MULTILOOP SCATTERING AMPLITUDES FOR COLLIDER PHYSICS

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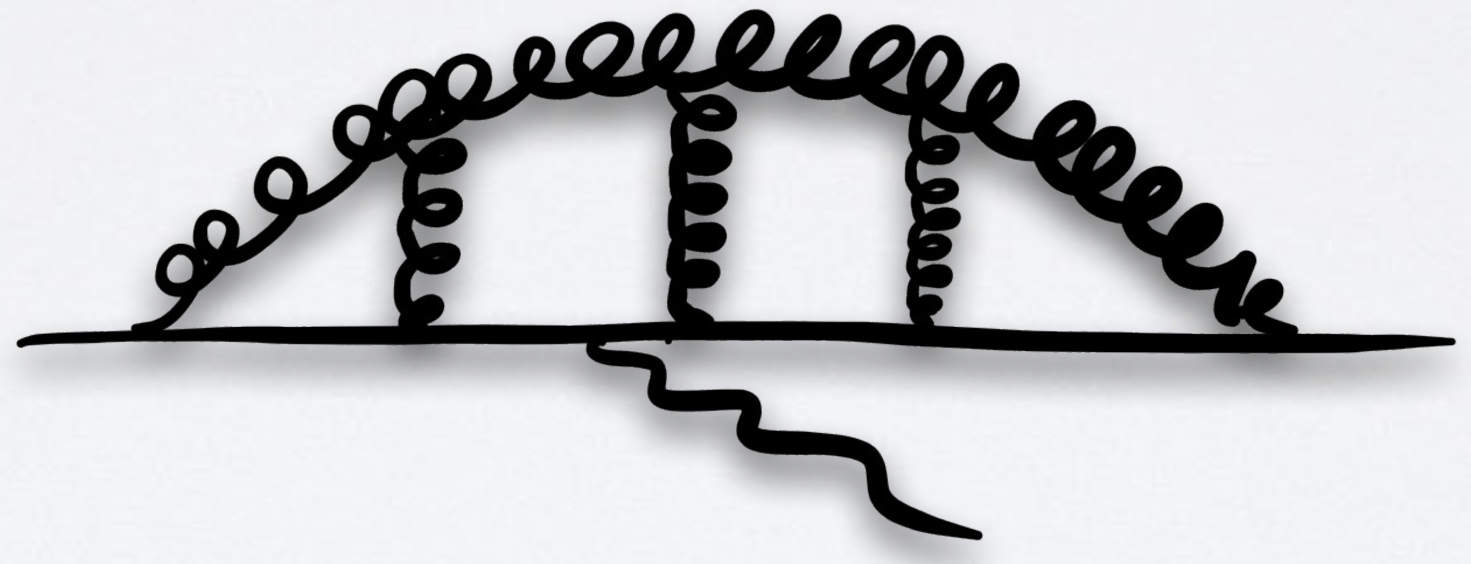


Theory Challenges in the Precision Era of the Large Hadron Collider

Kick-off Conference, August 28 - September 1, 2023, Galileo Galilei Institute

SCATTERING AMPLITUDES

- **Precision predictions at LHC** obtained with **higher order perturbative calculations**
- S matrix: $S_{fi} = \langle f, \text{out} \,|\, i, \text{in} \rangle = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p_f - p_i)\mathcal{M}_{fi}$
- $|\mathcal{M}_{fi}|^2$ has probabilistic interpretation
- Partonic cross section $d\hat{\sigma}_{fi} \propto |\mathcal{M}_{fi}|^2 d\Pi$
- Perturbative expansion of amplitude: $\mathcal{M}_{fi} = \sum_{k=0} \alpha^k \mathcal{M}^{(k)}$
- Higher order $\mathcal{M}^{(k)}$ require multi loop integrals, e.g. k loops

LOOP INTEGRANDS

- $\mathscr{M}^{(k)}$ integrand from projectors + Feynman diagrams, on-shell techniques, …
- Scheme to regularize UV and IR divergences + treatment of γ_5
- Helicity amplitude $\mathscr{M}^{(k)} = s_\lambda \mathscr{M}^{(k)}_\lambda$ where s_λ spinor factor and $\mathscr{M}^{(k)}_\lambda$ scalar
- $\mathscr{M}^{(k)}_\lambda = \sum_i c_i I_i$, different options:

I_i	c_i
Tensor integrals	Tensors
Scalar integrals	Rational scalar functions of kinematics, d
Master integrals (d-fact., phys. denom., canonical, finite, …)	Rational (algebraic,..) scalar functions
Special functions for Laurent exp. in eps (HPL, MPL, eMPL, problem specific, …)	Rational (algebraic,..) scalar functions

Requires IBP reduction

Resolving analytic features

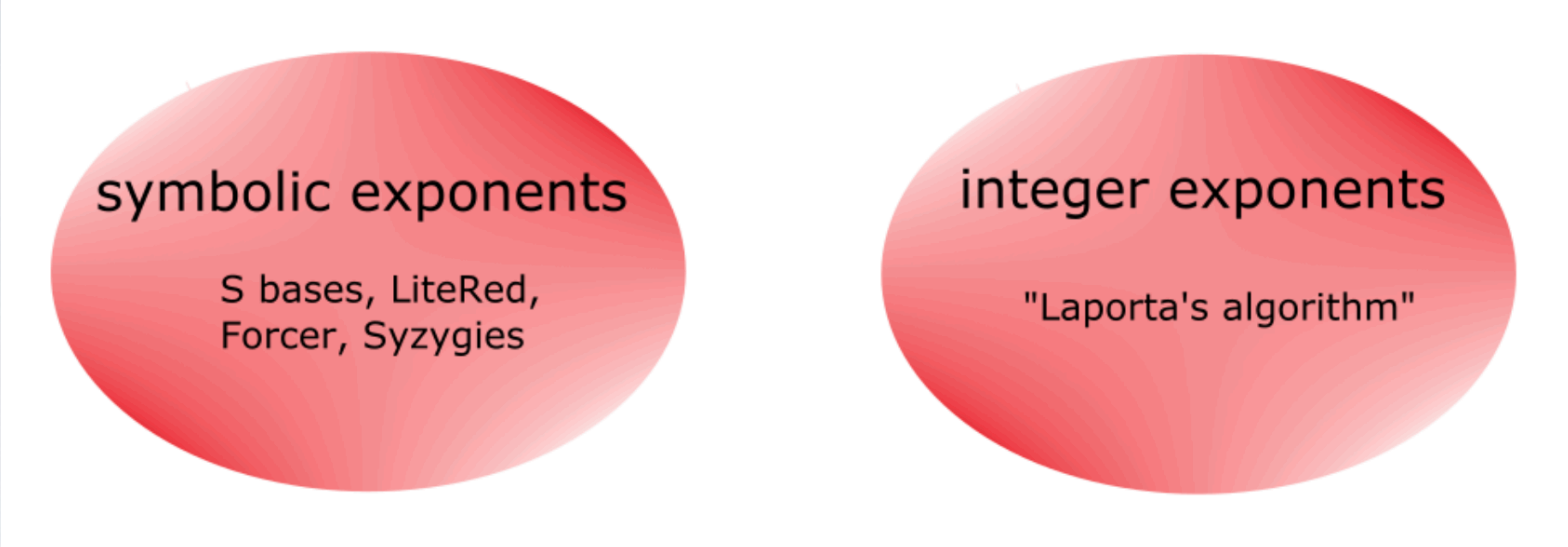
LOOP INTEGRALS

- $\mathscr{M}_\lambda^{(k)} = \sum_i c_i I_i$ amplitude through scalar loop integrals I_i (e.g. Feynman diagrams)
- $I(\nu_1, \ldots, \nu_N) = \int \frac{d^d k_1 \cdots d^d k_L}{D_1^{\nu_1} \cdots D_N^{\nu_N}}$ in dimensional regularization
- $D_j = q_j^2 - m_j^2 + i\delta$ propagator **denominator** ($\nu_j > 0$) or irreducible **numerator** ($\nu_j < 0$)
- $I(\nu_1, \ldots, \nu_N)$ for different arguments not independent, fulfill **linear relations**
- This talk:
  - **Loop integrals: reduction** through linear relations
  - **Loop integrals: evaluation**
  - **Rational coefficients** $c_i(s_{ij}, m_j^2; d)$: symbolic manipulations
  - **Results**

INTEGRAL REDUCTIONS

INTEGRATION-BY-PART (IBP) IDENTITIES

- IBP identities in dimensional regularization since integrals over total derivatives vanish:
  $$\int d^d k_1 \cdots d^d k_L \, \frac{\partial}{\partial k_i^\mu} \left(q^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right) = 0$$ where q^μ loop or ext. mom. *[Chetyrkin, Tkachov '81]*
- Expresses invariance under contin. generated lin. trans., there are also discrete symmetries
- Inserting integers for indices: **linear system of equations**, allows for systematic **reduction** *[Laporta '00]*
- Only finite number of integrals linearly independent: **basis** or **master integrals**
- In practice:
  - Reductions **computational bottleneck**
  - **Choice of basis** deserves discussion



- Various public reduction codes exists: Fire, Reduze, LiteRed, Kira, FiniteFlow, NeatIBP, …
- Calculations at the symbolic level: syzygies, Gröbner bases, …
- Calculations at the linear algebra level: finite fields, …
- Often very powerful in practice: combination of both
- Alternative approach: intersection theory

FINITE FIELDS AND RATIONAL RECONSTRUCTION

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}$$

aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients

$$\begin{array}{ccccccccc}
I_{\mathbb{Z}_p[x]} & \xrightarrow[\text{by number } x_i]{\text{sample } x} & I_{\mathbb{Z}_p, x_1} & \xrightarrow[\text{reduction}]{\text{row}} & O_{\mathbb{Z}_p, x_1} & \xrightarrow[\text{interpolation}]{\text{polynomial}} & O_{\mathbb{Z}_p[x]} & \xrightarrow[\text{reconstruction}]{\text{rational function}} & O_{\mathbb{Z}_p[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_p, x_2} & \longrightarrow & O_{\mathbb{Z}_p, x_2} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_p, x_3} & \longrightarrow & O_{\mathbb{Z}_p, x_3} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}$$

[AvM, Schabinger '14; Peraro '16; ...], note: parallelizable, multivariate e.g. by iteration

REDUCTIONS AND COMPLEXITY

ln (expression size)

$\mathscr{M} = \sum a_i I_i$ (unreduced integrals I_i)

by-pass with finite field sampling + rat-reconstr.

traditional symbolic algebra

IBP reduction

$\mathscr{M} = \sum b_i M_i$ (master integrals M_i)

partial fractioning

$\mathscr{M} = \sum pf(b_i) M_i$ (partial fractioned b_i)

analytical integration

$\mathscr{M} = \sum pf(c_i) P_i$ (transcendental functions P_i)

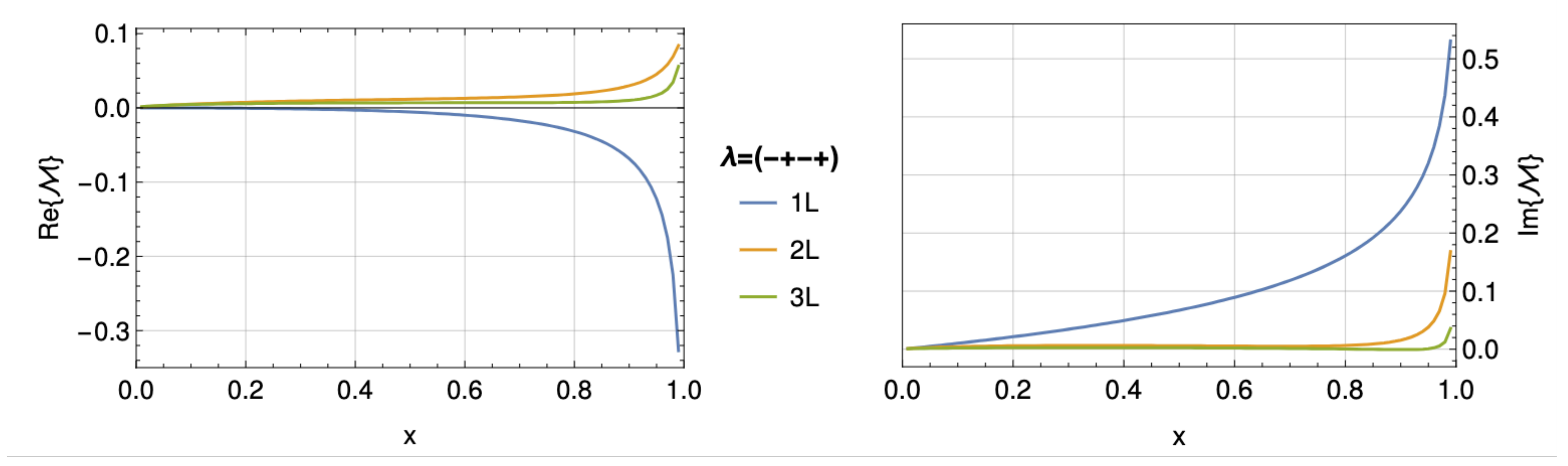
manipulations

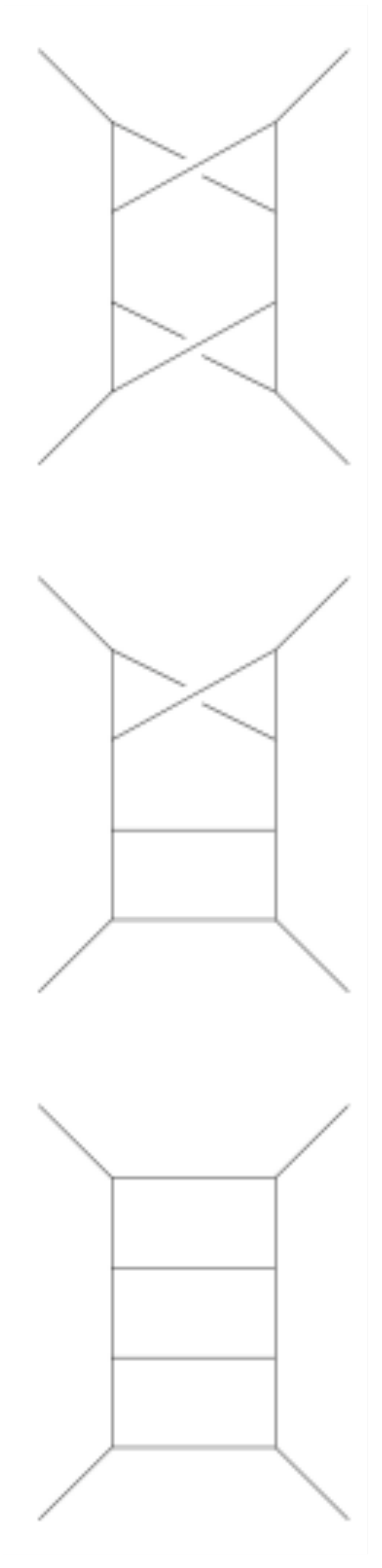
(Illustration idea by V. Sotnikov)

$gg \to \gamma\gamma$ @ 3 LOOPS

- Master integrals in terms of HPLs: *[Henn, Mistlberger, Smirnov, Wasser '20]*
- $gg \to \gamma\gamma$ helicity amplitudes: *[Bargiela, Caola, AvM, Tancredi '21]*
  - Symbolic intermediate expressions sizable but allow for easy crossings, simple workflow
  - Compact analytical results for amplitudes

	1L	2L	3L
Number of diagrams	6	138	3299
Number of inequivalent integral families	1	2	3
Number of integrals before IBPs and symmetries	209	20935	4370070
Number of master integrals	6	39	486
Size of the `Qgraf` result [kB]	4	90	2820
Size of the `Form` result before IBPs and symmetries [kB]	276	54364	19734644
Size of helicity amplitudes written in terms of MIs [kB]	12	562	304409
Size of helicity amplitudes written in terms of HPLs [kB]	136	380	1195





Syzygy Based IBPs Without Dots

Baikov's parametric representation of Feynman integrals:

$$\boxed{I(\nu_1,\ldots,\nu_N) = \mathcal{N}\int \mathrm{d}z_1\cdots\mathrm{d}z_m P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1}\cdots z_N^{\nu_N}}}$$

[Böhm, Georgoudis, Larsen, Schulze, Zhang '18]: useful for IBPs without dots

$$0 = \int \mathrm{d}z_1\cdots\mathrm{d}z_m \sum_{i=1}^{m} \frac{\partial}{\partial z_i}\left(a_i P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1}\cdots z_N^{\nu_N}}\right)$$

$$= \int \mathrm{d}z_1\cdots\mathrm{d}z_m \sum_{i=1}^{N}\left(\frac{\partial a_i}{\partial z_i} + \frac{d-L-E-1}{2P} a_i \frac{\partial P}{\partial z_i} - \frac{\nu_i a_i}{z_i}\right) P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1}\cdots z_N^{\nu_N}}$$

explicit solutions to constraint:

$$\left(\sum_{i=1}^{N} a_i \frac{\partial P}{\partial z_i}\right) + bP = 0 \qquad \text{(absence of dim. shifts)}$$

in addition, require for denominators of sector:

$$a_i = b_i z_i \qquad \text{(absence of dots)}$$

need intersection of two syzygy modules

Syzygies

- suppose that for given polynomials $f = (f_1, f_2, \ldots)$ one can find polynomials $s = (s_1, s_2, \ldots)$ such that $\sum_i f_i s_i = 0$, then s is called a syzygy
- if s is a syzygy, then $s \cdot g$ is a syzygy for any polynomial g
- the (infinite) set of syzygies for f is a syzygy module

- Reduction of numerators: “no-dot syzygies”
  [Gluza, Kajda, Kosower ‘11; Schabinger ’11; Ita ’15; Larsen, Zhang ’15; Böhm, Georgoudis, Larsen Schulze, Zhang ’18; …]

- Linear algebra approach: set degree restriction for monomials, use linear algebra with finite fields to determine syzygies (or intersections of syzygy modules) *[Agarwal, Jones, AvM ’20]*

Syzygy Based IBPs Without Numerators

[Lee-Pomeransky '13] representation:

$$I(\nu_1,\ldots,\nu_N) = \mathcal{N}\left[\prod_{i=1}^{N}\int_0^\infty \mathrm{d}x_i x_i^{\nu_i-1}\right] G^{-d/2} \qquad \text{with } G = \mathcal{U}+\mathcal{F}$$

[Bitoun, Bogner, Klausen, Panzer '17]: define (twisted) Mellin Transform

$$\mathcal{M}\{f\}(\nu) := \left(\prod_{k=1}^{N}\int_0^\infty \frac{x_k^{\nu_k-1}\mathrm{d}x_k}{\Gamma(\nu_k)}\right) f(x_1,\ldots,x_N)$$

Feynman integrals are Mellin transforms:

$$\tilde{I}(\nu) = \mathcal{M}\left\{G^{-d/2}\right\}(\nu)$$

with $\nu = (\nu_1,\ldots,\nu_N)$ and $\tilde{I}(\nu) = \Gamma[(L+1)d/2-\nu]I(\nu)$ (remark: similar for Baikov's rep.)

Properties of Mellin transform

1. $\mathcal{M}\{\alpha f+\beta g\}(\nu) = \alpha\mathcal{M}\{f\}(\nu)+\beta\mathcal{M}\{g\}(\nu)$
2. $\mathcal{M}\{x_i f\}(\nu) = \nu_i\mathcal{M}\{f\}(\nu+e_i\}$
3. $\mathcal{M}\{-\partial_i f\}(\nu) = \mathcal{M}\{f\}(\nu-e_i\}$ (proof: partial integration + surface term is zero)

Define shift operators

$$(\hat{i}^+F)(\nu_1,\ldots,\nu_N) = \nu_i F(\nu_1,\ldots,\nu_i+1,\ldots,\nu_N)$$
$$(\hat{i}^-F)(\nu_1,\ldots,\nu_N) = F(\nu_1,\ldots,\nu_i-1,\ldots,\nu_N)$$

which form Weyl algebra, $[\hat{i}^+,\hat{j}^-] = \delta_{ij}$

Shift relations from annihilators

[Lee '14; Bitoun, Bogner, Klausen, Panzer '17]: a differential operator P which annihilates $G^{-d/2}$

$$P\, G^{-d/2}$$

generates via the substitutions $x_i \to \hat{i}^+$, $\partial_i \to -\hat{i}^-$ a shift relation according to

$$\mathcal{M}\{P\, G^{-d/2}\} = 0$$

In fact, *every* shift relation is related in this way.

consider annihilators beyond linear order:

$$\left[c_0 + \sum_{i=1}^{N} c_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{N} c_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \dots \right] G^{-d/2} = 0$$

determine $c_0(x_1, \dots, x_N), \dots$ via syzygy equations:

$$c_0 \left[-\frac{2}{d} G^2\right] + \sum_{i=1}^{N} c_i \left[G \frac{\partial G}{\partial x_i}\right] + \sum_{i,j=1}^{N} c_{ij} \left[G \frac{\partial^2 G}{\partial x_i \partial x_j} + \left(-\frac{d}{2} - 1\right) \frac{\partial G}{\partial x_i} \frac{\partial G}{\partial x_j}\right] + \dots = 0$$

Syzygies generate linear relations for Feynman integrals:

$$\left(\left[c_0(\hat{1}^+, \dots, \hat{N}^+) - \sum_{i=1}^{N} c_i(\hat{1}^+, \dots, \hat{N}^+)\hat{i}^- + \sum_{i,j=1}^{N} c_{ij}(\hat{1}^+, \dots, \hat{N}^+)\hat{i}^- \hat{j}^- + \dots \right] \tilde{I}\right)(\nu_1, \dots, \nu_N) = 0$$

INTEGRAL SOLUTIONS


loopedia.mpp.mpg.de
Graph e12|e3|45|45|e|e| — Masses 000|00|00|00|1|1|
Edit
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Edge list: (e,0|0) (0,1|0) (0,2|0) (e,1|0) (1,3|0) (2,4|0) (2,5|0) (3,4|0) (3,5|0) (e,4|1) (e,5|1)
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External Leg E4 q² = m1²
Choose Configuration
View public records for this configuration ↓BELOW↓ or choose different configuration ↑ABOVE↑
Type: Linear Combination
Orders in ε: 0,1,2,3,4
Number of master integrals: 2
Reference: arXiv:1404.4853
Authors: Thomas Gehrmann, Andreas von Manteuffel, Lorenzo Tancredi, Erich Weihs
Description: The authors compute the full set of massless two-loop four-point functions with two off-shell legs with the same invariant mass relevant for diboson production at hadron colliders (qq̄→VV and gg→VV). The ε expansion is given in terms of multiple polylogarithms of uniform weight through to weight four. Results for physical scattering kinematics are optimized for numerical evaluations using Li2,2, Lin and log functions. In addition, expansions are provided at the production threshold and in the small mass limit.
Submitter: manteuffel@pa.msu.edu
Record 1503427764.GtvX
added 22 Aug 2017 18:49 UTC
last modified 22 Aug 2017 18:49 UTC
If you use these results in your calculation, please also cite arXiv:1709.01266.
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Results for loops = 2, legs = 4, all scales — Row 6»
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Ex.: Edge list [(1,2),(2,3),(2,3),(3,4)] or 1 2 2 3 2 3 3 4 — Nickel index e11|e|
Enter your graph by its edge list (adjacency list) or Nickel index
or browse:
Loops = any
Legs = any
Scales = any
Fulltext must contain:
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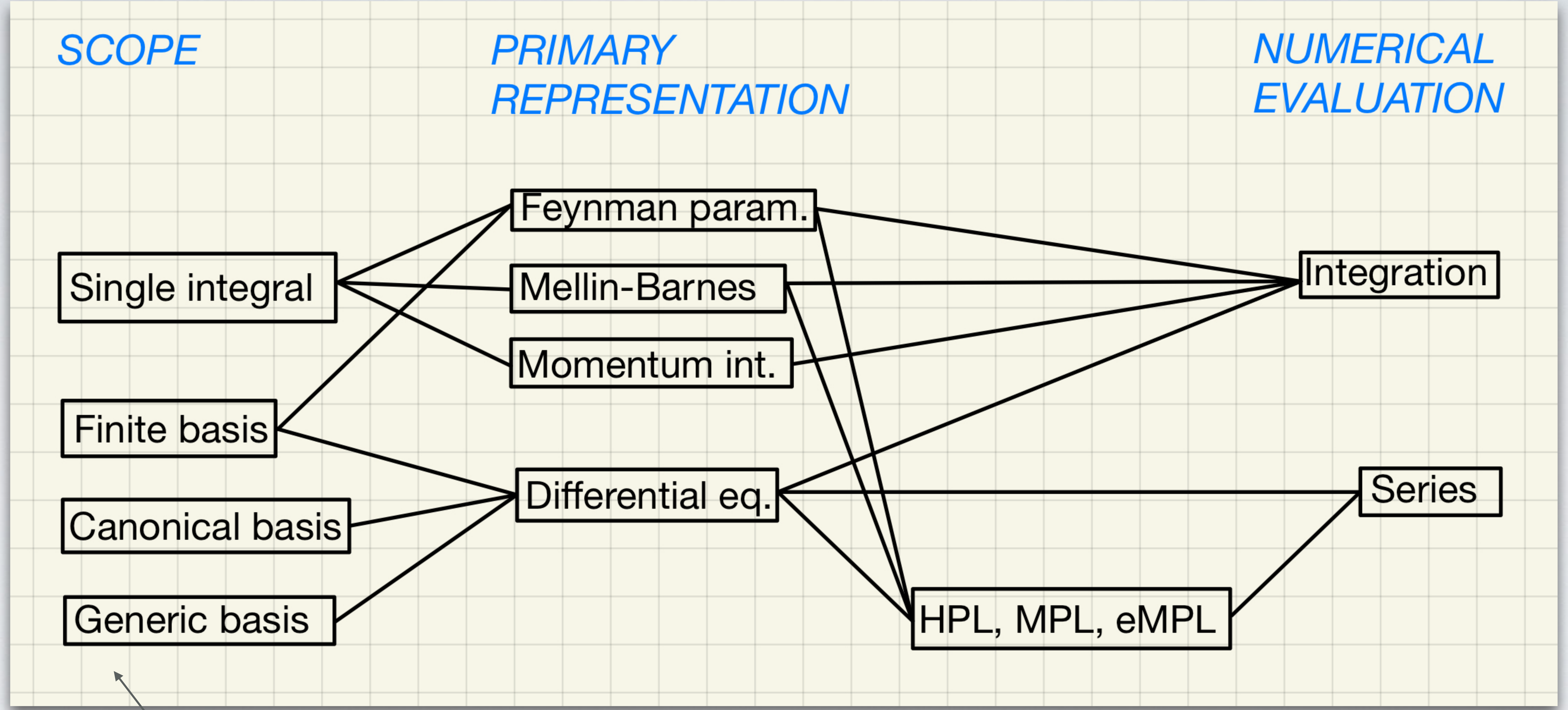
If you wish to add a new integral to the database, start by searching for its graph first.
The Loopedia Team is C. Bogner, S. Borowka, T. Hahn, G. Heinrich, S. Jones, M. Kerner, A. von Manteuffel, M. Michel, E. Panzer, V. Papara.
Software version of 29 Apr 2019 07:21 UTC. In case of technical difficulties with this site please contact Thomas Hahn.
This Web site uses the GraphState library [arXiv:1409.8227] for all graph-theoretical operations
and the neato component of Graphviz for drawing graphs.

EVALUATION OF INTEGRALS



Note: choice of basis also impacts performance. Also useful: just avoid mixed dep. of denominators on d + kinematics

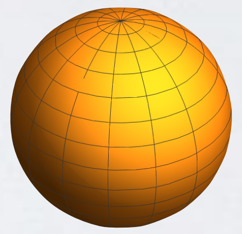
SOLVE INTEGRALS: DIFFERENTIAL EQUATIONS

- Integration of differential equations *[Kotikov '91, Remiddi '97]*:

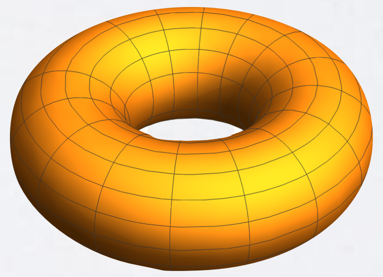
$$\boxed{\partial_x \vec{I}(x;\epsilon) = A(x;\epsilon)\vec{I}(x;\epsilon)}$$

where $\epsilon = (4-d)/2$ (analytical or through series expansions)

- Homogeneous solutions for $\epsilon = 0$ (leading singularities):
  - ***Rational number***, e.g. $1/2$



  - ***Rational functions***, e.g. $1/x$
  - ***Algebraic functions***, e.g. $\sqrt{x(x-4)}$
  - ***Elliptic integrals***, e.g. $K(x) = \int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-xz^2)}}, \ldots$



- Basis change involving homogenous solutions may allow to find ϵ-***form***:

$$\boxed{d\vec{m} = \epsilon\, \mathrm{d}\ln(l_a(x))\, A^{(a)}(x)\, \vec{m}}$$

[Kotikov '10, Henn '13]

SYMBOL CALCULUS

- ***Great for deriving diff. eq. and solving it:***

$$\mathrm{d}\vec{m} = \epsilon \sum_a \mathrm{d}\ln(l_a)\, A^{(a)}\, \vec{m}$$

- If letters l_i simple: iterated integration gives ***multiple polylogarithms***

  e.g. $l_1 = x, \quad l_2 = x - 1, \quad Li_2(x) = -\int_0^x \frac{dt}{t}\int_0^t \frac{dt'}{t'-1}$

- ***Symbol calculus*** *[Duhr, Gangl, Rhodes 2011]*: $S(Li_2(x)) = -\left(\ln(x) \otimes \ln(x-1)\right)$

- ***Letters*** form ***words*** with grammar (log law):

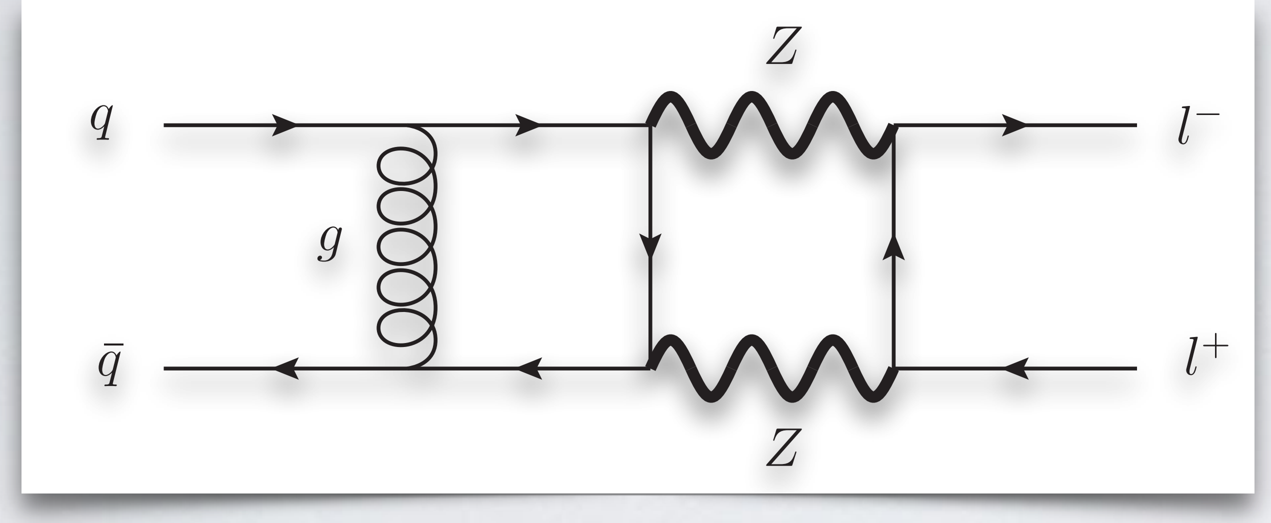
$$\ln(l_1 l_2) \otimes \ln(l_3) = \ln(l_1) \otimes \ln(l_3) + \ln(l_2) \otimes \ln(l_3)$$

- ***Computer algebra implementation*** allows to automatically derive functional identities, take limits, …

ALGEBRAIC LETTERS

- In mixed EW-QCD corrections to DY *[Heller, AvM, Schabinger '19]*

talk: Taushif Ahmed



provably non-rationalizable root in letters

e.g. $l_{13} = -(1-w)(z-w)(1-w\,z) + (1+w)\sqrt{4(1-w)^2 wz^2 + (w+z)^2(1+wz)^2}$

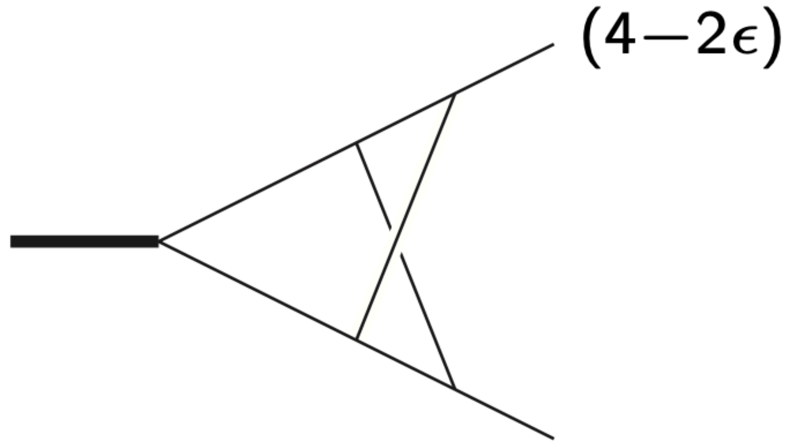
- Algebraic letters can be constructed from rational letters plus homogeneous solutions (just the bare root)

- Integrable in terms of multiple polylogarithms:

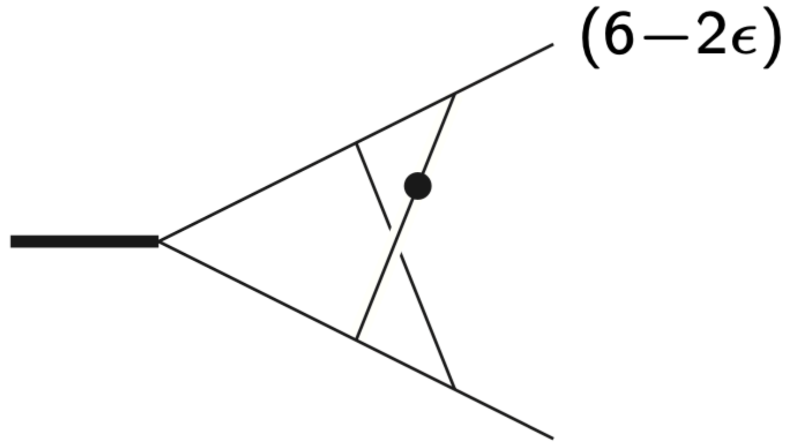
$$
\begin{aligned}
m_{32} = \epsilon^3 &\left[4\,\mathrm{Li}_3\left(\frac{l_1 l_2 l_6 l_7 l_{10} l_{13}}{l_{14} l_{15} l_{16}}\right) - 2\,\mathrm{Li}_3\left(\frac{l_2^3 l_6 l_7^2}{l_{15} l_{16}}\right) + \ldots + 4\,\mathrm{Li}_2\left(\frac{l_6 l_{14} l_{16}}{l_7 l_9 l_{15}}\right)\ln(l_3) + \ldots\right] \\
+\epsilon^4 &\left[-\mathrm{Li}_{2,2}\left(-\frac{l_1^2 l_3 l_{15}}{l_2^2 l_7 l_{14}}, \frac{l_2^2 l_7 l_{15}}{l_1 l_3 l_6 l_1 4}\right) + \ldots + \frac{701}{4}\,\mathrm{Li}_4\left(\frac{l_1 l_3^2 l_6^2 l_9 l_{14}}{l_2 l_7 l_{13} l_{15} l_{16}}\right) + \ldots\right] + O(\epsilon^5)
\end{aligned}
$$

- Such analytical solutions allow for fast and robust numerical evaluations in Monte Carlo programs

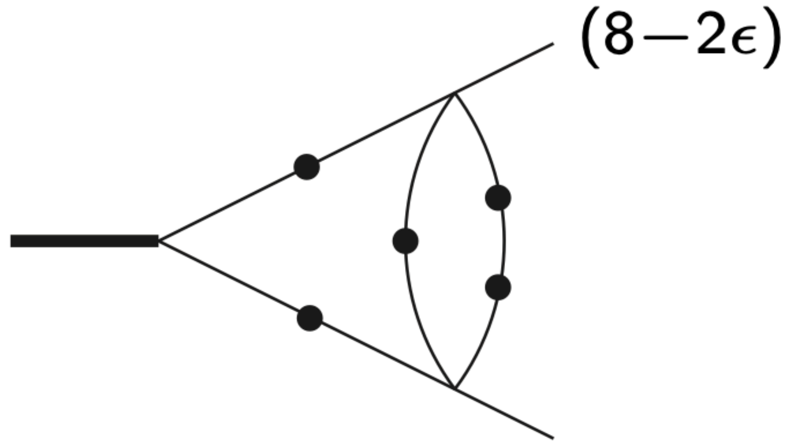
SOLVE INTEGRALS: PARAMETRIC INTEGRATION OF FINITE INTEGRALS



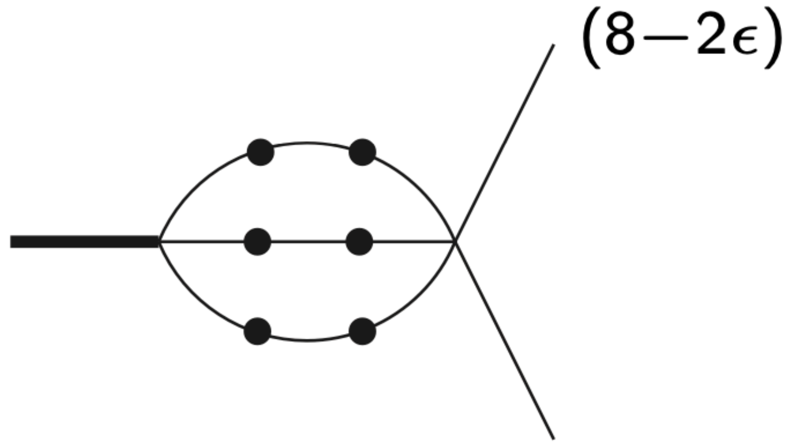
$$= -\frac{4(1-4\epsilon)}{\epsilon(1-\epsilon)q^2}$$



$$-\frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4(1-\epsilon)^2(2-\epsilon)^2q^2}$$



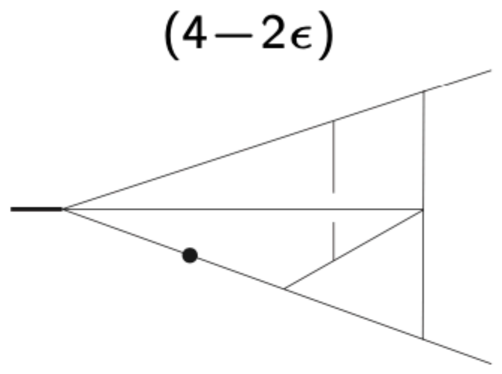
$$+\frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)(1-\epsilon)^2(2-\epsilon)^2q^2}$$



- General observation *[Panzer 2014; AvM, Panzer, Schabinger 2014]*:
  - any ***divergent*** loop integral can be expressed via **finite** basis integrals
  - Reduze 2 finds finite integrals
- Expand integrands of ***finite*** integrals around $\epsilon = (4-d)/2 \approx 0$
  - If linearly reducible: integrate ***analytically*** with HyperInt *[Panzer 2014]*
  - Improved ***numerical*** evaluations, used for HH *[Borowka, Greiner, Heinrich, Jones, Kerner '16]*, Hj *[Jones, Kerner, Lusioni '18]*, ZH *[Chen, Davies, Heinrich, Jones, Kerner, Mishima, Schlenk, Steinhauser '22]* …

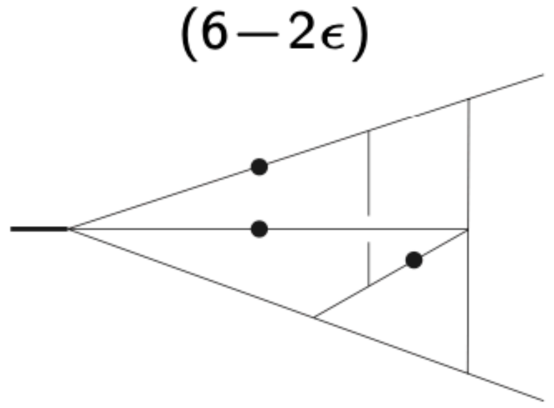
"NICE" FINITE INTEGRALS

- Example: 10 terms in ϵ for weight 6 in conventional basis:



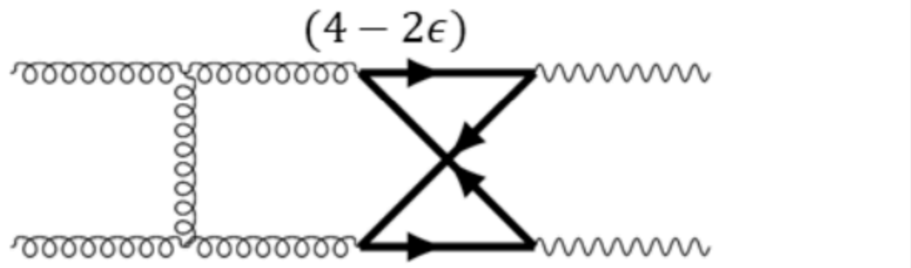
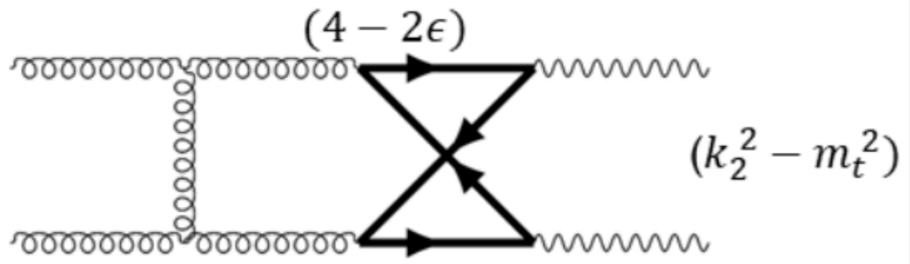
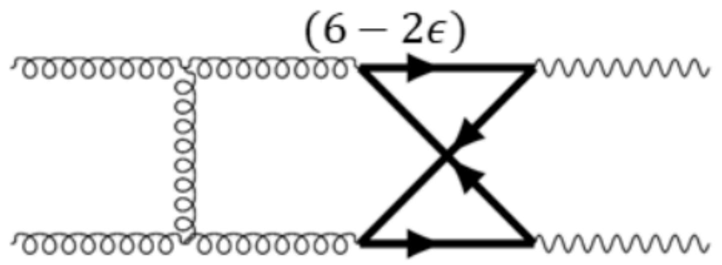
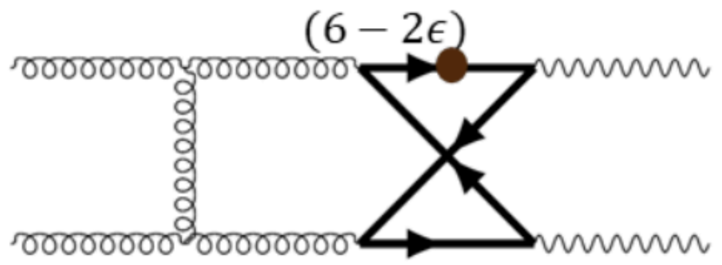
$$
\begin{aligned}
&= \frac{1}{\epsilon^8}\left(-\frac{1}{144}\right) + \frac{1}{\epsilon^7}\left(-\frac{1}{12}\right) + \frac{1}{\epsilon^6}\left(\frac{1}{24}\zeta_2 - \frac{7}{36}\right) + \frac{1}{\epsilon^5}\left(\frac{29}{24}\zeta_3 + \frac{1}{2}\zeta_2 - \frac{1}{72}\right) \\
&+ \frac{1}{\epsilon^4}\left(\frac{71}{16}\zeta_2^2 + \frac{29}{2}\zeta_3 + \frac{39}{16}\zeta_2 + \frac{335}{144}\right) + \frac{1}{\epsilon^3}\left(\frac{1819}{24}\zeta_5 - \frac{23}{6}\zeta_2\zeta_3 + \frac{213}{4}\zeta_2^2 + \frac{1211}{24}\zeta_3 + \frac{431}{48}\zeta_2\right. \\
&\left.+ \frac{47}{18}\right) + \frac{1}{\epsilon^2}\left(-\frac{1285}{24}\zeta_3^2 + \frac{80579}{1008}\zeta_2^3 + \frac{1819}{2}\zeta_5 - 46\zeta_2\zeta_3 + \frac{25787}{160}\zeta_2^2 + \frac{417}{8}\zeta_3 - \frac{1175}{48}\zeta_2 - \frac{7277}{72}\right) \\
&+ \frac{1}{\epsilon}\left(\frac{434203}{192}\zeta_7 - \frac{7139}{24}\zeta_2\zeta_5 - \frac{54139}{120}\zeta_2^2\zeta_3 - \frac{1285}{2}\zeta_3^2 + \frac{80579}{84}\zeta_2^3 + \frac{5571}{2}\zeta_5 - \frac{9005}{24}\zeta_2\zeta_3 + \frac{967}{480}\zeta_2^2\right. \\
&\left.- \frac{4045}{8}\zeta_3 - \frac{733}{24}\zeta_2 + \frac{57635}{72}\right) - \frac{2023}{12}\zeta_{5,3} - \frac{30581}{4}\zeta_3\zeta_5 - \frac{6829}{24}\zeta_2\zeta_3^2 + \frac{45893321}{100800}\zeta_2^4 + \frac{434203}{16}\zeta_7 \\
&- \frac{7139}{2}\zeta_2\zeta_5 - \frac{54139}{10}\zeta_2^2\zeta_3 - \frac{10706}{3}\zeta_3^2 + \frac{7987951}{3360}\zeta_2^3 + \frac{1309}{12}\zeta_5 - \frac{30317}{24}\zeta_2\zeta_3 - \frac{43847}{96}\zeta_2^2 + \frac{32335}{24}\zeta_3 \\
&+ \frac{2553}{4}\zeta_2 - \frac{334727}{72} + \mathcal{O}(\epsilon)\,.
\end{aligned}
$$

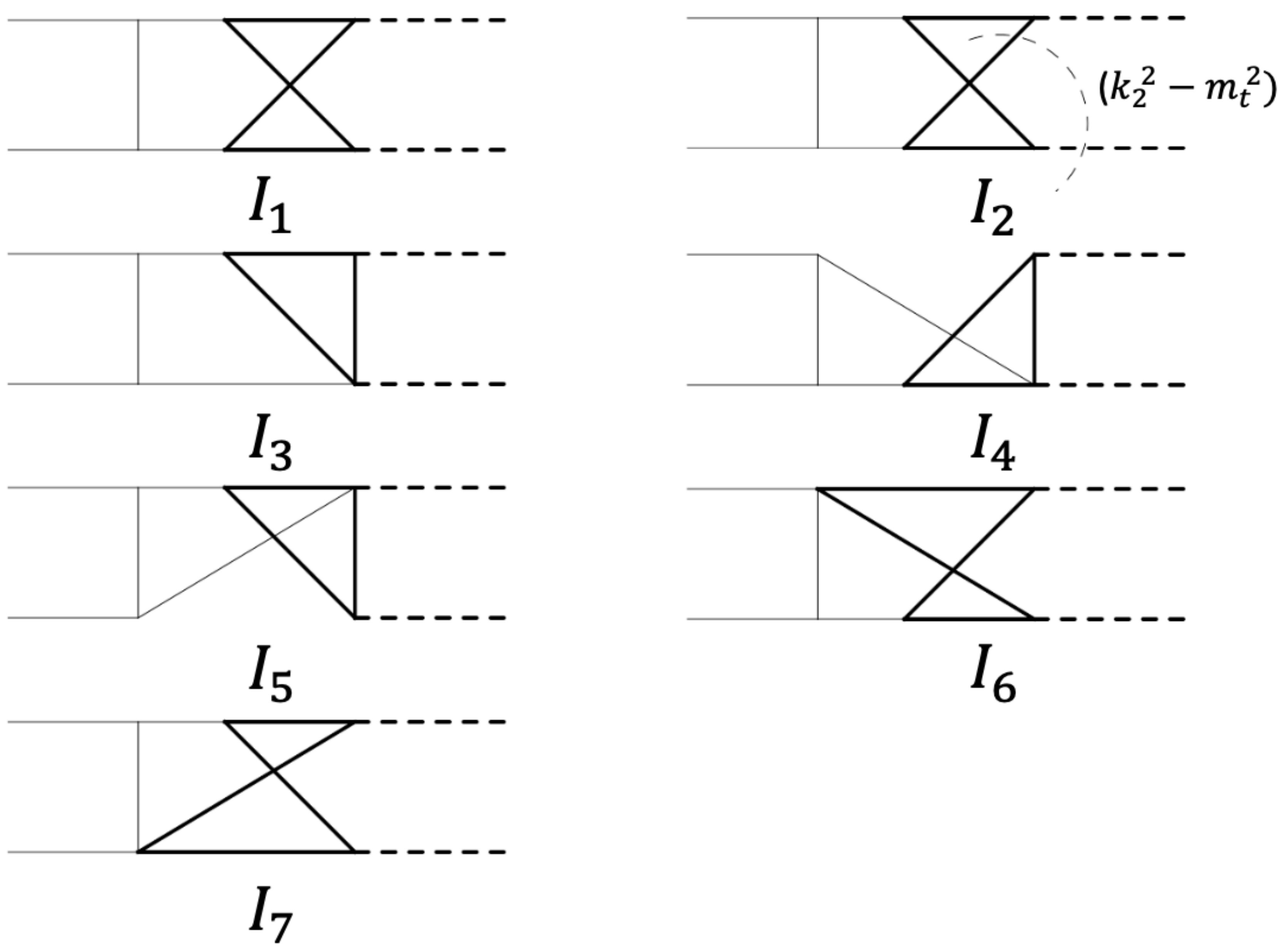
- Only 1 term for weight 6 for a nice finite integral:



$$= -\frac{3}{2}\zeta_3^2 - \frac{4}{3}\zeta_2^3 + 10\zeta_5 + 2\zeta_2\zeta_3 - \frac{1}{5}\zeta_2^2 - 6\zeta_3 + \mathcal{O}(\epsilon)$$

GENERALIZED FINITE INTEGRALS

Integral	Rel.Err.	Timing(s)
	~2*10^-3	45
	~4*10^-2	63
	~8*10^-6	55
	~8*10^-4	60
Linear combination	~1*10^-4	18



$$I = (m_z^2 - s - t)(sI_1 - I_6) + s(I_2 + I_3 - I_4 - I_5) - (m_z^2 - t)I_7$$

$$I(\nu_1,\ldots,\nu_N) = (-1)^{r+\Delta t}\,\Gamma(\nu - L\,d/2)\int\left(\prod_{j\in\mathcal{N}_T}\mathrm{d}x_j\right)\left(\prod_{j\in\mathcal{N}_t}\frac{x^{\nu_j-1}}{\Gamma(\nu_j)}\right)\delta\left(1-\sum_{j\in\mathcal{N}_T}x_j\right)$$
$$\left[\left(\prod_{j\in\mathcal{N}_{\setminus T}}\frac{\partial^{|\nu_j|}}{\partial x_j^{|\nu_j|}}\right)\left(\prod_{j\in\mathcal{N}_{\Delta t}}\frac{\partial^{|\nu_j|+1}}{\partial x_j^{|\nu_j|+1}}\right)\frac{\mathcal{U}^{\nu-(L+1)d/2}}{\mathcal{F}^{\nu-L\,d/2}}\right]_{x_j=0\ \forall\ j\in\mathcal{N}_{\setminus T}}\qquad(\nu_j\in\mathbb{Z}).$$

[Agarwal, AvM, Jones 2020]

Numerical integration used pySecDec
[Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke 2017]

MULTIVARIATE PARTIAL FRACTIONS

- Univariate partial fraction decomposition separates singularities:

$$\frac{x}{(x-1)(x+1)^2} = -\frac{1}{4(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{4(x-1)}$$

- Iterated partial fractioning introduces spurious poles in multivariate case:

$$\frac{1}{(x-f(y))(x-g(y))} = \frac{1}{(f(y)-g(y))}\frac{1}{(x-f(y))} - \frac{1}{(f(y)-g(y))}\frac{1}{(x-g(y))},$$

  for example:

$$\frac{1}{(x+y)(x-y)} = \frac{1}{2y}\frac{1}{(x-y)} - \frac{1}{2y}\frac{1}{(x+y)}.$$

- Our approach to be discussed in the following: MultivariateApart [*Heller, AvM '21*]

- Related work: *[Pak '11, Abreu et al '19, Boehm et al '20, Bendle et al '21]*

- Motivation: (non-planar) amplitudes sometimes reduced by factor $O(100)$ in size with respect to common denominator representation

PARTIAL FRACTIONS VIA POLYNOMIAL REDUCTIONS

- Algorithm: write inverse denominators as $q_i = 1/d_i$ and reduce polynomial in $q_1, \ldots, x_1, \ldots$ with respect to ideal
$$I = \langle q_1 d_1(x_1, \ldots) - 1, \ldots, q_m d_m(x_1, \ldots) - 1 \rangle$$
- Here, polynomial reduction means
$$p' = p - u \cdot g$$
such that p' "smaller" than p for some monomial ordering, u is an arbitrary polynomial and $g \in I$
- Result will always be unique if we consider all g from a "Gröbner basis".
- Depending on monomial ordering we can ensure specific features of output form:
  - Sorting $q_1, \ldots$ before $x_1, \ldots$ guarantees Leinartas (i)
  - A lexicographic ordering of the $q_1, \ldots$ and $x_1, \ldots$ (separately) guarantees also Leinartas (ii), but this is just some choice.

PARTIAL FRACTIONS FOR AMPLITUDES

multivariateapart.nb 100%

Univariate partial fractions separate terms with different poles:

In[1]:= $\text{Apart}\left[\frac{1}{x\,(1+x)},\, x\right]$

Out[1]= $\frac{1}{x} - \frac{1}{1+x}$

Let's consider a multivariate example:

In[2]:= $\text{multi} = \frac{2y - x}{y\,(x+y)\,(y-x)};$

Naive iteration introduces spurious poles (here 1/x) for multivariate case:

In[3]:= `Apart[multi, y]`

Out[3]= $\frac{1}{x\,y} + \frac{1}{2\,x\,(-x+y)} - \frac{3}{2\,x\,(x+y)}$

Solution: multivariate partial fractions using methods from polynomial ideal theory:

In[4]:= ``<< MultivariateApart` ``

```
MultivariateApart -- Multivariate partial fractions. By Matthias Heller and Andreas von Manteuffel.
```

In[5]:= `MultivariateApart[multi]`

Out[5]= $-\frac{1}{2\,(x-y)\,y} + \frac{3}{2\,y\,(x+y)}$

Note: minimize denominator degrees ($\neq$ Leinartas)

- PFD: significant ***reduction in size***
- Easy to identify ***linear relations*** between coefficients
- Easy to ***generate fast code*** even for complicated amplitudes
- Representation can be tuned for ***numerical stability*** !

  see $q\bar{q} \to \gamma\gamma j$ @ 2-loops

  [Agarwal, Buccioni, AvM, Tancredi '21]

DENOMINATOR GUESSING

- Suppose we want to reconstruct

$$r = \frac{\mathcal{N}}{d_1^{\alpha_1} \cdots d_m^{\alpha_m}}$$

  and we have a guess for the multivariate denominator factors, e.g.

  $d_1 = x - y, \quad d_2 = y, \quad d_3 = x + y$

- Sample variables such that prime factor decomposition of each denominator factor has unique signature, e.g. x=17, y=31:

  $d_1 = -7 \cdot 2, \qquad d_2 = 31, \qquad d_3 = 3 \cdot 2^4$

- Suppose the denominator of the rational function r is

  $d = 31^2 \cdot 7 \cdot 3^3 \cdot 2^{13}$

  What are the denominator factors ?

- What if we found

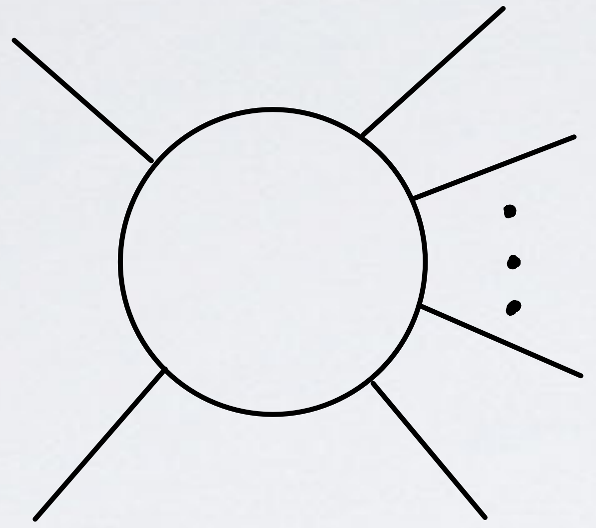
  $d = 97 \cdot 31^2 \cdot 7 \cdot 3^3 \cdot 2^{13}$?

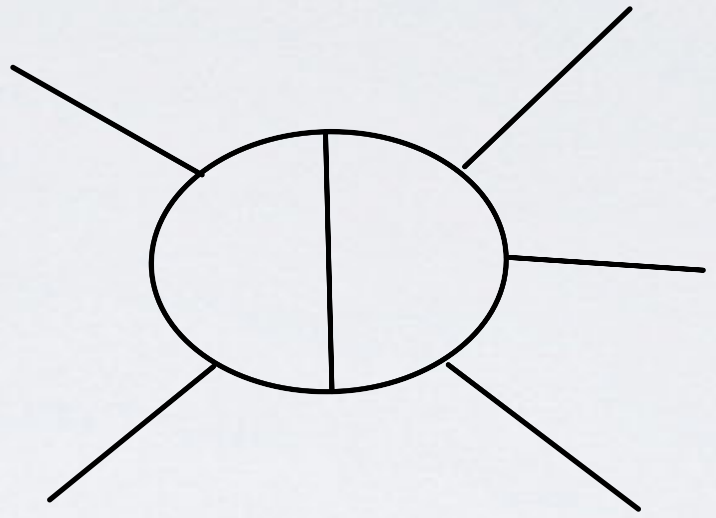
- Improves rational reconstruction (fewer samples !), guides partial fraction decomposition (no symbolic common denominator form)

- Method above from *[Heller, AvM '21]*, see also *[Abreu, Dormans, Febres Cordero, Ita, Page '18, De Laurentis, Page '22]*

p-adic techniques: talk by Herschel Chaudry

FULL-COLOR MASSLESS QCD AMPLITUDES





$q\bar{q} \to \gamma\gamma j$

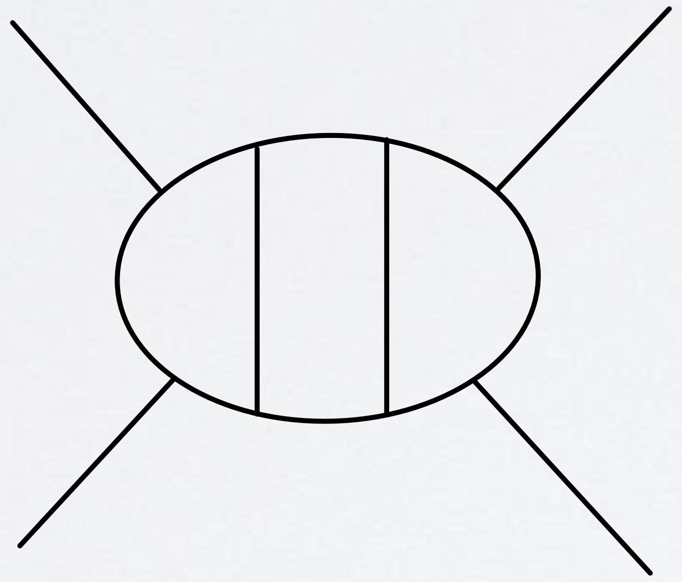
[Agarwal, Buccioni, AvM, Tancredi '21]

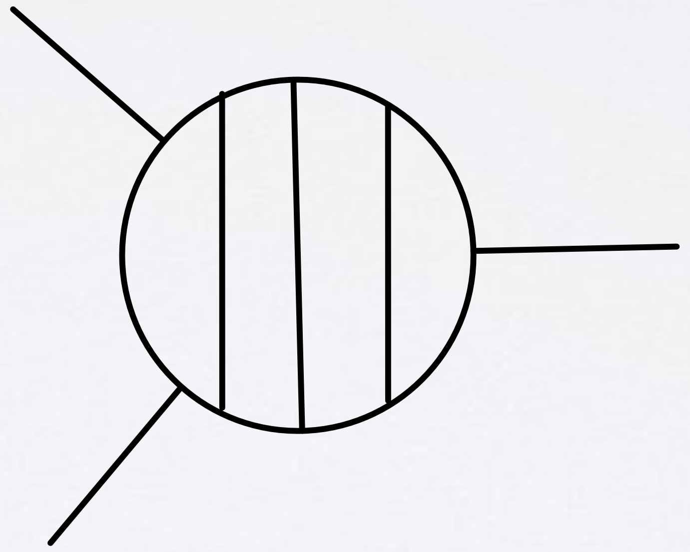
$gg \to \gamma\gamma j$

[Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]

$q\bar{q} \to \gamma\gamma\gamma$

[Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov '23]





$q\bar{q} \to \gamma\gamma$

[Caola, AvM, Tancredi '20]

$gg \to \gamma\gamma$

[Bargiela, Caola, AvM, Tancredi '21]

$q\bar{q} \to q'\bar{q}'$, $gg \to gg$, $q\bar{q} \to gg$:

[Caola, Chakraborty, Gambuti, AvM, Tancredi '21,'21,'22]

$q\bar{q} \to \gamma g$:

[Bargiela, Chakraborty, Gambuti '22]

$q\bar{q} \to \gamma^*$, $gg \to H$

[Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]

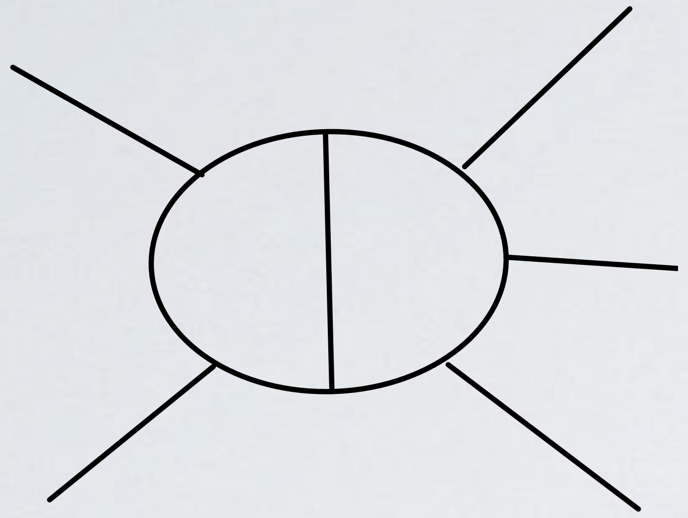
$b\bar{b} \to H$

[Chakraborty, Huber, Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]

ACCURACY OF LEADING COLOR APPROXIMATIONS

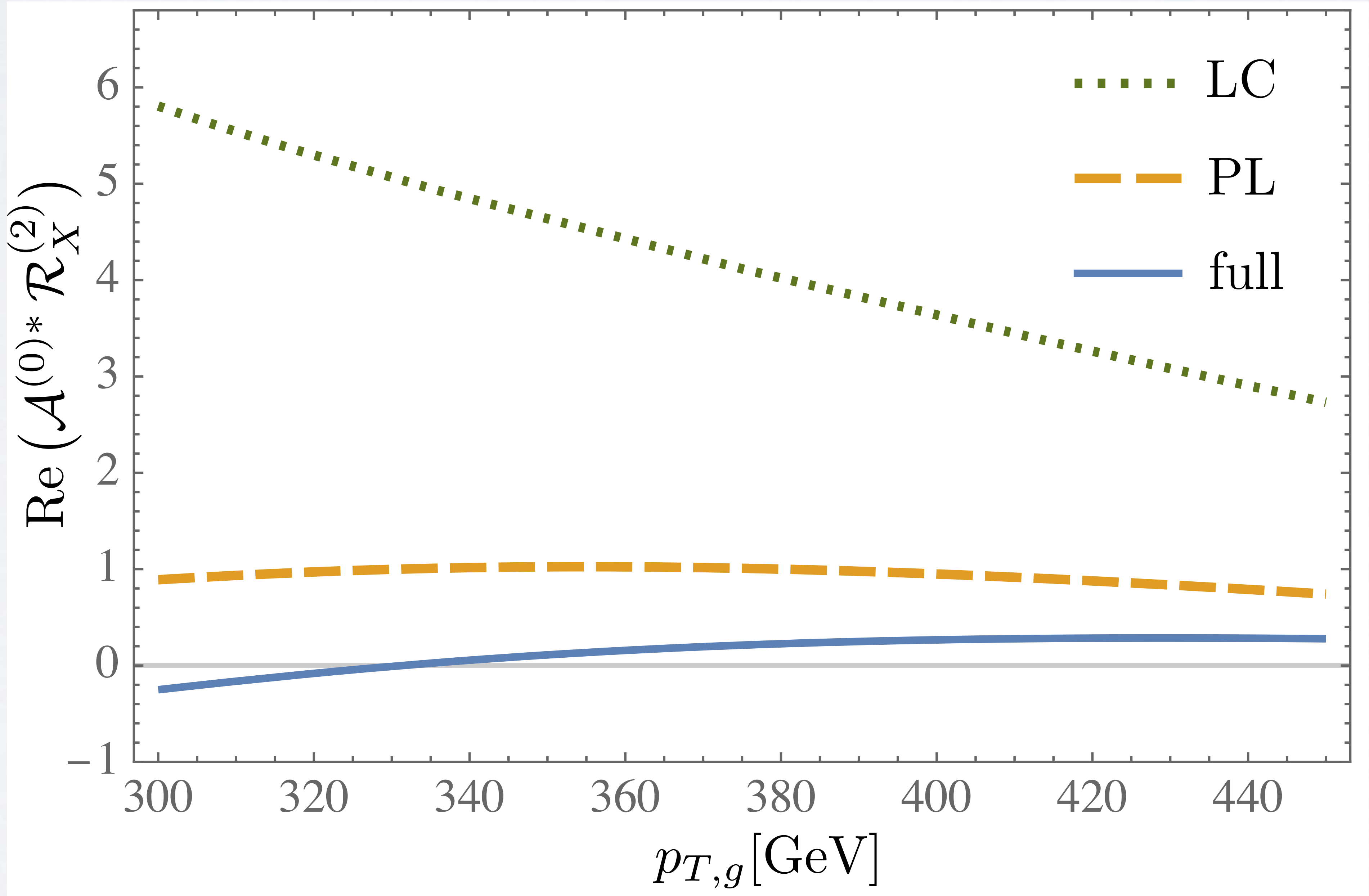
Ex.: $\gamma\gamma j$ @ NNLO: result w/ leading color virtual: *[Chawdhry, Czakon, Mitov, Poncelet 2021]*

Public library for master integrals: PentagonFunctions *[Chicherin, Sotnikov '20]*

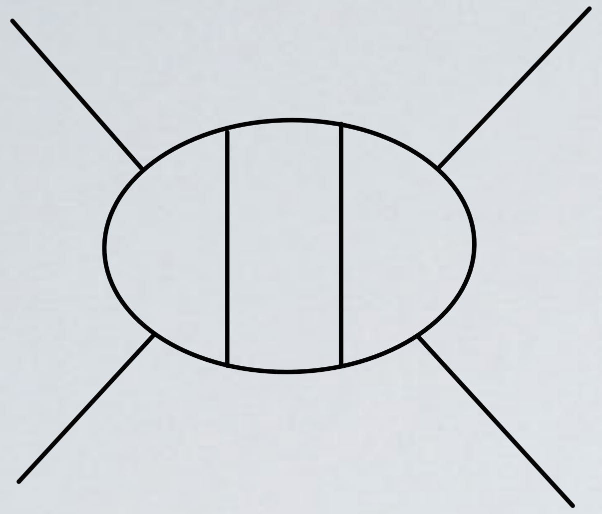


Leading color not always a good approximation:

e.g. 2-loop finite remainder for $u\bar{u} \to g\gamma\gamma$ in Catani's scheme:



[Agarwal, Buccioni, AvM, Tancredi PRL '21]



IR BEYOND DIPOLES

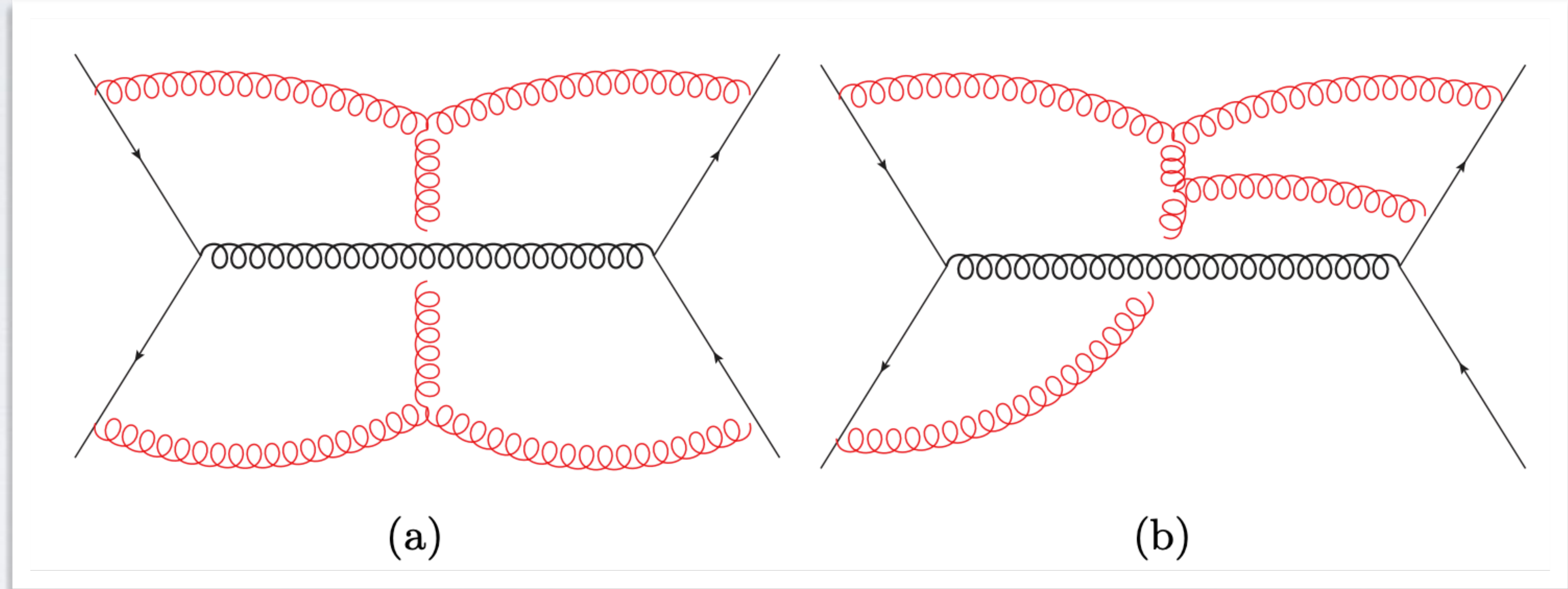
soft anomalous dimension matrix @ 3 loops
[Almelid, Duhr, Gardi '15]

$$\mathbf{\Gamma}(\{p\},\mu) = \mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) + \mathbf{\Delta}_4(\{p\})$$

$$\mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) = \sum_{1\le i<j\le 4} \frac{\mathbf{T}_i^a\,\mathbf{T}_j^a}{2}\,\gamma^{\text{cusp}}(\alpha_s)\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_s)$$

$$\mathbf{\Delta}_4^{(3)} = 128\, f_{abe}\, f_{cde}\left[\mathbf{T}_1^a\,\mathbf{T}_2^c\,\mathbf{T}_3^b\,\mathbf{T}_4^d\,D_1(x) - \mathbf{T}_4^a\,\mathbf{T}_1^b\,\mathbf{T}_2^c\,\mathbf{T}_3^d\,D_2(x)\right] - 16\,C\sum_{i=1}^{4}\sum_{\substack{1\le j<k\le 4\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\,\mathbf{T}_k^c\,,$$

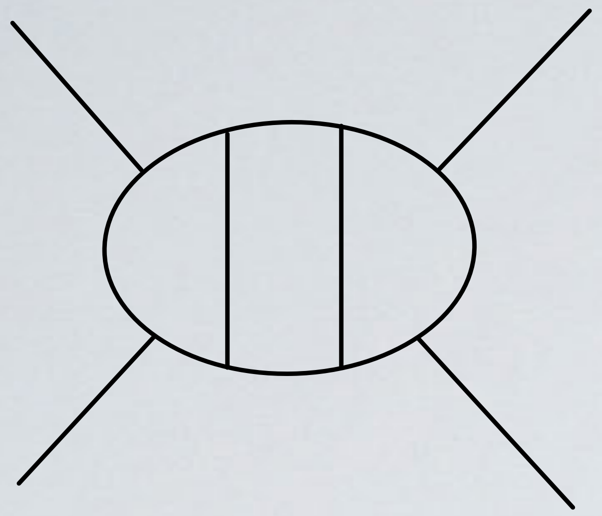
confirmed for N=4 four-point amplitude
[Henn, Mistlberger '16]



Our calculations confirm the predicted quadrupole structure for QCD in all partonic channels

$$q\bar{q}\to q'\bar{q}',\ gg\to gg,\ q\bar{q}\to gg$$

[Caola, Chakraborty, Gambuti, AvM, Tancredi '21,'21,'22]



HIGH ENERGY LIMIT

talk: *Einan Gardi*

- Interesting to study high-energy (Regge) limit of amplitudes beyond fixed order
- Regge-cut description to define Regge trajectory beyond 3-loops *[Falcioni, Gardi, Maher, Milloy, Vernazza; Nov ’21]*

$$\mathcal{H}_{\mathrm{ren},\pm} = Z_g^2\, e^{L\mathbf{T}_t^2\tau_g} \sum_{n=0}^{3} \bar{\alpha}_s^n \sum_{k=0}^{n} L^k \mathcal{O}_k^{\pm,(n)} \mathcal{H}^{(0)}_{\mathrm{ren}},$$

- Our $q\bar{q} \to q'\bar{q}'$, $gg \to gg$, $q\bar{q} \to gg$ calculations *[Caola, Chakraborty, Gambuti, AvM, Tancredi ’21,’21,’22]* allowed us to validate the framework and determine missing parameters:
- We extracted 3-loop gluon Regge trajectory, last building block for single-Reggeon exchanges at NNLL

$$\tau_3 = K_3 + N_c^2\left(16\zeta_5 + \frac{40\zeta_2\zeta_3}{3} - \frac{77\zeta_4}{3} - \frac{6664\zeta_3}{27} - \frac{3196\zeta_2}{81} + \frac{297029}{1458}\right) + \frac{n_f}{N_c}\left(-4\zeta_4 - \frac{76\zeta_3}{9} + \frac{1711}{108}\right)$$
$$+ N_c n_f\left(\frac{412\zeta_2}{81} + \frac{2\zeta_4}{3} + \frac{632\zeta_3}{9} - \frac{171449}{2916}\right) + n_f^2\left(\frac{928}{729} - \frac{128\zeta_3}{27}\right) + \mathcal{O}(\epsilon),$$

:

indep. extraction: *[Falcioni, Gardi, Maher, Milloy, Vernazza; Dec ’21]*

- Gluon Regge trajectory and gluon and quark impact factors extracted from different partonic 3-loop amplitudes agree

TOWARDS ALL-N, FOUR-LOOP DGLAP EVOLUTION

SPLITTING FUNCTIONS

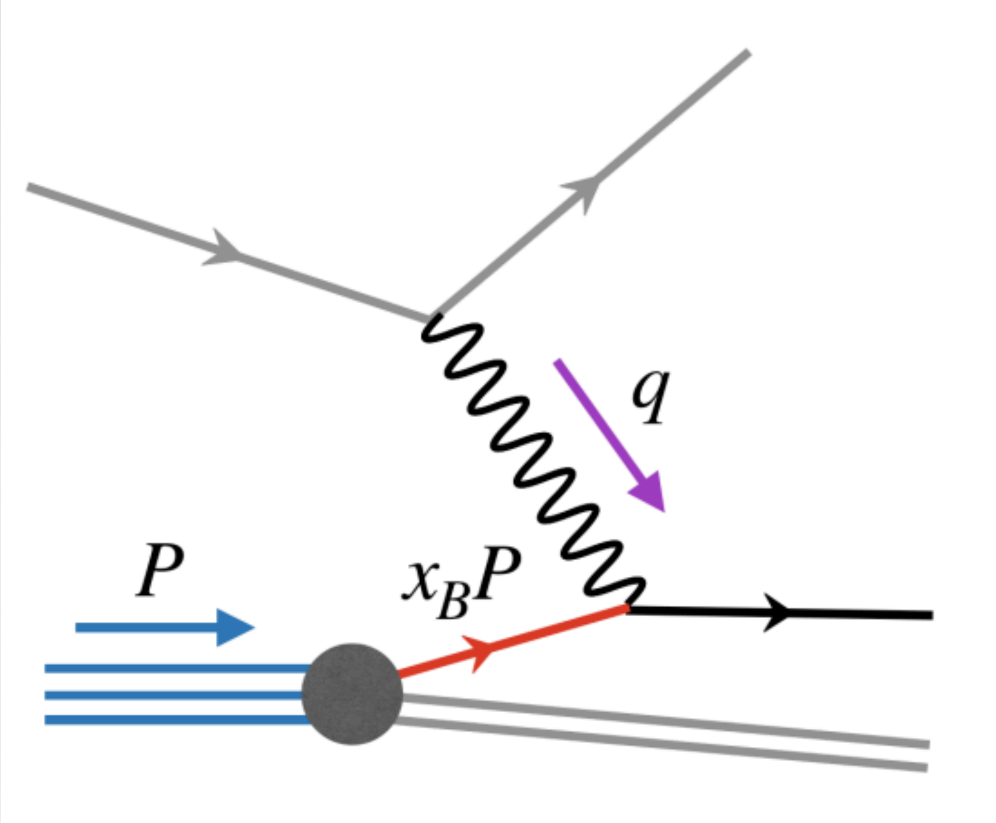
- Factorization of hadronic cross section:

$$\sigma \sim \sum_k f_{k|N}(x) \otimes \sigma_k(x) \quad \text{with } x = -\frac{q^2}{2P \cdot q}$$

- Splitting functions P_{ik} govern DGLAP evaluations of PDFs:

$$\frac{df_{i|N}}{d\ln\mu} = 2\sum_k P_{ik} \otimes f_{k|N}$$

- Consistent N3LO cross section requires 4-loop splitting functions, only partially known:
  - Large n_f limit *[Gracey '94,'96; Davies, Vogt, Ruijl, Ueda, Vermageren '16]*
  - Non-singlet $n \leq 16$ from off-shell OMEs *[Moch, Ruijl, Ueda, Vermaseren, Vogt '17]*
  - Singlet $n \leq 8$ from DIS *[Moch, Ruijl, Ueda, Vermaseren, Vogt '21]*
  - Pure-singlet, gluon-quark $n \leq 20$ from off-shell OMEs *[Falcioni, Herzog, Moch, Vogt '23, '23]*
  - Approximate N3LO PDF fits *[McGowan, Cridge, Harland-Lang, Thorne '22; Hekhorn, Magni '23]*
  - This talk: all-n results for pure-singlet n_f^2 splitting functions



[Image credit: Tong-Zhi Yang]

talk: Giulio Falcioni

SPLITTING FUNCTIONS FROM OPERATORS

- With Mellin transform $f_q(n) = -\int_0^1 dx\, x^{n-1} f_q(x)\,,\ \ \gamma_{ij}(n) = -\int_0^1 dx\, x^{n-1} P_{ij}(x)$

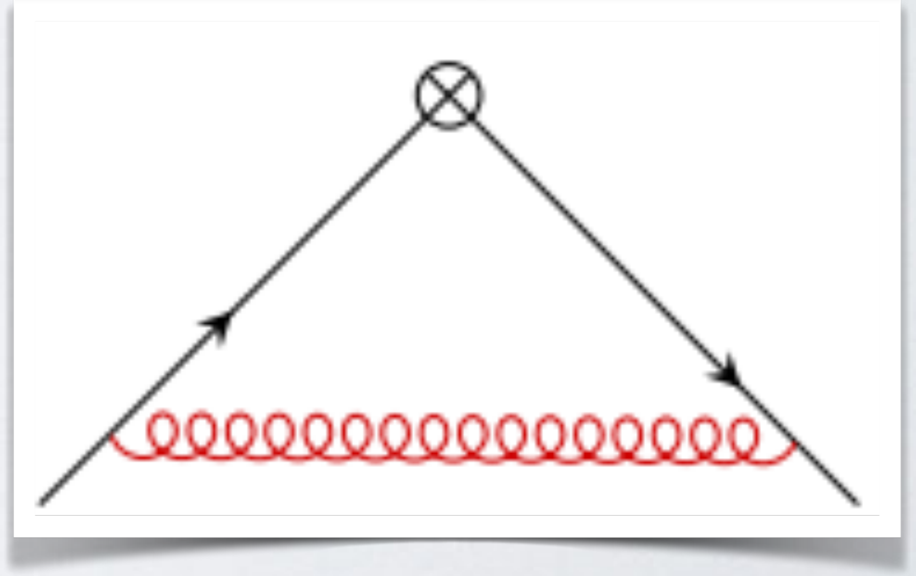
  DGLAP becomes $\dfrac{df_i(n,\mu)}{d\ln\mu} = -2\sum_j \gamma_{ij}(n) f_j(n,\mu)$

- The $\gamma_{ij}(n)$ appear as **anomalous dimensions** of **twist-two operators**,

  e.g. flavor non-singlet: $O_{q,k} = \dfrac{i^{n-1}}{2}\left[\overline{\psi}\Delta_\mu\gamma^\mu(\Delta\cdot D)^{n-1}\dfrac{\lambda_k}{2}\psi\right]$

  with multiplicative renormalization $O^R_{q,k} = Z^{ns} O^B_{q,k}$ where $\dfrac{dZ^{ns}}{d\ln\mu} = -2\gamma^{ns} Z^{ns}$

- Poles of (off-shell) operator matrix elements: **efficient** way to find $f_q(n)$



SINGLET CASE AND OPERATOR MIXING

- Singlet twist-two operators:

$$O_q = \frac{i^{n-1}}{2}\left[\overline{\psi}\Delta_\mu\gamma^\mu(\Delta\cdot D)^{n-1}\psi\right]$$

$$O_g = -\frac{i^{n-2}}{2}\left[\Delta_\mu G^{a\,\mu}{}_\nu(\Delta\cdot D)^{n-2}_{ab}\Delta_\kappa G_b{}^{\kappa\nu}\right]$$

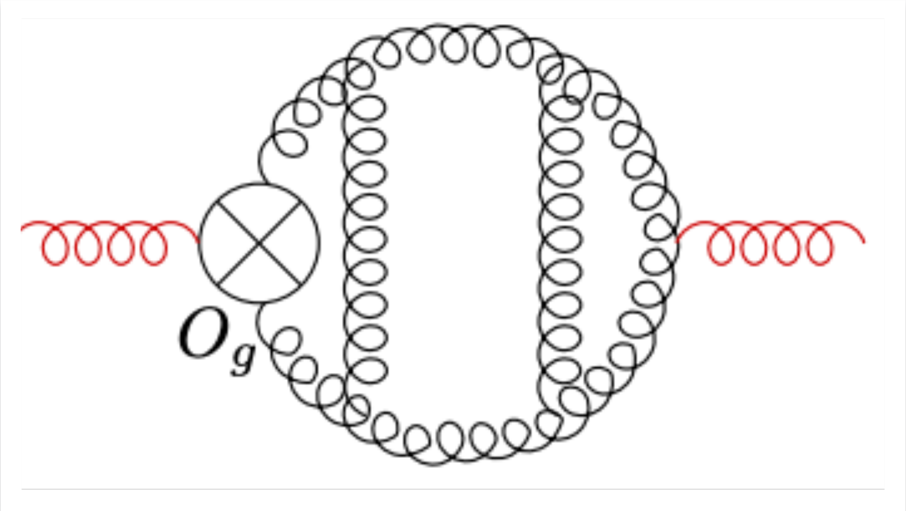
- Singlet operators mix under renormalization
- For off-shell OME, also new, unknown gauge-variant operators contribute
- Gauge-variant operators caused confusion in early literature
- Construction of operators for fixed Mellin moment n from generalized BRST: [Falcioni, Herzog '22]
- Our goal: **all-n results**
- Our method: directly compute **counter term Feynman rules** from multi-leg off-shell OMEs *[Gehrmann, AvM, Yang '23]*

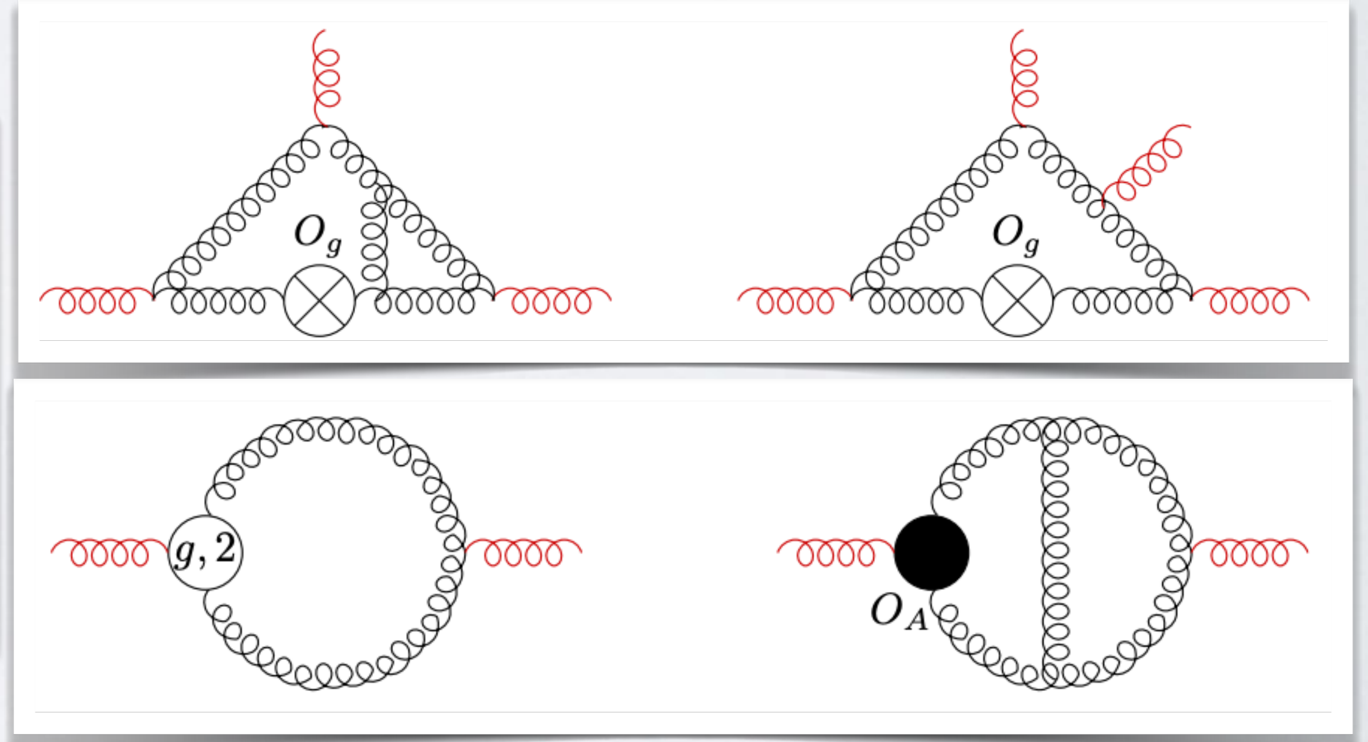
COUNTER TERMS FROM MULTI-LEG OMES

- Renormalization: $\begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^R = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ Z_{Aq} & Z_{Ag} & Z_{AA} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^B + \begin{pmatrix} [ZO]_q^{GV} \\ [ZO]_g^{GV} \\ [ZO]_A^{GV} \end{pmatrix}^B$

- Take OMEs according to $\langle j | O | j + mg \rangle$ with $j = q, g, c$ and m additional gluons
- Expand $[ZO]^{GV} = \sum_l [ZO]^{GV,(l)} \alpha_s^l$, determine counter terms from OMEs with extra legs, e.g.:

Loops \ Legs	2	3	4	5
0		$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_q, O_g
1	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g	
2	O_{ABC}	O_g		
3	O_q, O_g			





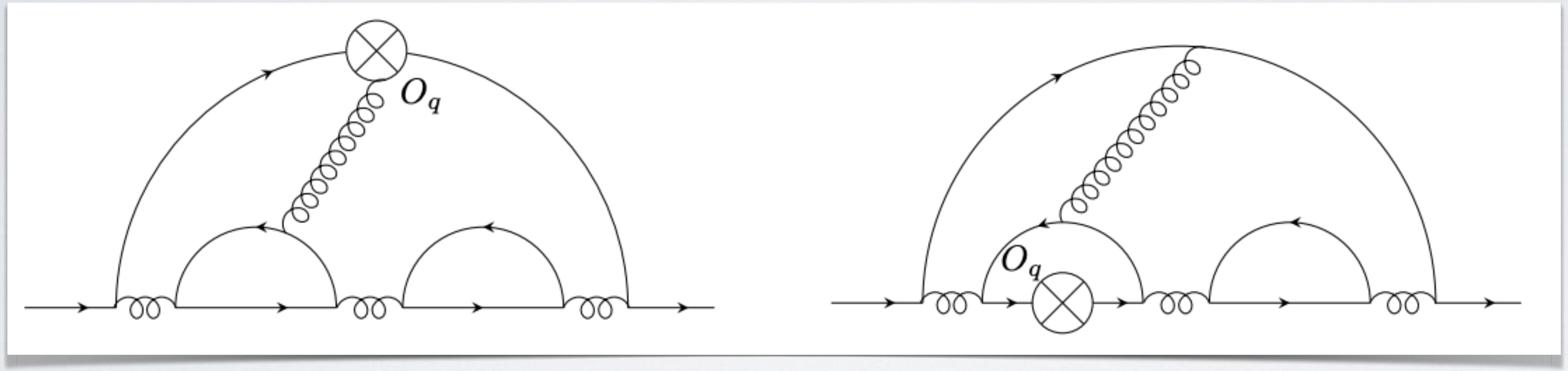
[Gehrmann, AvM, Yang ‘23]

THREE-LOOP SPLITTING FUNCTIONS

- Operator insertions introduce n dependent powers of scalar products
- Use **tracing parameter** t to map to standard linear propagators
  [Ablinger, Blümlein, Hasselhuhn, Schneider, Wissbrock '12]
  $$(\Delta \cdot p)^{n-1} \to \sum_{n=1}^{\infty} t^n (\Delta \cdot p)^{n-1} = \frac{t}{1 - t\,\Delta \cdot p}$$
  allows to use standard IBP technology
- We applied our method to **3-loop splitting functions**, computation in general R_ξ gauge
- Differential equations in t, find ϵ factorized form using Canonica and Libra, boundary values known
- Complicated counter terms, involve generalized harmonic sums
- Gauge parameter ξ drops out, full agreement with *[Moch, Vermaseren, Vogt '04, '04]*

FOUR-LOOP PURE SINGLET: N_f^2, ALL-N

- Four-loop contributions for quark, with two or three closed fermion loops
  [Gehrmann, AvM, Sotnikov, Yang '23]

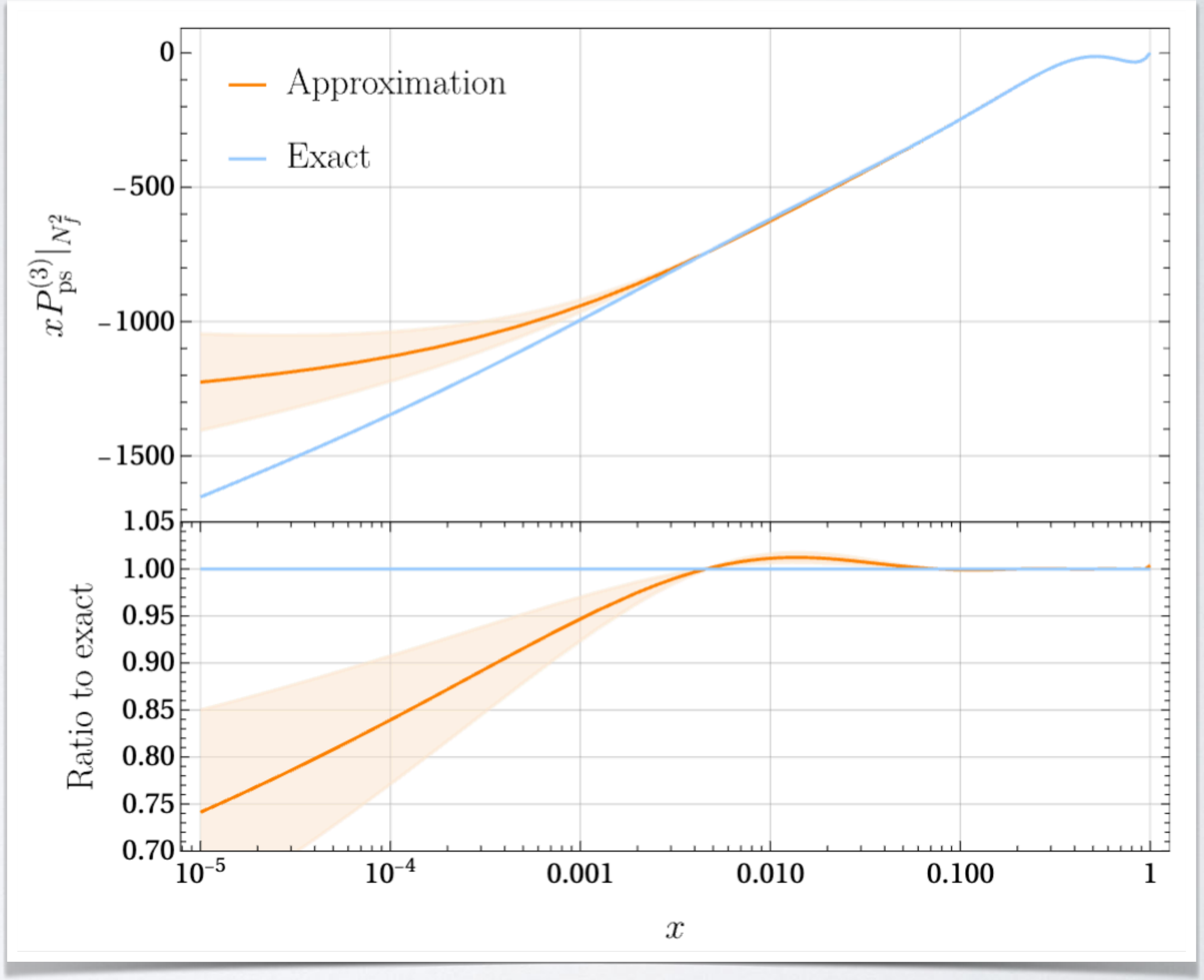


  (singlet and non-singlet, also non-planar)

- Use syzygies, compute with linear algebra
- Finred with finite field sampling to derive differential equations, reduction of amplitude
- Simple analytical result for splitting functions in terms of HPLs and powers of x

ALL-N RESULT IMPROVES SMALL X KNOWLEDGE

- $n \leq 20$ by *[Falcioni, Herzog, Moch, Vogt '23]*
- partial information for $x \to 0$: *[Catani, Hautmann '94; Davies, Kom, Moch, Vogt '22]*
- leading terms for $x \to 1$: *[Soar, Moch, Vermaseren, Vogt '09]*
- Generate fit similar to *[Falcioni, Herzog, Moch, Vogt '23]*, compare to all-n result:

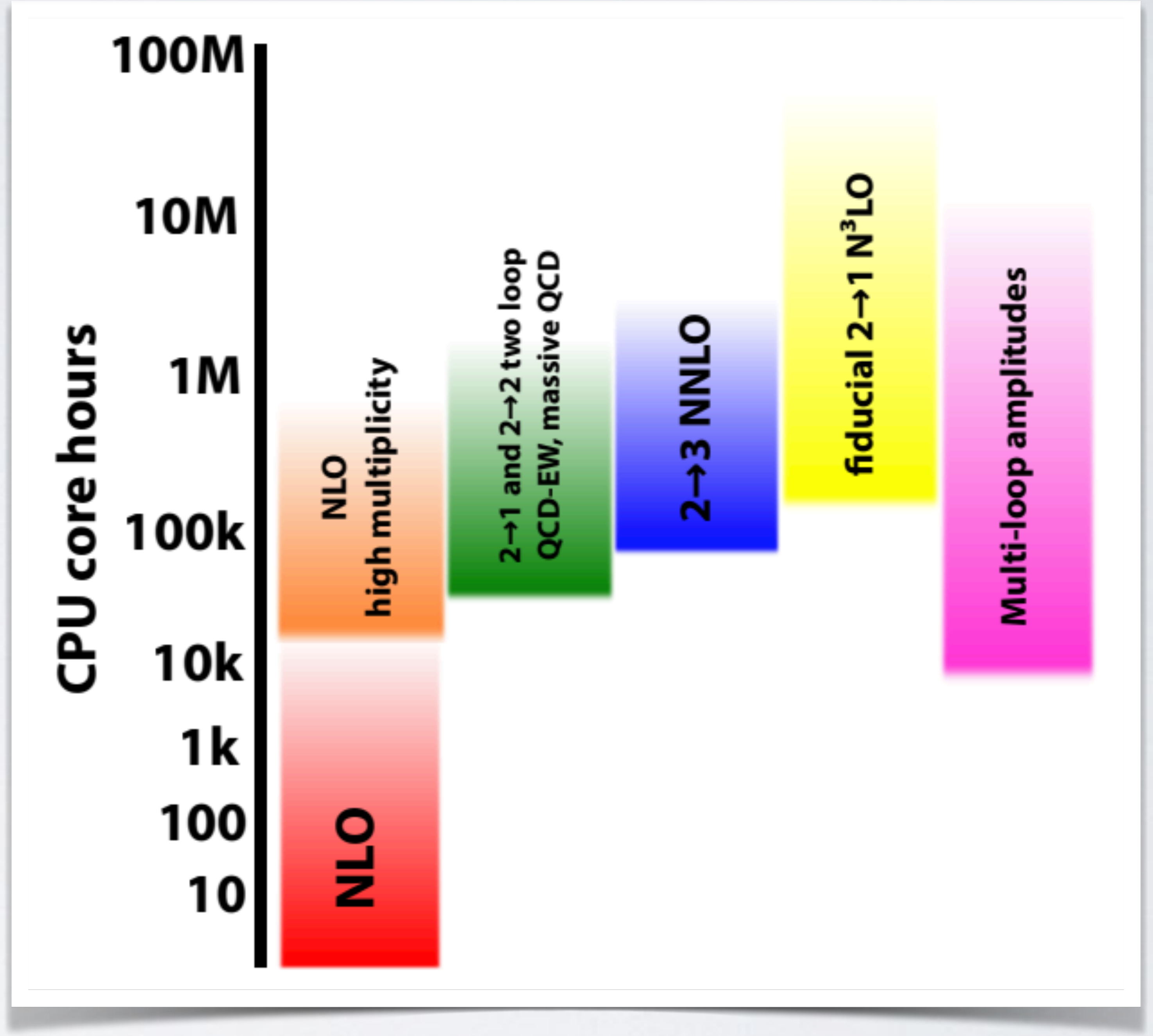


[Gehrmann, AvM, Sotnikov, Yang]

CONCLUSION & OUTLOOK

Status and Prospects For Future Precision Goals :

- **Numerical methods** easier to automate, avoid expression swell
- **Analytical insights** can enable much better numerical performance
- **Finite-field methods** crucial in many current calculations
- Challenging calculations (mixed EW-QCD, masses, higher loops):
  - Improved **IBP reductions** (fast, low memory, automated)
  - Improved **integral evaluation** (fast, reliable, automated)
  - Improved γ_5 **treatment** (rigorous, automated)



From: *Snowmass survey of 53 recent perturbative calculations* *[Febres-Cordero, AvM, Neumann '22]*