GGI, Florence, 2023

Theory Challenges in the Precision Era of the Large Hadron Collider

Regge pole and Regge cuts in full colour

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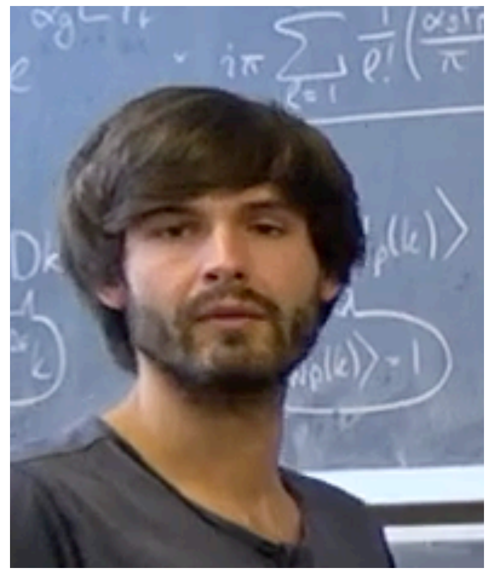
Based on collaboration with Simon Caron-Huot, Giulio Falcioni, Niamh Maher, Calum Milloy, Joscha Reichel and Leonardo Vernazza (2017-…)

Regge pole and Regge cuts in full colour

Motivation

- Understand the high-energy behaviour of partonic gauge-theory scattering amplitudes
  - considering the Regge limit (leading power) leads to **major simplifications**
  - unique access to multi-loop and multi-leg amplitudes
- Study multi-loop and multi-leg amplitudes in **full colour** (rather than the planar limit)
- Study properties of towers of high-energy logarithms to all orders and establish connection with the singularity structure in the complex angular momentum plane.
- Learn about IR factorisation

The high-energy limit of 2 → 2 gauge-theory amplitudes

- Simplification at leading power in t/s: **helicity** is conserved;
  gluon exchanges in the t-channel are dominant in any process gg→gg, qg→qg, qq→qq

Reggeization:
(Regge-pole)

$$\frac{s}{t} \longrightarrow \frac{s}{t}\left(\frac{s}{-t}\right)^{\alpha_g(t)}$$

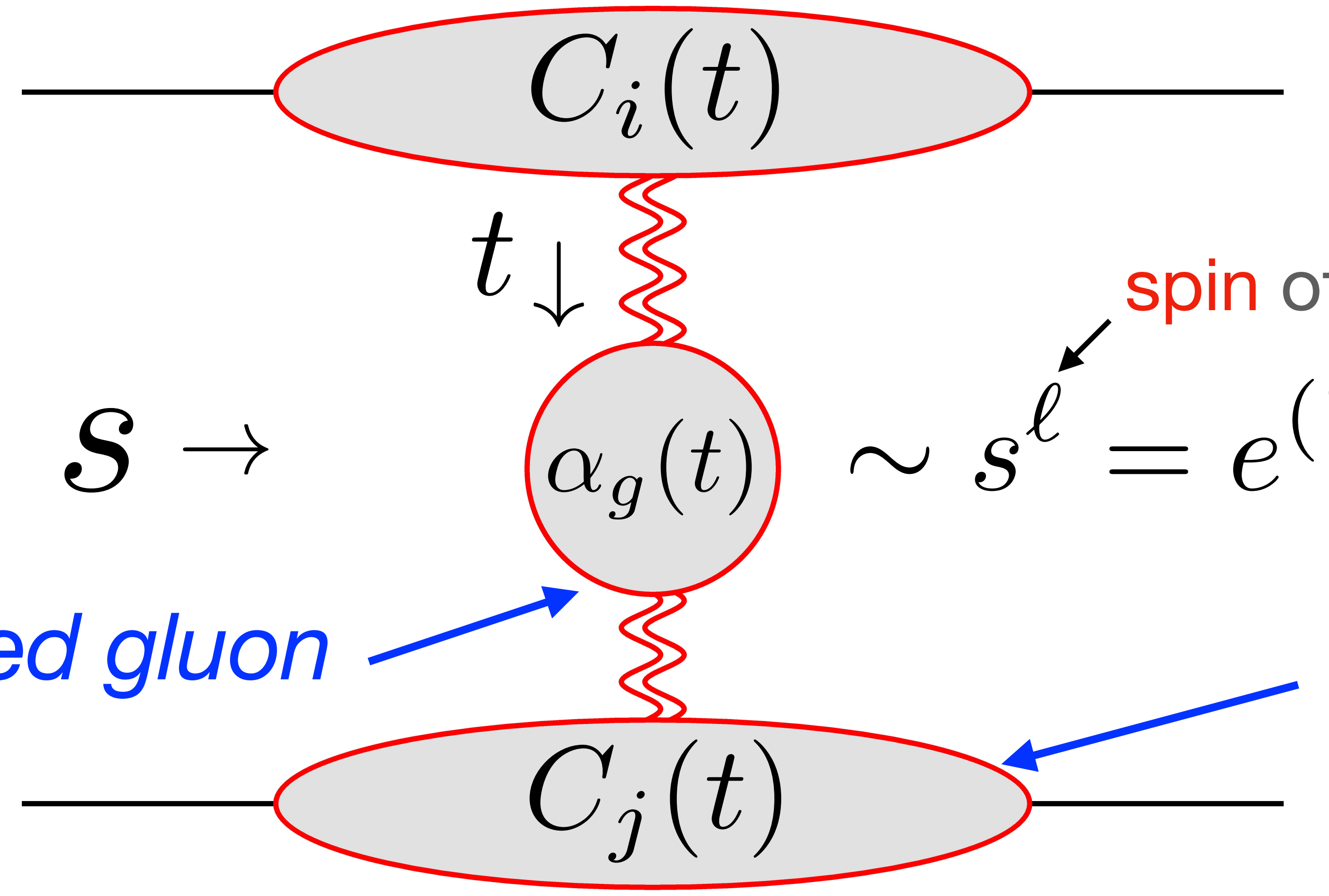
gluon Regge trajectory:

$$\alpha_g(t) = -\alpha_s \mathbf{T}_t^2 (\mu^2)^\epsilon \int \frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}} \frac{q_\perp^2}{k_\perp^2 (q_\perp - k_\perp)^2} + \mathcal{O}(\alpha_s^2)$$

$$= \frac{\alpha_s}{\pi}\mathbf{T}_t^2 \left(\frac{-t}{\mu^2}\right)^{-\epsilon} \frac{B_0(\epsilon)}{2\epsilon} + \mathcal{O}(\alpha_s^2)$$

$$B_0(\epsilon) = e^{\epsilon\gamma_E}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = 1 - \frac{\zeta_2}{2}\epsilon^2 - \frac{7\zeta_3}{3}\epsilon^3 + \dots$$

Factorization:



$$s \to \quad \sim s^{\ell} = e^{(1+\alpha_g(t))\Delta y} + \dots$$

spin of exchanged particle

- Regge-pole factorization amounts to a **relation** between gg→gg, qg→qg, qq→qq
- This holds for the **real part** of the amplitude through NLL.
  Beyond that it is violated by **non-planar** corrections associated with **multi-Reggeon** exchange forming **Regge cuts.** These effects are now much better understood.

$2 \to 2$ amplitudes: signature and reality properties

- Defining **signature** *even* and *odd* amplitudes under $s \leftrightarrow u$

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big)$$

- The spectral representation of the amplitude implies:

$$\mathcal{M}^{(+)}(s,t) = i\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{2\sin(\pi j/2)}\, a_j^{(+)}(t)\, e^{jL}\,,$$

$$\mathcal{M}^{(-)}(s,t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{2\cos(\pi j/2)}\, a_j^{(-)}(t)\, e^{jL}\,,$$

with $\left(a_{j^*}^{\pm}(t)\right)^* = a_j^{\pm}(t)$

$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right)$$

- Expanding the amplitude in the signature-symmetric log, L, the coefficients in $\mathcal{M}^{(+)}$ are **imaginary**, while in $\mathcal{M}^{(-)}$ **real**.

[See 1701.05241 Caron-Huot, EG, Vernazza]

The singularity structure of 2 → 2 amplitudes in the complex angular momentum plane: pole vs. cut

- The **signature-odd** amplitude admits

$$\mathcal{M}^{(-)}(s,t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{2\cos(\pi j/2)}\, a_j^{(-)}(t)\, e^{jL}\,,$$

	singularity	amplitude asymptotics
pole	$a_j^{(-)}(t) \simeq \dfrac{1}{j-1-\alpha(t)}$	$\mathcal{M}^{(-)}(s,t)\vert_{\text{Regge pole}} \simeq \dfrac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}}\,\dfrac{s}{t}\, e^{L\,\alpha(t)} + \ldots,$
cut	$a_j^{(-)}(t) \simeq \dfrac{1}{(j-1-\alpha(t))^{1+\beta(t)}}$	$\mathcal{M}^{(-)}(s,t)\vert_{\text{Regge cut}} \simeq \dfrac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}}\,\dfrac{s}{t}\,\dfrac{1}{\Gamma\left(1+\beta(t)\right)} L^{\beta(t)}\, e^{L\,\alpha(t)} + \text{subleading logs}$

- Reggeization of the **signature-odd** amplitude (NLL): a manifestation of a pure **Regge pole**.

- But what happens more generally?
  If both **Regge pole** and **Regge cut** are present - can we disentangle them at fixed order?

The origin of Regge cuts in 2 → 2 scattering: non-planar integrals

- Mandelstam (1963) has shown that Regge cuts exist in 2 → 2 scattering in certain **non-planar** diagrams starting from 3 loops.

1. – Introduction.

In the previous paper ([1]), hereafter referred to as I, it was shown that the cuts in the angular-momentum plane proposed by Amati, Fubini and Stanghellini (AFS) ([2]) were cancelled in the actual diagrams examined by them. We also pointed out that our reasoning could not be extended to diagrams such as Fig. 1. The distinguishing feature of these diagrams is that, if the ladder is contracted into a single line, both the left- and right-hand portions have third double-spectral functions in the t-channel, *i.e.* double-spectral functions A_{su}.

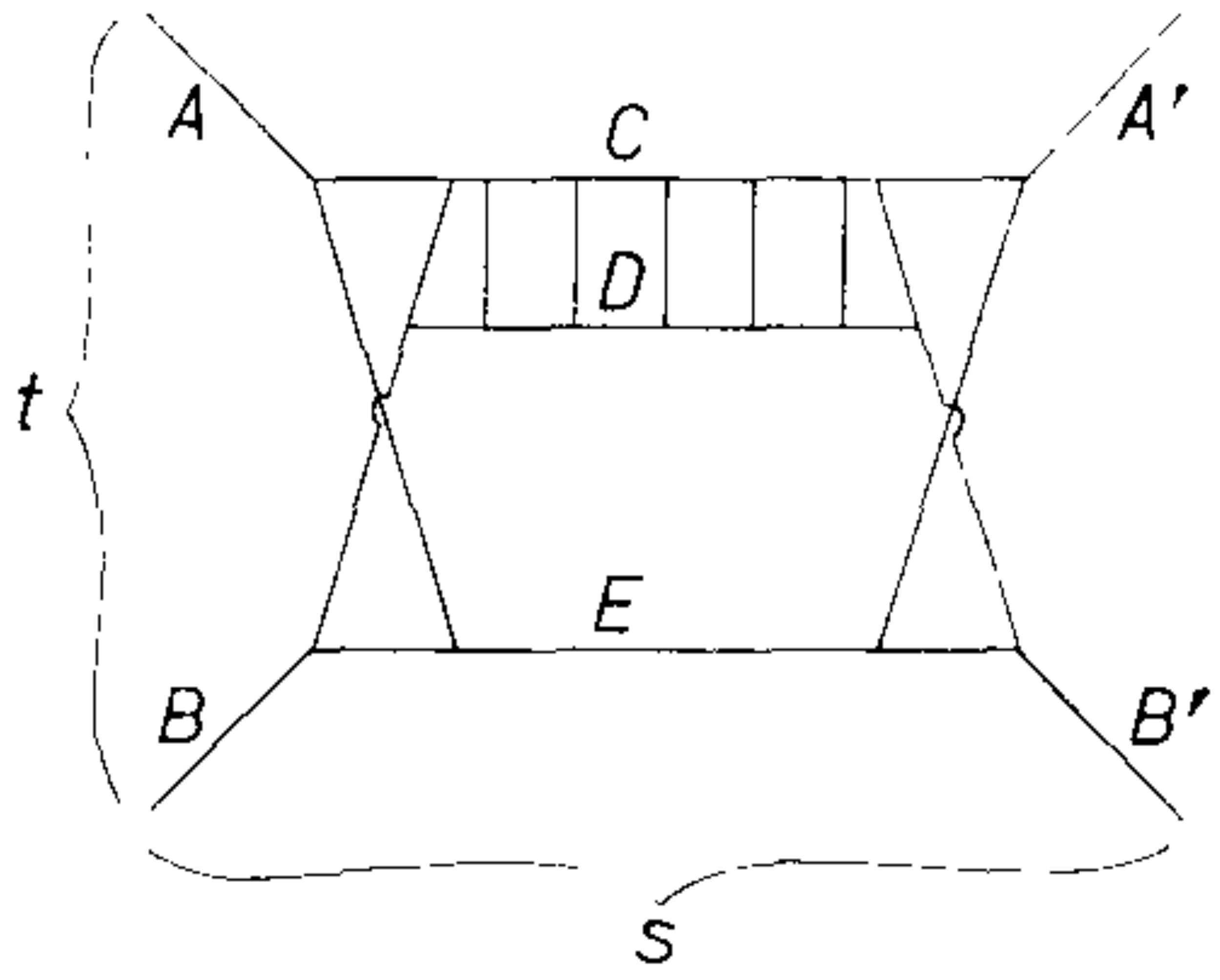


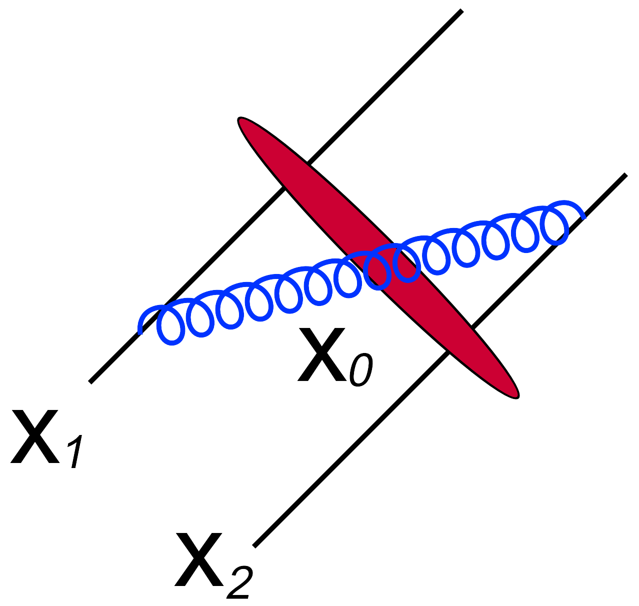
Fig. 1. – The simplest diagram where the cut in the l-plane is actually present.

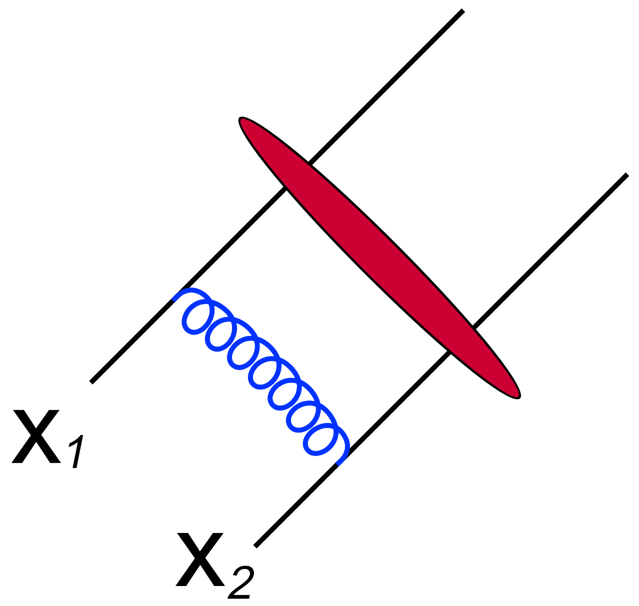
- Thus, Regge cuts exist, are associated with special *non-planar* diagrams, and with the exchange of at least two Reggeons in the t channel.

Non-linear rapidity evolution equations

- The colliding particles are replaced by (sets of) infinite lightlike Wilson lines

$$U(\mathbf{x}) = \mathcal{P}\exp\left\{ig_s\int_{-\infty}^{\infty} dx^+ A_+^a(x^+, x^- = 0; \mathbf{x})T^a\right\}$$





- Rapidity evolution equation [Balitsky-JIMWLK]

$$-\frac{d}{d\eta}\left[U(\mathbf{x}_1)\ldots U(\mathbf{x}_n)\right] = H\left[U(\mathbf{x}_1)\ldots U(\mathbf{x}_n)\right]$$

$$H = \frac{\alpha_s}{2\pi^2}\int d\mathbf{x}_i d\mathbf{x}_j d\mathbf{x}_0 \frac{\mathbf{x}_{0i}\cdot\mathbf{x}_{0j}}{\mathbf{x}_{0i}^2\mathbf{x}_{0j}^2}\left(T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\mathrm{adj}}(\mathbf{x}_0)(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R})\right)$$

$$T^a_{i,L} \equiv T^a U(\mathbf{x}_i)\frac{\delta}{\delta U(\mathbf{x}_i)}, \qquad T^a_{i,R} \equiv U(\mathbf{x}_i)T^a\frac{\delta}{\delta U(\mathbf{x}_i)}$$

Provides complete separation between the light-cone directions and the transverse plane: **2-dimensional dynamics**

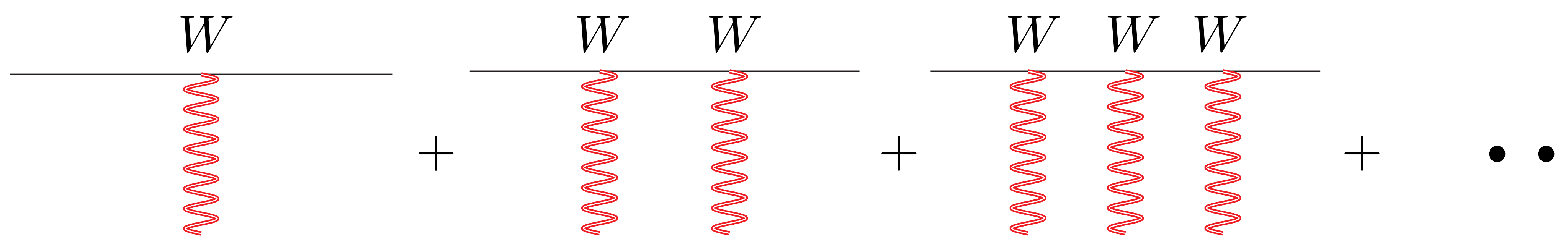
Defining the Reggeon

- In the perturbative regime $U(\mathbf{x}) \simeq 1$ it is natural to expand in terms of W Simon Caron-Huot (2013)

$$\boxed{U(\mathbf{x}) = \mathcal{P}\exp\left\{ig_s\int_{-\infty}^{\infty} dx^+ A^a_+(x^+, x^- = 0; \mathbf{x})T^a\right\} = e^{ig_s\, T^a W^a(\mathbf{x})}. \quad W \text{ sources a Reggeon}}$$

- Scattered particles are expanded in states of a definite number of Reggeons

$$|\psi_i\rangle \equiv \frac{Z_i^{-1}}{2p_1^+} a_i(p_4)a_i^\dagger(p_1)|0\rangle \sim g_s\,|W\rangle + g_s^2\,|WW\rangle + g_s^3\,|WWW\rangle + \ldots =$$



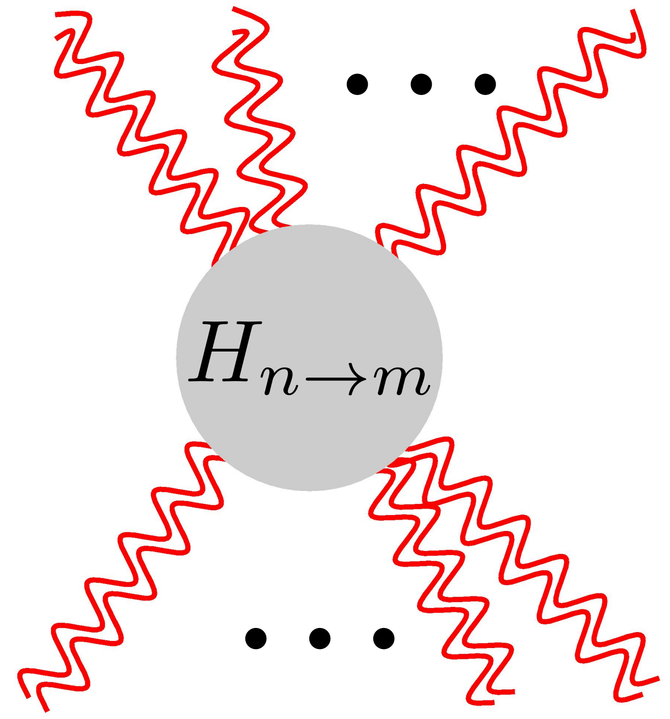
- Amplitudes are governed by rapidity evolution between the target and projectile:

$$\boxed{\frac{i(Z_iZ_j)^{-1}}{2s}\mathcal{M}_{ij\to ij} = \langle\psi_j|e^{-HL}|\psi_i\rangle}$$

$$-\frac{d}{d\eta}\,|\psi_i\rangle = H\,|\psi_i\rangle$$

$$H\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix} \equiv \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \cdots \\ 0 & H_{2\to2} & 0 & \cdots \\ H_{1\to3} & 0 & H_{3\to3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix}$$

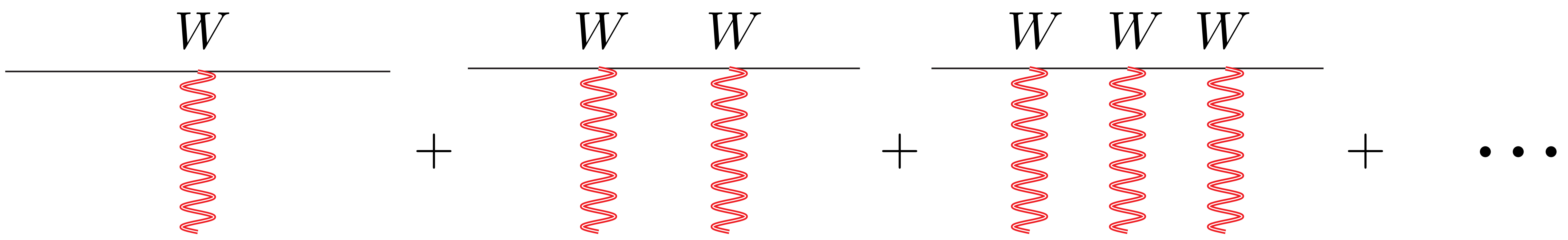
$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & \cdots \\ 0 & g_s^2 & 0 & \cdots \\ g_s^4 & 0 & g_s^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix}$$



Computing multi-Regge exchanges using non-linear rapidity evolution

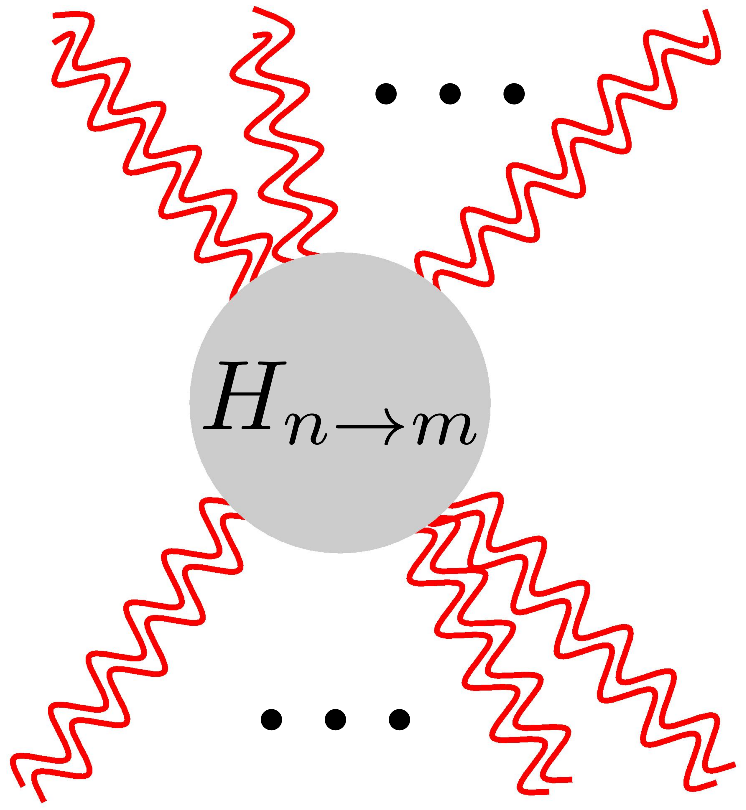
1701.05241 Caron-Huot, EG, Vernazza

Projectile $|\psi_i\rangle =$



n Reggeon to m Reggeon transition Hamiltonian [1701.05241]

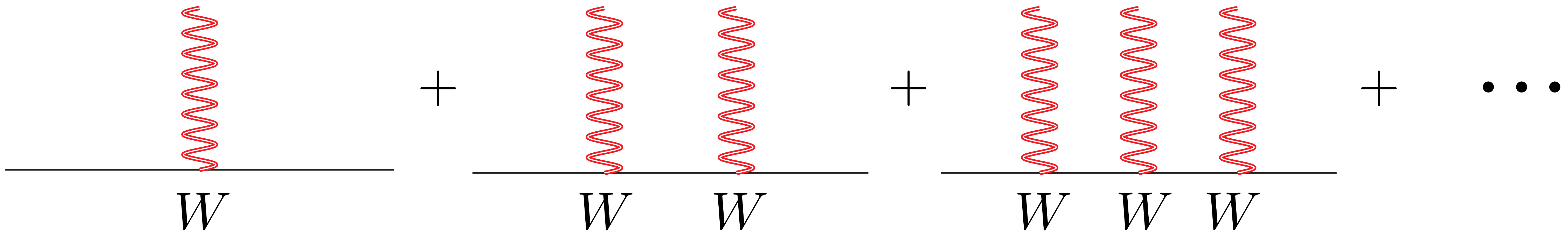
$\sum_{n,m}$



$$H\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix} \equiv \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \cdots \\ 0 & H_{2\to2} & 0 & \cdots \\ H_{1\to3} & 0 & H_{3\to3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix}$$

$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & \cdots \\ 0 & g_s^2 & 0 & \cdots \\ g_s^4 & 0 & g_s^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix}$$

Target $\langle\psi_j| =$



Signature, number of Reggeons and t-channel colour flow

- The signature odd and even sectors decouple

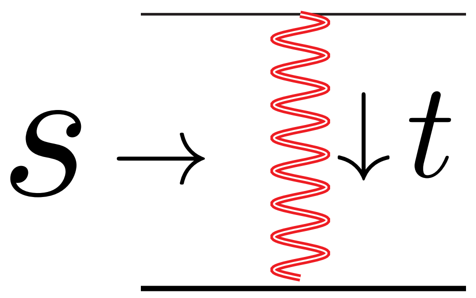
$$\frac{i}{2s}\mathcal{M}_{ij\to ij} \xrightarrow{\text{Regge}} \frac{i}{2s}\left(\mathcal{M}^{(-)}_{ij\to ij} + \mathcal{M}^{(+)}_{ij\to ij}\right) \equiv \langle\psi_j^{(-)}|e^{-HL}|\psi_i^{(-)}\rangle + \langle\psi_j^{(+)}|e^{-HL}|\psi_i^{(+)}\rangle$$

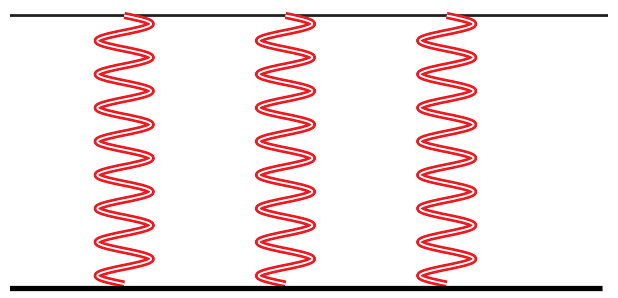
- odd/even signature amplitude is governed by the exchange of an odd/even number of Reggeons.

- Bose symmetry in gg → gg correlates odd/even signature with odd/even colour representations in the *t* channel.

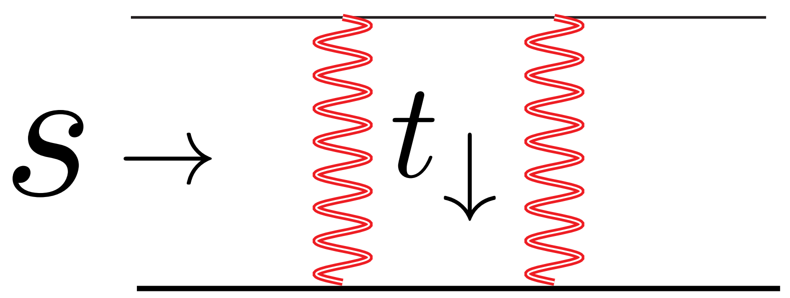
$$(\mathbf{8}\otimes\mathbf{8})_{gg}$$

odd $\quad 8_a \oplus (10 \oplus \overline{10})$





even $\quad 0 \oplus 1 \oplus 8_s \oplus 27$



More generally we use channel colour operators: $\mathbf{T}_t^2$ is even, $\quad \mathbf{T}^2_{s-u} \equiv \frac{\mathbf{T}_s^2 - \mathbf{T}_u^2}{2}$ is odd

The Soft Anomalous Dimension in the Regge limit

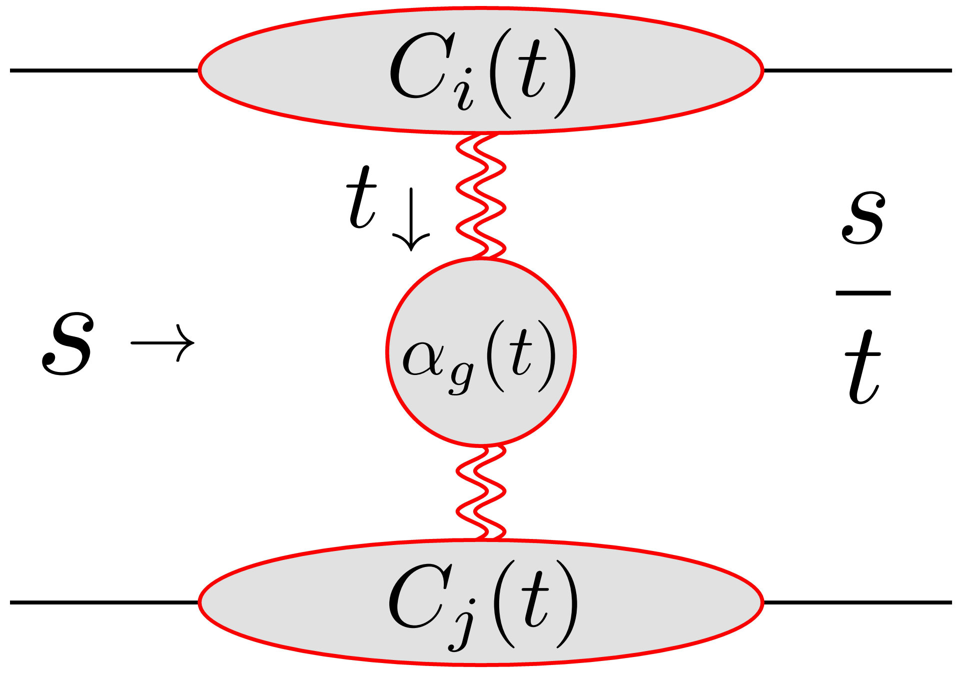
IR singularities in scattering amplitudes **exponentiate**

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s,\epsilon\right) = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}\left(\lambda,\alpha_s(\lambda^2,\epsilon)\right)\right\}\mathcal{H}\left(\frac{p_i}{\mu},\alpha_s\right)$$

in terms of the soft anomalous dimension Γ. In 2 → 2 scattering, at leading power in t/s:

$$\mathbf{\Gamma}_{ij\to ij}\left(\alpha_s,L,\frac{-t}{\lambda^2}\right) = \Gamma_i\left(\alpha_s,\frac{-t}{\lambda^2}\right)+\Gamma_j\left(\alpha_s,\frac{-t}{\lambda^2}\right) + \frac{1}{2}\gamma_K(\alpha_s)\left[L\mathbf{T}_{\mathbf{t}}^{\mathbf{2}}+i\pi\mathbf{T}_{\mathbf{s-u}}^{\mathbf{2}}\right]+\sum_{\ell=3}^{\infty}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^{\ell}\mathbf{\Delta}^{(\ell)}(L)$$

This may be compared with the **exponentiation** of rapidity logarithms (Reggeization)



$$\frac{s}{t}\longrightarrow\frac{s}{t}\,\mathrm{e}^{\alpha_g(t)\,L}$$

$$\alpha_g(t) = -\frac{1}{4}\int_0^{-t}\frac{d\lambda^2}{\lambda^2}\gamma_K(\alpha_s(\lambda^2)) + \mathcal{O}(\epsilon^0)$$ Korchemskaya & Korchemsky (1995)

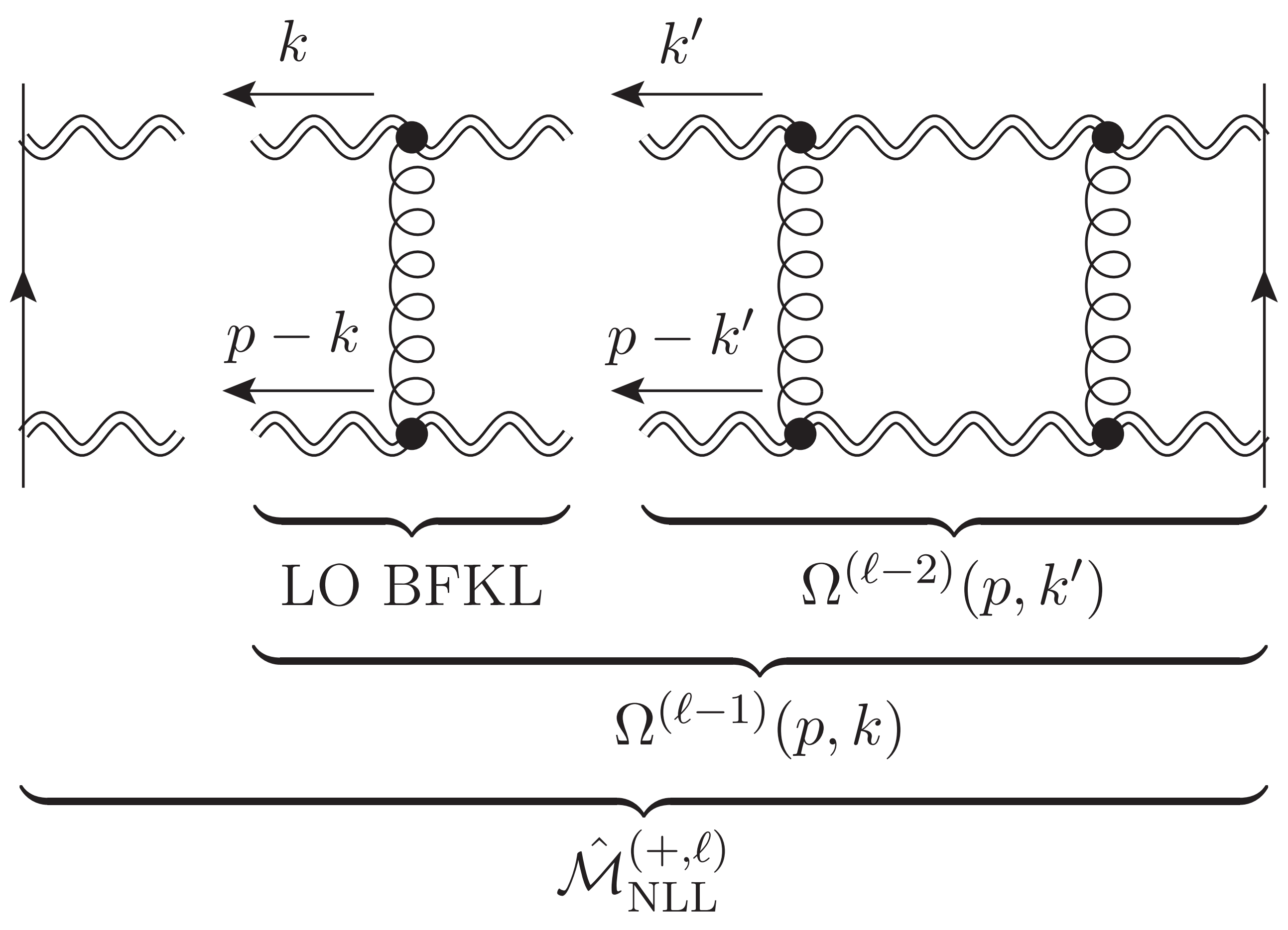
Reggeization violation is related to $i\pi\ \mathbf{T}_{\mathbf{s-u}}^{\mathbf{2}}\mathcal{M}_{\mathrm{tree}}$

And at higher orders also to non-dipole corrections $\mathbf{\Delta}$

Del Duca, Duhr, EG, Magnea, White (2011)

Signature-even 2 → 2 amplitude in full colour: NLL to any order from BFKL

Defining the *Reduced Amplitude*



$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} \equiv \langle\psi_j|e^{-(H-H_{1\to 1})L}|\psi_i\rangle = \langle\psi_j|e^{-\hat{H}X}|\psi_i\rangle$$

The evolution of the signature-even amplitude (NLL) is governed by the BFKL equation:

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\,\Omega^{(\ell-2)}(p,k), \qquad \hat{H} = \underbrace{(2C_A - \mathbf{T}_t^2)}_{C_1}\,\hat{H}_{\rm i} + \underbrace{(C_A - \mathbf{T}_t^2)}_{C_2}\,\hat{H}_{\rm m}$$

Integrate ($\hat{H}_{\rm i}$) Multiply ($\hat{H}_{\rm m}$)

$$\hat{H}_{\rm i}\,\Psi(p,k) = \int [{\rm D}k']\, f(p,k,k')\Big[\Psi(p,k') - \Psi(p,k)\Big],$$

$$\hat{H}_{\rm m}\,\Psi(p,k) = J(p,k)\,\Psi(p,k)$$

The wave function is IR safe!
This equation has been solved iteratively to high loop orders

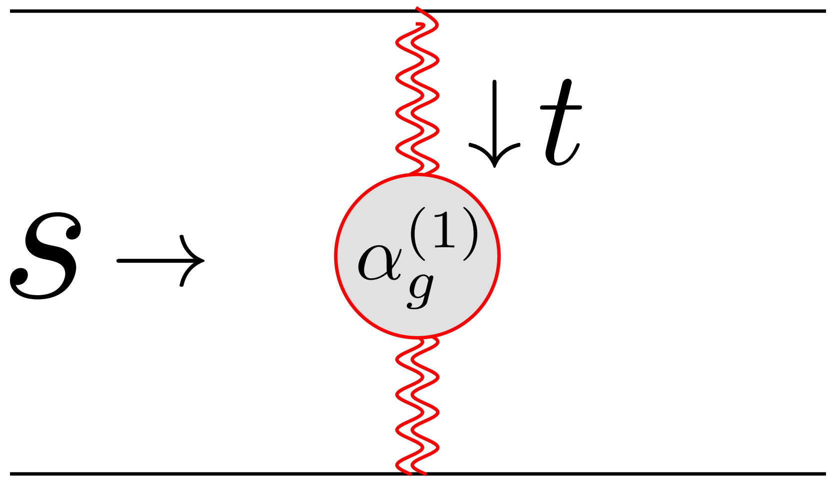
Caron-Huot, EG, Reichel, Vernazza, JHEP 1803 (2018) 098; JHEP 08 (2020) 116

Signature-odd amplitudes: Regge-pole factorisation and its breaking

$$\mathcal{M}^{(-)}_{ij\to ij} = C_i(t)\, e^{\alpha_g(t)\, C_A\, L}\, C_j(t) \mathcal{M}^{\text{tree}}_{ij\to ij} + \text{MR}$$

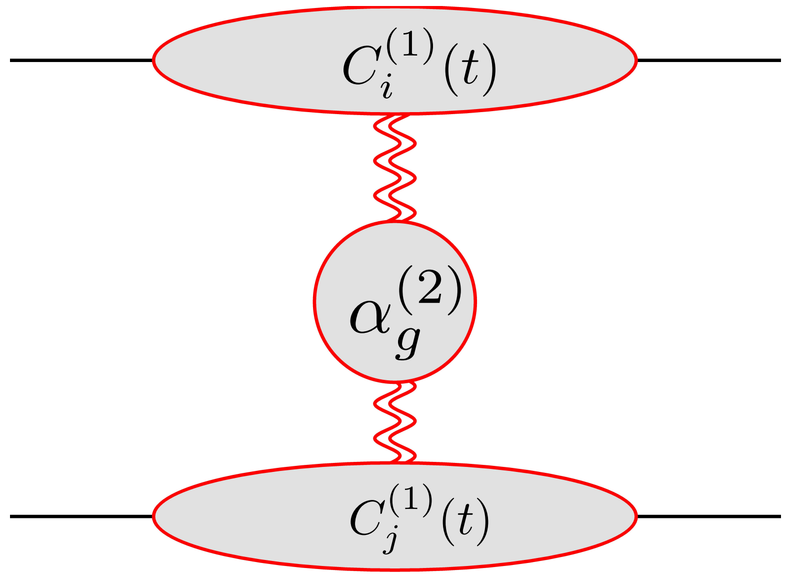
Colour **octet** exchange in the t channel: single Reggeon, factorize

LL $\alpha_s^n \log^n\left(\frac{s}{-t}\right)$



one-loop Regge trajectory

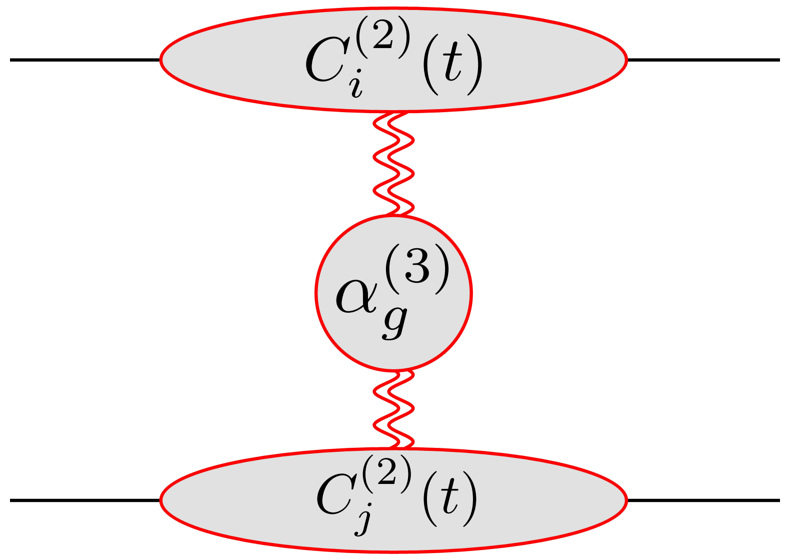
NLL $\alpha_s^n \log^{n-1}\left(\frac{s}{-t}\right)$



two-loop Regge trajectory

one-loop impact factors

NNLL $\alpha_s^n \log^{n-2}\left(\frac{s}{-t}\right)$



three-loop Regge trajectory

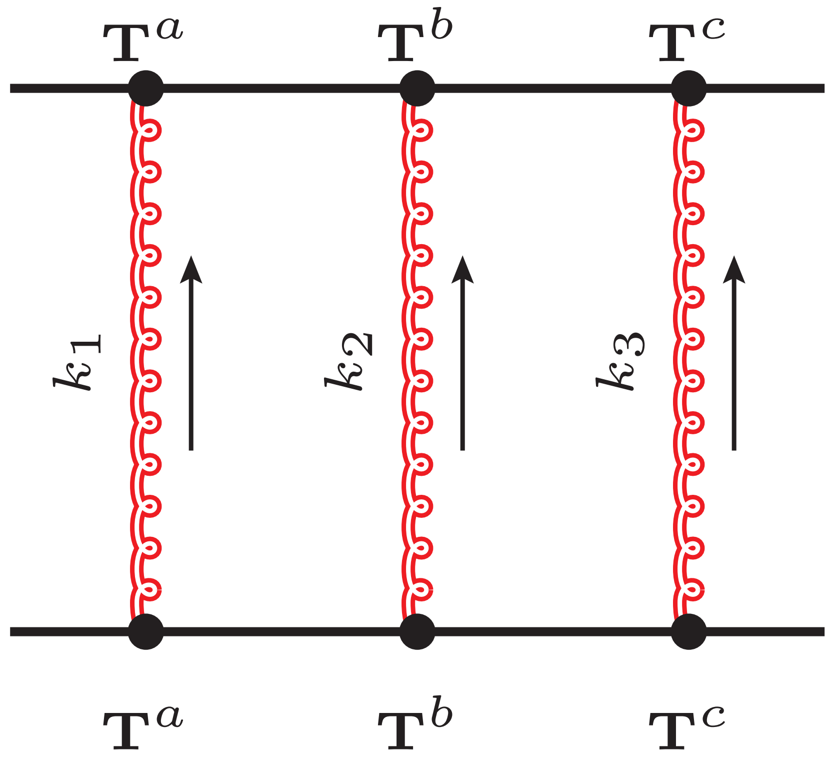
two-loop impact factors

Regge factorisation breaking (starting at 2 loops) can be inferred from comparing qq, qg, gg amplitudes

[Del Duca, Glover '01]

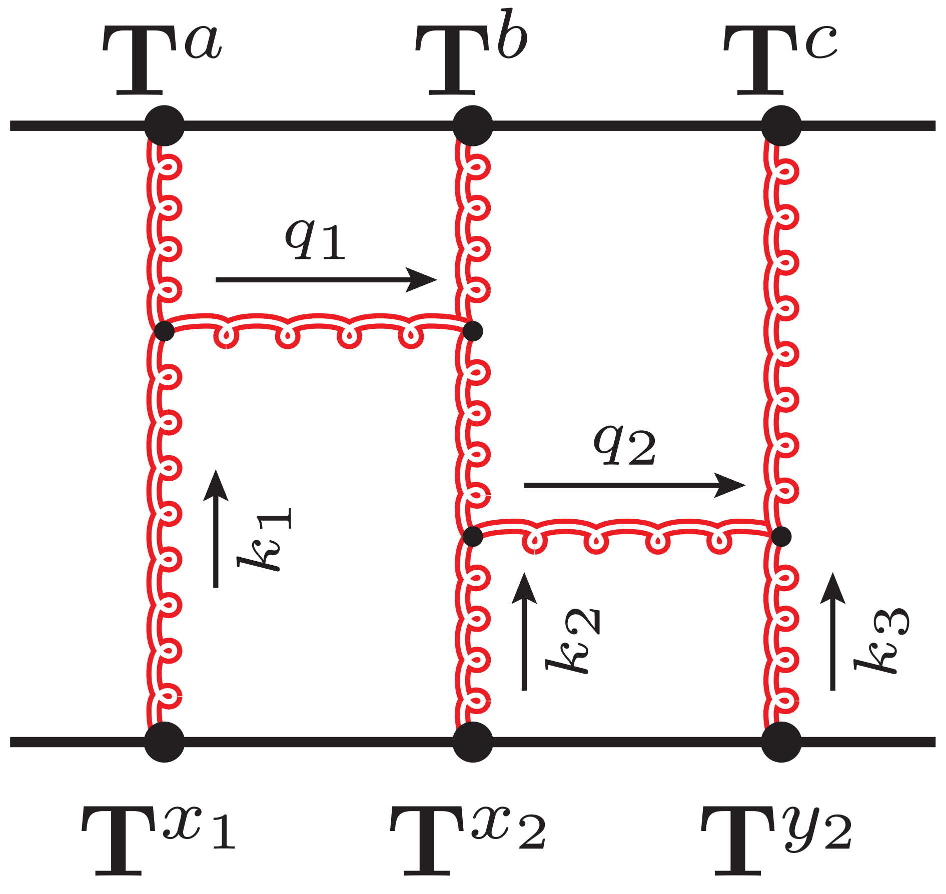
[Del Duca, Falcioni, Magnea, Vernazza '14]

But until recently unknown how to account for it



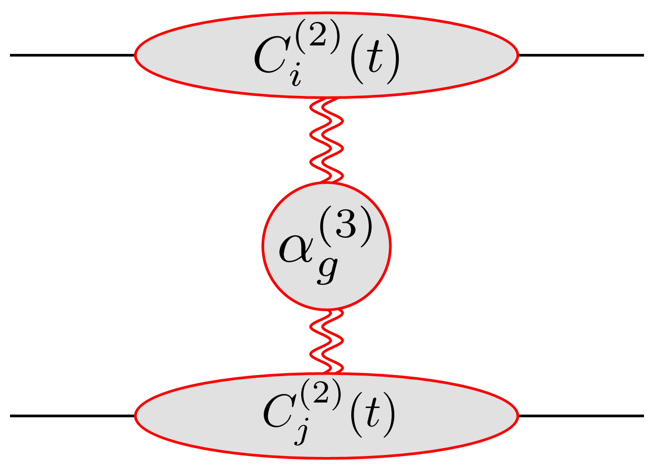
Signature-odd 2 → 2 amplitudes: understanding the NNLL tower

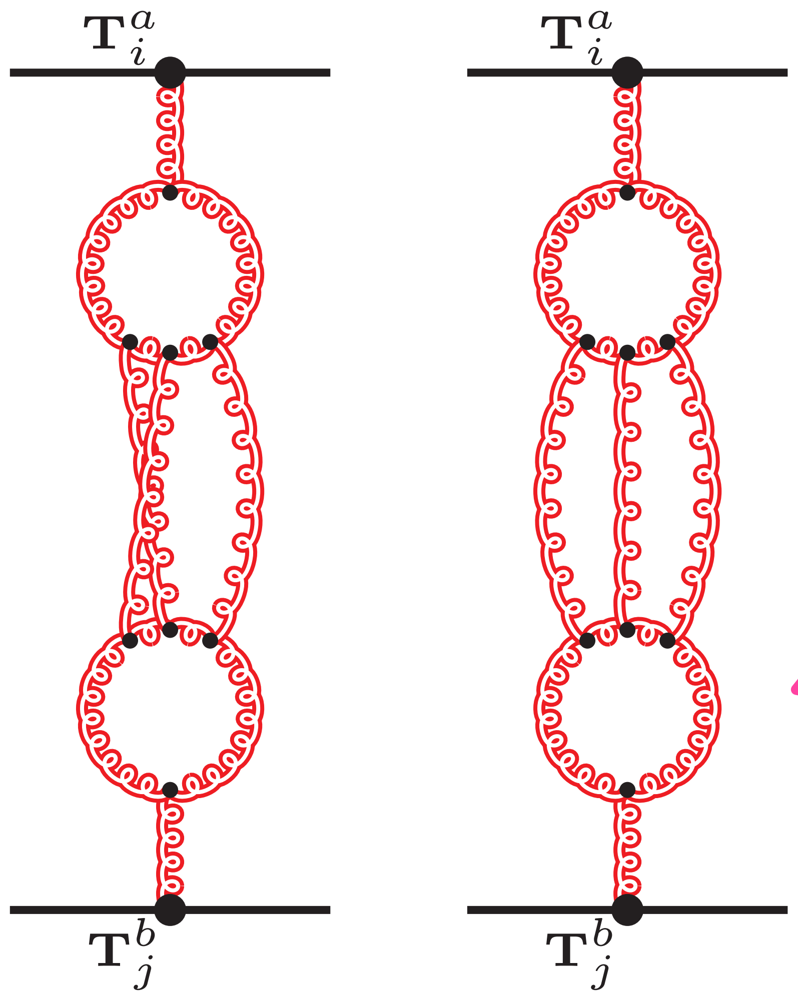
- Using **non-linear** rapidity evolution, the NNLL tower is determined to all orders in terms of **one** and **three** Reggeon exchanges

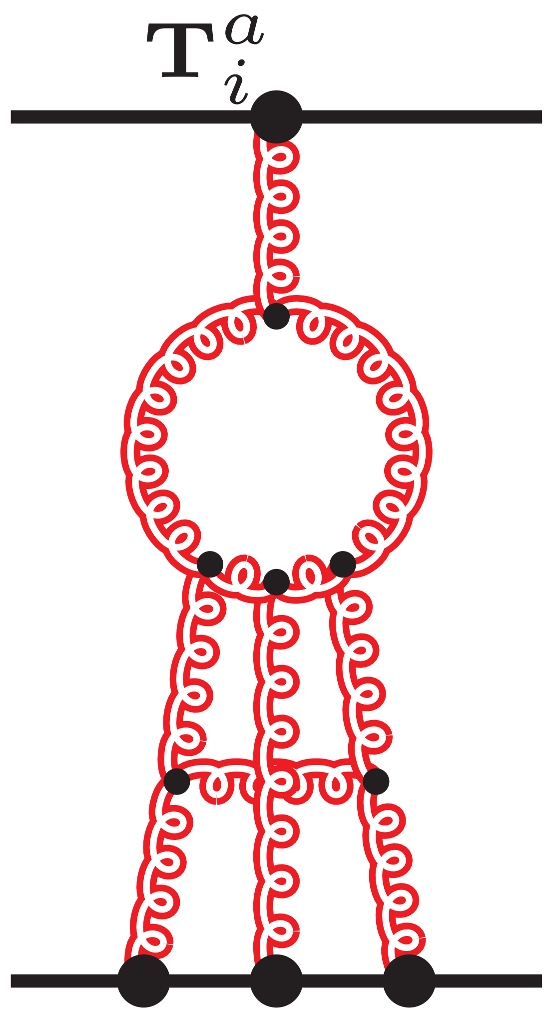


- Expanding in $X \equiv \dfrac{\alpha_s}{\pi} r_\Gamma L$

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-),\mathrm{NNLL}}_{ij\to ij} = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg\{ \boxed{\langle j_1|i_1\rangle^{\mathrm{NNLO}}} + r_\Gamma^2\pi^2 \sum_{k=0}^{\infty} \frac{(-X)^k}{k!} \Bigg[\boxed{\langle j_3|\hat{H}^k_{3\to3}|i_3\rangle} + \Theta(k\geq 1)\left[\langle j_1|\hat{H}_{3\to1}\hat{H}^{k-1}_{3\to3}|i_3\rangle + \boxed{\langle j_3|\hat{H}^{k-1}_{3\to3}\hat{H}_{1\to3}|i_1\rangle}\right] + \Theta(k\geq 2)\boxed{\langle j_1|\hat{H}_{3\to1}\hat{H}^{k-2}_{3\to3}\hat{H}_{1\to3}|i_1\rangle} \Bigg]^{\mathrm{LO}} \Bigg\}$$







- All diagrams computed to four loops

Caron-Huot, EG, Vernazza
JHEP 06 (2017) 016
Falcioni, EG, Milloy, Vernazza
Phys. Rev. D 103 (2021) L111501

Signature odd $2 \to 2$ amplitude at NNLL: Regge pole and cut, and the cancellation of planar contributions

Having defined a Reggeon $W^a(\mathbf{x})$ by $U(\mathbf{x}) = e^{i g_s T^a W^a(\mathbf{x})}$ we obtain definite predictions for MR

Requiring that the Regge cut is strictly non-planar fixes the separation between Regge pole vs. Regge cut

Falcioni, EG, Maher, Milloy, Vernazza
Phys.Rev.Lett. 128 (2022) 13, 13;
JHEP 03 (2022) 053

$$\begin{aligned}
\mathcal{M}^{(-)}_{ij\to ij} &= \underbrace{\mathcal{M}^{(-)\,\mathrm{SR}}_{ij\to ij} + \mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}} + \mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{nonplanar}} \\
&= \qquad\qquad \mathcal{M}^{(-)\,\mathrm{pole}}_{ij\to ij} \qquad\qquad + \mathcal{M}^{(-)\,\mathrm{cut}}_{ij\to ij}
\end{aligned}$$

$$\mathcal{M}^{(-)\,\mathrm{pole}}_{ij\to ij} = C_i(t)\, e^{\alpha_g(t)\, C_A\, L}\, C_j(t) \mathcal{M}^{\mathrm{tree}}_{ij\to ij}$$

$\mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}$ must be **universal** (gg, gq, qq) to be absorbed in the factorizing pole term.

$\mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}$ ***cannot*** contribute beyond 3 loops: the NNLL Regge pole term has **no** free parameters!

Indeed, at 4 loops **planar contributions conspire to cancel!**

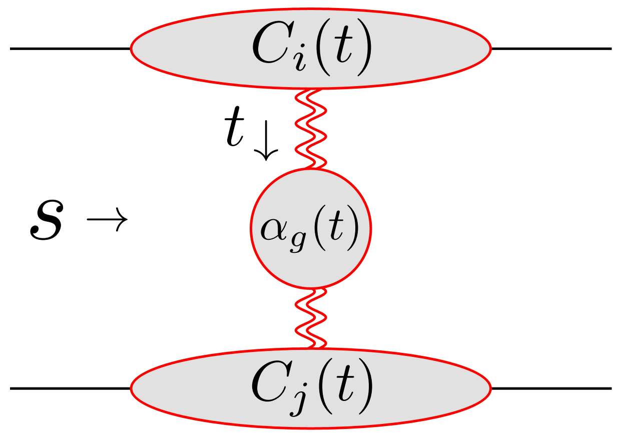
Partial cancellation of planar contributions between MR channels occurs at 3 loops, and becomes complete at 4 loops.
Prediction: MR at 5 loop and above must also vanish in planar limit.

Signature odd amplitude at NNLL: properties of Regge pole and cut

All-order structure through NNLL for any gauge theory, any representation:

$$\mathcal{M}^{(-)}_{ij\to ij} = Z_i(t)\,\bar{D}_i(t)\,Z_j(t)\,\bar{D}_j(t)\left[\left(\frac{-s}{-t}\right)^{C_A\alpha_g(t)} + \left(\frac{-u}{-t}\right)^{C_A\alpha_g(t)}\right]\mathcal{M}^{\text{tree}}_{ij\to ij} + \sum_{n=2}^{\infty} a^n L^{n-2}\mathcal{M}^{(\pm,n,n-2)\,\text{cut}}$$

Regge pole



- single Reggeon; colour octet
- dominant in planar limit
- factorizes; parameters at NNLL fully fixed at 3 loops in QCD upon matching to Caola et al. JHEP 10 (2021) 206

- IR singularities in impact factors Z_i, Z_j and trajectory fixed by the dipole formula. Singularities of the Regge trajectory are related to the cusp anomalous dimension, generalising the 2-loop relation by Korchemskaya & Korchemsky (1995).

$$\alpha_g(t) = -\frac{1}{4}\int_0^{-t}\frac{d\lambda^2}{\lambda^2}\gamma_K(\alpha_s(\lambda^2)) + \mathcal{O}(\epsilon^0)$$

Regge cut

- multiple Reggeons; various colour reps.
- suppressed in planar limit
- proportional to π^2
- no factorization: non-trivial at all loops
- no dependence on the matter content: the same for any gauge theory!

- Sensitive to soft singularities beyond the dipole formula.

Signature odd 2 → 2 amplitude at NNLL: results

- All-order structure through NNLL for any gauge theory, any representation:

$$\boxed{\mathcal{M}^{(-)}_{ij\to ij} = Z_i(t)\,\bar{D}_i(t)\,Z_j(t)\,\bar{D}_j(t)\left[\left(\frac{-s}{-t}\right)^{C_A\alpha_g(t)} + \left(\frac{-u}{-t}\right)^{C_A\alpha_g(t)}\right]\mathcal{M}^{\rm tree}_{ij\to ij} + \sum_{n=2}^{\infty} a^n L^{n-2}\mathcal{M}^{(\pm,n,n-2)\,{\rm cut}}}$$

- Explicit results for the **cut** through 4 loops:

$$\hat{\mathcal{M}}^{\rm cut}_{ij\to ij} = e^{-\mathbf{T}_t^2\,\alpha_g(t)\,L}\,\mathcal{M}^{\rm cut}_{ij\to ij}$$

$$\hat{\mathcal{M}}^{(-,2,0)\,{\rm cut}}_{ij\to ij} = \pi^2(r_\Gamma)^2 S^{(2)}(\epsilon)\left[(\mathbf{T}^2_{s-u})^2 - \frac{C_A^2}{4}\right]\mathcal{M}^{\rm tree}_{ij\to ij}\,,$$

$$\hat{\mathcal{M}}^{(-,3,1)\,{\rm cut}}_{ij\to ij} = -\pi^2(r_\Gamma)^3\left[S_A^{(3)}(\epsilon)\mathbf{T}^2_{s-u}[\mathbf{T}^2_{s-u},\mathbf{T}^2_t] + S_B^{(3)}(\epsilon)[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\mathbf{T}^2_{s-u}\right]\mathcal{M}^{\rm tree}_{ij\to ij}$$

$$\hat{\mathcal{M}}^{(-,4,2)\,{\rm cut}}_{ij\to ij} = \pi^2\,(r_\Gamma)^4\left[\frac{\mathbf{K}^{(4)}}{2\epsilon^4} + \left(\frac{1}{2\epsilon}\zeta_3 + \frac{3}{4}\zeta_4\right)\mathbf{K}^{(1)} + \mathcal{O}(\epsilon)\right]\mathcal{M}^{\rm tree}_{ij\to ij}$$

Confirmed using lower order soft anom. dim.

New constraint on the 4-loop soft anom. dim.

$$S^{(2)}(\epsilon) = -\frac{1}{8\epsilon^2} + \frac{3}{4}\epsilon\zeta_3 + \frac{9}{8}\epsilon^2\zeta_4 + \mathcal{O}(\epsilon^3)$$

$$S_A^{(3)}(\epsilon) = \frac{1}{48\epsilon^3} + \frac{37}{24}\zeta_3 + \frac{37}{16}\epsilon\,\zeta_4 + \mathcal{O}(\epsilon^2),$$

$$S_B^{(3)}(\epsilon) = \frac{1}{24\epsilon^3} + \frac{1}{12}\zeta_3 + \frac{1}{8}\epsilon\,\zeta_4 + \mathcal{O}(\epsilon^2)$$

$$\begin{aligned}\mathbf{K}^{(4)} &= \frac{1}{96}\Big[\mathbf{T}^2_{s-u},[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\Big]\mathbf{T}^2_t + \frac{7}{576}\mathbf{T}^2_t\left[\left(\mathbf{T}^2_{s-u}\right)^2,\mathbf{T}^2_t\right]\\ &\quad - \frac{1}{192}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t\mathbf{T}^2_{s-u} - \frac{5}{192}\mathbf{T}^2_{s-u}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t,\\ \mathbf{K}^{(1)} &= \frac{49}{48}\Big[\mathbf{T}^2_{s-u},[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\Big]\mathbf{T}^2_t - \frac{47}{288}\mathbf{T}^2_t\left[\left(\mathbf{T}^2_{s-u}\right)^2,\mathbf{T}^2_t\right]\\ &\quad + \frac{101}{96}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t\mathbf{T}^2_{s-u} - \frac{49}{48}\mathbf{T}^2_{s-u}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t + \frac{1}{24}\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\right)\end{aligned}$$

Regge pole and Regge cuts in full colour

Progress of the past few years:

- Simplification of multi-loop amplitude computation in the Regge limit using **rapidity evolution equations (2 dim!)**

- Results based on direct computation of **Multi-Reggeon contributions**
  - NLL for the signature even amplitudes (all orders)
  - NNLL for the signature odd amplitudes (currently to four loops).

- **The cut is non-planar**, allowing us to separate between Regge pole and cut contributions,
  and determine the Regge-pole parameters at NNLL from three-loop amplitudes.
  The singularities of the gluon Regge trajectory are related to the cusp anomalous dimension.
  Constraints on the soft anomalous dimension.

Future prospects:

- Scope for applications to 5-point two-loop amplitudes and beyond:
  Caron-Huot, Chicherin, Henn, Zhang, Zoia, JHEP 10 (2020) 188
  On-going work with Samuel Abreu, Giulio Falcioni, Calum Milloy and Leonardo Vernazza.

- Complementary approaches: Expansion by Regions, Glauber SCET [Rothstein, Stewart, …]

Regge-pole factorisation for multi-leg amplitudes in MRK

Multi-Regge Kinematics (MRK)

4-momentum $p = (p^+, p^-; \mathbf{p})$

target $p_1 = (0, p_1^-; \mathbf{0})$

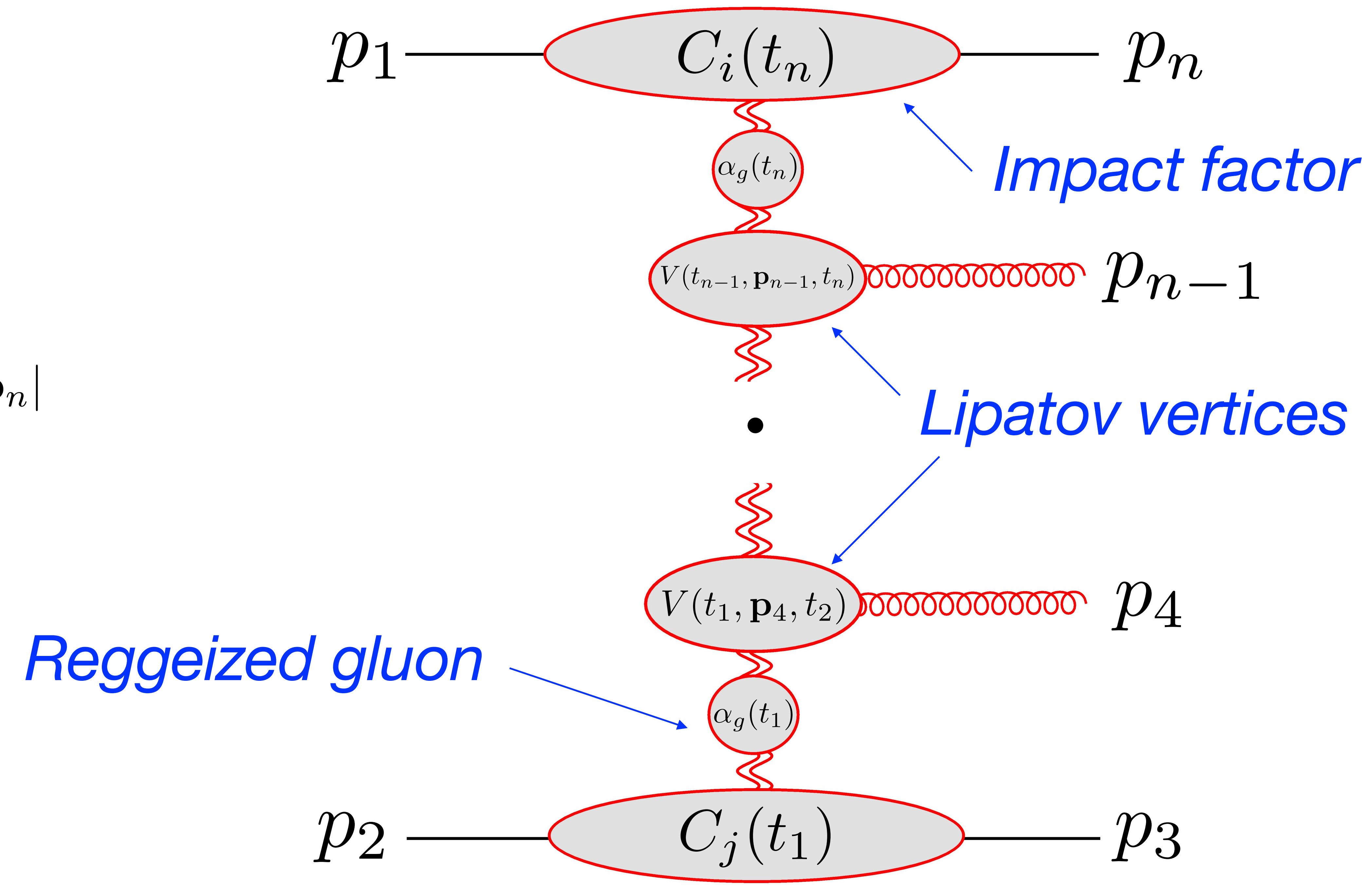
projectile $p_2 = (p_2^+, 0; \mathbf{0})$

strong hierarchy of light-cone components $p_3^+ \gg p_4^+ \gg ... \gg p_n^+$

$p_3^- \ll p_4^- \ll ... \ll p_n^-$

no ordering of transverse components $|\mathbf{p}_3| \sim |\mathbf{p}_4| \sim ... \sim |\mathbf{p}_n|$

Regge (pole) factorization in MRK

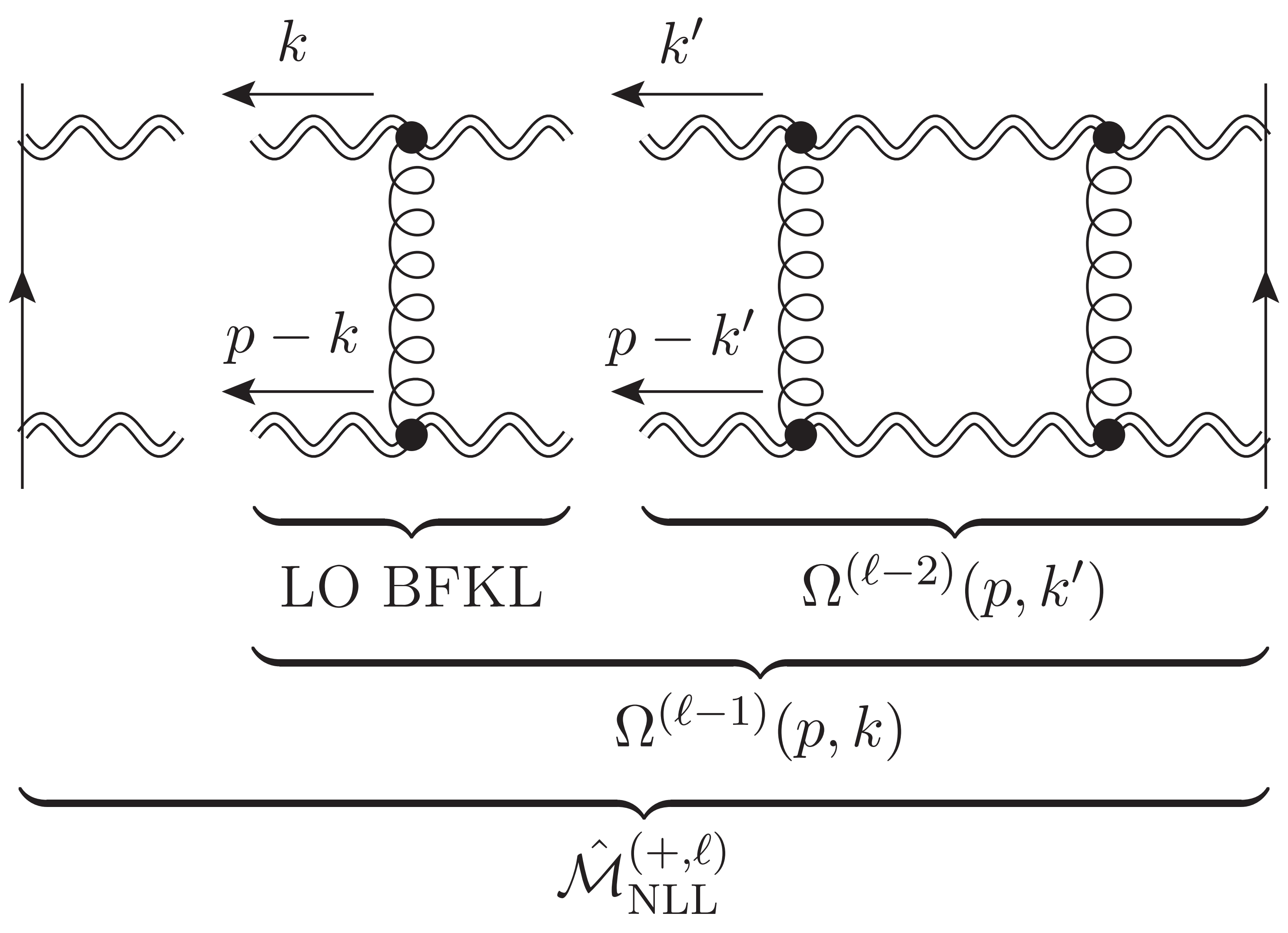


Regge (pole) factorization holds in MRK for the dispersive (real part) of the amplitudes through NLL; established using unitarity [Fadin et al. 2006]

Planar limit:

- Four- and five-point planar amplitudes have only Regge poles. Essential for the BDS ansatz in SYM.
- Six and higher point planar amplitudes have also Regge cuts in some (non-physical) kinematic regions [Bartels, Lipatov, Sabio Vera (2008)]

All orders solution for the soft approximation



Solving for the wavefunction in the soft approximation:

$$J_s(p,k) = \frac{1}{2\epsilon}\left[1-\left(\frac{p^2}{k^2}\right)^{\epsilon}\right] \qquad \xi \equiv \left(p^2/k^2\right)^{\epsilon}$$

$$\int[\mathrm{D}k']\,\frac{2(k\cdot k')}{k'^2(k-k')^2}\left(\frac{p^2}{k'^2}\right)^{n\epsilon} = -\frac{1}{2\epsilon}\frac{B_n(\epsilon)}{B_0(\epsilon)}\left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}$$

with $\quad B_n(\epsilon) = e^{\epsilon\gamma_{\rm E}}\dfrac{\Gamma(1-\epsilon)}{\Gamma(1+n\epsilon)}\dfrac{\Gamma(1+\epsilon+n\epsilon)\Gamma(1-\epsilon-n\epsilon)}{\Gamma(1-2\epsilon-n\epsilon)}\,.$

$$\hat{B}_n(\epsilon) = 1-\frac{B_n(\epsilon)}{B_0(\epsilon)} = 2n(2+n)\zeta_3\epsilon^3+3n(2+n)\zeta_4\epsilon^4+\ldots.$$

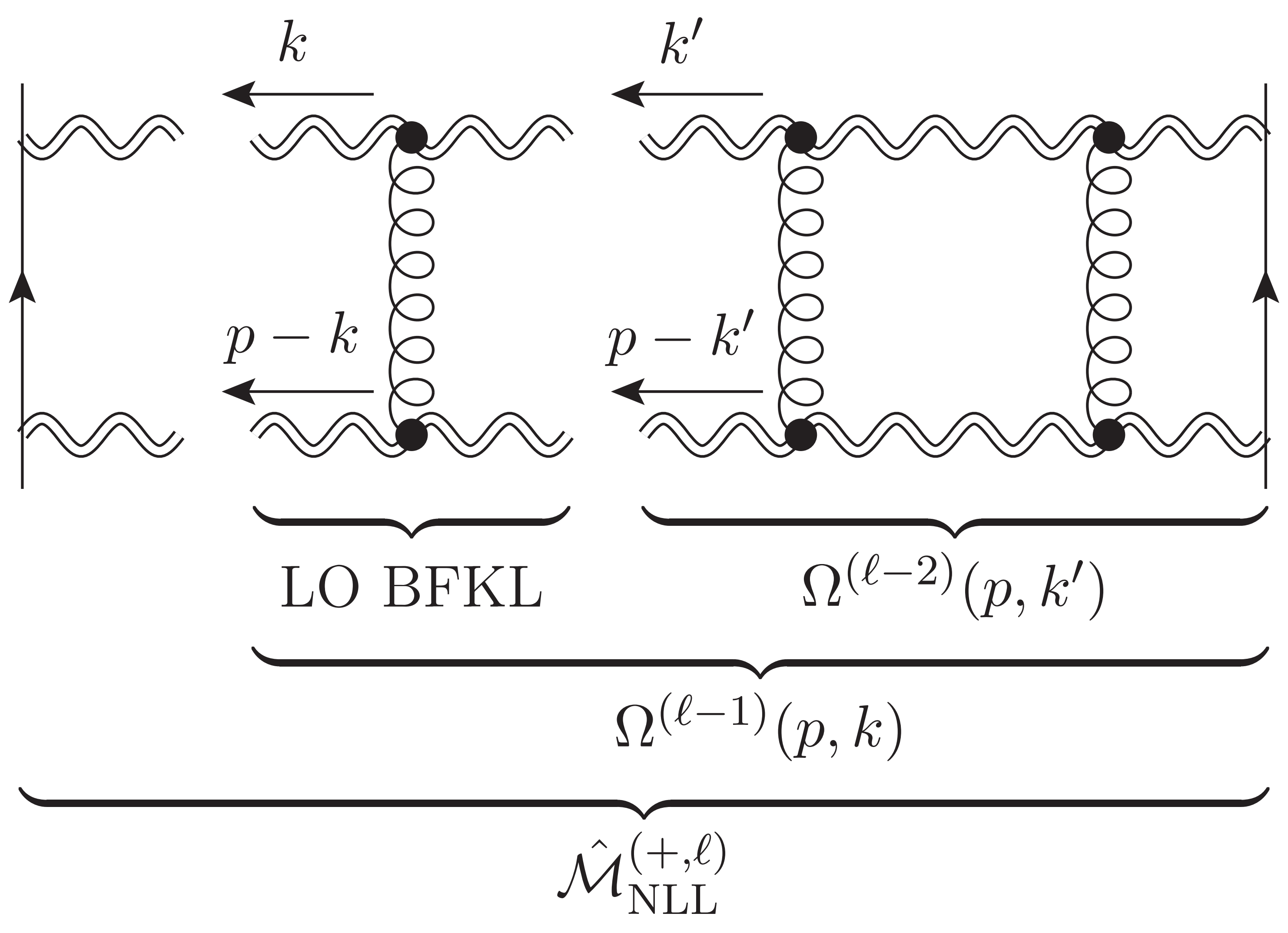
$$\begin{aligned}
\Omega^{(0)}(\xi) =& 1,\\
\Omega^{(1)}(\xi) =& \frac{(C_A-\mathbf{T}_t)}{2\epsilon}\Big(1-\xi\Big),\\
\Omega^{(2)}(\xi) =& \frac{(C_A-\mathbf{T}_t)^2}{(2\epsilon)^2}\left\{1-2\xi+\xi^2\left[1-\hat{B}_1(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\right\},\\
\Omega^{(3)}(\xi) =& \frac{(C_A-\mathbf{T}_t)^3}{(2\epsilon)^3}\left\{1-3\xi+3\xi^2\left[1-\hat{B}_1(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\right.\\
&\left.-\,\xi^3\left[1-\hat{B}_1(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\left[1-\hat{B}_2(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\right\}
\end{aligned}$$

All-order result:

$$\Omega^{(\ell-1)}(p,k) = \frac{(C_A-\mathbf{T}_t)^{\ell-1}}{(2\epsilon)^{\ell-1}}\sum_{n=0}^{\ell-1}(-1)^n\binom{\ell-1}{n}\left(\frac{p^2}{k^2}\right)^{n\epsilon}\prod_{m=0}^{n-1}\left\{1-\hat{B}_m(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right\}$$

The amplitude in the soft approximation

Having solved for the wavefunction we can compute the amplitude.



Summing over the two soft limits, we get (at any given order):

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!} \int [{\rm D}k] \, \frac{p^2}{k^2(p-k)^2} \, \Omega^{(\ell-1)}(p,k) \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(\rm tree)}$$

All IR divergences can be resumed into a closed form expression:

$$\boxed{\hat{\mathcal{M}}^{(+)}_{\rm NLL}\Big|_{\rm IR} = \frac{i\pi}{L(C_A-\mathbf{T}_t^2)} \left(1 - R(\epsilon)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1} \left[\exp\left\{\frac{B_0(\epsilon)}{2\epsilon}\frac{\alpha_s}{\pi} L(C_A-\mathbf{T}_t)\right\} - 1\right] \mathbf{T}^2_{s-u}\,\mathcal{M}^{(\rm tree)} + \mathcal{O}(\epsilon^0).}$$

$$R(\epsilon) \equiv \frac{B_0(\epsilon)}{B_{-1}(\epsilon)} - 1 = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1$$
$$= -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(10\zeta_6 - 2\zeta_3^2\right)\epsilon^6 + \mathcal{O}(\epsilon^7).$$

Signature-even 2 → 2 amplitude: Iterating the BFKL Hamiltonian in two dimensions

Caron-Huot, EG, Reichel, Vernazza, JHEP 08 (2020) 116

The 2d wavefunction computed in terms of pure Single-Valued Harmonic Polylogarithms (SVHPLs)

$$\Omega_{2d}^{(\ell-1)}(z,\bar{z}) = \hat{H}_{2d}\Omega_{2d}^{(\ell-2)}(z,\bar{z})$$

The action of the Hamiltonian on SVHPL amounts to the following DEs:

$$\begin{aligned}
\frac{d}{dz}\hat{H}_{2\mathrm{d,i}}\mathcal{L}_{0,\sigma}(z,\bar{z}) &= \frac{\hat{H}_{2\mathrm{d,i}}\mathcal{L}_{\sigma}(z,\bar{z})}{z} \\
\frac{d}{dz}\hat{H}_{2\mathrm{d,i}}\mathcal{L}_{1,\sigma}(z,\bar{z}) &= \frac{\hat{H}_{2\mathrm{d,i}}\mathcal{L}_{\sigma}(z,\bar{z})}{1-z} - \frac{1}{4}\frac{\mathcal{L}_{1,\sigma}(z,\bar{z})}{z} \\
&\quad - \frac{1}{4}\frac{\mathcal{L}_{0,\sigma}(z,\bar{z}) + 2\mathcal{L}_{1,\sigma}(z,\bar{z}) - [\mathcal{L}_{0,\sigma}(w,\bar{w}) + \mathcal{L}_{1,\sigma}(w,\bar{w})]_{w,\bar{w}\to\infty}}{1-z}
\end{aligned}$$

An algorithm is set up to iteratively determine the wavefunction to any loop order. Computed explicitly to 12 loops.

$$\begin{aligned}
\Omega_{2\mathrm{d}}^{(1)} &= \frac{1}{2}C_2\left(\mathcal{L}_0 + 2\mathcal{L}_1\right) \\
\Omega_{2\mathrm{d}}^{(2)} &= \frac{1}{2}C_2^2\left(\mathcal{L}_{0,0} + 2\mathcal{L}_{0,1} + 2\mathcal{L}_{1,0} + 4\mathcal{L}_{1,1}\right) + \frac{1}{4}C_1C_2\left(-\mathcal{L}_{0,1} - \mathcal{L}_{1,0} - 2\mathcal{L}_{1,1}\right) \\
\Omega_{2\mathrm{d}}^{(3)} &= \frac{3}{4}C_2^3\left(\mathcal{L}_{0,0,0} + 2\mathcal{L}_{0,0,1} + 2\mathcal{L}_{0,1,0} + 4\mathcal{L}_{0,1,1} + 2\mathcal{L}_{1,0,0} + 4\mathcal{L}_{1,0,1} + 4\mathcal{L}_{1,1,0} + 8\mathcal{L}_{1,1,1}\right) \\
&\quad + \frac{1}{4}C_1C_2^2\left(2\zeta_3 - 2\mathcal{L}_{0,0,1} - 3\mathcal{L}_{0,1,0} - 7\mathcal{L}_{0,1,1} - 2\mathcal{L}_{1,0,0} - 7\mathcal{L}_{1,0,1} - 7\mathcal{L}_{1,1,0} - 14\mathcal{L}_{1,1,1}\right) \\
&\quad + \frac{1}{16}C_1^2C_2\left(\mathcal{L}_{0,0,1} + 2\mathcal{L}_{0,1,0} + 4\mathcal{L}_{0,1,1} + \mathcal{L}_{1,0,0} + 4\mathcal{L}_{1,0,1} + 4\mathcal{L}_{1,1,0} + 8\mathcal{L}_{1,1,1}\right)
\end{aligned}$$

The full signature-even amplitude at NLL

Caron-Huot, EG, Reichel, Vernazza
JHEP 1803 (2018) 098
JHEP 08 (2020) 116

The **soft** wavefunction alone generates all IR singularities in the amplitude.
We can therefore split the wavefunction into **soft** and **hard**: $\Omega(p,k) = \Omega_{\rm hard}(p,k) + \Omega_{\rm soft}(p,k)$
and use dim. reg. only for the **soft**: $\Omega^{(2d)}_{\rm hard}(z,\bar z) \equiv \lim_{\epsilon\to 0}\Omega_{\rm hard} = \Omega^{(2d)}(z,\bar z) - \Omega^{(2d)}_{\rm soft}(z,\bar z)$

The full amplitude is recovered by summing two integrals:

$$\hat{\mathcal{M}}^{(+,\,\rm NLL)}_{ij\to ij}\left(\frac{s}{-t}\right) = -i\pi\left[\int [{\rm D}k]\frac{p^2}{k^2(p-k)^2}\Omega_{\rm soft}(p,k) + \frac{1}{4\pi}\int\frac{d^2z}{z\bar z}\Omega^{(2d)}_{\rm hard}(z,\bar z)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree)}_{ij\to ij}$$

We explicitly computed it to **13 loops**.
The first few orders are:

$$\hat{\mathcal{M}}^{(+,1,0)} = i\pi\, r_\Gamma\left\{\frac{1}{2\epsilon}\right\}\mathbf{T}^2_{s-u}\mathcal{M}^{\rm tree},$$

$$\hat{\mathcal{M}}^{(+,2,1)} = i\pi\,\frac{r_\Gamma^2}{2}\left\{-\frac{1}{4\epsilon^2}\right\}[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]\mathcal{M}^{\rm tree},$$

$$\hat{\mathcal{M}}^{(+,3,2)} = i\pi\,\frac{r_\Gamma^3}{3!}\left\{\frac{1}{8\epsilon^3}-\frac{11\zeta_3}{4}\right\}[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]]\mathcal{M}^{\rm tree},$$

$$\hat{\mathcal{M}}^{(+,4,3)} = i\pi\,\frac{r_\Gamma^4}{4!}\left\{-\left(\frac{\zeta_3}{8\epsilon}+\frac{3\zeta_4}{16}\right)[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]]\mathbf{T}^2_t - \frac{1}{16\epsilon^4}[\mathbf{T}^2_t,[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]]]\right\}\mathcal{M}^{\rm tree}$$

Constraint on the
4-loop soft anom. dim.

The soft amplitude can be resummed to all orders in $x = \frac{\alpha_s}{\pi}L$:

$$\hat{\mathcal{M}}_{\rm NLL,s} = \frac{i\pi}{L(C_A-\mathbf{T}^2_t)}\left\{\left(e^{\frac{B_0}{2\epsilon}(C_A-\mathbf{T}^2_t)x}-1\right)\left(1-\frac{C_A}{(C_A-\mathbf{T}^2_t)}R(\epsilon)\right)^{-1}+1\right.$$
$$\left. - e^{-\gamma_E(2C_A-\mathbf{T}^2_t)\,x}\,\frac{\Gamma\left(1-(C_A-\mathbf{T}^2_t)x\right)}{\Gamma\left(1+(C_A-\mathbf{T}^2_t)x\right)}\left(\frac{\Gamma\left(1+(C_A-\mathbf{T}^2_t)\frac{x}{2}\right)}{\Gamma\left(1-(C_A-\mathbf{T}^2_t)\frac{x}{2}\right)}\right)^{-\frac{\mathbf{T}^2_t}{C_A-\mathbf{T}^2_t}}\right\}\mathbf{T}^2_{s-u}\mathcal{M}^{\rm tree}$$

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1$$

A resummed result for the finite, hard part
(with SV MZVs) is yet unknown.

The soft anomalous dimension in the high-energy limit — three loops

$$\mathbf{\Gamma}_{ij\to ij}\left(\alpha_s, L, \frac{-t}{\lambda^2}\right) = \Gamma_i\left(\alpha_s, \frac{-t}{\lambda^2}\right) + \Gamma_j\left(\alpha_s, \frac{-t}{\lambda^2}\right) + \frac{1}{2}\gamma_K(\alpha_s)\left[L\mathbf{T_t^2} + i\pi\mathbf{T_{s-u}^2}\right] + \sum_{\ell=3}^{\infty}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^{\ell}\mathbf{\Delta}^{(\ell)}(L)$$

$$L \equiv \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right) = \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}$$

$$\mathbf{\Delta}^{(3)} = 0L^2 + i\pi\left[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]\right]\frac{11}{4}\zeta_3 L + O(L^0)$$

Ø. Almelid, C. Duhr, EG Phys. Rev. Lett. **117**, 172002

Absence of $\alpha_s^3 L^k$ for $k \geq 1$ in the *real part* and for $k \geq 2$ in the *imaginary part*, is a non-trivial prediction from rapidity evolution, which underpins the structure of corrections to the dipole formula.

Caron-Huot, EG, Vernazza JHEP 06 (2017) 016

The **only** term in the *real part* of the soft anom. dim. linear in the high-energy logarithm is the cusp anomalous dimension, **generalising** the Korchemskaya & Korchemsky relation between the gluon Regge trajectory and cusp to 3 loops.

The soft anomalous dimension in the high-energy limit — four loops

Falcioni, EG, Maher, Milloy, Vernazza (2021)

$$\mathbf{\Gamma}_{ij\to ij}\left(\alpha_s, L, \frac{-t}{\lambda^2}\right) = \Gamma_i\left(\alpha_s, \frac{-t}{\lambda^2}\right) + \Gamma_j\left(\alpha_s, \frac{-t}{\lambda^2}\right) + \frac{1}{2}\gamma_K(\alpha_s)\left[L\mathbf{T_t^2} + i\pi\mathbf{T_{s-u}^2}\right] + \sum_{\ell=3}^{\infty}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^{\ell}\mathbf{\Delta}^{(\ell)}(L)$$

$$L \equiv \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right) = \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}$$

$$\begin{aligned}\mathbf{\Delta}^{(4)}(L) = & -L^3 i\pi\frac{\zeta_3}{24}\Big[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\Big]\mathbf{T}_t^2 + L^2\mathbf{\Delta}^{(-,4,2)} \\ & + L^2\zeta_2\zeta_3\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24} - \frac{1}{4}\mathbf{T}_t^2[(\mathbf{T}_{s-u}^2)^2,\mathbf{T}_t^2] + \frac{3}{4}[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2\right) \\ & + \mathcal{O}(L)\end{aligned}$$

All Regge-limit constraints at four loops:

Signature even				Signature odd			
	L^3	L^2	L^1 (conj.)		L^3	L^2	L^1
$\mathcal{F}_A^{(+,4)}$	0	$-\frac{C_A}{8}\zeta_2\zeta_3$	0	$\mathcal{F}_A^{(-,4)}$	$i\pi\frac{C_A}{24}\zeta_3$	?	?
$\mathcal{F}_F^{(+,4)}$	0	0	0	$\mathcal{F}_F^{(-,4)}$	0	?	?
$\mathcal{G}_A^{(+,4)}$	0	$\frac{1}{2}\zeta_2\zeta_3$	$\frac{1}{6}g_A^{(4)}$				
$\mathcal{G}_F^{(+,4)}$	0	0	$\frac{1}{6}g_F^{(4)}$				
$\mathcal{H}_1^{(+,4)}$	0	0	0	$\mathcal{H}_1^{(-,4)}$	0	?	?
				$\tilde{\mathcal{H}}_1^{(-,4)}$	0	?	?