get the relation $\frac{dE^{\mathrm{rad}}_{\mathrm{soft}}}{d\omega} \simeq \lim_{\epsilon\to 0}[-4\epsilon\,\mathrm{Im}\delta_2]$

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By using the expression above for $\mathrm{Im}\delta_2$, one obtains Smarr's result Smarr 1977

Is the $\cosh^{-1}\sigma$ term problematic?

Another energy crisis

The soft spectrum is reliable till $\omega^* \lesssim \frac{1}{b}$

The PM-expanded energy emitted by soft gravitons diverges in the UR limit: $E^{\rm rad}_{\rm soft} \simeq E(c_1 \log(\sigma) + c_2)$ (c_i are constant $\sim \Theta^3$, c_2 is not universal)

Thus we should go back to exact soft answer (ignoring non-linear memory effects)

$$\frac{dE^{\rm rad}_{\rm soft}}{d\omega} \simeq \frac{4G}{\pi}\Bigg[2m_1m_2\left(\sigma^2-\frac{1}{2}\right)\frac{\cosh^{-1}\sigma}{\sqrt{\sigma^2-1}} - 2m_1m_2\left(\sigma_Q^2-\frac{1}{2}\right)\frac{\cosh^{-1}\sigma_Q}{\sqrt{\sigma_Q^2-1}} \quad \text{with } \sigma_Q=\sigma-\tfrac{Q^2}{2m_1m_2}$$

$$+\frac{m_1^2}{2} - m_1^2\left(\left(1+\frac{Q^2}{2m_1^2}\right)^2-\frac{1}{2}\right)\frac{\cosh^{-1}\left(1+\frac{Q^2}{2m_1^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_1^2}\right)^2-1}} + \frac{m_2^2}{2} - m_2^2\left(\left(1+\frac{Q^2}{2m_2^2}\right)^2-\frac{1}{2}\right)\frac{\cosh^{-1}\left(1+\frac{Q^2}{2m_2^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_2^2}\right)^2-1}}\Bigg]$$

There is no singularity at $Q=0$, but, there is one at $\frac{Q^2}{2m_i^2} = -2$: it is an unphysical singularity in the t-channel corresponding to the exchange of two on-shell BH's $(p_1+p_4)^2 = -(2m_1)^2$ or $(p_2+p_3)^2 = -(2m_2)^2$

It fixes the radius of convergence of the $Q \ll m_i^2$ expansion for $\dfrac{dE^{\rm rad}}{d\omega}$

A UR threshold

In GR notation, the threshold matches that found by D'Eath D'Eath 1978

$$\frac{Q}{2m_i} = \frac{p \sin\left(\frac{\Theta}{2}\right)}{m_i} \sim \frac{p}{m_i}\Theta \sim \sqrt{\frac{m_j \sigma}{2m_i}}\Theta \sim 1$$

When $1/\sqrt{\sigma}$ is of the order of Θ the PM expansion breaks down

In the UR limit, the exact formula yields finite results. The fraction of energy carried away by soft radiation is finite

$$\frac{dE^{\text{rad}}_{\text{soft}}}{d\omega} \simeq \frac{Gs\Theta^2_{\text{1PM}}}{\pi}\left[1 + \log\frac{4}{\Theta^2_{\text{1PM}}}\right], \quad \frac{E^{\text{rad}}_{\text{soft}}}{E} \simeq \frac{\Theta^3_{\text{1PM}}}{4\pi}\left[1 + \log\frac{4}{\Theta^2_{\text{1PM}}}\right]$$

Gruzinov, Veneziano; Ciafaloni, Colferai, Veneziano

Again the UR limit seems to be universal: the soft energy spectrum of GR and $\mathcal{N} = 8$ supergravity agree above the UR threshold

Notice the non-analytic dependence of the exact answer on $\Theta_{\text{1PM}} \sim G$