Towards the N^{3}LO evolution of the parton distribution functions

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Theory Challenges in the Precision Era of the Large Hadron Collider

Based on 2203.11181, 2302.07593, 2307.04158, . . .

ongoing collaboration with

Franz Herzog, **Sven Moch**, **Andreas Vogt** and **Andrea Pelloni**

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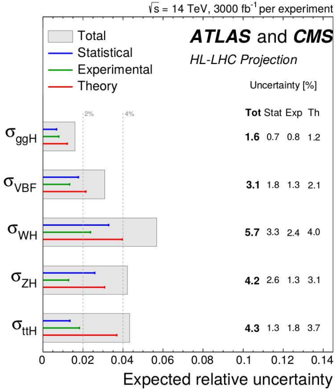
1%-level phenomenology at the LHC

The HL-LHC will push the precision frontier to the %-level
E.g. Higgs measurements



- Systematic uncertainties (e.g. luminosity determination, resolution) around 1%
- Statistical errors reduced by 20-fold increase of data collected
- Theory will be the most important source of errors.

CERN Yellow report

Theory errors

	Q [GeV]	$\delta\sigma^{\mathrm{N^3LO}}$	δ(scale)	δ(PDF-TH)
$gg \to$ Higgs	m_H	3.5%	$^{+0.21\%}_{-2.37\%}$	$\pm 1.2\%$
$b\bar{b} \to$ Higgs	m_H	-2.3%	$^{+3.0\%}_{-4.8\%}$	$\pm 2.5\%$
NCDY	30	-4.8%	$^{+1.53\%}_{-2.54\%}$	$\pm 2.8\%$
	100	-2.1%	$^{+0.66\%}_{-0.79\%}$	$\pm 2.5\%$
CCDY(W^+)	30	-4.7%	$^{+2.5\%}_{-1.7\%}$	$\pm 3.2\%$
	150	-2.0%	$^{+0.5\%}_{-0.5\%}$	$\pm 2.1\%$
CCDY(W^-)	30	-5.0%	$^{+2.6\%}_{-1.6\%}$	$\pm 3.2\%$
	150	-2.1%	$^{+0.6\%}_{-0.5\%}$	$\pm 2.13\%$

J. Baglio, C. Duhr, B. Mistlberger, R. Szafron 2209.06138

$$\delta(\text{PDF-TH}) = \frac{1}{2}\,\frac{\left|\sigma^{\mathrm{NNLO}}(\mathrm{NNLO\ PDF}) - \sigma^{\mathrm{NNLO}}(\mathrm{NLO\ PDF})\right|}{\sigma^{\mathrm{NNLO}}(\mathrm{NNLO\ PDF})}$$

Towards N^{3}LO PDFs

Scale evolution of the PDFs

(Gribov,Lipatov 1972; Lipatov 1975; Altarelli,Parisi 1977; Dokshitzer 1977)

$$\mu^2 \frac{d}{d\mu^2} f_i\left(x,\mu^2\right) = \int_x^1 \frac{dy}{y} P_{ij}(\alpha_s, y)\, f_j\left(\frac{x}{y},\mu^2\right), \qquad i = g, u, d, s, \ldots$$

Flavour decomposition of the quark contribution

$$f^{\pm}_{\mathrm{NS},ik} = (f_i \pm f_{\bar{i}}) - (f_k \pm f_{\bar{k}}), \qquad i,k = u,d,s,\ldots$$
$$f_{\mathrm{S}} = \sum_i (f_i + f_{\bar{i}}), \qquad i = u,d,s,\ldots,$$

Perturbative expansion

$$P_{ij}(\alpha_s, x) = \underbrace{a\, P^{(0)}_{ij}}_{\text{LO}} + \underbrace{a^2\, P^{(1)}_{ij}}_{\text{NLO}} + \underbrace{a^3\, P^{(2)}_{ij}}_{\text{NNLO}} + \underbrace{a^4\, P^{(3)}_{ij}}_{\text{N}^3\text{LO}}, a = \frac{\alpha_s}{4\pi}$$

Recent progress at N^3LO

- Large-n_f limit (Gracey 1994, 1996; Davies,Vogt,Ruijl,Ueda,Vermaseren 2016)
- Flavour non-singlet: complete planar limit and approximate full QCD (Ruijl,Ueda,Vermaseren,Vogt 2017)
- Four Mellin moments of the splitting kernels (Moch,Ruijl,Ueda,Vermaseren,Vogt 2021)
- Approximate N^3LO PDF fits (McGowan,Cridge,Harland-Lang,Thorne 2022; Hekhorn, Magni 2023)
- Complete n_f^2 term in $P_{qq}^{(3)}$ (Gehrmann,von Manteuffel,Sotnikov,Yang 2023)

In this talk

- Theory framework to compute the Mellin moments of P_{ij}
- Results for $P_{qq}^{(3)}$ and $P_{qg}^{(3)}$ up to 10 moments.

Operators of leading twist

The Mellin moments of the PDFs are operator matrix elements (Collins,Soper 1981)

$$\langle H(P)|\mathcal{O}^{(N),\text{bare}}_{\underbrace{i;+,\dots,+}_{N\text{ times}}}|H(P)\rangle = \left(P^{+}\right)^{N} \int_0^1 dx\, x^{N-1}\, f_i^{\text{bare}}(x)$$

$|H(P)\rangle$ proton state of momentum P, $\mathcal{O}^{(N)}_{i;\mu_1\dots\mu_N}$ operators of leading twist

$$\mathcal{O}^{(N)}_{g;\mu_1\dots\mu_N} = \frac{1}{2}\,\mathcal{S}_T\left\{F^{a_1}_{\rho\mu_1}\,D^{a_1a_2}_{\mu_2}\,\dots\,D^{a_{N-2}a_{N-1}}_{\mu_{N-1}}\,F^{a_N;\rho}{}_{\mu_N}\right\},$$
$$\mathcal{O}^{(N)}_{q;\mu_1\dots\mu_N} = \frac{1}{2}\mathcal{S}_T\left\{\bar{\psi}_{i_1}\,\gamma_{\mu_1}\,D^{i_1i_2}_{\mu_2}\,\dots\,D^{i_{N-1}i_N}_{\mu_N}\,\psi_{i_N}\right\},$$
$$\mathcal{O}^{(N),\rho}_{\text{ns};\mu_1\dots\mu_N} = \frac{1}{2}\mathcal{S}_T\left\{\bar{\psi}_{i_1}\,(\lambda^{\rho})\,\gamma_{\mu_1}\,D^{i_1i_2}_{\mu_2}\,\dots\,D^{i_{N-1}i_N}_{\mu_N}\,\psi_{i_N}\right\},$$

$\lambda^{\rho} \to$ generator of $\mathrm{SU}(n_f)$.

$\mathcal{S}_T \to$ symmetrise over $\mu_1\dots\mu_N$ and remove trace terms.

Scale dependence upon renormalisation

Scale dependence given by a **matrix** of renormalisation constants

$$\mathcal{O}_i^{(N),\mathrm{ren}}(\mu^2) = Z_{ij}^{(N)}(\alpha_s,\mu^2)\,\mathcal{O}_j^{(N),\mathrm{bare}}$$

The anomalous dimensions of $\mathcal{O}_i^{(N),\mathrm{ren.}}$ control the evolution of the PDFs
(Gross,Wilczek 1974; Politzer,Georgi 1974)

$$\gamma_{ij}^{(N)} \equiv -\left(\mu^2 \frac{d}{d\mu^2} Z_{ik}^{(N)}\right) Z_{kj}^{-1} = -\int_0^1 dx\, x^{N-1}\, P_{ij}^{(N)}(x,\alpha_s).$$

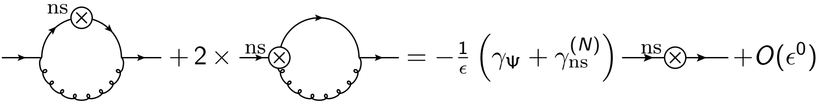
In minimal subtraction

$$\gamma_{ij}^{(N)} = a\,\frac{\partial}{\partial a} Z_{ij}^{(N)}\Big|_{\frac{1}{\epsilon}}$$

Computing the anomalous dimensions

Non-singlet case

$$\mathcal{O}_{\rm ns}^{(N),R}(\mu^2) = Z_{\rm ns}^{(N)}(\mu^2)\, \mathcal{O}_{\rm ns}^{(N),\text{bare}}$$



Computing the anomalous dimensions

Non-singlet case

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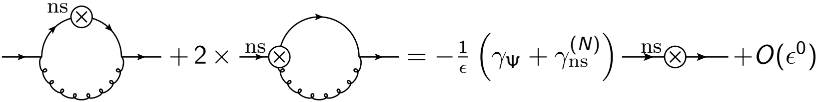
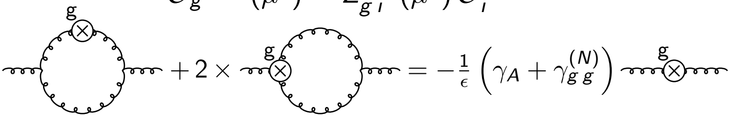


Singlet operators $\rightarrow$ the alien issue (Gross,Wilczek 1974)

$$\mathcal{O}_{g}^{(N),R}(\mu^2) = Z_{g\,i}^{(N)}(\mu^2)\,\mathcal{O}_{i}^{(N),\text{bare}}$$



$$-\sum_{i\neq g}\frac{\gamma_{g\,i}^{(N)}}{\epsilon}$$

$$+\,O\left(\epsilon^0\right)$$

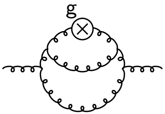
Alien operators

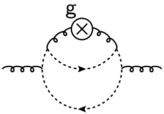
Multiloop renormalisation

Π_g^{ren}, i.e. renormalised 2-point functions with an insertion of $\mathcal{O}_g^{(N)}$, is finite

$$Z_3 \left(Z_{g\,i} \, \Pi_i \left(g_{\text{bare}}(g), \xi_{\text{bare}}(g, \xi) \right) \right) = \text{finite}$$

Diagrammatically





Multiloop renormalisation

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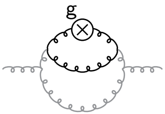
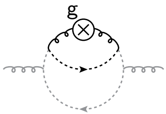
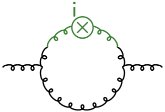
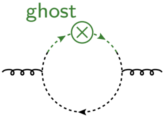




Alien operators, including **ghost** operators enter in subdivergences

$Z_{g\,\mathrm{i}}\,\cdot$ 

$Z_{g\,\mathrm{ghost}}\,\cdot$ 

Facts about aliens

Required aliens at 2-loop level (Dixon and Taylor 1974). Defining

$$\partial \equiv \partial_+ = \partial_\mu \Delta^\mu, \quad D = D_\mu \Delta^\mu, \quad A^a = A^a_\mu \Delta^\mu, \quad F^a_\nu = F^a_{\nu\mu}\Delta^\mu,$$

$$\mathcal{O}_A^{(N)} = \eta\, F^{a;\alpha} D_\alpha^{ab} \partial^{N-2} A^b - g\, f^{abc} F^a_\alpha \sum_{i=1}^{N-2} \kappa_i \partial^\alpha \left[\left(\partial^{i-1} A^b\right) \left(\partial^{N-2-i} A^c\right)\right] + O(g^2),$$

$$\mathcal{O}_c^{(N)} = -\eta\, \bar{c}^a \partial^N c^a - g\, f^{abc} \sum_{i=1}^{N-2} \eta_i(\eta, \kappa_i)\, \bar{c}^a \partial \left[\left(\partial^{N-2-i} A^b\right) \left(\partial^i c^c\right)\right] + O(g^2),$$

- η, κ_i chosen to cancel the divergences (Hamberg,van Neerven 1993)
- Joglekar and Lee (1976): generalisation of BRST implies that aliens are
  - Operators proportional to the equation of motion
  - BRST-exact operators

Reminder: BRST invariance of the Yang-Mills lagrangian

$$\mathcal{L} = \underbrace{-\frac{1}{4} F^{a;\mu\nu} F^a_{\mu\nu}}_{\mathcal{L}_0} + \underbrace{s \left[\bar{c}^a \left(\partial^\mu A^a_\mu - \frac{\xi_L}{2} b^a \right) \right]}_{\text{Gauge fixing + ghost}}$$

- $\mathcal{L}_0$ **invariant** under

$$\delta A^a_\mu = (D_\mu \omega)^a,$$

with ω^a scalar function.

- s is the BRST transformation obtained by $\omega^a \to c^a$

$$s A^a_\mu = (D_\mu c)^a,$$

c^a, $\bar{c}^a$ and b^a transform such that

$$s^2(\text{anything}) = 0.$$

Introducing leading twist operators

$$\widetilde{\mathcal{L}} = \underbrace{\mathcal{L}_0 + c_g\, \mathcal{O}_g^{(N)} + \mathcal{O}_{\text{EOM}}^{(N)}}_{\mathcal{L}_{\text{GGI}}} + \mathbf{s'} \underbrace{\left[\bar{c}^a \left(\partial^\mu A_\mu^a - \frac{\xi_L}{2} b^a \right) \right]}_{\text{Gauge fixing + ghost}}$$

$\mathcal{O}_{\text{EOM}}^{(N)}$ takes care of *gluonic* divergent (sub)diagrams

$$\mathcal{O}_{\text{EOM}}^{(N)} = (D^\mu F_\mu)^a \Big[\underbrace{\eta\, \partial^{N-2} A^a}_{\mathcal{O}_g^{I}} + g f^{a a_1 a_2} \sum_{i_1+i_2=N-3} \underbrace{\kappa_{i_1 i_2} (\partial^{i_1} A^{a_1})(\partial^{i_2} A^{a_2})}_{\mathcal{O}_g^{II}}$$

$$+ g^2 \sum_{\substack{i_1+i_2+i_3 \\ N-4}} \underbrace{\left(\kappa^{(1)}_{i_1 i_2 i_3} f^{a a_1 z} f^{a_2 a_3 z} + \kappa^{(2)}_{i_1 i_2 i_3} d_4^{a a_1 a_2 a_3} + \kappa^{(3)}_{i_1 i_2 i_3} d_{\widehat{4ff}}^{a a_1 a_2 a_3} \right) (\partial^{i_1} A^{a_1})..(\partial^{i_3} A^{a_3})}_{\mathcal{O}_g^{III}}$$

$$+ g^3 \sum_{\substack{i_1+..+i_4 \\ N-5}} \underbrace{\left(\kappa^{(1)}_{i_1 \dots i_4} (f\, f\, f)^{a a_1 a_2 a_3 a_4} + \kappa^{(2)}_{i_1 \dots i_4} d_{4f}^{a a_1 a_2 a_3 a_4} \right) (\partial^{i_1} A^{a_1})..(\partial^{i_4} A^{a_4})}_{\mathcal{O}_g^{IV}} + O(g^4) \Big]$$

Generalised Gauge and BRST transformations

$\mathcal{L}_{\mathrm{GGI}}$ is invariant under *generalised gauge transformations*. Given

$$\mathcal{O}^{(N)}_{\mathrm{EOM}} = (D^{\mu} F_{\mu\nu})^{a}\, \mathcal{G}^{a}_{\nu}(A^{b}, \partial A^{b}, \partial^{2} A^{b}, \dots),$$

the generalised transformation $A^{a}_{\mu} \to A^{a}_{\mu} + \delta' A^{a}_{\mu}$ is shown to be

$$\boxed{\delta' A^{a}_{\mu} = \delta A^{a}_{\mu} - \delta \mathcal{G}^{a}_{\mu} + g\, f^{abc}\, \mathcal{G}^{b}_{\mu}\, \omega^{c}}$$

This defines immediately the **generalised BRST transformations**

$$\boxed{s'(A^{a}_{\mu}) = s(A^{a}_{\mu}) - s(\mathcal{G}^{a}_{\mu}) + g\, f^{abc}\, \mathcal{G}^{b}_{\mu}\, c^{c} \equiv s(A^{a}_{\mu}) + s_{\Delta}(A^{a}_{\mu})}$$

such that $s'^{2}(\text{anything}) = 0$.

Generalised BRST symmetry at work

The general ansatz of $\mathcal{O}^{(N)}_{\text{EOM}}$ fixes the structure of the aliens

- Example: first moment $N = 2$

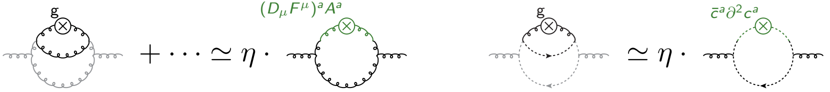
$$\mathcal{G}^a_\mu = \eta \Delta_\mu \, A^a, \qquad s_\Delta \left[\bar{c}^a \partial^\rho A^a_\rho\right] = -\bar{c}^a \partial \left[-\eta \,(Dc)^a + \eta g f^{abc} A^b c^c\right],$$

$$\mathcal{O}^I_g = \eta \,(D^\nu F_\nu)^a A^a, \qquad \mathcal{O}^I_c = \eta \;\bar{c}^a \partial^2 c^a.$$

There is a **single** alien operator

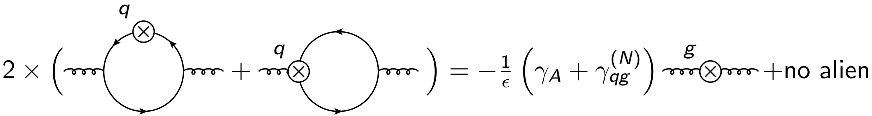
$$\mathcal{O}^I_A = \eta \left[(D^\nu F_\nu)^a A^a + \bar{c}^a \partial^2 c^a\right].$$

η mixes the physical operator into **gluon** and **ghost** aliens, Z_{galien}.

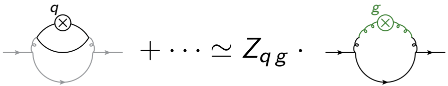


Quark operators

$\mathcal{O}_q^{(N)}$ has a much simpler mixing structure



The quark 2-point functions renormalise easily. Aliens occur only at 3 loops



Note: the aliens must include now also a **quark** contribution in the EOM.

Pure singlet: aliens

$$O_g^I = \eta\,(D^\nu F_\nu)^a \partial^{N-2} A^a, \qquad O_g^{II} = g\, f^{abc}\,(D^\nu F_\nu)^a \sum_{i_1+i_2=N-3} \kappa_{i_1 i_2} (\partial^{i_1} A^b)(\partial^{i_2} A^c),$$

$$O_q^I = \eta g (\bar\psi \Delta\!\!\!/\, t^a \psi) \partial^{N-2} A^a, \qquad O_q^{II} = g^2 (\bar\psi \Delta\!\!\!/\, t^a \psi) \sum_{i_1+i_2=N-3} \kappa_{i_1 i_2} (\partial^{i_1} A^b)(\partial^{i_2} A^c),$$

$$O_c^I = \eta \bar c^a \partial^N c^a, \qquad O_c^{II} = -(\partial \bar c^a) \sum_{i_1+i_2=N-3} \eta_{i_1 i_2} (\partial^{i_1} A^b)(\partial^{i_2+1} c^c),$$

BRST and antiBRST symmetry impose relations

$$\eta_{ij} = 2\kappa_{ij} + \eta \begin{pmatrix} N-2 \\ i \end{pmatrix} = -\sum_{s=0}^{i} (-1)^{s+j} \begin{pmatrix} s+j \\ s \end{pmatrix} \eta_{i-s,j+s}$$

The mixing constant are found to factorise up to 2 loops

$$\kappa_{ij} = \frac{\eta(N)}{8} \left[(-1)^i - 3 \begin{pmatrix} N-2 \\ i \end{pmatrix} + 3 \begin{pmatrix} N-2 \\ i+1 \end{pmatrix} \right]$$

η renormalise ghost 2pt functions. Agreement with (Gehmann,von Manteuffel,Yang 2023).

Pure singlet anomalous dimensions

The required 2pt functions computed with FORCER (Ruijl,Ueda,Vermaseren 2017) for moments up to $N = 20$

$$\begin{aligned}
\gamma_{\rm ps}^{(3)}(N{=}2) &= -691.5937093\, n_f + 84.77398149\, n_f^2 + 4.466956849\, n_f^3\,, \\
\gamma_{\rm ps}^{(3)}(N{=}4) &= -109.3302335\, n_f + 8.776885259\, n_f^2 + 0.306077137\, n_f^3\,, \\
\gamma_{\rm ps}^{(3)}(N{=}6) &= -46.03061374\, n_f + 4.744075766\, n_f^2 + 0.042548957\, n_f^3\,, \\
&\dots \\
\gamma_{\rm ps}^{(3)}(N{=}20) &= -0.442681568\, n_f + 0.805745333\, n_f^2 - 0.020918264\, n_f^3\,.
\end{aligned}$$

- Agreement with results up to $N = 8$ (Moch,Ruijl,Ueda,Vermasersen,Vogt 2021), extended up to $N = 12$ (Moch,Ruijl,Ueda,Vermaseren,Vogt to appear).
- Leading terms in the large-n_f limit agree with (Davies,Moch,Ruijl,Ueda,Vermaseren,Vogt 2016)
- Terms n_f^2 agree with (Gehrmann,von Manteuffel,Sotnikov,Yang 2023)

Approximations of $P_{qq}^{(3)}(x)$ (I)

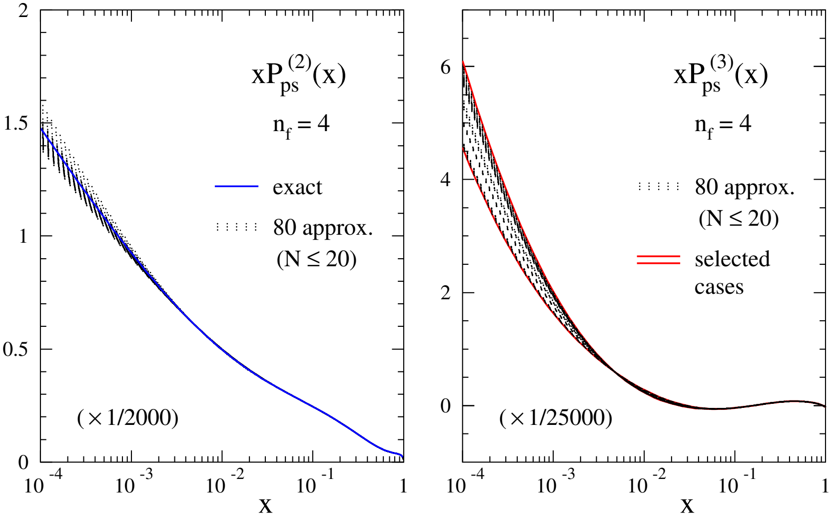
Following (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017): approximations of the x-space results from 80 trial functions matching

- Moments up to $N = 20$
- Small-x limits
  - Coefficients of $\frac{\log^2 x}{x}$ (Catani,Hautmann 1994)
  - Coefficients of $\log^k x$ with $k = 6, 5, 4$ (Davies,Kom,Moch,Vogt 2022)
- Large-x limits
  - Coefficients of $(1-x)^j \log^k(1-x)$ with $k = 4, 3$ and $\forall j \geq 1$ (Soar,Moch,Vermaseren,Vogt 2010)

while unknown coefficients are fitted

- Small-x: $\frac{\log x}{x}$, $1/x$, $\log^k x$ with $k = 3, 2, 1$
- Large-x: $(1-x)\log^k x$, $k = 2, 1$

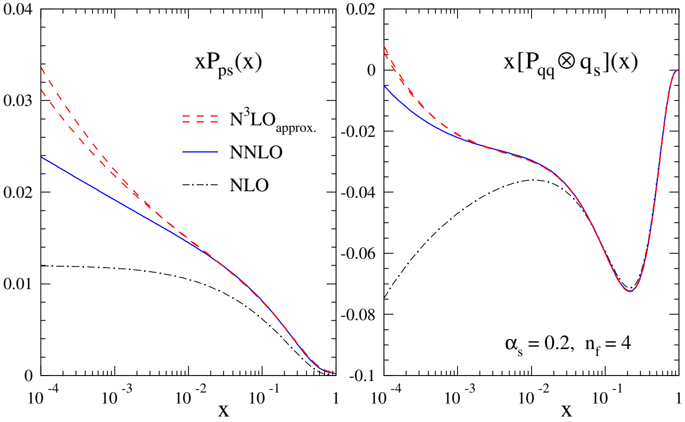
Approximations of $P_{qq}^{(3)}(x)$ (II)



Impact of the N^{3}LO corrections

$P_{qq}(x)$ including approximate N^{3}LO corrections for fixed $\alpha_s = 0.2$ (left).
$P_{qq} \otimes f_S$ (right), where

$$x\, f_S(x) = 0.6\, x^{-0.3}\, (1-x)^{3.5}\, (1 + 5.0x^{0.8})$$



Results for $\gamma_{qg}^{(3)}$

Moments up to $N = 20$ of $\gamma_{qg}^{(3)}$ were computed in the same approach

$$
\begin{aligned}
\gamma_{\mathrm{qg}}^{(3)}(N=2) &= -654.4627782\, n_f + 245.6106197\, n_f^2 - 0.924990969\, n_f^3 \,, \\
\gamma_{\mathrm{qg}}^{(3)}(N=4) &= 290.3110686\, n_f - 76.51672403\, n_f^2 - 4.911625629\, n_f^3 \,, \\
\gamma_{\mathrm{qg}}^{(3)}(N=6) &= 335.8008046\, n_f - 124.5710225\, n_f^2 - 4.193871425\, n_f^3 \,, \\
&\dots \\
\gamma_{\mathrm{qg}}^{(3)}(N=20) &= 52.24329555\, n_f - 109.3424891\, n_f^2 - 2.153153725\, n_f^3 \,.
\end{aligned}
$$

- Agreement with moments up to $N = 8$ computed in (Moch,Ruijl,Ueda,Vermaseren,Vogt 2021), extended to $N = 10$ (Moch,Ruijl,Ueda,Vermaseren,Vogt to appear)
- Agreement with the large-n_f limit (Davies,Vogt,Ruijl,Ueda,Vermaseren 2016)

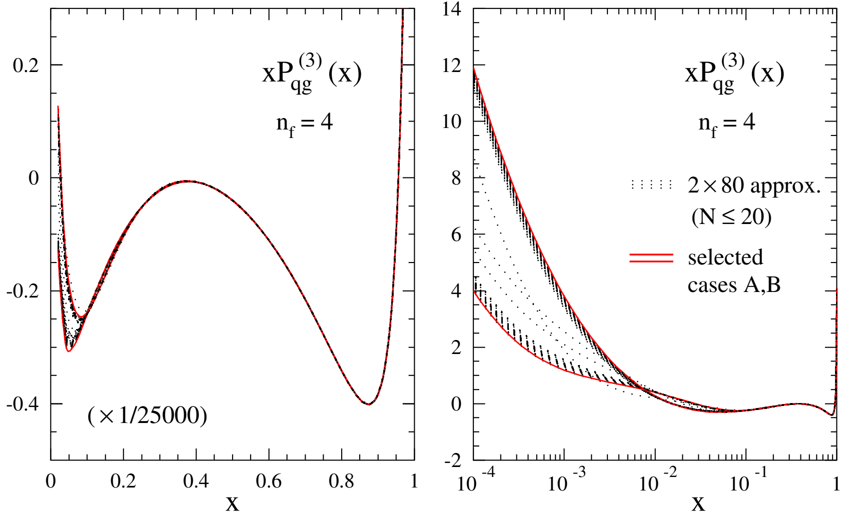
Approximations of $P_{qg}^{(3)}$ (I)

The trial functions for $P_{qg}^{(3)}$ are constrained by the limits at

- Small-x:
  - Coefficients of $\frac{\log^2 x}{x}$ (Catani,Hautmann 1994)
  - Coefficients of $\log^k x$ with $k = 6, 5, 4$ (Davies,Kom,Moch,Vogt 2022)
- Large-x:
  - Coefficients of $\log^k(1-x)$ with $k = 6, 5, 4$ (Soar,Moch,Vermaseren,Vogt 2010; Vogt 2010; Almasy,Soar,Vogt 2011)
  - Coefficients of $(1-x)\log^k(1-x)$ with $k = 6, 5, 4$ (Soar,Moch,Vermaseren,Vogt 2010)

The coefficients of $\log^k(1-x)$ with $k = 1, 2, 3$ are now **unknown**
$\rightarrow$ uncertainties are larger compared to P_{qq}.

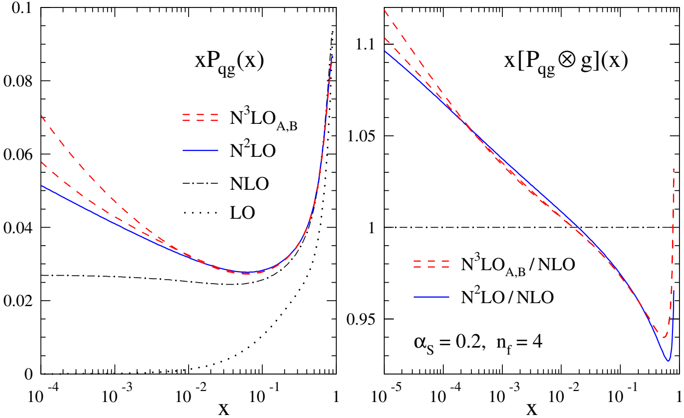
Approximations of $P_{qg}^{(3)}$ (II)



Impact of the N^{3}LO corrections

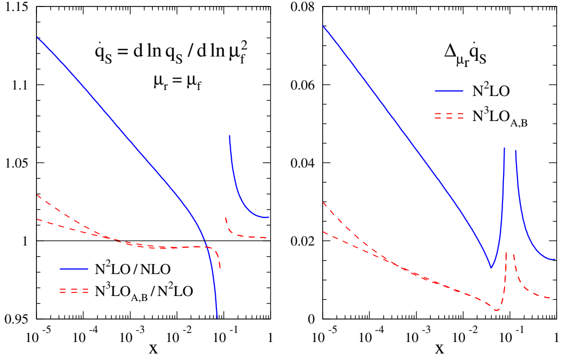
$P_{qg}(x)$ including approximate N^{3}LO corrections fixing $\alpha_s = 0.2$ (left).
$P_{qg} \otimes f_g$ (right), where

$$x\, f_g(x) = 1.6\, x^{-0.3}\,(1-x)^{4.5}\,(1-0.6x^{0.3})$$



Scale evolution of the quark distribution

Using approximate P_{qq} and P_{qg} one derives $\mu_f^2 \frac{d}{d\mu_f^2} f_S \equiv \dot{q}_S = P_{qq} \otimes f_S + P_{qg} \otimes f_g$



The stability under variations of the renormalisation scale are quantified via

$$\Delta_{\mu_r} \dot{q}_S = \frac{1}{2} \frac{\max[\dot{q}_S(\mu_r^2 = \lambda \mu_f^2)] - \min[\dot{q}_S(\mu_r^2 = \lambda \mu_f^2)]}{\text{average}[\dot{q}_S(\mu_r^2 = \lambda \mu_f^2)]}, \qquad \lambda = \frac{1}{4} \dots 4$$

Conclusions

Theory

- The moments of $P_{ij}(x)$ are computed *efficiently* from the renormalisation of 2-point correlators **provided** we take into account the mixing with alien operators.

- A generalised BRST symmetry fixes the structure of the aliens.
  - Classification in towers of contributions: O_k^I, O_k^{II}, O_k^{III}, . . . with $k = g, c, q$
    O_k^I include 2-, 3- and 4-point vertices,
    O_k^{II} include 3-, 4- and 5-point vertices, . . .
  - The renormalisation of quark operators requires few classes of terms.

Results

- The moments of $P_{qq}^{(3)}$ and $P_{qg}^{(3)}$ were computed up to $N = 20$.
- The approximate expressions of $P_{qq}(x)$ and $P_{qg}(x)$ at N^{3}LO are characterised by
  - Small uncertainties at large-x, growing at small-x. E.g. $\delta P_{qg}(x = 10^{-4}) \sim \mathcal{O}(10\%)$
  - The convolution with PDFs dampens the uncertainty at small-x.
- Effect of the N^{3}LO corrections to $\mu^2 \frac{d}{d\mu^2} f_S(x, \mu^2) = \dot{q}_S$

$$\delta_{\mathrm{N^3LO}} \dot{q}_S(x = 10^{-4}) \lesssim 1\%.$$

- Renormalisation scale uncertainties are small, e.g. $x = 10^{-4}$ 2% vs compared to 5% at NNLO.

Outlook

- Ongoing work to compute the moments of $P_{gq}^{(3)}(x)$ and $P_{gg}^{(3)}(x)$.
- Can we obtain the exact expressions? This requires results for all N.
  Only coefficient of Riemann-ζ numbers were reconstructed from the available moments
  - $\gamma_{qq}(N) \rightarrow$ Coefficients of ζ_5, ζ_4 and of $\zeta_3 n_f \frac{d_{RR}}{n_c}$ and $\zeta_3 {n_f}^2 C_F^2$
  - $\gamma_{qg}(N) \rightarrow$ Coefficients of ζ_5 and $\zeta_3 n_f \frac{d_{RA}}{n_A}$ and $\zeta_3 n_f^2 \frac{d_{RR}}{n_A}$
  - These do not translate to the same coefficients of ζ in x-space.
- All fits would improve significantly with knowledge of the terms $\sim \frac{\log x}{x}$ in $P_{ij}^{(3)}$.
- Different methods to attack the all-N problem.

Thank you!