Hopf algebroids, Atiyah sequences and noncommutative gauge theories

Giovanni Landi

Trieste

Emergent Geometries from Strings and Quantum Fields

Galileo Galilei Institute - Arcetri

18 July 2023

recent papers

Paolo Aschieri, GL, Chiara Pagani,

Xiao Han, GL, Yang Liu,

Xiao Han, GL,

Background

A (commutative) Hopf Algebroid is somehow the dual of a groupoid

(like Hopf algebras vs groups)

Extension of scalars
(similarly to the passage from Hilbert space to Hilbert module):

the ground field k (the complex numbers $\mathbb{C}$)
gets replaced by a (noncommutative) algebra B

so a Hopf algebra over a noncommutative base algebra

Not all structures survive:

(dual) source and target maps, (partial) coproduct, a counit

but in general there is no antipode or there are more than one

(dual) bisections make sense (a version of gauge transformations)

Abstract

Try to work out a gauge algebroid for a noncommutative principal bundle

Try to get a suitable class of (infinitesimal) gauge transformations

some natural structures

braiding Lie algebras to get bigger classes

a sequenze of braided Lie algebras; its splitting as a connection

Weil algebra

Chern–Weil homomorphism and braided Lie algebra cohomology

upgrade it to Hopf algebra cyclic cohomology

The classical gauge groupoid

$\pi : P \to M$ a G-principal bundle over M

The diagonal action of G on $P \times P$ $(u, v)g := (ug, vg)$;

$[u, v]$ is the orbit of (u, v) and $\Omega = P \times_G P$ the collection of orbits

Ω is a groupoid over M, — the gauge or Ehresmann groupoid of the bundle

Source and target projections"

$$s([u, v]) := \pi(v), \qquad t([u, v]) := \pi(u).$$

the object inclusion $M \to P \times_G P$:

$$m \mapsto \mathrm{id}_m := [u, u], \qquad u \in \pi^{-1}(m)$$

Partial multiplication $[u, v'] \cdot [v, w]$, defined when $\pi(v') = \pi(v)$:

$$[u, v] \cdot [v', w] = [u, wg], \qquad v = v'g$$

with inverse $[u, v]^{-1} = [v, u]$.

A bisection: a map $\sigma : M \to \Omega$, which is right-inverse to the source projection, $s \circ \sigma = \mathrm{id}_M$, and such that $t \circ \sigma : M \to M$ is a diffeo of M

The collection of bisections, $\mathcal{B}(\Omega)$, form a group

$$\sigma_1 * \sigma_2(m) := \sigma_1\big((t \circ \sigma_2)(m)\big)\sigma_2(m), \qquad \text{for} \quad m \in M.$$

The identity is the object inclusion $m \mapsto \mathrm{id}_m$, with inverse

$$\sigma^{-1}(m) = \big(\sigma\big((t \circ \sigma)^{-1}(m)\big)\big)^{-1};$$

$(t \circ \sigma)^{-1}$ as a diffeomorphism of M; the second inversion is the one in Ω.

The subset $\mathcal{B}_{P/G}(\Omega)$ of vertical bisections, the ones that are right-inverse to the target projection as well, $t \circ \sigma = \mathrm{id}_M$, form a subgroup of $\mathcal{B}(\Omega)$.

There is a group isomorphism between $\mathcal{B}(\Omega)$ and the group of principal (G-equivariant) bundle automorphisms of the principal bundle,

$$\mathrm{Aut}_G(P) := \{\varphi : P \to P\, ;\ \varphi(pg) = \varphi(p)g\}\, ,$$

while $\mathcal{B}_{P/G}(\Omega)$ is isomorphic to the subgroup of gauge transformations, principal bundle automorphisms which are vertical,

$$\mathrm{Aut}_{P/G}(P) := \{\varphi : P \to P\, ;\ \varphi(pg) = \varphi(p)g\, ,\ \pi(\varphi(p)) = \pi(p)\}.$$

The classical sequences   Atiyah 1957

$\pi : P \to M$ a G-principal bundle over M

at level of groups

$$1 \to \text{Aut}_{P/G}(P) \to \text{Aut}_G(P) \to \text{Diff}(M) \to 1$$

at level of derivations

$$0 \to \mathfrak{g} \to \mathcal{X}(P)_G \to \mathcal{X}(M) \to 0$$

$\mathfrak{g} = \mathcal{X}(P)_G^{ver}$: vertical and invariant; infinitesimal gauge transformation

a splitting of this sequence is a way to give a connection

(horizontal lift or a vertical projection)

an obstruction: $H^1(M, \mathfrak{g} \otimes \Omega^1(M))$

Noncommutative principal bundles

- H a Hopf algebra
- A a right H-comodule algebra with coaction $\delta^A : A \to A \otimes H$; $\quad \delta(a) = a_{(0)} \otimes a_{(1)}$

$\Rightarrow$ the subalgebra of coinvariant elements

$$B := A^{coH} = \left\{ b \in A \mid \delta^A(b) = b \otimes 1_H \right\}$$

The extension $B \subseteq A$ is H-Hopf–Galois if the canonical Galois map

$$\chi : A \otimes_B A \longrightarrow A \otimes H, \quad a' \otimes_B a \mapsto a' a_{(0)} \otimes a_{(1)}$$

is an isomorphism

χ is left A-linear, its inverse is determined by the restriction $\tau := \chi^{-1}_{|_{1_A \otimes H}}$

$$\tau = \chi^{-1}_{|_{1_A \otimes H}} : H \to A \otimes_B A \ , \quad h \mapsto \tau(h) = h^{<1>} \otimes_B h^{<2>} \, .$$

the translation map; thus by definition:

$$h^{<1>} {h^{<2>}}_{(0)} \otimes {h^{<2>}}_{(1)} = 1_A \otimes h$$

Everything algebraic

G be a semisimple affine algebraic group

$\pi : P \to P/G$ be a principal G-bundle with P and P/G affine varieties

$H = \mathcal{O}(G)$ the dual coordinate Hopf algebra

$A = \mathcal{O}(P)$, $B = \mathcal{O}(P/G)$ the dual coordinate algebras

$B \subseteq A$ be the subalgebra of functions constant on the fibers.

Then $B = A^{coH}$ and $\mathcal{O}(P \times_{P/G} P) \simeq A \otimes_B A$

Bijectivity of $P \times G \to P \times_{P/G} P$, $(p,g) \mapsto (p,pg)$, characterizing principal bundles, corresponds to the bijectivity of the canonical map $\chi : A \otimes_B A \to A \otimes H$

thus $B = A^{coH} \subseteq A$ is a Hopf–Galois extension

An important notion is that of the classical translation map

$t : P \times_{P/G} P \to G, \quad (p,q) \mapsto t(p,q)$ where $q = p\, t(p,q)$

the dual to τ before

Gauge transformations

Classical

The group $\mathcal{G}_P$ of gauge transformations of a principal G-bundle $\pi : P \to P/G$ is the group (for point-wise product) of G-equivariant maps

$$\mathcal{G}_P := \{\sigma : P \to G;\ \sigma(pg) = g^{-1}\sigma(p)g\}$$

Equivalently, is the subgroup (for map composition) of principal bundle automorphisms which are vertical (project to the identity on the base space):

$$\mathrm{Aut}_{P/G}(P) := \{\varphi : P \to P;\ \varphi(pg) = \varphi(p)g,\ \pi(\varphi(p)) = \pi(p)\},$$

These definitions can be dualised for algebras rather than spaces.

For $A = \mathcal{O}(P)$, $B = \mathcal{O}(P/G)$, $H = \mathcal{O}(G)$, the gauge group $\mathcal{G}_P$ of G-equivariant maps corresponds to H-equivariant maps that are also algebra maps

$$\mathcal{G}_A := \{\mathsf{f} : H \to A;\ \delta^A \circ \mathsf{f} = (\mathsf{f} \otimes \mathsf{id}) \circ \mathsf{Ad}\,,\ \mathsf{f} \text{ algebra map}\}\ .$$

The group structure is the convolution product.

Similarly, the vertical automorphisms description leads to H-equivariant maps

$$\mathsf{Aut}_B\, A = \{\mathsf{F} : A \to A;\ \delta^A \circ \mathsf{F} = (\mathsf{F} \otimes \mathsf{id}) \circ \delta^A\,,\ \mathsf{F}|_B = \mathsf{id} : B \to B\,,\ \mathsf{F} \text{ algebra map}\}\ .$$

The noncommutative case

Let $B = A^{coH} \subseteq A$ be a faithfully flat Hopf–Galois extension

The collection $\mathsf{Aut}_H(A)$ of unital algebra maps of A into itself, which are H-equivariant,

$$\delta^A \circ \mathsf{F} = (\mathsf{F} \otimes \mathsf{id}) \circ \delta^A \qquad F(a)_{(0)} \otimes F(a)_{(1)} = F(a_{(0)}) \otimes a_{(1)}$$

and restrict to the identity on the subalgebra B, is a group by map composition

with inverse operation

$$F^{-1}(a) = a_{(0)} F({a_{(1)}}^{<1>})\, {a_{(1)}}^{<2>}$$

H.P. Schneider: vertical H-equivariant algebra maps are invertible

Bialgebroids

B an algebra

B-ring : a triple (A, μ, η) M. Takeuchi, G. Böhm ....

A a B-bimodule with B-bimodule maps $\mu : A \otimes_B A \to A$ and $\eta : B \to A$

associativity and unit conditions:

$$\mu \circ (\mu \otimes_B \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes_B \mu), \quad \mu \circ (\eta \otimes_B \mathrm{id}_A) = \mathrm{id}_A = \mu \circ (\mathrm{id}_A \otimes_B \eta).$$

Dually, B-coring : a triple (C, Δ, ε)

C is a B-bimodule with B-bimodule maps $\Delta : C \to C \otimes_B C$ and $\varepsilon : C \to B$

coassociativity and counit conditions:

$$(\Delta \otimes_B \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes_B \Delta) \circ \Delta, \quad (\varepsilon \otimes_B \mathrm{id}_C) \circ \Delta = \mathrm{id}_C = (\mathrm{id}_C \otimes_B \varepsilon) \circ \Delta$$

A left B-bialgebroid $\mathcal{C}$:

a $(B \otimes B^{op})$-ring and a B-coring structure on $\mathcal{C}$ with compatibility conditions

There are source and target maps (with commuting ranges)

$$s := \eta(\,\cdot\, \otimes_B 1_B) : B \to \mathcal{C} \quad \text{and} \quad t := \eta(1_B \otimes_B \,\cdot\,) : B^{op} \to \mathcal{C}$$

The compatibility conditions for a left B-bialgebroid $\mathcal{C}$

(i) The bimodule structures in the B-coring $(\mathcal{C}, \Delta, \varepsilon)$ and those of the $B \otimes B^{op}$-ring $(\mathcal{C}, s, t)$ are related as

$$b \triangleright a \triangleleft \tilde{b} := s(b)t(\tilde{b})a \qquad \text{for } b, \tilde{b} \in B,\ a \in \mathcal{C}.$$

(ii) The coproduct Δ corestricts to an algebra map from $\mathcal{C}$ to

$$\mathcal{C} \times_B \mathcal{C} := \Big\{ \sum\nolimits_j a_j \otimes_B \tilde{a}_j \mid \sum\nolimits_j a_j t(b) \otimes_B \tilde{a}_j = \sum\nolimits_j a_j \otimes_B \tilde{a}_j s(b),\ \ \forall\, b \in B \Big\},$$

(iii) The counit $\varepsilon : \mathcal{C} \to B$ satisfies the properties,

(1) $\varepsilon(1_{\mathcal{C}}) = 1_B$,

(2) $\varepsilon(s(b)a) = b\varepsilon(a)$,

(3) $\varepsilon(as(\varepsilon(\tilde{a}))) = \varepsilon(a\tilde{a}) = \varepsilon(at(\varepsilon(\tilde{a})))$, for all $b \in B$ and $a, \tilde{a} \in \mathcal{C}$.

A Hopf algebroid with invertible antipode G. Böhm

For a left bialgebroid $(\mathcal{C}, \Delta, \varepsilon, s, t)$ over the algebra B, an invertible antipode $S : \mathcal{C} \to \mathcal{C}$ in an algebra anti-homomorphism with inverse $S^{-1} : \mathcal{C} \to \mathcal{C}$ s.t.

$$S \circ t = s$$

and compatibility conditions with the coproduct:

$$(Sh_{(1)})_{(1')} h_{(2)} \otimes_B S(h_{(1)})_{(2')} = 1_{\mathcal{C}} \otimes_B Sh$$

$$(S^{-1}h_{(2)})_{(1')} \otimes_B (S^{-1}h_{(2)})_{(2')} h_{(1)} = S^{-1}h \otimes_B 1_{\mathcal{C}}$$

These then imply $S(h_{(1)})\, h_{(2)} = t \circ \varepsilon \circ Sh$.

The above similar to a Hopf algebra with an algebra B as the ground field.

source of difficulties/interest : there is no unique antipode in general

A weaker condition P. Schauenburg

A bialgebroid $\mathcal{C}$ is a Hopf algebroid if the map

$$\lambda : \mathcal{C} \otimes_{B^{op}} \mathcal{C} \to \mathcal{C} \otimes_B \mathcal{C}, \qquad \lambda(p \otimes_{B^{op}} q) = p_{(1)} \otimes_B p_{(2)} q$$

is invertible

$$\otimes_{B^{op}} \quad pt(b) \otimes_{B^{op}} q = p \otimes_{B^{op}} t(b) q \qquad \otimes_B \quad t(b) p \otimes_B q = p \otimes_B s(b) q$$

For $B = k$, this reduces to the map

$$\lambda : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}, \qquad p \otimes q \mapsto p_{(1)} \otimes p_{(2)} q$$

which for a usual Hopf algebra with an antipode has inverse

$$p \otimes q \mapsto p_{(1)} \otimes S(p_{(2)}) q$$

Also here, if there is an invertible antipode S as before Böhm one constructs an inverse for the map λ ; for $X, Y \in \mathcal{C}$,

$$\lambda^{-1}(X \otimes_B Y) = S^{-1}(S(X)_{(2)}) \otimes_{B^{op}} S(X)_{(1)} Y$$

No claim that S here is unique

The noncommutative gauge bialgebroid aka Ehresmann–Schauenburg

$B = A^{co\,H} \subseteq A$ be a Hopf–Galois extension

right coaction : $\delta(a) = a_{(0)} \otimes a_{(1)}$

translation map : $\tau(h) = h^{<1>} \otimes_B h^{<2>}$

The B-bimodule $\mathcal{C}(A, H)$ of coinvariant elements for the diagonal coaction,

$$(A \otimes A)^{coH} = \{a \otimes \tilde{a} \in A \otimes A\,;\; a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)} = a \otimes \tilde{a} \otimes 1_H\}$$

is a B-coring with coproduct and counit:

$$\Delta(a \otimes \tilde{a}) = a_{(0)} \otimes \tau(a_{(1)}) \otimes \tilde{a} = a_{(0)} \otimes a_{(1)}{}^{<1>} \otimes_B a_{(1)}{}^{<2>} \otimes \tilde{a},$$

$$\varepsilon(a \otimes \tilde{a}) = a\tilde{a}.$$

One see $\mathcal{C}(A, H)$ is a subalgebra of $A \otimes A^{op}$ and it is indeed a (left) B-bialgebroid

Product $\quad (x \otimes \tilde{x}) \bullet_{\mathcal{C}(A,H)} (y \otimes \tilde{y}) = xy \otimes \tilde{y}\tilde{x}$

Target and source maps $\quad t(b) = 1_A \otimes b \quad$ and $\quad s(b) = b \otimes 1_A$

Han, L. ; Han Majid - 2022

The Ehresmann–Schauenburg bialgebroid $\mathcal{C}(A, H)$ of a Hopf–Galois extension is a Hopf algebroid :

If the Hopf algebra H is coquasitriangular with R matrix (a convolution invertible map) $\mathcal{R} : H \otimes H \to k$ (+ conditions),

there is an antipode: the inverse of the braiding induced by $\mathcal{R}$:

$$\Psi(a \otimes \tilde{a}) = a_{(0)} \otimes \tilde{a}_{(0)} \otimes \mathcal{R}(a_{(1)} \otimes \tilde{a}_{(1)})$$

this is an invertible H-comodule map with inverse

$$\Psi^{-1}(a \otimes \tilde{a}) = a_{(0)} \otimes \tilde{a}_{(0)} \otimes \mathcal{R}^{-1}(a_{(1)} \otimes \tilde{a}_{(1)})$$

both map restrict to the invariant subspace $\mathcal{C}(A, H)$.

Then $S = \Psi^{-1}$ obeys all properties of an antipode for $\mathcal{C}(A, H)$.

The bialgebroid $\mathcal{C}(A,H)$ of a Hopf–Galois extension as a quantization (of the dualization) of the classical gauge groupoid principal bundle

Its bisections correspond to gauge transformations

$\mathcal{C}(A,H)$ the gauge bialgebroid of a Hopf–Galois extension $B = A^{coH} \subseteq A$

A bisection is a B-bilinear unital left character on the B-ring $(\mathcal{C}(A,H), s)$.

The collection $\mathcal{B}(\mathcal{C}(A,H))$ of bisections of the bialgebroid $\mathcal{C}(A,H)$ is a group with convolution product :

$$\sigma_1 * \sigma_2(x \otimes \tilde{x}) := \sigma_1((x \otimes \tilde{x})_{(1)})\, \sigma_2((x \otimes \tilde{x})_{(2)}) = \sigma_1(x_{(0)} \otimes x_{(1)}{}^{<1>})\, \sigma_2(x_{(1)}{}^{<2>} \otimes \tilde{x})$$

using the B-coring coproduct $\Delta(x \otimes \tilde{x}) = (x \otimes \tilde{x})_{(1)} \otimes_B (x \otimes \tilde{x})_{(2)}$

A group isomorphism

$$\alpha : \mathsf{Aut}_H(A) \to \mathcal{B}(\mathcal{C}(A, H))$$

between gauge transformations and bisections:

$$\mathcal{B}(\mathcal{C}(A, H)) \ni \sigma \quad \mapsto \quad F_\sigma(a) := \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})\, a_{(1)}{}^{<2>}, \quad F_\sigma \in \mathsf{Aut}_H(A)$$

$$F \in \mathsf{Aut}_H(A) \ni F \quad \mapsto \quad \sigma_F(a \otimes \tilde{a}) := F(a)\tilde{a}, \quad \sigma_F \in \mathcal{B}(\mathcal{C}(A, H))$$

Bisection can be given for any bialgebroid

For the general case one would need additional requirements so to get a proper composition law for bisections

Explicit examples

the monopole bundles over the quantum S^2_q

a not faithfully flat example from SL(2)

the SU(2) - bundle $S^7_\theta \to S^4_\theta$

the $SO_\theta(2n)$ bundle $SO_\theta(2n+1) \to S^{2n}_\theta$

some example from q-geometry

change from automorphisms to derivations
(infinitesimal gauge transformations)

Lie algebras of suitable ‘bisections’

braided versions of them

Atiyah sequences of braided Lie algebras of derivations

Braiding then

K a Hopf algebra

K-equivariant H-Hopf–Galois extension $B \subseteq A^H$:

A carries a left action $\triangleright : K \otimes A \to A$ of K, compatible with the H-coaction:

$$(k \triangleright a)_{(0)} \otimes (k \triangleright a)_{(1)} = k \triangleright (a_{(0)} \otimes a_{(1)}) \ .$$

Recall: K is quasitriangular if there exists an invertible element $\mathsf{R} \in K \otimes K$ with respect to which the coproduct Δ of K is quasi-cocommutative

$$\Delta^{cop}(k) = \mathsf{R}\Delta(k)\overline{\mathsf{R}} \qquad \Delta^{cop} := \tau \circ \Delta$$

and $\overline{\mathsf{R}} \in K \otimes K$ the inverse of R, $\mathsf{R}\overline{\mathsf{R}} = \overline{\mathsf{R}}\mathsf{R} = 1 \otimes 1$.

R is required to satisfy (these allow for a good representation theory) ,

$$(\Delta \otimes \mathrm{id})\mathsf{R} = \mathsf{R}_{13}\mathsf{R}_{23} \qquad \text{and} \qquad (\mathrm{id} \otimes \Delta)\mathsf{R} = \mathsf{R}_{13}\mathsf{R}_{12}.$$

The Hopf algebra K is triangular when $\overline{\mathsf{R}} = \mathsf{R}_{21} = \tau(\mathsf{R})$, $\quad \tau$ the flip.

We further assume the Hopf algebra K to be triangular.

This allows for the study of braided Lie algebras.

A braided Lie algebra associated with a triangular Hopf algebra (K, R), is a K-module $\mathfrak{g}$ with a bilinear map

$$[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

that satisfies the following conditions.

(i) K-equivariance: for $\Delta(k) = k_{(1)} \otimes k_{(2)}$ the coproduct of K,

$$k \triangleright [u, v] = [k_{(1)} \triangleright u, k_{(2)} \triangleright v]$$

(ii) braided antisymmetry:

$$[u, v] = -[\mathsf{R}_\alpha \triangleright v, \mathsf{R}^\alpha \triangleright u],$$

(iii) braided Jacobi identity:

$$[u, [v, w]] = [[u, v], w] + [\mathsf{R}_\alpha \triangleright v, [\mathsf{R}^\alpha \triangleright u, w]]$$

Infinitesimal gauge transformations

$B = A^{coH} \subseteq A$ a K-equivariant Hopf–Galois extension, for (K, R) triangular.

Inside the braided Lie algebra Der(A) consider the subspace of braided derivations that are H-equivariant

$$\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) = \big\{u \in \mathrm{Hom}(A, A) \mid \delta(u(a)) = u(a_{(0)}) \otimes a_{(1)},$$

$$u(aa') = u(a)a' + (\mathsf{R}_\alpha \triangleright a)(\mathsf{R}^\alpha \triangleright u)(a')\,, \text{ for all } a, a' \in A\big\}$$

and then those derivations that are vertical,

$$\mathrm{aut}^{\mathsf{R}}_B(A) := \{u \in \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \mid u(b) = 0, \text{ for all } b \in B\} \ .$$

Elements of $\mathrm{aut}^{\mathsf{R}}_B(A)$ are regarded as infinitesimal gauge transformations

of the K-equivariant Hopf–Galois extension $B = A^{coH} \subseteq A$.

Atiyah sequences and their splittings

A K-equivariant Hopf–Galois extension $B = A^{coH} \subseteq A$

The braided Lie algebra of vertical equivariant derivations

$$\mathsf{aut}^{\mathsf{R}}_{B}(A) := \{u \in \mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \mid u(b) = 0,\ b \in B\}$$

is a braided Lie subalgebra of equivariant derivations

$$\mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) = \{u \in \mathsf{Der}(A) \mid \delta \circ u = (u \otimes \mathsf{id}) \circ \delta\}\ .$$

Each derivation in $\mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A)$, being H-equivariant, restricts to a derivation on the subalgebra of coinvariant elements $B = A^{coH}$

A sequence of braided Lie algebras $\mathsf{aut}^{\mathsf{R}}_{B}(A) \to \mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \to \mathsf{Der}^{\mathsf{R}}(B)$

When exact,

$$0 \to \mathsf{aut}^{\mathsf{R}}_{B}(A) \to \mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \to \mathsf{Der}^{\mathsf{R}}(B) \to 0$$

is a version of the Atiyah sequence of a (commutative) principal fibre bundle.

An H-equivariant splitting of the sequence is a connection on the bundle

The general construction

(K, R) a triangular Hopf algebra ; an exact sequence of K-braided Lie algebras

$$0 \to \mathfrak{g} \xrightarrow{\imath} P \xrightarrow{\pi} T \to 0$$

For B an algebra; take (B, T) a braided Lie–Rinehart pair:

T is a B-module with a braided Lie algebra morphism $T \to \mathsf{Der}^{\mathsf{R}}(B)$;

B is a T-module and T acts as braided derivations of B,

$$X(bb') = X(b)b' + (\mathsf{R}_\alpha \triangleright b)(\mathsf{R}^\alpha \triangleright X)(b')\,, \qquad b, b' \in B, \quad X \in T\,,$$

and

$$[X, bX']_{\mathsf{R}} = X(b)X' + (\mathsf{R}_\alpha \triangleright b)[(\mathsf{R}^\alpha \triangleright X), X']_{\mathsf{R}}\,, \qquad b \in B, \quad X, X' \in T.$$

A connection on the sequence is a splitting: a B-module map,

$$\rho : T \to P, \qquad \pi \circ \rho = \mathsf{id}_T$$

the 'vertical projection', is the B-module map $\omega_\rho : P \to \mathfrak{g}$,

$$\omega_\rho(Y) = Y - \rho(Y^\pi)\,, \qquad Y \in P$$

The extend to which ρ or ω_ρ fail to be braided Lie algebra morphisms is measured by the *(basic)* curvature

$$\Omega(X, X') := \rho([X, X']_{\mathsf{R}}) - [\rho(X), \rho(X')]_{\mathsf{R}}\,, \qquad X, X' \in T.$$

Ω is a $\mathfrak{g}$-valued braided two-form on T.

The curvature can also be given as a basic $\mathfrak{g}$-valued braided two-form on P (*spatial* curvature):

$$\Omega_{\omega_\rho}(Y, Y') := \Omega(Y^\pi, Y'^\pi), \qquad Y, Y' \in P.$$

$$\Omega_{\omega_\rho}(Y, Y') = [Y, \omega_\rho(Y')]_{\mathsf{R}} + [\omega_\rho(Y), Y']_{\mathsf{R}} - \omega_\rho([Y, Y']_{\mathsf{R}}) - [\omega_\rho(Y), \omega_\rho(Y')]_{\mathsf{R}}.$$

This expression can be read as a *structure equation*:

$$d\omega_\rho = \Omega_{\omega_\rho} + [\omega_\rho, \omega_\rho]_{\mathsf{R}}\ .$$

Here

$$d\zeta(Y, Y') := [Y, \zeta(Y')]_{\mathsf{R}} + [\zeta(Y), Y']_{\mathsf{R}} - \zeta([Y, Y']_{\mathsf{R}}), \qquad Y, Y' \in P.$$

(generalised to higher forms)

There is a Bianchi identity:

$$d\Omega_{\omega_\rho} + [\Omega_{\omega_\rho}, \omega_\rho]_{\mathsf{R}} = 0 \; .$$

when the connection is equivariant: $k \triangleright \omega_\rho = \varepsilon(k)\omega_\rho$

this is true ‘the way it is written

in general one needs a suitable interpretation of the curvature as a derivation of the braided Lie algebra $\mathfrak{g}$ and of the above expression

The space of connections $C(T, \mathfrak{g})$:

an affine space modelled on B-module maps $\eta : T \to \mathfrak{g}$

with $\rho : T \to P$ a connection and $\eta : T \to \mathfrak{g}$, the sum $\rho' = \rho + \eta$ is a connection.

An action of the braided Lie algebra P: $\quad P \times C(T, \mathfrak{g}) \longrightarrow C(T, \mathfrak{g})$

$$(Y, \rho) \to \rho + \delta_Y \rho, \qquad (\delta_Y \rho)(X) := [Y, \rho(X)]_{\mathsf{R}} - \rho([Y^\pi, X]_{\mathsf{R}},$$

$(\delta_Y \rho)(X) \in \mathfrak{g}$ or $\delta_Y \rho : T \to \mathfrak{g}$.

For vertical elements $V \in \mathfrak{g}$, this is an infinitesimal gauge transformations:

$$(\delta_V \rho)(X) = [V, \rho(X)]_{\mathsf{R}}\,,$$

thus $\mathfrak{g}$ is the braided Lie algebra of such transformations.

The curvature of the transformed connection $\rho' = \rho + \delta_Y \rho$:

$$\Omega' = \Omega + \delta_Y \Omega - [\delta_Y \rho, \delta_Y \rho]_{\mathsf{R}}$$

for $V \in \mathfrak{g}$ an infinitesimal gauge transformation this reduces to

$$(\delta_V \Omega)(X, X') = [V, \Omega(X, X')]_{\mathsf{R}}.$$

Calabi pseudo-cohomology

Two sequences are equivalent if there is an isomorphism $P \to P'$ with commutative diagrams

$$\begin{array}{c} 0 \to \mathfrak{g} \to P \to T \to 0 \\ \downarrow \\ 0 \to \mathfrak{g} \to P' \to T \to 0 \end{array}$$

Classified by $\mathcal{H}^2(T, \mathfrak{g})$, the Calabi pseudo-cohomology of the Lie algebra T with values in $\mathfrak{g}$. If A is abelian $\mathcal{H}^2(T, \mathfrak{g})$ is the CE cohomology group $H^2(T, \mathfrak{g})$.

A pseudo-cochain: a pair (ϕ, Φ),

$\phi : T \to \mathrm{Der}(\mathfrak{g})$, Φ a $\mathfrak{g}$-valued skew map on $T \times T$, such that

$$\phi(X)\phi(X') - \phi(X')\phi(X) = \phi([X, X']) + \mathrm{ad}_{\Phi(X,X')} \qquad X, X' \in T.$$

Such a pair is a 2-pseudo-cocycle if $\delta_\phi(\Phi) = 0$, where

$$\delta_\phi(\Phi)(X, X', X'') = \phi(X) \triangleright \Phi(X', X'') - \Phi([X, X', X'') + c.p.$$

Two such pairs (ϕ, Φ), (ϕ', Φ') are equivalent if there is a map $\eta : T \to \mathfrak{g}$, s.t.

$$\phi'(X) = \phi(X) + +\mathsf{ad}_{\eta(X)}$$

$$\Phi'(X, X') = \Phi(X, X') + (\delta_\phi \eta)(X, X') + [\eta(X), \eta(X')].$$

Equivalent pseudo-cochains leads to equivalent pseudo-cocycles and the space of equivalent classes of 2-pseudo-cocycles is denoted $\mathcal{H}^2(T, \mathfrak{g})$, the order 2 Calabi pseudo-cohomology of the Lie algebra T with values in $\mathfrak{g}$.

Given a splitting of the sequence, that is given a connection $\rho : T \to P$, one construct a pseudo-cocycle (ϕ, Φ) by

$$\phi(X) \triangleright V = [\rho(X), V] \qquad X \in T, V \in \mathfrak{g}$$

$$\Phi(X, X') = \Omega(X, X') = \rho([X, X']) - [\rho(X), \rho(X')]\,, \qquad X, X' \in T.$$

Jacopi identity implies it is a pseudo-cocycle:

$$\delta_\phi(\Phi) = 0$$

this is the Bianchi identity.

Given two connections ρ and $\rho' = \rho + \eta$, the corresponding pseudo-cocycles (ϕ', Φ') and (ϕ, Φ) are equivalent, they belong to the same class in $\mathcal{H}^2(T, \mathfrak{g})$.

Pseudo-cocycles associated with equivalent extensions determine the same class in $\mathcal{H}^2(T, \mathfrak{g})$.

Conversely, given a pseudo-cocycle one construct a sequence of Lie algebras $0 \to \mathfrak{g} {\to} P {\to} T \to 0$

cohomologous pseudo-cocycle give equivalent sequences.

The space of equivalent classes of extensions of T by $\mathfrak{g}$ is in a bijective correspondence with $\mathcal{H}^2(T, \mathfrak{g})$.

$\mathcal{H}^2(T, \mathfrak{g})$ is a complicate object in general

An R-symmetric map of degree q

$$\varphi : \mathfrak{g} \otimes^{\mathsf{R}} \ldots \otimes^{\mathsf{R}} \mathfrak{g} \to B$$

which intertwining the representation $\mathsf{ad}_{\mathsf{R}} \otimes^{\mathsf{R}} \ldots \otimes^{\mathsf{R}} \mathsf{ad}_{\mathsf{R}}$ of P on $\mathfrak{g} \otimes^{\mathsf{R}} \ldots \otimes^{\mathsf{R}} \mathfrak{g}$ with the action of P on B (ad_{R} is the braided commutator).

$\mathcal{S}_{\mathsf{R}}$ the braided anti-symmetrization.

Then

$$\varphi_\rho = \mathcal{S}_{\mathsf{R}} \circ f(\Omega \otimes^{\mathsf{R}} \ldots \otimes^{\mathsf{R}} \Omega)$$

is a braided B-valued $2q$-form on T.

One has:

$$d\varphi_\rho = 0$$

For the cohomology classes:

$$[\varphi_\rho] = [\varphi_{\rho'}] \qquad \rho, \rho' \quad \text{two connections on the sequence}$$

$$\varphi_\rho = \varphi_{\rho'} + d(.....)$$

Consider:

$\mathsf{Inv}^q = \{$ all such φ as before $\}$ $\qquad$ $\mathsf{Inv} = \oplus_q \mathsf{Inv}^q$

H_{Ch} Chevalley cohomology of (T, B)

we get a linear map

$$\mathsf{cw} : \mathsf{Inv} \to H_{Ch} \qquad \varphi \to [\varphi_\rho]$$

When pulled back to P:

$$\pi^* \varphi_\rho = d(\text{ Chern Simons })$$

Twisting

The constructions survive under a Drinfeld twists

Examples from θ-deformations

$$F = e^{\pi i\theta(H_1\otimes H_2 - H_2\otimes H_1)} \qquad [H_1, H_2] = 0$$

$$\mathsf{R}_\mathsf{F} = \overline{\mathsf{F}}^2 = e^{-2\pi i\theta(H_1\otimes H_2 - H_2\otimes H_1)}$$

Jordanian twist . κ-Minkowski

$$\mathsf{F} = \exp\left(u\tfrac{\partial}{\partial u}\otimes\sigma\right) \qquad \sigma = \ln\left(1 + \frac{1}{\kappa}P_0\right)$$

$P_0 = iu\frac{\partial}{\partial x^0} \qquad [u\frac{\partial}{\partial u}, P_0] = P_0$

In particular $\mathcal{O}(S^4_\theta)$

with generators b_μ, $\mu = (\mu_1, \mu_2) = (0,0), (\pm 1, 0), (0, \pm 1)$

the weights for the action of H_1, H_2.

Their commutation relations are

$$b_\mu \bullet_\theta b_\nu = \lambda^{2\mu\wedge\nu}\, b_\nu \bullet_\theta b_\mu \qquad \lambda = e^{-\pi i\theta}\ .$$

with sphere relation $\sum_{b_\mu} b^*_\mu \cdot_\theta b_\mu = 1$.

$\mathrm{Der}^{\mathsf{R}_\mathsf{F}}(\mathcal{O}(S^4_\theta))$ is generated as an $\mathcal{O}(S^4_\theta)$-module by operators $\widetilde{H}_\mu$

defined on the algebra generators as

$$\widetilde{H}_\mu(b_\nu) := \delta_{\mu^*\nu} - b_\mu \bullet_\theta b_\nu$$

and extended to the whole algebra $\mathcal{O}(S^4_\theta)$ as braided derivations:

$$\widetilde{H}_\mu(b_\nu \bullet_\theta b_\tau) = \widetilde{H}_\mu(b_\nu) \bullet_\theta b_\tau + \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta \widetilde{H}_\mu(b_\tau).$$

They verify

$$\widetilde{H}_\mu(\sum_\nu b^*_\nu \bullet_\theta b_\nu) = 0\,, \qquad \sum\nolimits_\mu b^*_\mu \bullet_\theta \widetilde{H}_\mu = 0$$

In the classical limit $\theta = 0$, the derivations $\widetilde{H}_\mu$ reduce to

$$H_\mu = \partial_{\mu^*} - b_\mu \Delta, \qquad \Delta = \sum_\mu b_\mu \partial_\mu$$

the Liouville vector field.
The weights μ are those of the five dimensional representation of $so(5)$.

The bracket in $\mathrm{Der}^{\mathsf{R_F}}(\mathcal{O}(S^4_\theta))$ is the braided commutator

$$\begin{aligned}[][\widetilde{H}_\mu, \widetilde{H}_\nu]_{\mathsf{R_F}} &:= \widetilde{H}_\mu \circ \widetilde{H}_\nu - \lambda^{2\mu\wedge\nu} \widetilde{H}_\nu \circ \widetilde{H}_\mu \\ &= b_\mu \bullet_\theta \widetilde{H}_\nu - \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta \widetilde{H}_\mu\end{aligned}$$

The generators $\widetilde{H}_\mu$ can be expressed in terms of their commutators as

$$\widetilde{H}_\nu = \sum_\mu b^*_\mu \bullet_\theta [\widetilde{H}_\mu, \widetilde{H}_\nu]_{\mathsf{R_F}}$$

Denote $\quad \widetilde{H}^\pi_{\mu,\nu} := [\widetilde{H}_\mu, \widetilde{H}_\nu]_{\mathsf{R_F}} = -\lambda^{2\mu\wedge\nu} \widetilde{H}^\pi_{\nu,\mu}$

Their braided commutators close the braided Lie algebra $so_\theta(5)$:

$$[\widetilde{H}^\pi_{\mu,\nu}, \widetilde{H}^\pi_{\tau,\sigma}]_{\mathsf{R_F}} = \delta_{\nu^*\tau} \widetilde{H}^\pi_{\mu,\sigma} - \lambda^{2\mu\wedge\nu} \delta_{\mu^*\tau} - \lambda^{2\tau\wedge\sigma} (\delta_{\nu^*\sigma} \widetilde{H}^\pi_{\mu,\tau} - \lambda^{2\mu\wedge\nu} \delta_{\sigma^*\mu} \widetilde{H}^\pi_{\nu,\tau})$$

The instanton $\mathcal{O}(SU(2))$ Hopf–Galois extension $\mathcal{O}(S^4_\theta) \subset \mathcal{O}(S^7_\theta)$.

A short exact sequence of braided Lie algebras

$$0 \to \mathrm{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta)) \xrightarrow{\imath} \mathrm{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta)) \xrightarrow{\pi} \mathrm{Der}(\mathcal{O}(S^4_\theta)) \to 0$$

$\mathrm{Der}(\mathcal{O}(S^4_\theta))$ generated as before by elements $\widetilde{H}^{\pi}_{\mu,\nu}$

$\mathrm{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$ generated by (explicit) derivations $\widetilde{H}_{\mu,\nu}$ realising a representation of $so_\theta(5)$ as derivations on $\mathcal{O}(S^7_\theta)$ and

$$\pi(\widetilde{H}_{\mu,\nu}) = \widetilde{H}^{\pi}_{\mu,\nu}.$$

$\mathrm{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ vertical and equivariant (alternatively via a connection)

The horizontal lift: the $\mathcal{O}(S^4_\theta)$-module map $\rho : \mathrm{Der}(\mathcal{O}(S^4_\theta)) \to \mathrm{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$ defined on the generators $\widetilde{H}_\nu$ of $\mathrm{Der}^{\mathsf{R}_\mathsf{F}}(\mathcal{O}(S^4_\theta))$ as

$$\rho(\widetilde{H}_\nu) := \sum_\mu b^*_\mu \bullet_\theta \widetilde{H}_{\mu,\nu}$$

is a splitting of the sequence above .

The corresponding vertical projection is the $\mathcal{O}(S^4_\theta)$-module map

$\Psi : \mathrm{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))) \to \mathrm{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$

$$\Psi(\widetilde{H}_{\mu,\nu}) := \widetilde{H}_{\mu,\nu} - \rho(\widetilde{H}^\pi_{\mu,\nu}) = \widetilde{H}_{\mu,\nu} - \big(b_\mu \bullet_\theta \rho(\widetilde{H}_\nu) - \lambda^{2\mu\wedge\nu}\, b_\nu \bullet_\theta \rho(\widetilde{H}_\mu)\big)$$

These derivations generated the algebra $\mathrm{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$.

The curvature

$$\Omega(X,Y) := [\rho(X),\rho(Y)]_{\mathsf{R_F}} - \rho([X,Y]_{\mathsf{R_F}}) = \imath \circ \Psi[\rho(X),\rho(Y)]_{\mathsf{R_F}}$$

One finds $\qquad [\rho(\widetilde{H}_\mu),\rho(\widetilde{H}_\nu)]_{\mathsf{R_F}} = \widetilde{H}_{\mu,\nu}$

Then

$$\Omega(\widetilde{H}_\mu,\widetilde{H}_\nu) = \widetilde{H}_{\mu,\nu} - \big(b_\mu \bullet_\theta \rho(\widetilde{H}_\nu) - \lambda^{2\mu\wedge\nu}\, b_\nu \bullet_\theta \rho(\widetilde{H}_\mu)\big) = \imath \circ \Psi(\widetilde{H}_{\mu,\nu}).$$

There is also a connection 1-form; it is anti-selfdual.

An action of braided conformal transformations

$$so_\theta(5,1)$$

yields noncommutative families of anti-selfdual connections

Galois objects

of a Hopf algebra H (noncommutative principal bundle over a point)

An H-Hopf–Galois extension A of the ground field $\mathbb{C}$.

Examples:

Group Hopf algebras $H = \mathbb{C}[G]$: equivalence classes of $\mathbb{C}[G]$-Galois objects are in bijective correspondence with the cohomology group $H^2(G, \mathbb{C}^\times)$

$H^2(\mathbb{Z}^r, \mathbb{C}^\times) = (\mathbb{C}^\times)^{r(r-1)/2}$: infinitely many iso classes of $\mathbb{C}[\mathbb{Z}^r]$-Galois objects

Taft algebras : q a primitive N-th root of unity; T_N, neither commutative nor cocommutative Hopf algebra; generators x, g with relations:

$$x^N = 0\,, \quad g^N = 1\,, \quad xg - q\,gx = 0\,.$$

coproduct: $\Delta(x) := 1 \otimes x + x \otimes g, \qquad \Delta(g) := g \otimes g$

counit: $\varepsilon(x) := 0, \varepsilon(g) := 1$, and antipode: $S(x) := -xg^{-1}, S(g) := g^{-1}$.

Summing up:

Worked out a gauge algebroid for a noncommutative principal bundle

A suitable class of (infinitesimal) gauge transformations

Infinite dimensional Hopf algebra (of possibly braided derivations)

A Chern-Weil homomorphisms and characteristic classes

Chern-Simons terms

some natural structures but we are only at the beginning ...

Thanks