p-adic reconstruction of rational functions in multi-loop amplitude calculations

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Introduction (1)

- Multi-loop amplitudes are core ingredients in higher-order calculations (see Andreas von Manteuffel's talk)
- The last decade has seen huge progress in calculating these amplitudes at 2+ loops for multi-leg processes
- As we pursue ever more ambitious calculations, the computational difficulty grows very quickly
- More powerful and sophisticated techniques are vital for tackling these calculations
- Focus of this talk: the rational functions appearing in multi-loop amplitudes.

Introduction (2)

- Calculation of large rational functions is a central bottleneck in multi-loop amplitude computations
- In recent years, finite-field numeric methods have widely been employed to calculate multi-loop amplitudes and IBPs
- In parallel, it has been observed that symbolic expressions can be significantly simplified by partial fractioning
- This talk: can we reconstruct directly in partial-fractioned form?

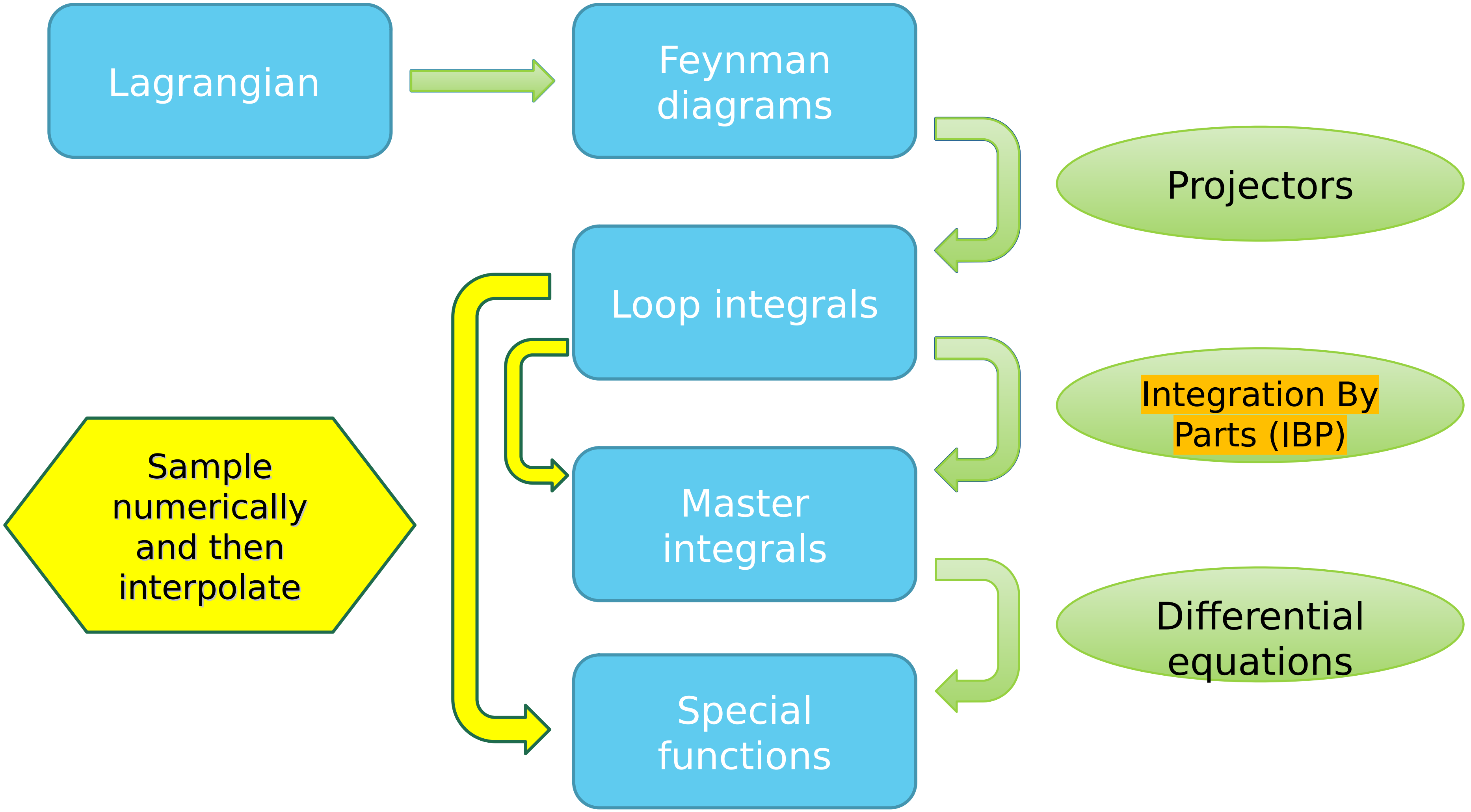
Outline

1. Introduction
   1. Why numerical reconstruction?
   2. Why partial-fractioned form?
   3. p-adic numbers
2. Details of interpolation strategy
3. Results
4. Conclusion

Why numerical (finite-field) reconstruction methods?

- Long-used in computer algebra (e.g. Mathematica), now also used in physics
  - e.g. [1406.4513 – Manteuffel, Schabinger], [1608.01902 – Peraro]
  - Has enabled calculation of many new multi-loop multi-scale amplitudes
- Core idea: perform repeated numerical calculations and then interpolate result
  - Bypasses large intermediate expressions
    - Generic feature of symbolic calculations (not specific to physics)
  - Use finite-field numbers instead of real numbers. (advantage: exact results)
- Most computing time is spent evaluating the numerical probes
  - Number of probes is determined by the polynomial degrees of the expressions in the final result
- Reconstruct analytical results using interpolation and Chinese remainder theorem
  - Various libraries e.g. FireFly, FiniteFlow

A typical multi-loop toolbox



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Partial fractioning

- Widespread use in recent years to simplify final (and intermediate) results of heavy calculations
- Popular libraries: Singular, MultivariateApart
- Example throughout this talk: the largest rational function in the (2nd-)largest IBP expression needed for 2-loop 5-point massless non-planar QCD amplitudes
  - Analytic expression courtesy of authors (Agarwal, Buccioni, von Manteuffel, Tancredi) of [2105.04585]
  - Partial-fractioned form is O(100) times smaller than common-denominator form
    - ~600 MB vs ~5 MB
    - ~1,400,000 free parameters vs ~14,000 free parameters
- This talk: from numeric evaluations, reconstruct such expressions directly in partial-fractioned form

Why reconstruct in partial-fractioned form?

- Surprise: the 125-times simplification doesn't occur if, prior to partial fractioning, we randomise the numerical coefficients in the numerator of common-denominator form
  - Therefore, simplification comes from physics, not computer algebra
    - Can we exploit this?
      - Yes, if we can reconstruct directly into partial-fractioned form
    - Can we (fully) explain this?
- We will reconstruct piece-by-piece
  - Added benefit: partial-fractioned terms have further structure, which we can spot - and (future work) exploit - on the fly

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A brief history of *p*-adic numbers

- Described/explored by Kurt Hensel in 1897
- Widely used in computer algebra for several decades
    - Finding rational solutions to various types of equations
        - Reconstruct a rational number from its *p*-adic expansion
- Appearance in particle physics too!
    - *p*-adic / adelic quantum mechanics / string theory [since '80s/'90s]
    - Ansatze for amplitudes [De Laurentis & Page, 2022]
        - Constrained ansatze in common-denominator form, to then be fitted with standard finite-field methods
- This talk: interpolate rational functions directly in partial-fractioned form, from p-adic evaluations.

Brief intro to *p*-adic numbers

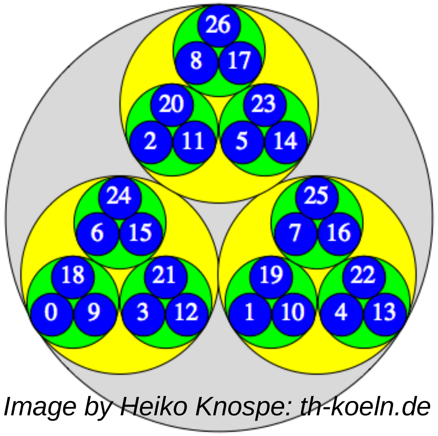

Image by Heiko Knospe: th-koeln.de

- p-adic numbers $\mathbb{Q}_p$ are an alternative completion of the rationals $\mathbb{Q}$
  - Alternative absolute value: $|(a * p^n / b)|_p = 1/p^n$, where p is prime and a,b,p are coprime
  - For each prime p, a separate field $\mathbb{Q}_p$
    - Nice results, e.g. Hasse's local-global principle: certain equations have solutions in $\mathbb{Q}$ iff they have solutions in $\mathbb{R}$ and in each $\mathbb{Q}_p$
- Can expand any rational number x as a power series in p
    - e.g. $80 = 3 + 4*7 + 1*7^2$
    - e.g. $-1 = 6 + 6*7 + 6*7^2 + 6*7^3 + O(7^4)$
    - e.g. $(2 / 21) = 3*7^{-1} + 2 + 2*7 + 2*7^2 + 2*7^3 + O(7^4)$
  - If x is integer then the coefficient of p^0 is (x mod p)
  - Expansion operation commutes with all arithmetical operations + * - /

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Strategy for interpolation

1. Assume denominator (in common-denominator form) is known, and factorised.
2. Make list of “candidates”: all possible subsets of the denominator factors
3. Use p-adic probes to <u>filter</u> the candidates
4. Use more p-adic probes to <u>reconstruct</u> numerator of a candidate
5. Repeat steps 3 and 4
   - Gives more information that just doing step 3 once.
   - See also [De Laurentis, Maitre, 1904.04067], which uses high-precision floating-point to calculate gggggg @ 1L. Also see [Campbell et al., 2203.17170]

Example of some reconstructed pieces:

$$\frac{-\frac{693}{400}\,\mathrm{t23}^2\,\mathrm{t51}^5-\frac{693}{200}\,\mathrm{t23}\,\mathrm{t51}^5\,\mathrm{x12}-\frac{693\,\mathrm{t51}^5\,\mathrm{x12}^2}{400}}{(-7+2\,\mathrm{d})\,\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}+$$

$$\frac{-\frac{28}{225}\,\mathrm{t23}^2\,\mathrm{t51}^5-\frac{56}{225}\,\mathrm{t23}\,\mathrm{t51}^5\,\mathrm{x12}-\frac{28\,\mathrm{t51}^5\,\mathrm{x12}^2}{225}}{(-8+3\,\mathrm{d})\,\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}+\frac{-\frac{1}{144}\,\mathrm{t23}^2\,\mathrm{t51}^5-\frac{1}{72}\,\mathrm{t23}\,\mathrm{t51}^5\,\mathrm{x12}-\frac{\mathrm{t51}^5\,\mathrm{x12}^2}{144}}{\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}+$$

$$\frac{\frac{3\,\mathrm{t23}^2\,\mathrm{t51}^5}{1400}+\frac{3}{700}\,\mathrm{t23}\,\mathrm{t51}^5\,\mathrm{x12}+\frac{3\,\mathrm{t51}^5\,\mathrm{x12}^2}{1400}}{(-1+\mathrm{d})\,\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}+\frac{\frac{160\,\mathrm{t23}^2\,\mathrm{t51}^5}{63}+\frac{320}{63}\,\mathrm{t23}\,\mathrm{t51}^5\,\mathrm{x12}+\frac{160\,\mathrm{t51}^5\,\mathrm{x12}^2}{63}}{(-10+3\,\mathrm{d})\,\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}$$

P-adic filtering

- Select a subset of denominator factors (ignoring powers)
- Generate a *p*-adic point that makes each of those factors become *p*-adically small (possibly with weights)
  - e.g. $\{s_{12}, s_{12}\text{-}s_{23}, s_{34}\} \sim \{O(p^2), O(p), O(p)\}$
- Evaluate the rational function at that *p*-adic point, and note the order of its "*p*-adic pole"
  - e.g. rational function ~ $O(1/p^4)$
  - Note: small primes suffice, e.g. p=101
- For safety, repeat with ~2-3 more points, keeping same weights. (Preferably change *p* each time)
- Filter out any candidate factors whose p-adic pole is too strong
  - e.g. $1/[s_{12}{}^2 * (s_{12} - s_{23}) * s_{34}] \sim O(1/p^5)$ at the above point

P-adic reconstruction

- Select a subset of denominator factors (ignoring powers) and weights
  - e.g. $\{s_{12}, s_{12}\text{-}s_{23}, s_{34}\} \sim \{O(p^2), O(p), O(p)\}$
- Identify all candidates that could generate highest pole.
  - Conjecture: with correct strategy, can always ensure there is only one candidate!
    - e.g. [unknown numerator] / $[s_{12} * (s_{12} - s_{23})^3 * s_{34}]$
- Write down ansatz for numerator of that candidate
  - Typically 1-50 free parameters
  - As we'll see, numerators of partial-fractioned terms often turn out to be simpler than naïve expectation. Future work: smarter ansatz.
- For fixed p, generate several points that give the p-adic weights chosen above
- Evaluate the rational function at those points.
  - Coefficient of leading pole = ansatz mod p
- Interpolate ansatz, mod p
- Repeat for other choices of p
  - In this work, typically used ~5 primes of O(100)
- Use Chinese Remainder Theorem (+rational reconstruction) to reconstruct ansatz in Q
  - Must do this before proceeding to other candidates

Choice of probe weights

- Simple choice: exponential weights
  - Naively, might expect to need large powers to uniquely pick out one partial-fractioned term.
    - e.g. if singularity degrees are known to be bounded to be below 10, we can set $(s_{12}, s_{23}, s_{34}) \sim (p^{100}, p^{10}, p)$. Then if the rational function diverges there like $1/p^{273}$, we know we have picked out the term $1/(s_{12}^2\, s_{23}^7\, s_{34})$
    - But this strategy would require evaluating to very high p-adic precision.
- Smart choice: low weights
  - Choose a limited set of small weights
    - e.g. $(s_{12}, s_{23}, s_{34}) \sim (p,p^2,p)$ or (p,p,p)
    - Repeatedly cycle through this set, trying to find a probe point that picks out a single candidate partial-fractioned term.
      - In this work, used ~6k probe points, with each kinematic weight always <5.
    - Heuristically, seems to work

A complication: relations / bases

- There are relations between partial-fractioned candidate terms
  e.g. $\frac{1}{x^2 y} - \frac{1}{x^2 (x+y)} - \frac{1}{x y (x+y)} == 0$
- Choice of basis:
  - Basis in MultivariateApart / Leinartas's decomposition
    - Chosen depending on a specified variable ordering
    - Avoids introducing new spurious factors, but can still introduce spurious higher powers of existing factors
    - Unique basis -> allows vectorised addition in symbolic calculations
  - Basis in this work: prioritises avoiding introducing spurious higher powers
    - Basis customised to given rational function, so that no partial-fractioned term has stronger divergence than the overall function
    - Further study needed to see which basis choices are "best"

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Results

- Reconstructed largest rational function in (2nd-)largest IBP expression needed for all non-planar 2-loop 5-point massless QCD amplitudes.
- Number of free parameters: 52.5k (vs 1.37M in common-denominator form)
  - Scope for further improvement: notice that out of the 52.5k, only 15.4k are non-zero
- Number of numerical probes: see later slide

Results – preliminary

Representation	Terms	Size (ByteCount)	Free parameters	Cost
Common denominator form (with denominator factorised, and numerator fully expanded)	1	605 MB	1.37M	1.37M finite-field probes per prime field
P-adic reconstruction (this work) (with each numerator fully expanded)	2.8k	5.5 MB	52.5k (of which 15.4k turn out to be non-zero)	#p-adic probes: see next slide
MultivariateApart (with default settings)	2.5k	4.7 MB	14.7k	Input must already be analytically known

Number of p-adic probes

- (preliminary) Number of p-adic probes during this calculation
  - Filtering: ~6k probes per p-adic field (but fewer probes would probably suffice)
  - Reconstruction: ~60k probes per p-adic field
    - But can greatly reduce this by recycling probes
      - e.g. probes with $(s_{23}, s_{34}) \sim (p^2, p^1)$ can be used to reconstruct $1/(s_{23}^3 s_{34}^3)$ but also $1/(s_{23}^3 s_{34}^2)$, $1/(s_{23}^2 s_{34}^3)$, $1/(s_{23} s_{34})$, etc
  - Number of p-adic fields used: typically 5, e.g. $\mathbb{Q}_{101}$, $\mathbb{Q}_{103}$, $\mathbb{Q}_{107}$, $\mathbb{Q}_{109}$, $\mathbb{Q}_{113}$

A closer look at the reconstructed result

- Recall that we must reconstruct only 52.5k coefficients (compared to 1.3M in common-denominator form). Of these 52.5k, find that only 15.4k are non-zero.
  - e.g. some *p*-adically reconstructed terms:

$$\frac{-\frac{693}{400}\,\mathrm{t23}^2\,\mathrm{t51}^5-\frac{693}{200}\,\mathrm{t23}\,\mathrm{t51}^5\,\mathrm{x12}-\frac{693\,\mathrm{t51}^5\,\mathrm{x12}^2}{400}}{(-7+2\,\mathrm{d})\,\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}+$$

(Notice: only terms ~ t_{51}^5 are non-zero)

$$\frac{-\frac{28}{225}\,\mathrm{t23}^2\,\mathrm{t51}^5-\frac{56}{225}\,\mathrm{t23}\,\mathrm{t51}^5\,\mathrm{x12}-\frac{28\,\mathrm{t51}^5\,\mathrm{x12}^2}{225}}{(-8+3\,\mathrm{d})\,\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}+\frac{-\frac{1}{144}\,\mathrm{t23}^2\,\mathrm{t51}^5-\frac{1}{72}\,\mathrm{t23}\,\mathrm{t51}^5\,\mathrm{x12}-\frac{\mathrm{t51}^5\,\mathrm{x12}^2}{144}}{\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}+$$

$$\frac{\frac{3\,\mathrm{t23}^2\,\mathrm{t51}^5}{1400}+\frac{3}{700}\,\mathrm{t23}\,\mathrm{t51}^5\,\mathrm{x12}+\frac{3\,\mathrm{t51}^5\,\mathrm{x12}^2}{1400}}{(-1+\mathrm{d})\,\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}+\frac{\frac{160\,\mathrm{t23}^2\,\mathrm{t51}^5}{63}+\frac{320}{63}\,\mathrm{t23}\,\mathrm{t51}^5\,\mathrm{x12}+\frac{160\,\mathrm{t51}^5\,\mathrm{x12}^2}{63}}{(-10+3\,\mathrm{d})\,\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}$$

  - Furthermore, these pieces can be combined to become:

$$-\frac{(-2+\mathrm{d})^2\,\mathrm{d}\,(2+\mathrm{d})\,\mathrm{t51}^5\,(\mathrm{t23}+\mathrm{x12})^2}{8\,(-1+\mathrm{d})\,(-7+2\,\mathrm{d})\,(-10+3\,\mathrm{d})\,(-8+3\,\mathrm{d})\,\mathrm{t45}^3\,(\mathrm{t34}-\mathrm{t51}-\mathrm{x12})^3}$$

    - Future work: exploit this to further simplify/speed up?
      - Does this simplicity appear only at the highest poles?

Further technical details: some options for performing *p*-adic probes

- One option: work directly with power-series in p up to some chosen p-adic order.
  - Possible loss of precision during probe (albeit better controlled than in floating-point real numbers)
- Another option: evaluate at integer points which happen to match the desired p-adic series at the desired p-adic order, then re-expand result as series in p
  - No loss of precision at intermediate stages of calculation
  - Size of integer probe result scales linearly with number of digits, and so with p-adic order
  - Can use finite fields to perform integer probes
    - The size of the finite field does not need to match the p of the p-adic field
      - e.g. use 64-bit finite fields to evaluate at an integer point that is special when viewed in Q_{101}

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Summary and Outlook

- Method for *p*-adic reconstruction of rational functions directly in partial-fractioned form
  - Demonstrated by reconstructing the largest rational function in largest IBP coefficient needed for non-planar 2-loop 5-point massless QCD amplitudes.
- Harnesses the major simplification of rational functions under partial fractioning
  - This comes from physics, not from computer algebra
  - (preliminary) Requires fewer numerical evaluations
  - Produces simpler expressions
  - Promising tool for exploring even further simplification.
    - Seek sufficient analytic understanding of the source of this simplification, so that it can be used to further improve speed/reach/elegance of future calculations