Quantum Principal Bundles over non affine bases

Rita Fioresi, FaBiT, Unibo

July 10, 2023

CaLIGOLA Dissemination Activity

Marie Curie Staff Exchange CaLIGOLA:

Title: Cartan geometry, Lie and representation theory, Integrable Systems, quantum Groups and quantum computing towards the understanding of the geometry of deep Learning and its Applications

Abstract: CaLIGOLA aims at advancing the research in Cartan Geometry, Lie Theory, Integrable Systems and Quantum Groups to provide insight into a variety of multidisciplinary fields oriented towards the applications with a special interest in machine learning and quantum computing.

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Unibo website:

https://site.unibo.it/caligola/en

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$$\hat{a} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = aa' = \chi \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \hat{a} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

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$$(1 \otimes \pi)\Delta(f) = \chi^{-1} \otimes f, \quad \text{for} \quad \pi : \mathbb{C}[G] \longrightarrow \mathbb{C}[P]$$

Line Bundles and projective embeddings of G/P

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t is the object we will quantize to obtain a quantum homogeneous projective space.

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We say that $\mathcal{O}_q(X)$ is a **quantum homogeneous variety**, if $\mathcal{O}_q(X)$ admits a coaction of the quantum group $\mathbb{C}_q[G]$, reducing to the coaction of $\mathbb{C}[G]$ on $\mathcal{O}(X)$ when $q = 1$.

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Quantum Section and Projective homogeneous spaces

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$\mathcal{O}_q(G/P)$ is a *projective homogeneous quantum variety*

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$$\mathbb{C}_q[M_n] = \mathbb{C}_q\langle a_{ij}\rangle / I_M \tag{1}$$

where I_M is the ideal of the Manin relations:

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The generalization to n dimensions is immediate!

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The General theory

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If furtherly $j_1 : \mathcal{O}_q(P) \longrightarrow \mathcal{F}(U_1)$ is a cleaving map and we have a family $\phi_{1i} : \mathcal{F}(U_1) \longrightarrow \mathcal{F}(U_i)$ of $\mathcal{O}_q(P)$ comodule isomorphisms compatible with restrictions, then $\mathcal{F}$ is a quantum principal bundle.

Quantum Special Linear group

Define the *quantum matrices*

$$\mathbb{C}_q[M_n] = \mathbb{C}_q\langle a_{ij}\rangle / I_M \tag{2}$$

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The quantum matrix algebra $\mathcal{O}_q(M_n)$ is a bialgebra, with:

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A: H (right) comodule, $\delta_H : A \longrightarrow A \otimes H$

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Theorem (Aschieri-F.-Latini-Weber 2021).

Compatible f.o.d.c. $G \longrightarrow G/P$

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