Automatic computation of Soft anomalous dimension matrices for the Heavy Quark Hadroproduction

Based on hep-ph/2206.10977 + work in progress

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Bakar Chargeishvili, Maria Vittoria Garzelli, Sven-Olaf Moch

II. Institut für Theoretische Physik
Universität Hamburg

Theory Challenges in the Precision Era of the Large Hadron Collider
Galileo Galilei Institute, Firenze

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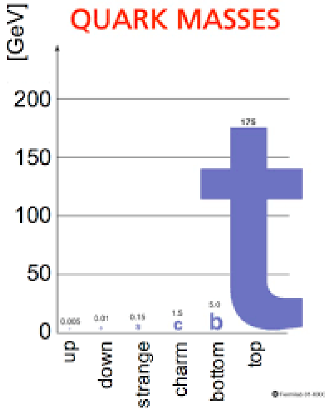
Motivation

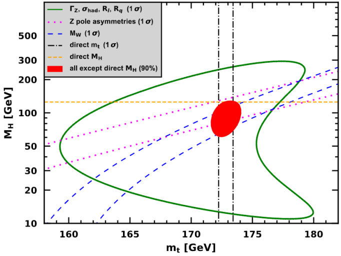
Top quarks are special

- Pointlike particle with mass of gold atom, 35x heavier than b-quark $\rightarrow$ Why?
- Current mass estimate (pole mass, from cross-section measurement):
  - $m_{pole} = 173.1 \pm 0.9$ GeV
- Participates in all type of interactions and has a strongest coupling with the Higgs boson.
- Higgs vev: $v/\sqrt{2} \simeq 175$ GeV $\rightarrow$ A role in EW symmetry breaking or coincidence?
- Top = the only "free" quark $\rightarrow$ no bound states.
  - Spin/polarization passed on decay products without dilution $\rightarrow$ direct access to quark properties
- Events containing top quarks are backgrounds to new physics searches



Having precise measurement of the top quark properties is important!

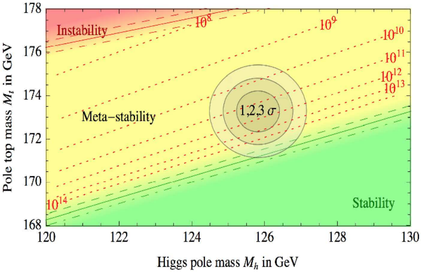
- The mass of the top is an important input to the global electroweak fits, which assess the self-consistancy within SM.



[PDG 2022]

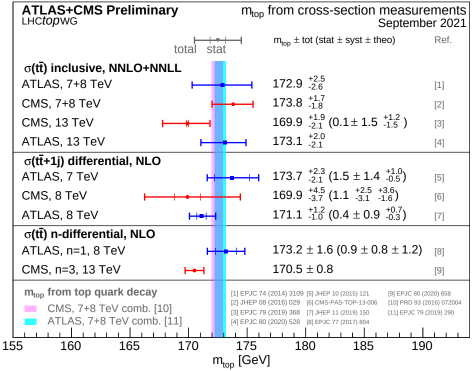
- Top-quark mass is related together with the Higgs-boson mass to the vacuum stability of the SM.

Precise value of the top quark mass might determine the fate of the universe!



[Alekhin, Djouadi, Moch, B716 (2012) 214]

The indirect top-quark mass measurement are becoming a clear competitor of the direct methods.



[ATL-PHYS-PUB-2022-016]

- It has been shown that in case of tt+jet, the observable $\rho_s = 2m_t/\sqrt{s_{t\bar{t}j}}$ shows very good sensitivity on the top quark mass [hep-ph/1303.6415]
- Latest studies using tt+jet cross-section use NLO (NLO+PS) predictions for the mass determination
- Elevating the accuracy of the theory predictions beyond NLO would improve the current m_t estimates

Challenges of the NNLO calculation

Fully accurate NNLO computation for $2 \to 3$ process with partons present in both initial and final states imposes a serious technical challenge:

- 2-loop master integrals for $2 \to 3$ kinematics with massive partons are mostly unknown.
- No easy way to deal with soft and collinear singularities at NNLO level. Multiple approaches are under development:
  - Antenna subtraction [Gehrmann et al.]
  - CoLoRFul subtraction [Del Duca et al.]
  - Sector-improved residue subtraction [Czakon et al.]
  - Local analytic sector subtraction [Magnea et al.]
  - qT-slicing [Catani et al.]
  - ...
- Automation of these procedures will take some time...

Meanwhile, it is possible to work in the threshold limit of the $t\bar{t}$+jet production and by summing the large logarithms to all orders of the perturbation theory, estimate the NNLO correction.

The problem of large logarithms

- The perturbative expansion of observable relies on the smallness of α_s and expansion coefficients c_k:

$$\hat{O}(\alpha_s) = \hat{O}_0 \left[1 + \frac{\alpha_s}{2\pi} c_1 + \left(\frac{\alpha_s}{2\pi} \right)^2 c_2 + \dots \right]$$

- At higher orders the cross-section is given as sum of virtual and real contributions
- In dimensional regularization $(D = 4 - \epsilon, \epsilon < 0)$ one finds cancellation of the form

$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + \underbrace{\left(Q^2\right)^{\epsilon} \int_0^{m_{\text{jet}}^2} \frac{dk^2}{\left(k^2\right)^{1+\epsilon}}}_{\text{real}} \implies \ln\left(m_{\text{jet}}^2 / Q^2\right)$$

- When $m_{\text{jet}}^2 \ll Q^2$ these logs can become quite large
- Parametrized using the threshold variable $z = \frac{M}{\hat{s}}$, at the n-th order of the perturbation theory there are terms proportional to:

$$-\frac{\alpha_s^n}{n!} \left[\frac{\ln^m (1-z)^{-1}}{1-z} \right]_+, \qquad m \leq 2n - 1$$

- Action of $-\frac{\alpha_s^n}{n!}\left[\frac{\ln^m(1-z)}{1-z}\right]_+$ on a smooth function $\mathcal{F}$:

$$-\frac{\alpha_s^n}{n!}\int_0^1 dz\frac{\mathcal{F}(z)-\mathcal{F}(1)}{1-z}\ln^{2n-1}\left((1-z)^{-1}\right)=\frac{\alpha_s^n}{n!}\int_0^1 dz\mathcal{F}'(1)\ln^{2n-1}\left((1-z)^{-1}\right)+\dots$$
$$\sim\frac{\alpha_s^n}{n!}(2n-1)!+\dots$$

- At n-th order these contributions grow faster than $n!$ and spoil the convergence of the perturbative series

- Solution $\Longrightarrow$ Resummation

- Sum up these logs order-by-order and rearrange them into exponentials:

$$\frac{\sigma^N}{\sigma^N_B} = 1 + \sum_{n=1}^{\infty} \alpha_s^n \sum_{m=0}^{2n} \tilde{c}_{n,m} L^m = \exp\left\{ \sum_{n=1}^{\infty} \alpha_s^n \sum_{m=1}^{n+1} c_{n,m} L^m \right\} \underbrace{C(\alpha_s)}_{\text{constants}}$$

$$= \exp\left[\text{LL} + \text{NLL} + \text{NNLL} + \ldots\right] C(\alpha_s)$$

	LL	NLL	NNLL	...
LO	1			
NLO	$\alpha_s L^2$	$\alpha_s L$		
NNLO	$\alpha_s^2 L^4$	$\alpha_s^2 L^{\{3,2\}}$	$\alpha_s^2 L$	
$\vdots$	...	...	...	$\ddots$

- In the threshold limit the partonic cross-section can be factorized as [Collins, Soper, Sterman, 1983]:

$$\hat{\sigma} = \psi_i \otimes \psi_j \otimes H \otimes S \otimes J$$

- ψ_i - initial state jet functions, modeling the initial state collinear radiation
- S - soft gluon exchange
- H - hard amplitude matrix
- J - final state jet function

General strategy:

- Evaluate each of these integrals perturbatively at scales at which they are free of large logs
- By means of the RGEs evolve everything back to the common scale
- Expand the result to the desired fixed order
- The strategy has been successfully applied on $t\bar{t}$ production processes with associated bosons: $t\bar{t} + W/Z/H$ [Kulesza et al.], [Broggio et al.].

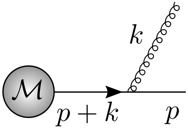
In case of $t\bar{t}$+jet complications come from the:

(i) Involved calculations because of richer color structure

(ii) Appearance of the final state jet, which has highly nontrivial soft singularity structure

Currently we focus on calculation of the soft function.

- Soft emissions factorize:

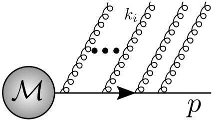


$$= \mathcal{M}\frac{\not{p}+\not{k}}{(p+k)^2}\gamma^\mu T^a u(p) = \mathcal{M}\frac{\not{p}+\not{k}}{2p\cdot k}\gamma^\mu T^a u(p) \overset{k\ll p}{\approx} \mathcal{M}\frac{p^\mu}{p\cdot k}T^a u(p)$$

Emission of multiple gluons:

- Define a Wilson line:

$$\Phi_\beta^{(f)}(\lambda_2,\lambda_1;x) = \mathcal{P}\exp\left[-ig\mu^\epsilon\int_{\lambda_1}^{\lambda_2} d\eta\beta\cdot A^{(f)}(\eta\beta+x)\right]$$



$$\sim \mathcal{M}\langle 0|\Phi_\beta(\infty,0)|0\rangle u(p)$$

$$\Phi_\beta(\infty,0) = 1 + g\mu^\epsilon\int\frac{d^Dk}{(2\pi)^D}\frac{\beta^\mu}{k\cdot\beta}\tilde{A}_\mu(k) + \frac{g^2\mu^{2\epsilon}}{2}\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{\beta^\mu}{k_1\cdot\beta}\frac{\beta^\nu}{k_2\cdot v}\tilde{A}_\mu(k_1)\tilde{A}_\nu(k_2) + \ldots$$

- For the process $a(\beta_a)b(\beta_b) \to 1(\beta_1)2(\beta_2)3(\beta_3)$ define a eikonal nonlocal operator ω_I:

$$\omega_I^{(f)}(x)_{\{c_k\}} = \sum_{d_i} \Phi_{\beta_3}^{f_3}(\infty, 0; x)_{c_3,d_3} \Phi_{\beta_2}^{f_2}(\infty, 0; x)_{c_2,d_2} \Phi_{\beta_1}^{f_1}(\infty, 0; x)_{c_1,d_1} \left(c_I^{(f)}\right)_{d_3 d_2 d_1, d_b d_a}$$
$$\times \Phi_{\beta_a}^{f_a}(0, -\infty; x)_{d_a,c_a} \Phi_{\beta_b}^{f_b}(0, -\infty; x)_{d_b,c_b}$$

- Eikonal cross-section is given as:

$$\sigma_{LI}^{(f),\mathrm{eik}}(\alpha_s, \epsilon) = \sum_{\xi} \delta(w - w(\xi)) \times \left\langle 0 \left| \overline{T} \left[\left(\omega_L^{(f)}(0) \right)^{\dagger}_{\{b_i\}} \right] \right| \xi \right\rangle \left\langle \xi \left| T \left[\omega_I^{(f)}(0)_{\{b_i\}} \right] \right| 0 \right\rangle$$

- Define the soft function as the part of this cross-section which is free of collinear divergences

$$\sigma_{LI}^{(f),eik^N} = S_{JI}^N j_a^N j_b^N j_1^N j_2^N j_3^N$$

Evolution equation for the soft function

- Because the soft matrix is defined as a product of two operators it has to be renormalized multiplicatively:

$$S_{LI}^{(f)^{(B)}} = \left(Z_S^{(f)^\dagger}\right)_{LB} S_{BA}^{(f)} \left(Z_S^{(f)}\right)_{AI} \tag{1}$$

- Derive RGE:

$$\mu \frac{d}{d\mu} S_{LI}^{(f)} = \left(\mu \frac{\partial}{\partial \mu} + \beta\left(\alpha_s\right) \frac{\partial}{\partial \alpha_s}\right) S_{LI}^{(f)} = -\left(\Gamma_S^{(f)}\right)^\dagger_{LB} S_{BI}^{(f)} - S_{LA}^{(f)} \left(\Gamma_S^{(f)}\right)_{AI}, \tag{2}$$

- where at 1-loop level:

$$\left(\Gamma_S^{(f)}\right)_{LI}(g) = -\alpha_s \frac{\partial}{\partial \alpha_s} \mathrm{Res}_{\epsilon \to 0} \left(Z_S^{(f)}\right)_{LI}(g, \epsilon) \tag{3}$$

- Z_S are UV-divergent parts of the eikonal amplitudes. Solution of eq. (2):

$$S(\mu) = \bar{\mathcal{P}} \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma_S^\dagger(\alpha_s(\mu'^2))\right] S(\mu_0) \mathcal{P} \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma_S(\alpha_s(\mu'^2))\right] \tag{4}$$

The strategy of choosing the color basis is explained in [Sjödahl 2008], here we quote the result:

gg-channel

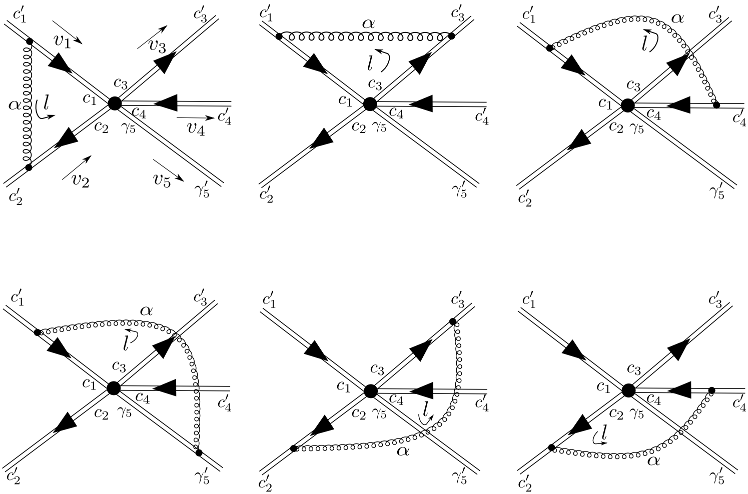
$$\mathbf{c}^1_{abcde} = t^e_{\ cd}\delta_{ab}$$
$$\mathbf{c}^2_{abcde} = if_{abe}\delta_{cd}$$
$$\mathbf{c}^3_{abcde} = id_{abe}\delta_{cd}$$
$$\mathbf{c}^4_{abcde} = if_{abn}if_{men}t^m_{\ cd}$$
$$\mathbf{c}^5_{abcde} = d_{abn}if_{men}t^m_{\ cd}$$
$$\mathbf{c}^6_{abcde} = if_{abn}d_{men}t^m_{\ cd}$$
$$\mathbf{c}^7_{abcde} = d_{abn}d_{men}t^m_{\ cd}$$
$$\mathbf{c}^8_{abcde} = P^{10+\overline{10}}_{abme}t^m_{\ cd}$$
$$\mathbf{c}^9_{abcde} = P^{10-\overline{10}}_{abme}t^m_{\ cd}$$
$$\mathbf{c}^{10}_{abcde} = -P^{27}_{abme}t^m_{\ cd}$$
$$\mathbf{c}^{11}_{abcde} = P^{0}_{abme}t^m_{\ cd}$$

$q\bar{q}$-channel

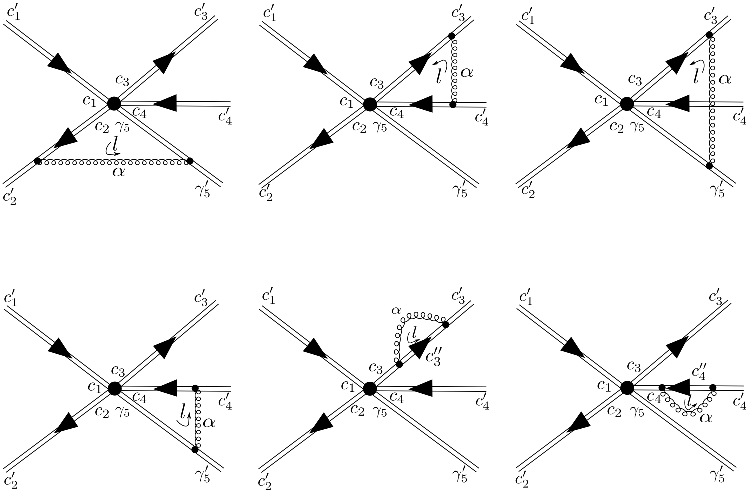
$$\mathbf{c}^1_{abcde} = t^e_{\ cd}\delta_{ab}$$
$$\mathbf{c}^2_{abcde} = t^e_{\ ab}\delta_{cd}$$
$$\mathbf{c}^3_{abcde} = t^m_{\ ba}t^n_{\ cd}if_{mne}$$
$$\mathbf{c}^4_{abcde} = t^m_{\ ba}t^n_{\ cd}d_{mne}$$

P^i are projectors: $P^i_{ABmn}P^j_{mnCD} = \delta_{ij}P^i_{ABCD}$

Wilson webs


c'_1 c'_3 v_1 v_3 c_3 c_1 α l c_4 c'_4 v_4 c_2 γ_5 v_2 v_5 c'_2 γ'_5
c'_1 α c'_3 l c_3 c_1 c_4 c'_4 c_2 γ_5 c'_2 γ'_5
c'_1 α c'_3 l c_3 c_1 c_4 c'_4 c_2 γ_5 c'_2 γ'_5
c'_1 α c'_3 l c_3 c_1 c_4 c'_4 c_2 γ_5 c'_2 γ'_5
c'_1 c'_3 c_3 c_1 c_4 c'_4 c_2 γ_5 l α c'_2 γ'_5
c'_1 c'_3 c_3 c_1 c_4 c'_4 c_2 γ_5 l α c'_2 γ'_5

Wilson webs



Eikonal amplitudes

Connection $(i-j)$	Kinematical part (κ_{ij}) before integration	Color part $(\mathscr{F}_{ij})$
$1-2$	$\frac{v_1^\mu}{-v_1\cdot l+i\epsilon}\,\frac{-v_2^\nu}{v_2\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_1c_1'}\,T^\alpha_{c_2'c_2}\,\delta_{c_3c_3'}\delta_{c_4c_4'}\delta_{\gamma_5\gamma_5'}$
$1-3$	$\frac{v_1^\mu}{v_1\cdot l+i\epsilon}\,\frac{v_3^\nu}{v_3\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_1c_1'}\,T^\alpha_{c_3'c_3}\,\delta_{c_2c_2'}\delta_{c_4c_4'}\delta_{\gamma_5\gamma_5'}$
$1-4$	$\frac{v_1^\mu}{v_1\cdot l+i\epsilon}\,\frac{-v_4^\nu}{v_4\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_1c_1'}\,T^\alpha_{c_4c_4'}\,\delta_{c_2c_2'}\delta_{c_3c_3'}\delta_{\gamma_5\gamma_5'}$
$1-5$	$\frac{v_1^\mu}{v_1\cdot l+i\epsilon}\,\frac{-v_5^\nu}{v_5\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_1c_1'}(-if^{\alpha\gamma_5\gamma_5'})\delta_{c_2c_2'}\delta_{c_3c_3'}\delta_{c_4c_4'}$
$2-3$	$\frac{-v_2^\mu}{-v_2\cdot l+i\epsilon}\,\frac{v_3^\nu}{-v_3\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_2'c_2}\,T^\alpha_{c_3'c_3}\,\delta_{c_1c_1'}\delta_{c_4c_4'}\delta_{\gamma_5\gamma_5'}$
$2-4$	$\frac{-v_2^\mu}{-v_2\cdot l+i\epsilon}\,\frac{-v_4^\nu}{-v_4\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_2'c_2}\,T^\alpha_{c_4c_4'}\,\delta_{c_1c_1'}\delta_{c_3c_3'}\delta_{\gamma_5\gamma_5'}$
$2-5$	$\frac{-v_2^\mu}{-v_2\cdot l+i\epsilon}\,\frac{v_5^\nu}{-v_5\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_2'c_2}(-if^{\alpha\gamma_5'\gamma_5})\delta_{c_1c_1'}\delta_{c_3c_3'}\delta_{c_4'c_4}$
$3-4$	$\frac{v_3^\mu}{-v_3\cdot l+i\epsilon}\,\frac{-v_4^\nu}{v_4\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_3'c_3}\,T^\alpha_{c_4c_4'}\,\delta_{c_1c_1'}\delta_{c_2c_2'}\delta_{\gamma_5\gamma_5'}$
$3-5$	$\frac{v_3^\mu}{-v_3\cdot l+i\epsilon}\,\frac{-v_5^\nu}{v_5\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_3'c_3}(-if^{\alpha\gamma_5\gamma_5'})\delta_{c_1c_1'}\delta_{c_2c_2'}\delta_{c_4c_4'}$
$4-5$	$\frac{v_4^\mu}{-v_4\cdot l+i\epsilon}\,\frac{-v_5^\nu}{v_5\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_4c_4'}(-if^{\alpha\gamma_5\gamma_5'})\delta_{\gamma_1\gamma_1'}\delta_{\gamma_2\gamma_2'}\delta_{c_3c_3'}$
$3-3$	$\frac{v_3^\mu}{v_3\cdot l+i\epsilon}\,\frac{v_3^\nu}{-v_3\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_3''c_3}\,T^\alpha_{c_3'c_3''}\,\delta_{c_1c_1'}\delta_{c_2c_2'}\delta_{c_4c_4'}\delta_{\gamma_5\gamma_5'}$
$4-4$	$\frac{-v_4^\mu}{v_4\cdot l+i\epsilon}\,\frac{-v_4^\nu}{-v_4\cdot l+i\epsilon}\,\frac{N_{\mu\nu}(l)}{l^2+i\epsilon}$	$T^\alpha_{c_4''c_4}\,T^\alpha_{c_4'c_4''}\,\delta_{c_1c_1'}\delta_{c_2c_2'}\delta_{c_3c_3'}\delta_{\gamma_5\gamma_5'}$

The gluon propagator in a general axial gauge is given as:

$$N^{\mu\nu}(k) = g^{\mu\nu} - \frac{n^\mu k^\nu + n^\nu k^\mu}{n \cdot k} + n^2 \frac{k^\mu k^\nu}{(n \cdot k)^2} \tag{5}$$

To deal with the unphysical singularity introduced by the axial gauge we use the principal value prescription [Leibbrandt 1987]:

$$\frac{\mathcal{P}}{(l \cdot n)^\beta} = \frac{1}{2}\left(\frac{1}{(l \cdot n + i\epsilon)} + (-1)^\beta \frac{1}{(-l \cdot n + i\epsilon)}\right) \tag{6}$$

As a result, each integral over the kinematical part can be reduced to the following form:

$$\begin{aligned}\omega_{ij}(\delta_i v_i, \delta_j v_j, \Delta_i, \Delta_j) = \Delta_i \Delta_j \delta_i \delta_j \Bigg(& I_1(\delta_i v_i, \delta_j v_j) - \frac{1}{2} I_2(\delta_i v_i, n) - \frac{1}{2} I_2(\delta_i v_i, -n) \\ & - \frac{1}{2} I_3(\delta_j v_j, n) - \frac{1}{2} I_3(\delta_j v_j, -n) + I_4(n^2)\Bigg)\end{aligned} \tag{7}$$

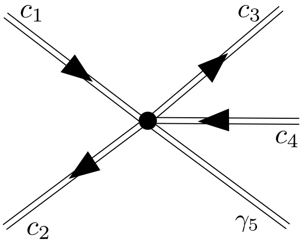
The integrals $I_1 - I_4$ are evaluated in [Kidonakis 1997].
For example, when both partons are massless:

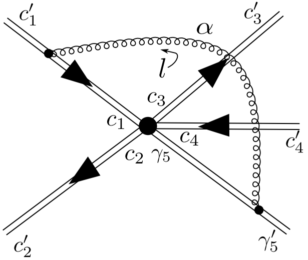
$$I_1^{\text{UV pole}} = \frac{\alpha}{2\pi}\left\{\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left[\gamma + \ln\left(\delta_i \delta_j \frac{v_{ij}}{2}\right) - \ln(4\pi)\right]\right\}$$
$$I_2^{\text{UV pole}} = \frac{\alpha}{2\pi}\left\{\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left[\gamma + \ln(\nu_i) - \ln(4\pi)\right]\right\}$$
$$I_3^{\text{UV pole}} = \frac{\alpha}{2\pi}\left\{\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left[\gamma + \ln(\nu_j) - \ln(4\pi)\right]\right\}$$
$$I_4^{\text{UV pole}} = -\frac{\alpha}{\pi}\frac{1}{\epsilon}.$$

where

$$\nu_a = \frac{(\nu_a \cdot n)^2}{|n|^2} \tag{8}$$

and $v_{ij} = v_i \cdot v_j = \frac{2p_i p_j}{s}$.





Before the gluon exchange:

$$\mathbf{c}_1 = \delta_{c_1 c_2} T^{\gamma_5}_{c_3 c_4}$$
$$\mathbf{c}_2 = \delta_{c_3 c_4} T^{\gamma_5}_{c_2 c_1}$$
$$\mathbf{c}_3 = T^{\alpha}_{c_3 c_4} T^{\beta}_{c_3 c_4} i f^{\alpha\beta\gamma_5}$$
$$\mathbf{c}_4 = T^{\alpha}_{c_3 c_4} T^{\beta}_{c_3 c_4} d^{\alpha\beta\gamma_5}$$

After the gluon exchange:

$$\mathbf{c}_1' = \delta_{c_1' c_2'} T^{\gamma_5'}_{c_3' c_4'}$$
$$\mathbf{c}_2' = \delta_{c_3' c_4'} T^{\gamma_5}_{c_2' c_1'}$$
$$\mathbf{c}_3' = T^{\alpha}_{c_3' c_4'} T^{\beta}_{c_3' c_4'} i f^{\alpha\beta\gamma_5'}$$
$$\mathbf{c}_4' = T^{\alpha}_{c_3' c_4'} T^{\beta}_{c_3' c_4'} d^{\alpha\beta\gamma_5'}$$

The vertex correction $\mathscr{F}_{15} = T^{\alpha}_{c_1 c'_1} i f^{\alpha \gamma'_5 \gamma_5} \delta_{c_2 c'_2} \delta_{c_3 c'_3} \delta_{c_4 c'_4}$, modifies the Born basis in the following way:

$$\mathbf{c}_1 \mathscr{F}_{15} = -\mathbf{c}'_3 \,, \tag{9}$$

$$\mathbf{c}_2 \mathscr{F}_{15} = -\frac{N_c}{2} \mathbf{c}'_2 \,, \tag{10}$$

$$\mathbf{c}_3 \mathscr{F}_{15} = -\frac{1}{2} \mathbf{c}'_1 - \frac{N_c}{4} \mathbf{c}'_3 - \frac{N_c}{4} \mathbf{c}'_4 \,, \tag{11}$$

$$\mathbf{c}_4 \mathscr{F}_{15} = \left(\frac{1}{N_c} - \frac{N_c}{4} \right) \mathbf{c}'_3 - \frac{N_c}{4} \mathbf{c}'_4 \,. \tag{12}$$

The linear transformation, which describes the modification of the Born-color structure caused by the soft gluon exchange between two partons:

$$\mathcal{F}_{15} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{N_c}{2} & 0 & 0 \\ -1 & 0 & -\frac{N_c}{4} & \frac{1}{N_c} - \frac{N_c}{4} \\ 0 & 0 & -\frac{N_c}{4} & -\frac{N_c}{4} \, , \end{pmatrix} \tag{13}$$

Let us define the following tensor:

$$G_{AB} = \mathrm{tr}\left(\mathbf{c}_A \mathbf{c}_B'^{\dagger} \sum_{i,j;i\leq j} \omega_{ij}^{(\mathrm{UV})} \mathscr{F}_{ij}\right). \tag{14}$$

The soft anomalous dimension matrix is then given as:

$$\Gamma_{IJ} = \left(S^{(0)}\right)^{-1}_{IK} G_{KJ}\,, \tag{15}$$

where $S^{(0)}$ is a Born-level soft matrix, defined as:

$$S^{0}_{LI} = \left\langle \mathbf{c}_L \middle| \mathbf{c}_I \right\rangle. \tag{16}$$

This procedure is fully automatized:

- Wilson diagrams and corresponding amplitudes are generated using a `C`-package `WilsonWebs`.
- The color decomposition is done using a `FORM` routine `FORMSoft`.

- Example components from $q\bar{q} \to t\bar{t}g$-channel

$$\Gamma^{(1)}_{1,1} = \frac{1}{2N_c}\left[2L_\beta + N_c^2\left(-2\log(\nu_5) + 2\log(v_{35}) + 2\log(v_{45}) + 2\log\left(\frac{s}{m_t^2}\right) - \log(16) + 2\right)\right.$$
$$\left. + \left(N_c^2 - 1\right)\left(-\log(\nu_1) - \log(\nu_2) + 2i\pi\right) + \log(4)\right]$$

$$\Gamma^{(1)}_{1,2} = \frac{1}{N_c}\left[\log(v_{13}) - \log(v_{14}) - \log(v_{23}) + \log(v_{24})\right]$$

- Example components from $gg \to t\bar{t}g$-channel

$$\Gamma^{(1)}_{2,9} = 0$$

$$\Gamma^{(1)}_{2,10} = \frac{N_c + 3}{4N_c + 4}\left[\log(v_{13}) - \log(v_{14}) - \log(v_{23}) + \log(v_{24})\right]$$

$$\Gamma^{(1)}_{11,8} = \frac{N_c(N_c+1) - 2}{2N_c}\left[\log(v_{13}) + \log(v_{14}) - 2\log(v_{15}) - \log(v_{23}) - \log(v_{24}) + 2\log(v_{25})\right]$$

$$\Gamma^{(1)}_{11,9} = \frac{N_c(N_c+1) - 2}{2N_c}\left[-\log(v_{13}) + \log(v_{14}) + \log(v_{23}) - \log(v_{24})\right]$$

4-top production:

- The multiplet bases for this process have been previously analyzed in [Keppeler, Sjödahl 2012]
- A simple call:

```
./WilsonWebs -L 1 -i q qbar -f t tbar t tbar | ./FORMSoft
```

can generate 1-Loop soft anomalous dimension matrices in less than 1 millisecond.

- Example results:

$$\Gamma^{(1)}_{1,1} = \frac{N_c^2 - 1}{N_c}\left[-L_\beta^{(34)} - L_\beta^{(56)} - \frac{\log(\nu_1)}{2} - \frac{\log(\nu_2)}{2} + \log(-v_{12}) - 1 - \frac{\log(4)}{2}\right]$$

$$\begin{aligned}\Gamma^{(1)}_{5,5} = \frac{1}{2N_c}\Bigg[& 2L_\beta^{(34)} - 6L_\beta^{(35)} - 6L_\beta^{(46)} + 2L_\beta^{(56)} + N_c^2 \cdot \left(2\log\left(\frac{s}{m_t^2}\right) - 2\right) + \\ & \left(N_c^2 - 6\right)\left(-L_\beta^{(36)} - L_\beta^{(45)} + \log(v_{13}) + \log(v_{15}) + \log(v_{24}) + \log(v_{26})\right) + \\ & \left(N_c^2 - 1\right)\left(-\log(\nu_1) - \log(\nu_2) - \log(4)\right) - 2\log(-v_{12}) + 6\log(v_{14}) + 6\log(v_{16}) + \\ & 6\log(v_{23}) + 6\log(v_{25}) + 2\Bigg]\end{aligned}$$

Multi-jet production:

- Following command:

```
./WilsonWebs -L 1 -i g g -f t tbar g g g g g g g g g g
```

successfully constructs all the necessary ingredients for 1-Loop anomalous dimension calculation of $t\bar{t}$ in association with 10 jets in gg-channel within less than 1 minute

- The flag `-L` controls the loop count and can be set to an arbitrary high value (bearing in mind the computational time limitations)
  - Multiloop calculations have not been tested so far using these tools
  - The master integrals needed for the multiloop calculations are also not included in these packages

The following checks are done:

(i) Correctness of the color decomposition procedure:

- $\mathrm{Tr}\left(H^{(0)}S^{(0)}\right)$ should evaluate back to the squared Born matrix element

(ii) Correctness of the 1-loop pole structure:

- The analytic pole structure at NLO is known, e.g. by evaluating the Catani-Seymour subtraction terms.
- The same structure can be generated using $H^{(0)}$ and $\Gamma^{(1)}$ matrices.

$$V_{\text{poles}} = 2\left\langle M_0 \middle| I^{(1)} \middle| M_0 \right\rangle ,$$

where (for massless case):

$$\boldsymbol{I}^{(1)}\left(\epsilon, \mu^2; \{p\}\right) = \frac{1}{2}\frac{e^{-\epsilon\psi(1)}}{\Gamma(1-\epsilon)}\sum_i \frac{1}{\boldsymbol{T}_i^2}\mathcal{V}_i^{\text{sing}}(\epsilon)\sum_{j\neq i}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left(\frac{\mu^2 e^{-i\lambda_{ij}\pi}}{2p_i\cdot p_j}\right)^{\epsilon}$$

Both of these checks have been successfully passed.

It is possible to perform the same calculation at 2 loop level using the general result from [Becher, Neubert, 2009]:

$$\begin{aligned}\mathbf{\Gamma}(\{p\},\{m\},\mu) = &\sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \gamma_{\text{cusp}}\ (\alpha_s) \ln\left(\frac{\mu^2}{-s_{ij}}\right) + \sum_i \gamma^i (\alpha_s) \\ &- \sum_{(I,J)} \frac{\mathbf{T}_I \cdot \mathbf{T}_J}{2} \gamma_{\text{cusp}}\ (\beta_{IJ}, \alpha_s) + \sum_I \gamma^I (\alpha_s) \\ &+ \sum_{(I,j)} \mathbf{T}_I \cdot \mathbf{T}_j \gamma_{\text{cusp}}\ (\alpha_s) \ln\left(\frac{m_I \mu}{-s_{Ij}}\right) \\ &+ \sum_{(I,J,K)} \mathrm{i} f^{abc} \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_K^c F_1 (\beta_{IJ}, \beta_{JK}, \beta_{KI}) \\ &+ \sum_{(I,J)} \sum_k \mathrm{i} f^{abc} \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_k^c f_2 \left(\beta_{IJ}, \ln\left(\frac{-\sigma_{Jk} v_J \cdot p_k}{-\sigma_{Ik} v_I \cdot p_k}\right)\right).\end{aligned}$$

- The procedure has been implemented in the package WilsonWebs2

- WilsonWebs2 can automatically generate the orthogonal multiplet color basis for any $2 \to n$ process

- Construct the 2-loop expression for soft anomalous dimension following the formalis of [Becher, Neubert, 2009].

- Perform the color decomposition

- Following command:

```
./WilsonWebs2 -i g g -f t tbar g g g g g g g g g g
```

  successfully calculated the soft anomalous dimension at 2 loops in gluonic channel for $t\bar{t}$ production in association with 10 gluonic jets.

- Working on the implementation of the consistency checks

2-loop resulst for $q\bar{q} \to t\bar{t}g$

$$\Gamma^{(1)}_{1,1} = \frac{1}{2N_c} N_c^2 \left(\mathrm{LL}(3,5) - \mathrm{LL}(4,5) \right) \gamma_{cusp}(\alpha_s) + \left(N_c^2 - 1 \right) L(1,2)\gamma_{cusp}(\alpha_s) + \gamma_{cusp}(\beta_{34}, \alpha_s)$$

$$\Gamma^{(1)}_{1,2} = \frac{1}{4N_c} N_c \left(F_2(3,4,1) + F_2(3,4,2) - F_2(4,3,1) - F_2(4,3,2) \right) + 2\,\mathrm{LL}(3,1)\gamma_{cusp}(\alpha_s) + 2\,\mathrm{LL}(3,2)\gamma_{cusp}(\alpha_s) + 2\,\mathrm{LL}(4,1)\gamma_{cusp}(\alpha_s) + 2\,\mathrm{LL}(4,2)\gamma_{cusp}(\alpha_s)$$

$$\Gamma^{(1)}_{1,3} = -\frac{L(1,5)}{2} - \frac{L(2,5)}{2} - \frac{\mathrm{LL}(3,1)}{4} - \frac{\mathrm{LL}(3,2)}{4} + \frac{\mathrm{LL}(4,1)}{4} + \frac{\mathrm{LL}(4,2)}{4}\gamma_{cusp}(\alpha_s)$$

$$\Gamma^{(1)}_{1,4} = \frac{1}{4N_c^2} \left(N_c^2 - 4 \right) \left(\mathrm{LL}(3,1) + \mathrm{LL}(3,2) + \mathrm{LL}(4,1) + \mathrm{LL}(4,2) \right) \gamma_{cusp}(\alpha_s)$$

$$\Gamma^{(1)}_{2,1} = \frac{1}{4N_c} N_c \left(-F_2(3,4,1) - F_2(3,4,2) + F_2(4,3,1) + F_2(4,3,2) \right) + 2\,\mathrm{LL}(3,1)\gamma_{cusp}(\alpha_s) + 2\,\mathrm{LL}(3,2)\gamma_{cusp}(\alpha_s) + 2\,\mathrm{LL}(4,1)\gamma_{cusp}(\alpha_s) + 2\,\mathrm{LL}(4,2)\gamma_{cusp}(\alpha_s)$$

$$\Gamma^{(1)}_{2,2} = \frac{1}{2N_c} N_c^2 \left(-L(1,5) + L(2,5) \right) \gamma_{cusp}(\alpha_s) - \left(N_c^2 - 1 \right) \gamma_{cusp}(\beta_{34}, \alpha_s) - L(1,2)\gamma_{cusp}(\alpha_s)$$

$$\Gamma^{(1)}_{2,3} = N_c \left(\frac{F_2(3,4,1)}{8} - \frac{F_2(3,4,2)}{8} - \frac{F_2(3,4,5)}{4} - \frac{F_2(4,3,1)}{8} + \frac{F_2(4,3,2)}{8} + \frac{F_2(4,3,5)}{4} \right) - \frac{\mathrm{LL}(3,1)\gamma_{cusp}(\alpha_s)}{4} + \frac{\mathrm{LL}(3,2)\gamma_{cusp}(\alpha_s)}{4} + \frac{\mathrm{LL}(3,5)\gamma_{cusp}(\alpha_s)}{2} - \ldots$$

$$\Gamma^{(1)}_{2,4} = \frac{N_c^2 - 4}{8N_c^2} N_c \left(-F_2(3,4,1) - F_2(3,4,2) + F_2(4,3,1) + F_2(4,3,2) \right) + 2 \left(\mathrm{LL}(3,1) + \mathrm{LL}(3,2) + \mathrm{LL}(4,1) + \mathrm{LL}(4,2) \right) \gamma_{cusp}(\alpha_s)$$

$$\Gamma^{(1)}_{3,1} = -L(1,5) - L(2,5) - \frac{\mathrm{LL}(3,1)}{2} - \frac{\mathrm{LL}(3,2)}{2} + \frac{\mathrm{LL}(4,1)}{2} + \frac{\mathrm{LL}(4,2)}{2}\gamma_{cusp}(\alpha_s)$$

$$\Gamma^{(1)}_{3,2} = N_c \left(-\frac{F_2(3,4,1)}{4} + \frac{F_2(3,4,2)}{4} + \frac{F_2(3,4,5)}{2} + \frac{F_2(4,3,1)}{4} - \frac{F_2(4,3,2)}{4} - \frac{F_2(4,3,5)}{2} \right) - \frac{\mathrm{LL}(3,1)\gamma_{cusp}(\alpha_s)}{2} + \frac{\mathrm{LL}(3,2)\gamma_{cusp}(\alpha_s)}{2} + \mathrm{LL}(3,5)\gamma_{cusp}(\alpha_s) - \ldots$$

$$\Gamma^{(1)}_{3,3} = \frac{1}{4N_c} N_c^2 \left(-L(1,5) + L(2,5) + \mathrm{LL}(3,5) - \mathrm{LL}(4,5) \right) \gamma_{cusp}(\alpha_s) + \left(N_c^2 - 2 \right) \left(\mathrm{LL}(3,1)\gamma_{cusp}(\alpha_s) + \mathrm{LL}(4,2)\gamma_{cusp}(\alpha_s) \right) - 2L(1,2)\gamma_{cusp}(\alpha_s) - 2\,\mathrm{LL}(3,2)\gamma_{cusp}(\alpha_s) - \ldots$$

$$\Gamma^{(1)}_{3,4} = \frac{1}{4N_c} \left(N_c^2 - 4 \right) \left(-L(1,5) - L(2,5) - \mathrm{LL}(3,1) + \mathrm{LL}(3,5) + \mathrm{LL}(4,2) + \mathrm{LL}(4,5) \right) \gamma_{cusp}(\alpha_s)$$

$$\Gamma^{(1)}_{4,1} = \frac{\mathrm{LL}(3,1)}{2} + \frac{\mathrm{LL}(3,2)}{2} + \frac{\mathrm{LL}(4,1)}{2} + \frac{\mathrm{LL}(4,2)}{2}\gamma_{cusp}(\alpha_s)$$

$$\Gamma^{(1)}_{4,2} = N_c \left(\frac{F_2(3,4,1)}{4} + \frac{F_2(3,4,2)}{4} - \frac{F_2(4,3,1)}{4} - \frac{F_2(4,3,2)}{4} \right) + \frac{\mathrm{LL}(3,1)\gamma_{cusp}(\alpha_s)}{2} + \frac{\mathrm{LL}(3,2)\gamma_{cusp}(\alpha_s)}{2} + \frac{\mathrm{LL}(4,1)\gamma_{cusp}(\alpha_s)}{2} + \frac{\mathrm{LL}(4,2)\gamma_{cusp}(\alpha_s)}{2}$$

$$\Gamma^{(1)}_{4,3} = \frac{1}{4} N_c \left(-L(1,5) - L(2,5) - \mathrm{LL}(3,1) + \mathrm{LL}(3,5) + \mathrm{LL}(4,2) + \mathrm{LL}(4,5) \right) \gamma_{cusp}(\alpha_s)$$

Untested...

There are 2 main strategies to generate color bases:

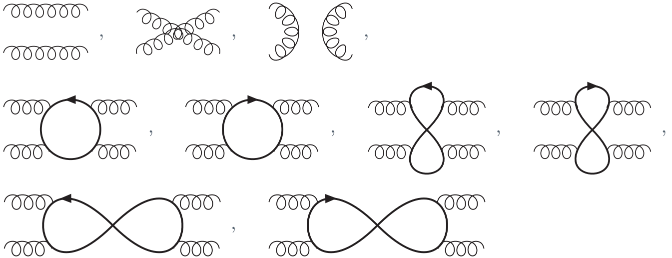
- **Trace bases**: easy to generate, but they are overcomplete and make calculations different.
  - Number of algorithmic implementations exist
- **Multiplet bases**: harder to generate, but they build minimal and orthogonal set, making further calculations easier
  - No algorithmic implementation exist so far
  - The strategy is discussed in [Cvitanović, 1984] and [Keppeler, Sjödahl, 2012]

Trace bases: example for $gg \to gg$

$$c_1 = \frac{1}{4}\delta_{r_A r_1}\delta_{r_B r_2}, \quad c_2 = \frac{1}{4}\delta_{r_A r_2}\delta_{r_B r_1}, \quad c_3 = \frac{1}{4}\delta_{r_A r_B}\delta_{r_1 r_2},$$

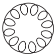
$$c_4 = \mathrm{Tr}\left(T_F^{r_A} T_F^{r_B} T_F^{r_2} T_F^{r_1}\right), \quad c_5 = \mathrm{Tr}\left(T_F^{r_A} T_F^{r_B} T_F^{r_1} T_F^{r_2}\right), \quad c_6 = \mathrm{Tr}\left(T_F^{r_A} T_F^{r_1} T_F^{r_2} T_F^{r_B}\right),$$

$$c_7 = \mathrm{Tr}\left(T_F^{r_A} T_F^{r_1} T_F^{r_B} T_F^{r_2}\right), \quad c_8 = \mathrm{Tr}\left(T_F^{r_A} T_F^{r_2} T_F^{r_1} T_F^{r_B}\right), \quad c_9 = \mathrm{Tr}\left(T_F^{r_A} T_F^{r_2} T_F^{r_B} T_F^{r_1}\right)$$



Diagrams taken from [Keppeler, 2017]

- **The issue**: Easy to generate, but they are not *orthogonal*, e.g.:

$$\langle c1, c3 \rangle = \Big\langle$$  $$\Big\rangle =$$  $$= N_c^2 - 1 \neq 0\,.$$

Multiplet color basis for $q\bar{q} \to t\bar{t}g$

Consider the process $q_a\bar{q}_b \to t_c\bar{t}_d g_e$:

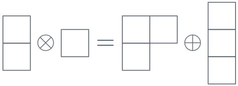
$$\text{Initial state:} \quad 3 \otimes \bar{3} = 1 \oplus 8$$

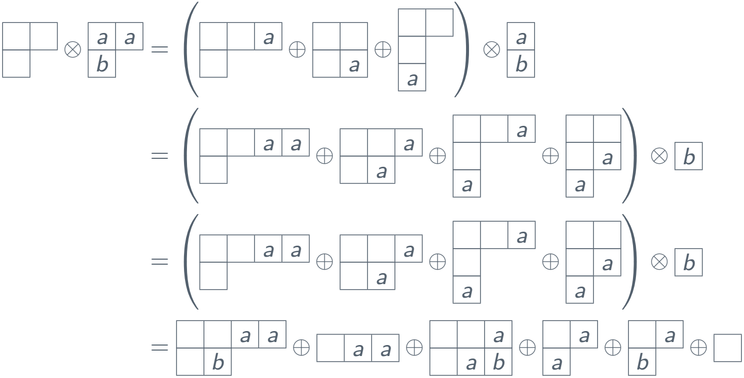
$$\text{Final state:} \quad 3 \otimes \bar{3} \otimes 8 = (1 \oplus 8) \otimes 8 = 1 \oplus 3 \times 8 \oplus 10 \oplus \overline{10} \oplus 27 \oplus 0$$

- Initial state singlet $\Rightarrow$ final state is singlet $\Rightarrow$ gluon octet must matched by $t\bar{t}$ octet $\Rightarrow \delta_{ab} t^e_{cd}$
- Initial state octet $\Rightarrow$ final state is octet $\Rightarrow$
  - gluon matches the intial octet $\Rightarrow$ $t\bar{t}$ is singlet $\Rightarrow$ $t^e_{ab}\delta_{cd}$
  - if $t\bar{t}$ is octet $\Rightarrow$ $8 \otimes 8$ contains 2 octets to match the inital octet:
    - 8^a: $t^m_{\ ba} t^n_{\ cd} i f_{mne}$
    - 8^s: $t^m_{\ ba} t^n_{\ cd} d_{mne}$

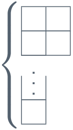
Decomposition of any state into irreducible representations

Such decompositions can be achieved using the formalism of Young tableau:





Automatic generation of color bases

- There are number of `Mathematica` packages for Young tableau manipulation
  - `LieART` [Feger et al, 2019]
  - `xActs` [Nutma, 2013]
  - ...
- They only work for fixed value of N_c
  - Impossible to generate $N_c - 2$  , which is 0 for $SU(3) \otimes SU(3)$
- We came up with *our implementation* of this procedure
  - Uses `z3 theorem prover` [de Moura,Bjørner,2008] for generation of Young tableau
  - Can work with abstract N_c, hence generates every possible representation for $N_c \to \infty$
  - Because of `z3`-"magic" avoids generation of not allowed Young tableau, as opposed to "generate and check validity" strategy used by other packages
- Because the inital state is restricted to $q\bar{q}$, gg, qg or $\bar{q}g$ there are only limited number of projectors required to parametryze the basis
  - All these projectors are known e.g. from [Cvitanović, 1984].
- This way we have fully automatic procedure to generate orthogonal multiplet basis for any $2 \to n$ process.

Demonstration of `z3-solver` in action:

```
from z3 import *
Coffee, Conference = Bools('Coffee Conference')
s = Solver()
s.add(Or(Coffee, Conference),
Or(Not(Coffee), Conference),
Or(Not(Coffee), Not(Conference)))
print(s.check())
print(s.model())
```

Output:

```
sat
[Coffee = False, Conference = True]
```

- z3 is very efficient at solving constraint based models, like sudoku tables or construction of Young tableau.

- For any $2 \to n$ process the tools presented in this work are capable of *automatically* calculating:
  1. The multiplet orthogonal color basis
  2. Soft anomalous dimension calculation at 1-loop
  3. Soft anomalous dimension calculation at 2-loop

- Analytical results of the soft anomalous dimension matrices at 1-loop for all partonic channels of $t\bar{t}j$ and $t\bar{t}t\bar{t}$ have been calculated and checked for the consistency
  - 2-loop results are also available and currently under testing

- The software tool for performing this calculation are openly available and they can be applicable to other projects as well:
  - WilsonWebs - automatic generation of n-loop Wilson diagrams.
  - WilsonWebs2 - generation of 2-Loop soft anomalous dimension using the approach of [Becher, Neubert, 2008]
  - FORMSoft - Soft anomalous dimension matrices at 1-loop.
  - FORMHard - Hard matrix generator (with or without the color decomposition).
  - pyDipole - NLO pole structure generator using Catani-Seymour formalism and $\Gamma^{(1)}$ and $H^{(0)}$ matrices.

Thanks for your attention!

Universität Hamburg

DER FORSCHUNG | DER LEHRE | DER BILDUNG

Backup

- The leading-order soft function:

$$S^{(0)}_{IJ} = \mathrm{tr}\left(c_I^{\dagger} c_J\right) \tag{17}$$

- The lowest-order hard function:

$$H^{0}_{ij,IJ} = h^{0}_{ij,I} h^{*0}_{ij,J} \tag{18}$$

- $h^{(0)}_{ij,J}$ are color projected amplitudes (A):

$$h_I^{(0)} = \left(S^{(0)}\right)^{-1}_{IK} \mathrm{tr}\left(c_K^{\dagger} A\right), \qquad h_I^{*(0)} = \mathrm{tr}\left(A^{\dagger} c_K\right) \left(S^{(0)}\right)^{-1}_{IK} \tag{19}$$

- Consistency check for the color decomposition procedure:

$$\mathrm{tr}\left(H^{(0)}_{AB} S^{(0)}_{BC}\right) = AA^{\dagger} \tag{20}$$