FACTORISATION AND SUBTRACTION BEYOND NLO

Lorenzo Magnea

University of Torino - INFN Torino

Theory Challenges - GGI - Florence - 31/09/23

UNIVERSITÀ DI TORINO

The Galileo Galilei Institute For Theoretical Physics

Arcetri, Firenze

INFN

Outline

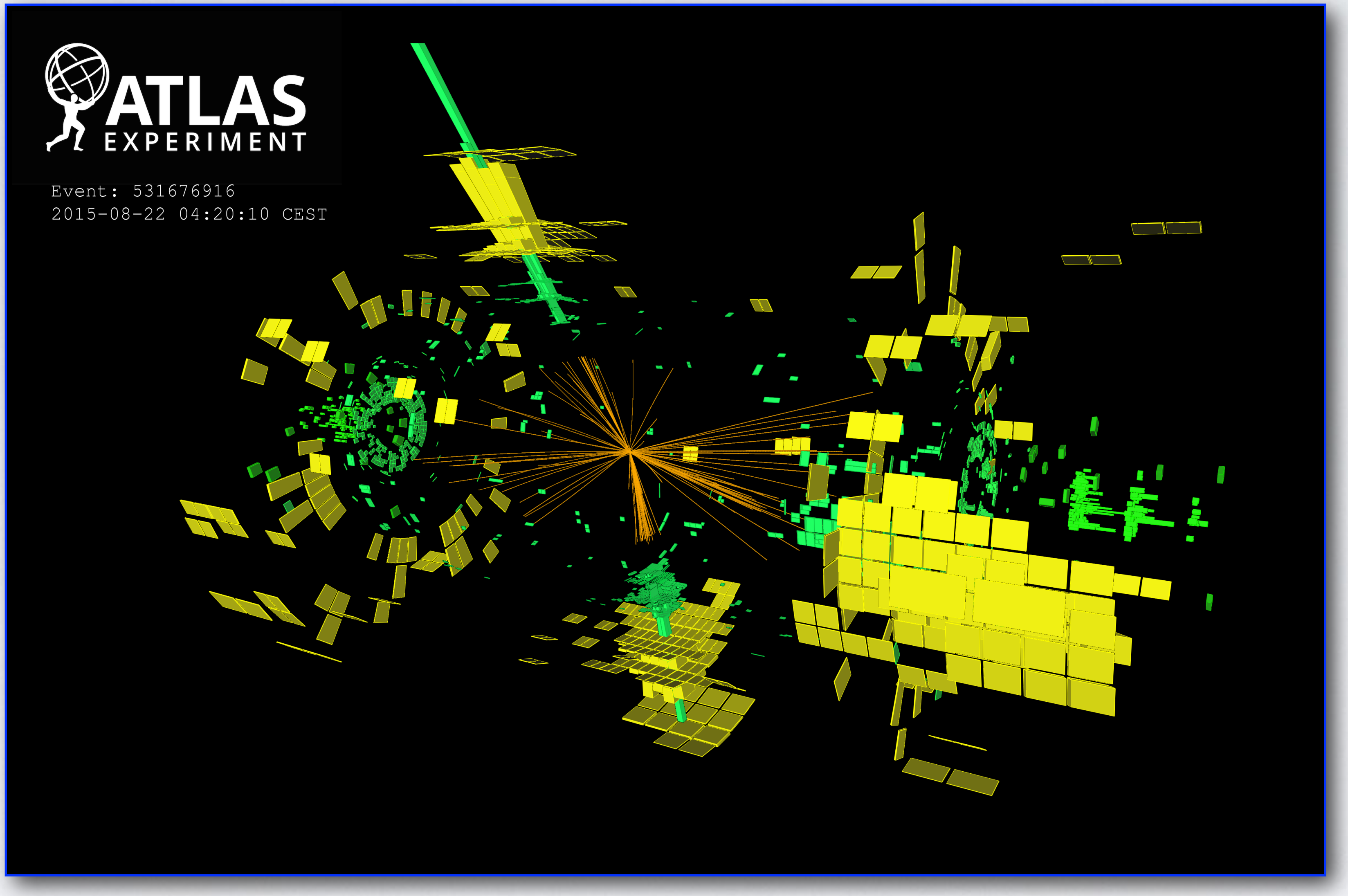
- Introduction
- Architecture of Subtraction
- Engineering Counterterms
- Building Strongly Ordered Limits
- Outlook

Outline

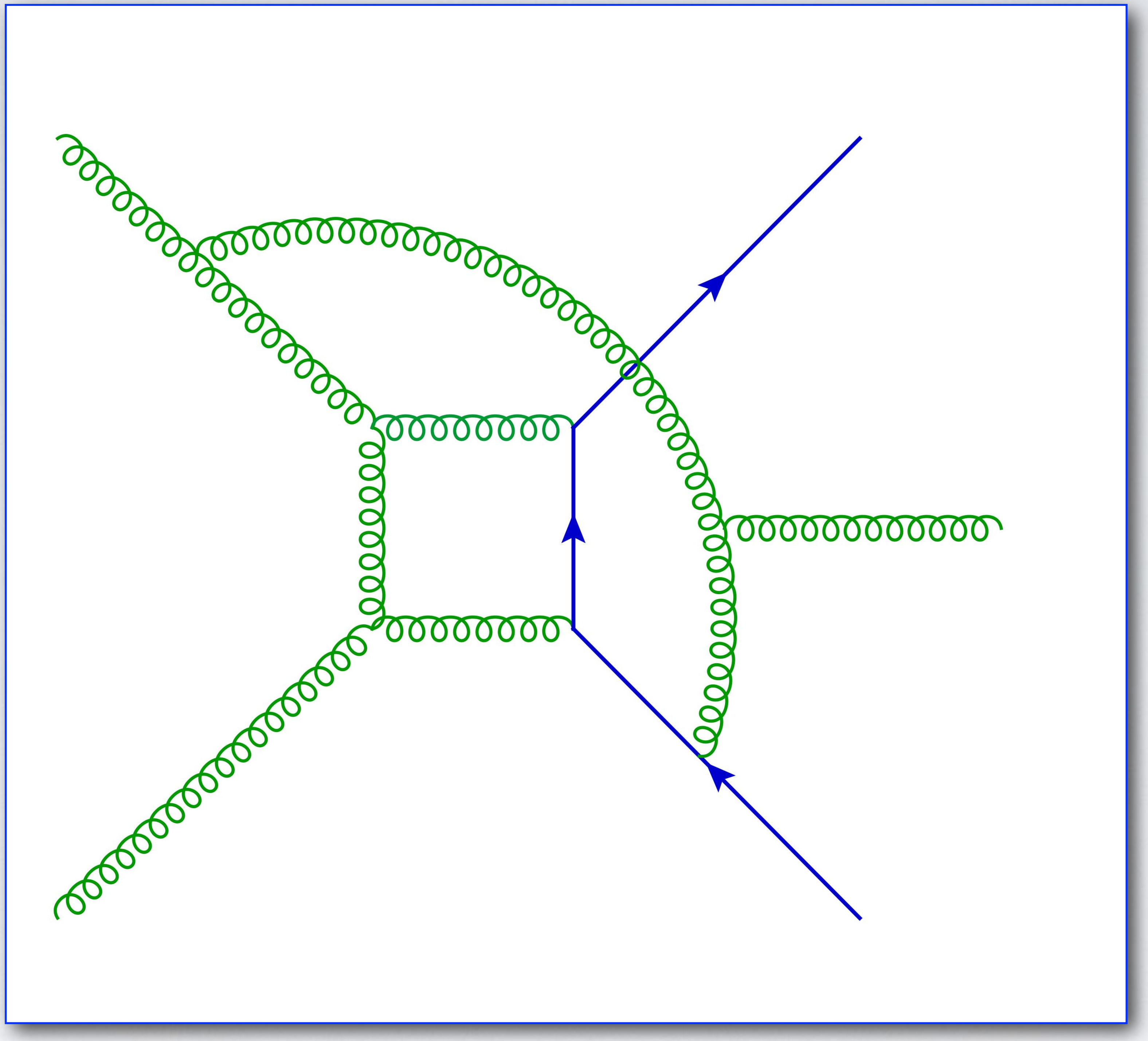
- Introduction
- Architecture of Subtraction
- Engineering Counterterms
- Building Strongly Ordered Limits
- Outlook

In collaboration with
Calum Milloy
Chiara Signorile-Signorile
Paolo Torrielli
Sandro Uccirati

INTRODUCTION

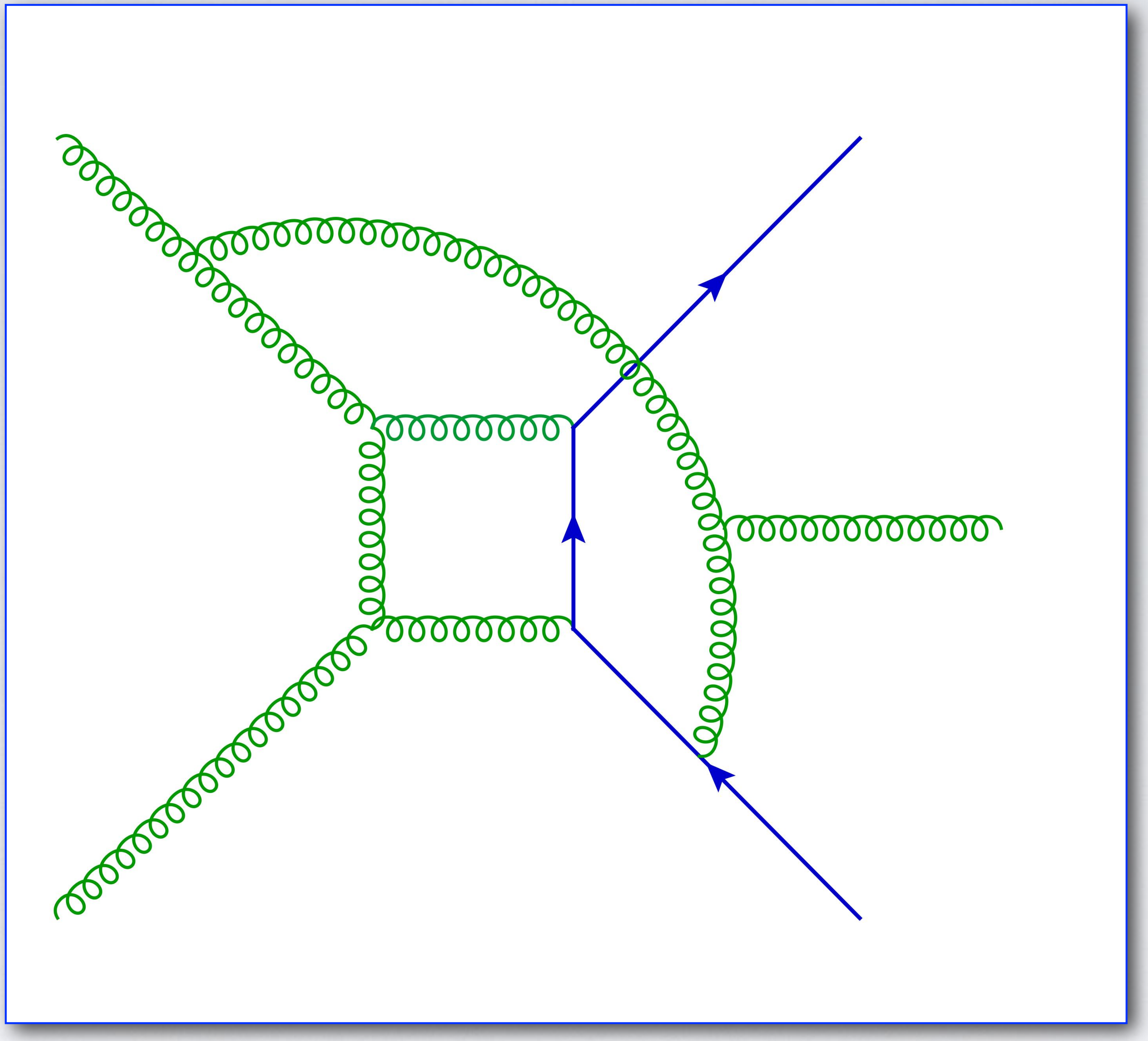


Pictorial infrared



A diagram contributing a double-virtual NNLO correction to t-tbar-jet production

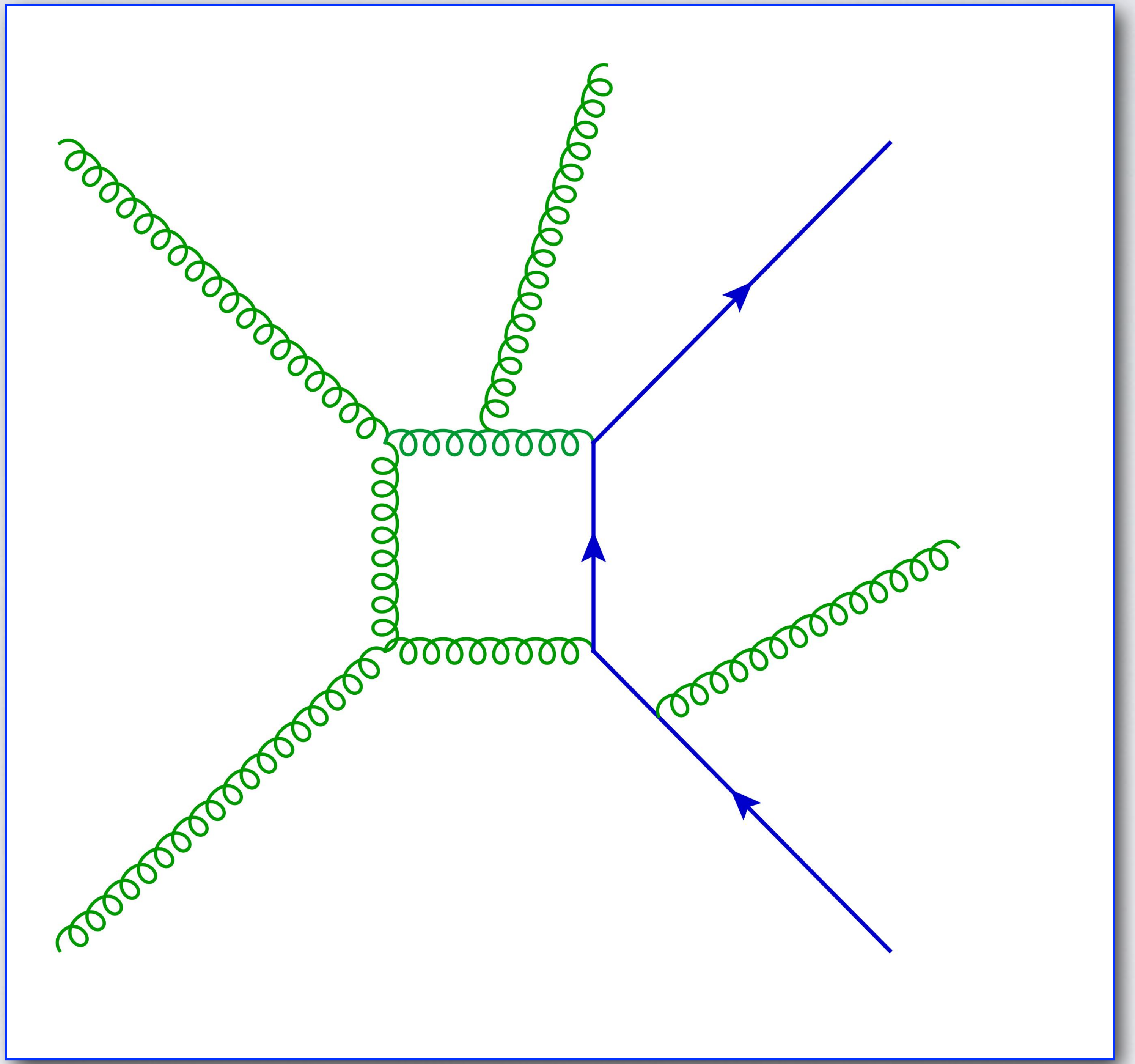
Pictorial infrared



$$\frac{1}{\epsilon^4}$$

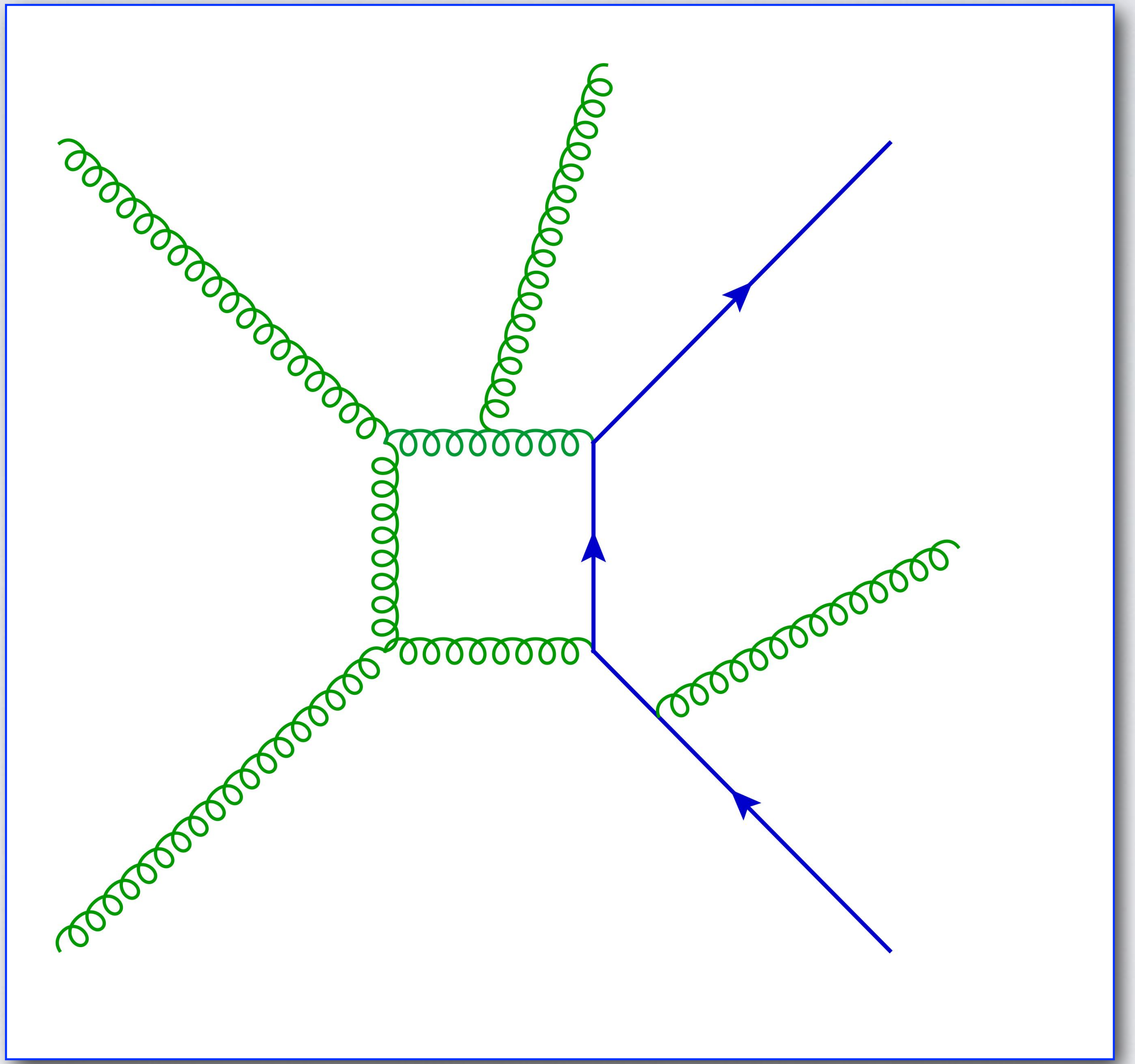
A diagram contributing a double-virtual NNLO correction to t-tbar-jet production

Pictorial infrared



A diagram contributing a real-virtual NNLO correction to t-tbar-jet production

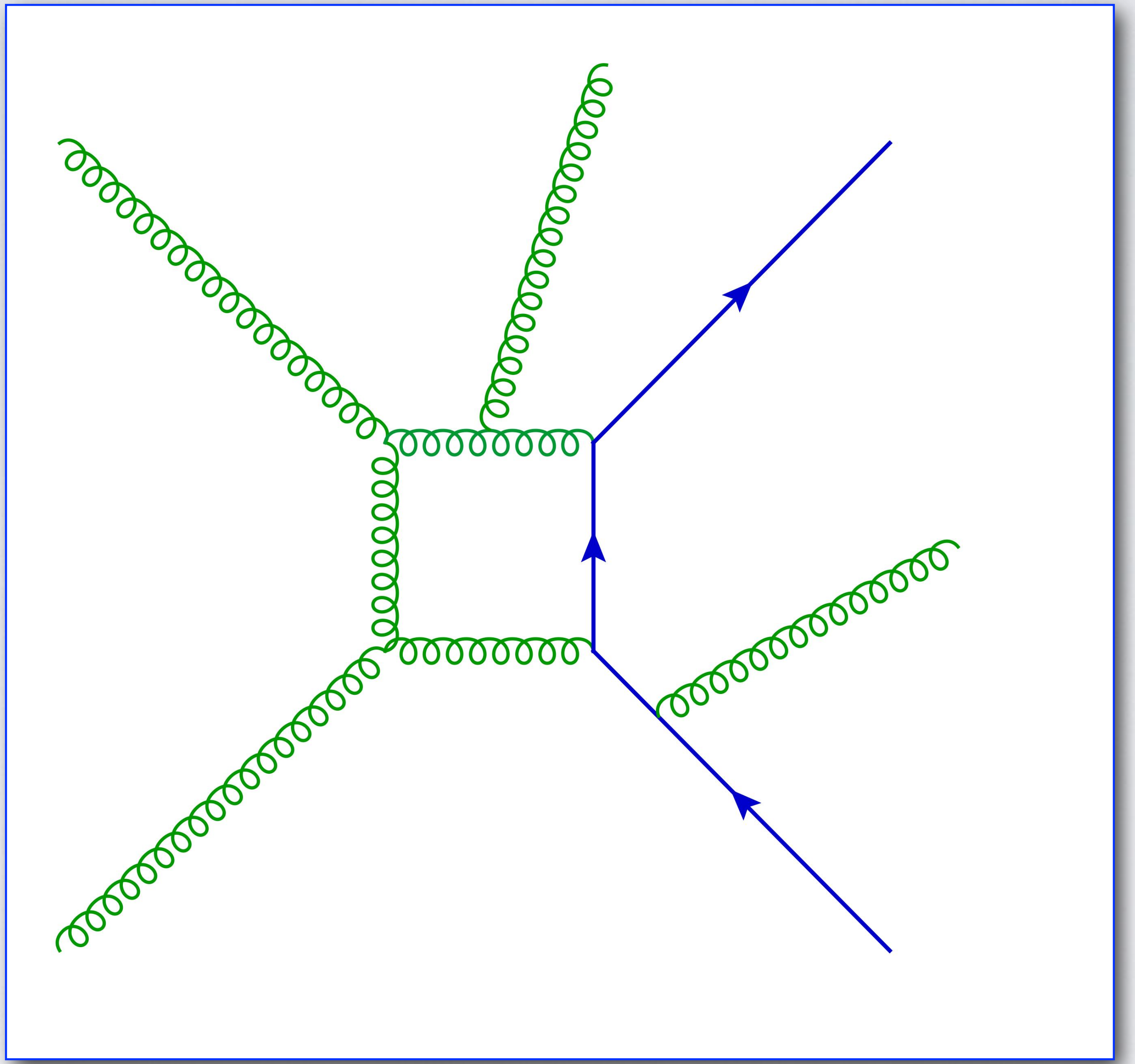
Pictorial infrared



$$\frac{1}{\epsilon^2}$$

A diagram contributing a real-virtual NNLO correction to t-tbar-jet production

Pictorial infrared

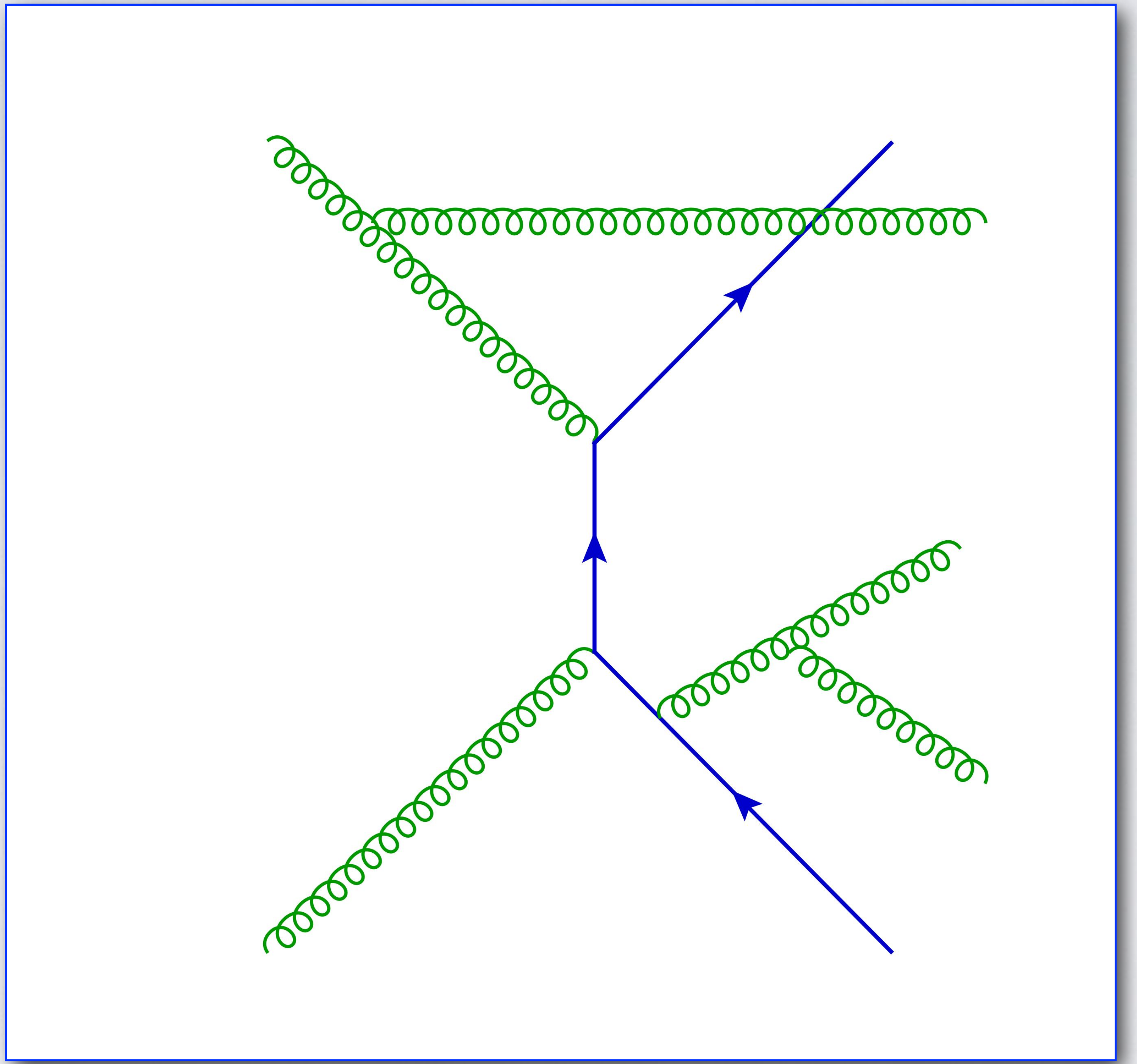


$$\frac{1}{\epsilon^2}$$

$$\frac{dE}{E}\,\frac{dk_\perp}{k_\perp}$$

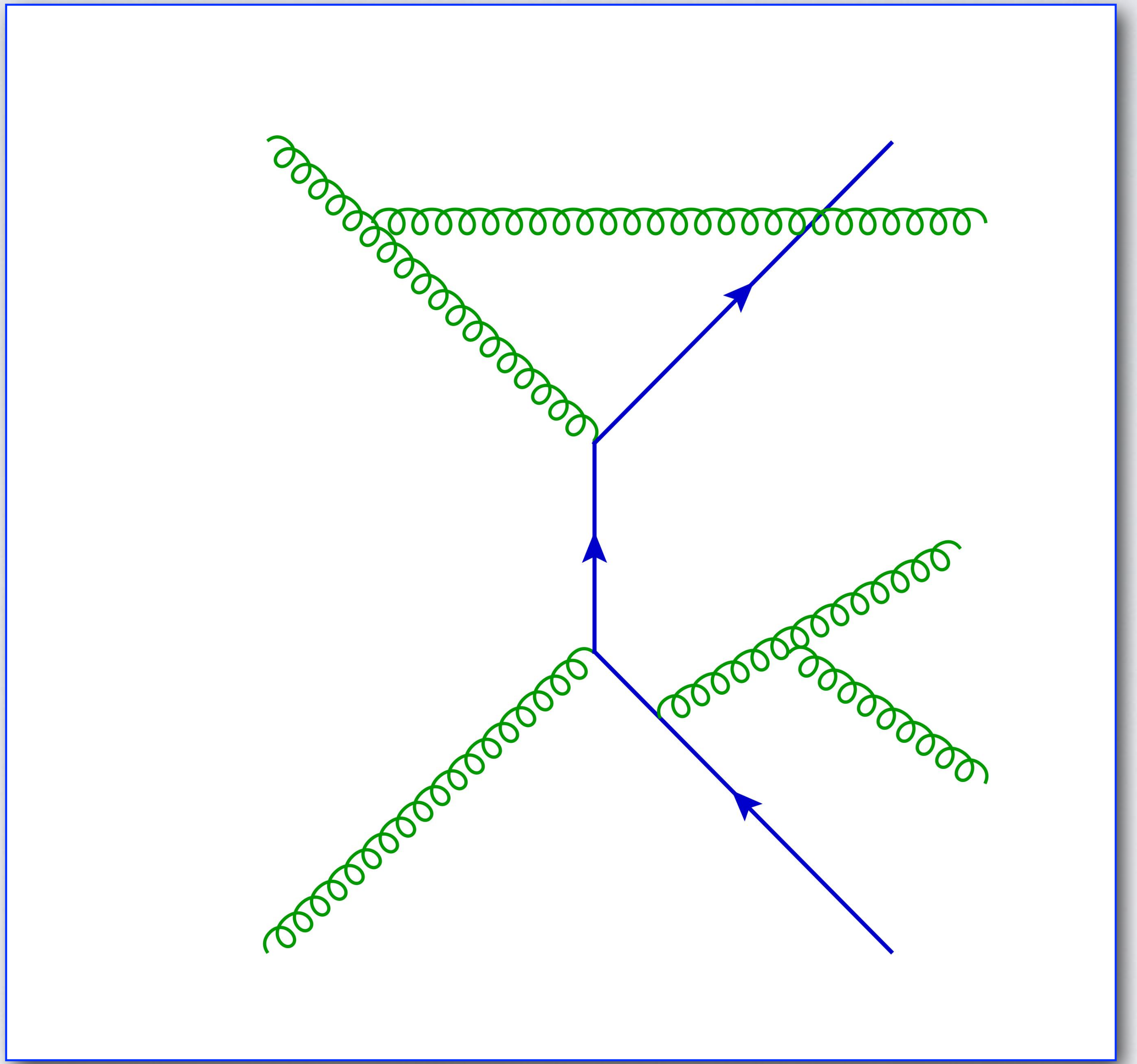
A diagram contributing a real-virtual NNLO correction to t-tbar-jet production

Pictorial infrared



A diagram contributing a double-real NNLO correction to t-tbar-jet production

Pictorial infrared



$$\left(\frac{dE}{E}\,\frac{dk_\perp}{k_\perp}\right)^2$$

A diagram contributing a double-real NNLO correction to t-tbar-jet production

A shopping list

What do we wish for an optimal subtraction algorithm at $\mathrm{N^kLO}$?

Here is a possible shopping list

- Complete generality across all IR-safe observables with arbitrary numbers of final state partons.
- Exact locality of the IR and collinear counterterms.
- Exact independence on external slicing parameters.
- Complete analytical results for all integrated counterterms.
- Overall computational efficiency, including interfacing with MC codes.

Such an algorithm would allow a user to input a process and an observable, and get out a prediction at the available order. The only external inputs would be the matrix elements.

This wish list has essentially been accomplished at NLO (CS, FKS, NS+, CKR, LASS, …)

Decades of effort at NNLO

The subtraction problem at NLO is completely solved, with efficient algorithms applicable to any process for which matrix elements are known.

At NNLO after twenty years of efforts several groups have working algorithms, successfully applied to processes with up to four legs. Five legs imply heavy computational costs.

- Antenna Subtraction
- Sector-Improved Residue Subtraction
- Nested Soft-Collinear Subtraction
- ColourfulNNLO
- N-Jettiness Slicing
- Q_T Slicing
- Geometric Slicing
- Unsubtraction
- Projection to Born
- Local Unitarity
- Local Analytic Sector Subtraction ……

1806.09570 - 2010.14493 - 2209.09123 - 2212.11190

LASS status

- So far the formalism is developed for massless partons.
- At NLO we have a full-fledged subtraction formalism, and simple integrals.
- NLO numerical implementation is under way.
- At NNLO Local Analytic Subtraction has been achieved for final state radiation.
    - A complete set of NNLO sector functions with the desired sum rules is available.
    - Flexible phase space mappings for single and double unresolved limits exist.
    - Phase space mappings have been checked not to misalign nested limits.
    - All integrals for final state radiation are done analytically, without IBP techniques.
- The numerical implementation at NNLO is the natural next step, also soon under way.
- Generalisation to initial state radiation requires work but no new concepts.
- More `interesting' integrals may arise with massive partons.

ARCHITECTURE

NLO Subtraction

The computation of a generic IR-safe observable at NLO requires the combination

$$\frac{d\sigma_{\mathrm{NLO}}}{dX} = \lim_{d\to 4}\left\{\int d\Phi_n\, V_n\, \delta_n(X) + \int d\Phi_{n+1}\, R_{n+1}\, \delta_{n+1}(X)\right\},$$

The necessary numerical integrations require finite ingredients in d=4. Define counterterms

$$K_{n+1}^{(\mathbf{1})} = \mathbf{L}^{(\mathbf{1})}\, R_{n+1}\,.$$

$$I_n^{(\mathbf{1})} \equiv \int d\Phi_{\mathrm{r},\,1}^{n+1}\, K_{n+1}^{(\mathbf{1})}\,,$$

Add and subtract the same quantity to the observable: each contribution is now finite.

$$\frac{d\sigma_{\mathrm{NLO}}}{dX} = \int d\Phi_n\Big(V_n + \underline{I_n^{(\mathbf{1})}}\Big)\,\delta_n(X) + \int d\Phi_{n+1}\Big(R_{n+1}\,\delta_{n+1}(X) - \underline{K_{n+1}^{(\mathbf{1})}\,\delta_n(X)}\Big)\,,$$

Search for the simplest fully local integrand K_{n+1} with the correct singular limits.

NLO Subtraction

The computation of a generic IRC-safe observable at NLO requires the combination

$$\frac{d\sigma_{\rm NLO}}{dX} = \lim_{d\to 4}\left\{\int d\Phi_n\, V_n\, \delta_n(X) + \int d\Phi_{n+1}\, R_{n+1}\, \delta_{n+1}(X)\right\},$$

The necessary numerical integrations require finite ingredients in d=4. Define counterterms

$$K^{(\mathbf{1})}_{n+1} = \mathbf{L}^{(\mathbf{1})} R_{n+1}\,.$$

$$I^{(\mathbf{1})}_n \equiv \int d\Phi^{n+1}_{\mathrm{r},\,1}\, K^{(\mathbf{1})}_{n+1}\,,$$

Add and subtract the same quantity to the observable: each contribution is now finite.

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n\Big(V_n + \underline{I^{(\mathbf{1})}_n}\Big)\,\delta_n(X) + \int d\Phi_{n+1}\Big(R_{n+1}\,\delta_{n+1}(X) - \underline{K^{(\mathbf{1})}_{n+1}\,\delta_n(X)}\Big)\,,$$

Search for the simplest fully local integrand K_{n+1} with the correct singular limits.

Defining $\mathbf{L}^{(1)}$ with sectors

Minimize complexity: split phase space in sectors with sector function $\mathcal{W}_{ij}$ in order to have at most one soft (i) and one collinear (ij) singularity in each sector (FKS).

- Sector functions must form a partition of unity.
- In order not to appear in analytic integrations, sector functions must obey sum rules. Denoting with $\mathbf{S}_i$ the soft limit for parton i and $\mathbf{C}_{ij}$ the collinear limit for the ij pair,

$$\mathbf{S}_i \sum_{k \neq i} \mathcal{W}_{ik} = 1\,, \qquad \mathbf{C}_{ij} \sum_{ab \,\in\, \mathrm{perm}(ij)} \mathcal{W}_{ab} = 1\,, \qquad \longleftarrow \quad \text{sum rules}$$

- Sector functions are defined in terms of Lorentz invariants before choosing an explicit parametrisation of phase space. A possible choice is

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum\limits_{k,\, l \neq k} \sigma_{kl}}\,, \qquad \text{with} \qquad \sigma_{ij} = \frac{1}{e_i\, w_{ij}}\,, \qquad e_i = \frac{s_{qi}}{s}\,, \qquad w_{ij} = \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}\,.$$

- With the help of sector functions, one can now define a candidate counterterm

$$\mathbf{L}^{(1)} R_{n+1} = \sum_i \sum_{j \neq i} \Big(\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij} \Big) R_{n+1} \mathcal{W}_{ij}\,.$$

Phase-space mappings at NLO

In order to factorise a Born matrix element B_n with n on-shell particles conserving momentum, we need a mapping from the (n+1)-particle to the Born phase spaces. We use (CS)

$$\bar{k}_i^{(abc)} = k_i, \quad \text{if } i \neq a, b, c,$$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}} k_c \,, \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}} k_c \,,$$

We can now redefine soft and collinear limits to include the re-parametrisation. Explicitly

$$\overline{\mathbf{S}}_i \, R(\{k\}) = -\mathcal{N}_1 \sum_{l,\,m} \delta_{f_i g} \frac{s_{lm}}{s_{il}\, s_{im}} B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big) \,,$$

$$\overline{\mathbf{C}}_{ij} \, R(k) = \frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij} \, B\Big(\{\bar{k}\}^{(ijr)}\Big) + Q_{ij}^{\mu\nu} \, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big) \Big] \,,$$

$$\overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, R(\{k\}) = 2\mathcal{N}_1 \, C_{f_j} \, \delta_{f_i g} \frac{s_{jr}}{s_{ij}\, s_{ir}} B\Big(\{\bar{k}\}^{(ijr)}\Big) \,,$$

Note that we have assigned parametrisation triplets differently in different terms. Then

$$\overline{K} = \sum_{i,j \neq i} \overline{K}_{ij} \,, \qquad \overline{K}_{ij} \equiv \big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \big) \, R \, \mathcal{W}_{ij} \,,$$

Far from trivial beyond NLO!
Systematics needed.
(Del Duca and Lionetti 1910.01024)

Phase-space mappings at NLO

In order to factorise a Born matrix element B_n with n on-shell particles conserving momentum, we need a mapping from the (n+1)-particle to the Born phase spaces. We use (CS)

$$\bar{k}_i^{(abc)} = k_i, \quad \text{if } i \neq a, b, c,$$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}} k_c, \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}} k_c,$$

We can now redefine soft and collinear limits to include the re-parametrisation. Explicitly

$$\overline{\mathbf{S}}_i \, R(\{k\}) = -\mathcal{N}_1 \sum_{l,m} \delta_{f_i g} \frac{s_{lm}}{s_{il}\, s_{im}} B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big),$$

$$\overline{\mathbf{C}}_{ij} \, R(k) = \frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij} \, B\Big(\{\bar{k}\}^{(ijr)}\Big) + Q_{ij}^{\mu\nu} \, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\Big],$$

$$\overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, R(\{k\}) = 2\mathcal{N}_1 \, C_{f_j} \, \delta_{f_i g} \frac{s_{jr}}{s_{ij}\, s_{ir}} B\Big(\{\bar{k}\}^{(ijr)}\Big),$$

Note that we have assigned parametrisation triplets differently in different terms. Then

$$\overline{K} = \sum_{i, j \neq i} \overline{K}_{ij}, \qquad \overline{K}_{ij} \equiv \left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij}\right) R \, \mathcal{W}_{ij},$$

NNLO Subtraction

The pattern of cancellations is more intricate at higher orders

$$\frac{d\sigma_{\rm NNLO}}{dX} = \lim_{d\to 4}\left\{\int d\Phi_n\, VV_n\, \delta_n(X) + \int d\Phi_{n+1}\, RV_{n+1}\, \delta_{n+1}(X) + \int d\Phi_{n+2}\, RR_{n+2}\, \delta_{n+2}(X)\right\},$$

More counterterm functions need to be defined

$$K^{(\mathbf{1})}_{n+2} = \mathbf{L}^{(1)}\, RR_{n+2}\,, \qquad K^{(\mathbf{2})}_{n+2} = \mathbf{L}^{(2)}\, RR_{n+2}\,, \qquad K^{(\mathbf{12})}_{n+2} = \mathbf{L}^{(1)}\,\mathbf{L}^{(2)}\, RR_{n+2}\,, \qquad K^{(\mathbf{RV})}_{n+1} = \mathbf{L}^{(1)}\, RV_{n+1}\,.$$

$$I^{(\mathbf{1})}_{n+1} = \int d\Phi^{n+2}_{\rm r,\,1}\, K^{(\mathbf{1})}_{n+2}\,, \quad I^{(\mathbf{12})}_{n+1} = \int d\Phi^{n+2}_{\rm r,\,1}\, K^{(\mathbf{12})}_{n+2}\,, \quad I^{(\mathbf{2})}_{n} = \int d\Phi^{n+2}_{\rm r,\,2}\, K^{(\mathbf{2})}_{n+2}\,, \quad I^{(\mathbf{RV})}_{n} = \int d\Phi^{n+1}_{\rm r,\,1}\, K^{(\mathbf{RV})}_{n+1}\,.$$

A finite expression for the observable in d=4 must combine several ingredients

$$\begin{aligned}\frac{d\sigma_{\rm NNLO}}{dX} = & \int d\Phi_n\Big[VV_n + \underline{I^{(\mathbf{2})}_n} + \underline{I^{(\mathbf{RV})}_n}\Big]\,\delta_n(X) \\ & + \int d\Phi_{n+1}\Big[\Big(RV_{n+1} + \underline{I^{(\mathbf{1})}_{n+1}}\Big)\,\delta_{n+1}(X) - \Big(\underline{K^{(\mathbf{RV})}_{n+1}} + \underline{I^{(\mathbf{12})}_{n+1}}\Big)\,\delta_n(X)\Big] \\ & + \int d\Phi_{n+2}\Big[RR_{n+2}\,\delta_{n+2}(X) - \underline{K^{(\mathbf{1})}_{n+2}}\,\delta_{n+1}(X) - \Big(\underline{K^{(\mathbf{2})}_{n+2}} - \underline{K^{(\mathbf{12})}_{n+2}}\Big)\,\delta_n(X)\Big]\end{aligned}$$

N^3LO Subtraction

A systematic generalisation to higher orders is possible. At three loops one finds

$$
\begin{aligned}
\frac{d\sigma_{\rm N3LO}}{dX} = & \int d\Phi_n \left[VVV_n + I_n^{(\mathbf{3})} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\,\mathbf{2})} \right] \delta_n(X) \\
+ & \int d\Phi_{n+1} \left[\left(RVV_{n+1} + I_{n+1}^{(\mathbf{2})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{1})} \right) \delta_{n+1}(X) \right. \\
& \qquad \left. - \left(K_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(\mathbf{23})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{12})} \right) \delta_n(X) \right] \\
+ & \int d\Phi_{n+2} \left\{ \left(RRV_{n+2} + I_{n+2}^{(\mathbf{1})} \right) \delta_{n+2}(X) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(\mathbf{12})} \right) \delta_{n+1}(X) \right. \\
& \qquad \left. - \left[\left(K_{n+2}^{(\mathbf{RRV},\,\mathbf{2})} + I_{n+2}^{(\mathbf{13})} \right) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{12})} + I_{n+2}^{(\mathbf{123})} \right) \right] \delta_n(X) \right\} \\
+ & \int d\Phi_{n+3} \left[RRR_{n+3}\, \delta_{n+3}(X) - K_{n+3}^{(\mathbf{1})}\, \delta_{n+2}(X) - \left(K_{n+3}^{(\mathbf{2})} - K_{n+3}^{(\mathbf{12})} \right) \delta_{n+1}(X) \right. \\
& \qquad \left. - \left(K_{n+3}^{(\mathbf{3})} - K_{n+3}^{(\mathbf{13})} - K_{n+3}^{(\mathbf{23})} + K_{n+3}^{(\mathbf{123})} \right) \delta_n(X) \right],
\end{aligned}
$$

A general formula for N^kLO subtraction is available, involving $p = 2^{(k+1)} - 2 - k$ counterterms.

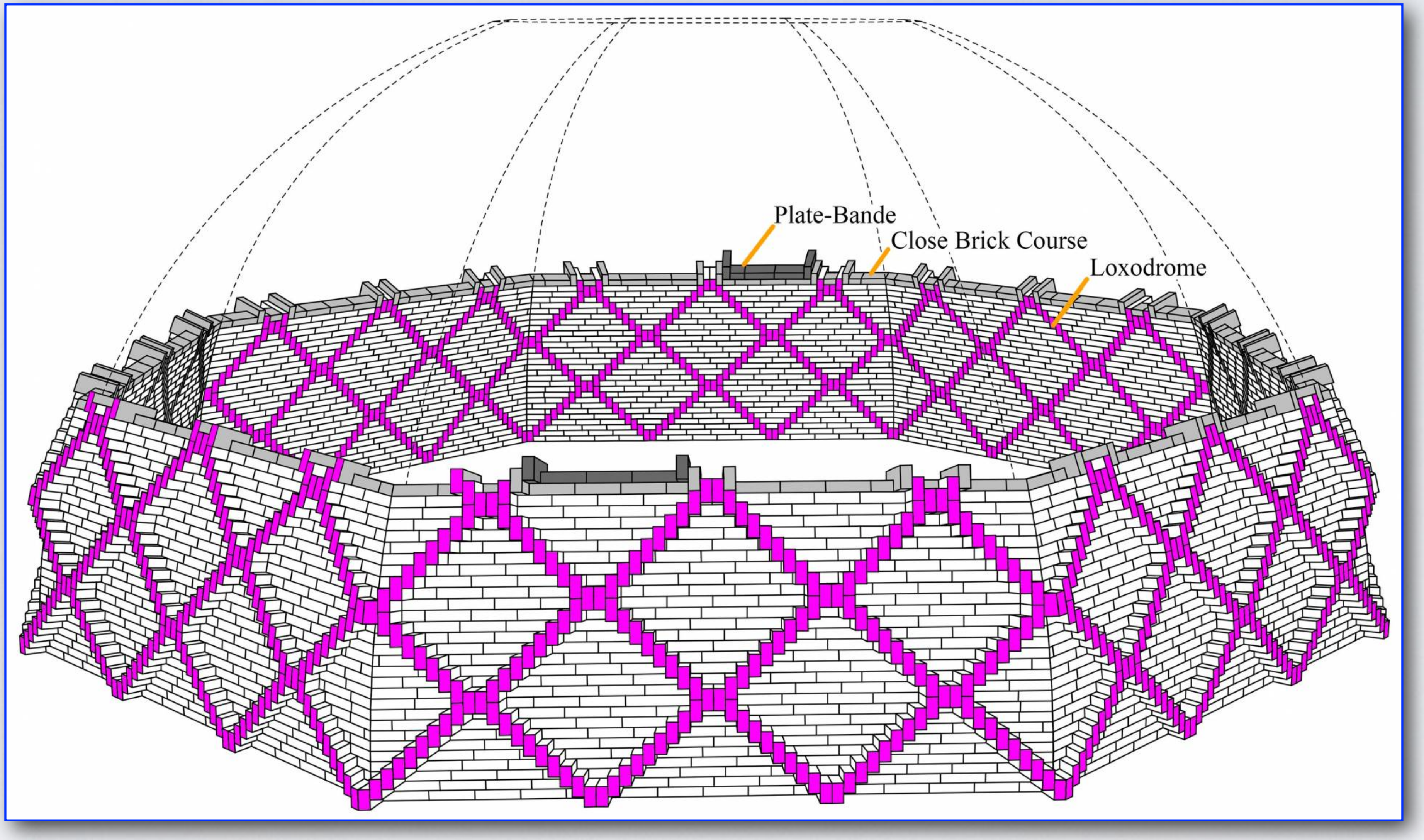
N^3LO Subtraction

A systematic generalisation to higher orders is possible. At three loops one finds

$$
\begin{aligned}
\frac{d\sigma_{\rm N3LO}}{dX} &= \int d\Phi_n \left[VVV_n + I_n^{(\mathbf{3})} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\,\mathbf{2})} \right] \delta_n(X) \\
&+ \int d\Phi_{n+1} \left[\left(RVV_{n+1} + I_{n+1}^{(\mathbf{2})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{1})} \right) \delta_{n+1}(X) \right. \\
&\qquad \left. - \left(K_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(\mathbf{23})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{12})} \right) \delta_n(X) \right] \\
&+ \int d\Phi_{n+2} \left\{ \left(RRV_{n+2} + I_{n+2}^{(\mathbf{1})} \right) \delta_{n+2}(X) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(\mathbf{12})} \right) \delta_{n+1}(X) \right. \\
&\qquad \left. - \left[\left(K_{n+2}^{(\mathbf{RRV},\,\mathbf{2})} + I_{n+2}^{(\mathbf{13})} \right) - \left(\underline{K_{n+2}^{(\mathbf{RRV},\mathbf{12})}} + I_{n+2}^{(\mathbf{123})} \right) \right] \delta_n(X) \right\} \\
&+ \int d\Phi_{n+3} \left[RRR_{n+3}\, \delta_{n+3}(X) - K_{n+3}^{(\mathbf{1})}\, \delta_{n+2}(X) - \left(K_{n+3}^{(\mathbf{2})} - \underline{K_{n+3}^{(\mathbf{12})}} \right) \delta_{n+1}(X) \right. \\
&\qquad \left. - \left(K_{n+3}^{(\mathbf{3})} - \underline{K_{n+3}^{(\mathbf{13})}} - \underline{K_{n+3}^{(\mathbf{23})}} + \underline{K_{n+3}^{(\mathbf{123})}} \right) \delta_n(X) \right],
\end{aligned}
$$

A general formula for N^kLO subtraction is available, involving $p = 2^{(k+1)} - 2 - k$ counterterms.

ENGINEERING



The virtual amplitude

A. Sen, A.H. Mueller, J. Collins, G. Sterman, J. Botts, LM, S. Catani, L. Dixon, E. Gardi, M. Neubert, T. Becher, I. Feige, M. Schwartz, O. Erdogan, Y. Ma, …

Infrared divergences in fixed-angle multi-particle scattering amplitudes factorise

$$\mathcal{A}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \mathcal{F}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) ,$$

The infrared factor is a colour operator determined by a finite anomalous dimension matrix

$$\mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{P} \exp\left[\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\, \Gamma_n\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2, \epsilon)\right)\right] ,$$

All infrared poles arise from the scale integration, through the d-dimensional running coupling

$$\lambda\frac{\partial\alpha_s}{\partial\lambda} \equiv \beta(\alpha_s, \epsilon) = -2\epsilon\alpha_s - \frac{\alpha_s^2}{2\pi}\sum_{k=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^k b_k \, .$$

For massless theories, the all-order structure of the anomalous dimension in known, up to corrections due to higher-order Casimir operators of the gauge algebra

$$\Gamma_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_n^{\rm dip}\left(\frac{s_{ij}}{\mu^2}, \alpha_s(\mu^2)\right) + \Delta_n\left(\rho_{ijkl}, \alpha_s(\mu^2)\right) ,$$

$$\rho_{ijkl} = \frac{p_i\cdot p_j\, p_k\cdot p_l}{p_i\cdot p_l\, p_j\cdot p_k} = \frac{s_{ij}s_{kl}}{s_{il}s_{jk}} \, .$$

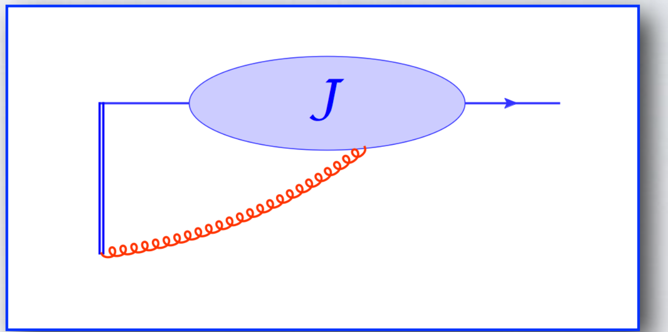
Operator Definitions

The precise functional form of this graphical factorisation is

$$\mathcal{A}_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^{n}\left[\frac{\mathcal{J}_i\Big((p_i\cdot n_i)^2/(n_i^2\mu^2)\Big)}{\mathcal{J}_{E,\,i}\Big((\beta_i\cdot n_i)^2/n_i^2\Big)}\right]\mathcal{S}_n\left(\beta_i\cdot\beta_j\right)\;\mathcal{H}_n\left(\frac{p_i\cdot p_j}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}\right)$$

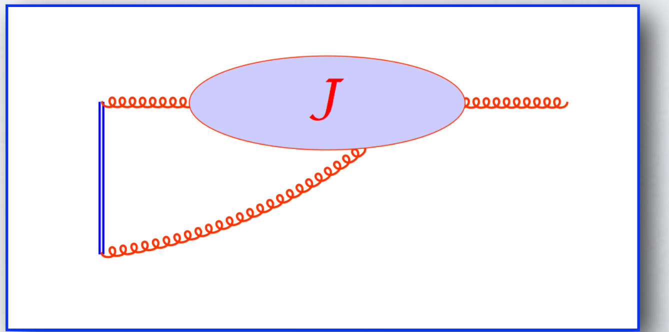
Here we introduced dimensionless four-velocities $\beta_i = p_i/Q$, and factorisation vectors n_i^μ, $n_i^2 \neq 0$ to define the jets in a gauge-invariant way. For outgoing quarks

$$\overline{u}_s(p)\,\mathcal{J}_q\left(\frac{(p\cdot n)^2}{n^2\mu^2}\right) = \langle p,s\,|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle$$



where Φ_n is the Wilson line operator along the direction n. For outgoing gluons

$$g_s\,\varepsilon_\mu^{*\,(\lambda)}(k)\mathcal{J}_g^{\mu\nu}\left(\frac{(k\cdot n)^2}{n^2\mu^2}\right) \equiv \langle k,\lambda|\,\Big[\Phi_n(\infty,0)\,\mathrm{i}D^\nu\,\Phi_n(0,\infty)\Big]\,|0\rangle\;,$$

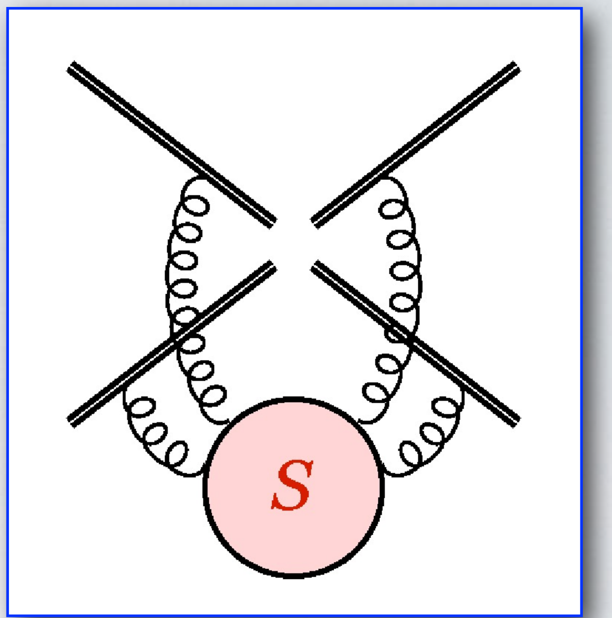


T. Becher, G. Bell.

Wilson line correlators

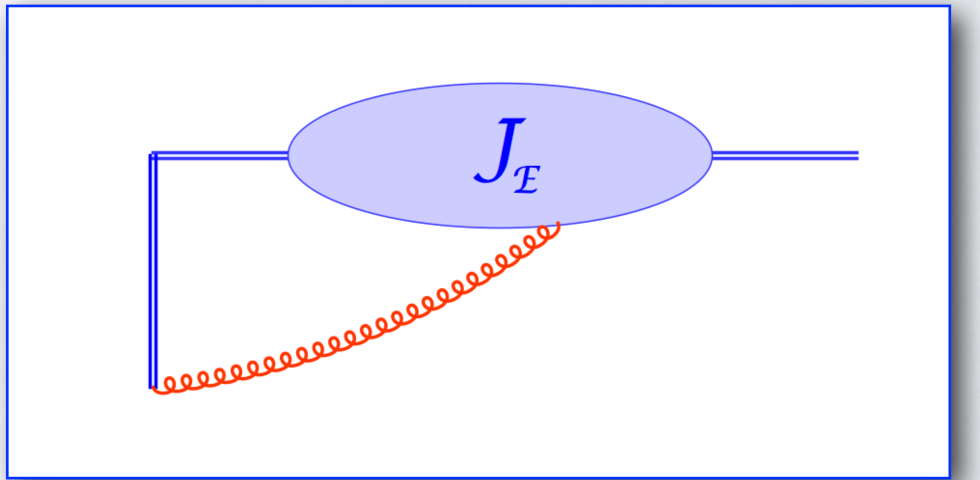
The soft function S is a color operator, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$\mathcal{S}_n\left(\beta_i \cdot \beta_j\right) = \langle 0| \prod_{k=1}^{n} \Phi_{\beta_k}(\infty, 0)\,|0\rangle$$



The soft jet function $J_{\mathcal{E}}$ contains soft-collinear poles: it is defined by replacing the field in the ordinary jet J with a Wilson line in the appropriate color representation.

$$\mathcal{J}_{\mathrm{E}}\left(\frac{(\beta \cdot n)^2}{n^2}\right) = \langle 0|\,\Phi_\beta(\infty, 0)\,\Phi_n(0, \infty)\,|0\rangle$$



Wilson-line matrix elements exponentiate non-trivially and have tightly constrained functional dependence on their arguments. They are known to three loops.

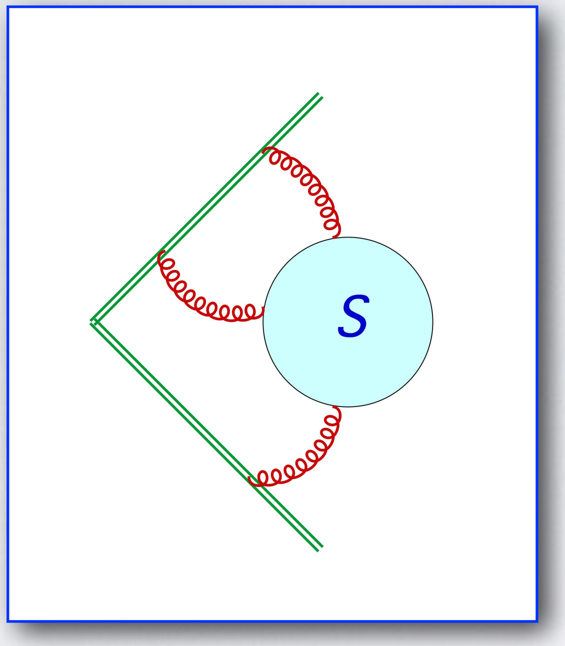
Soft cross sections: pictorial

Consider first the (academic) case of purely soft final state divergences.

Soft cross sections: pictorial

Consider first the (academic) case of purely soft final state divergences.

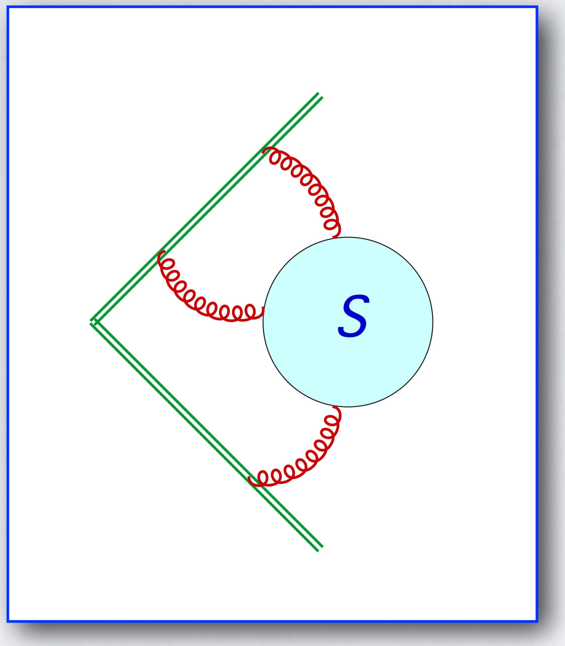
At amplitude level poles factorise and exponentiate.



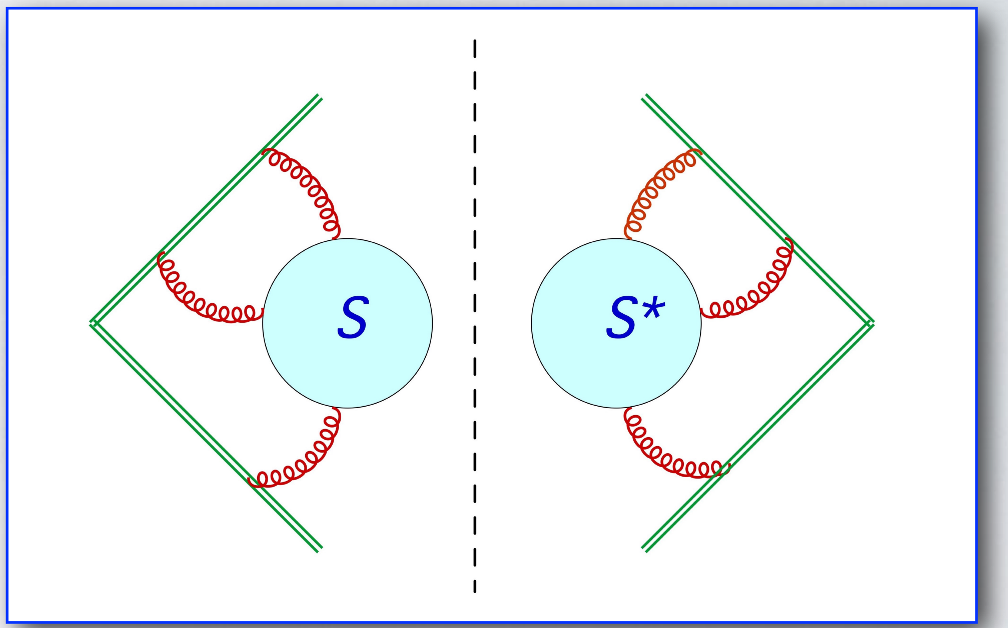
Soft cross sections: pictorial

Consider first the (academic) case of purely soft final state divergences.

At amplitude level poles factorise and exponentiate.



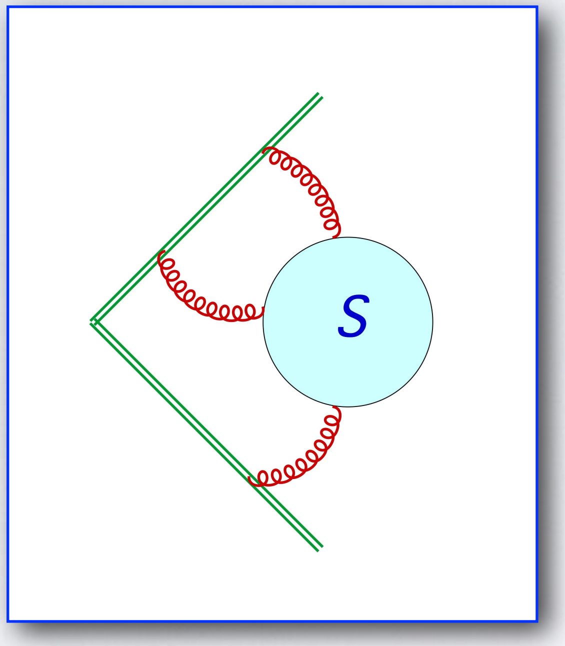
We need to build cross-section level quantities.



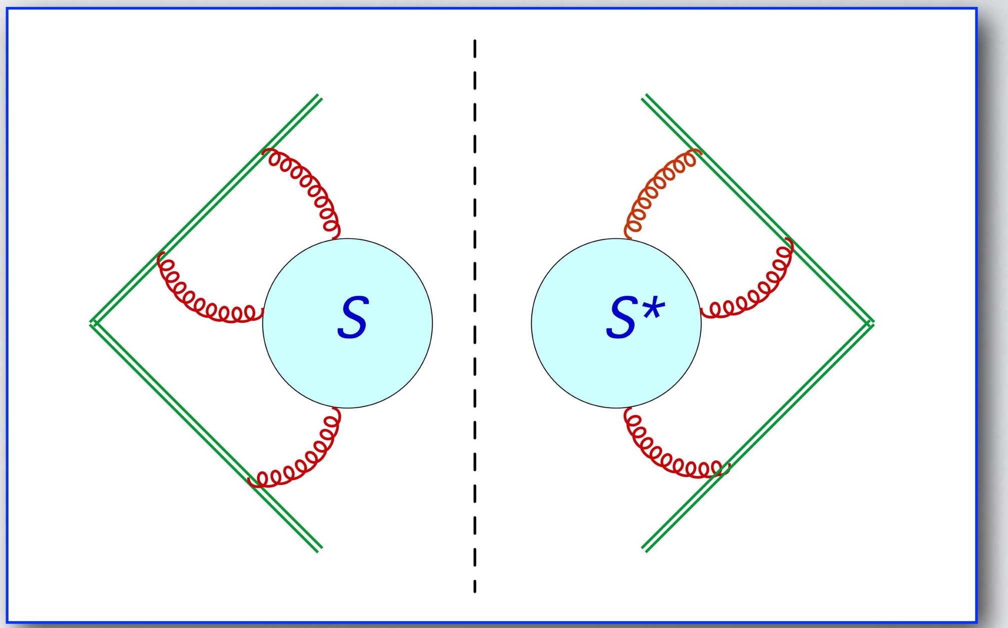
Soft cross sections: pictorial

Consider first the (academic) case of purely soft final state divergences.

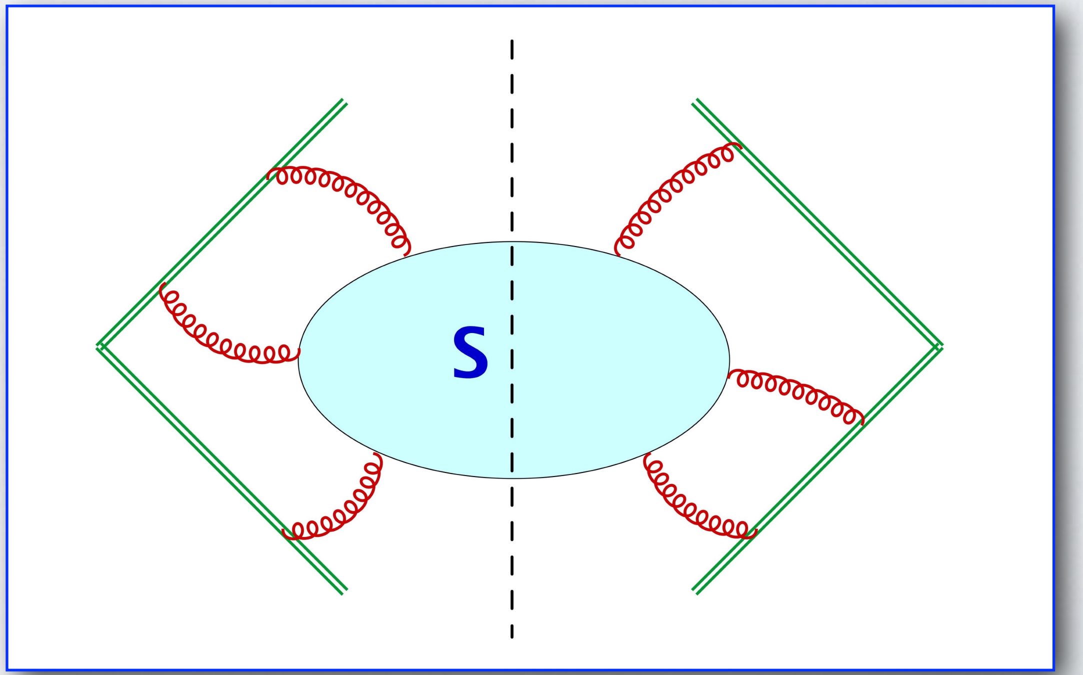
At amplitude level poles factorise and exponentiate.



We need to build cross-section level quantities.



- Inclusive eikonal cross sections are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local soft counterterms.



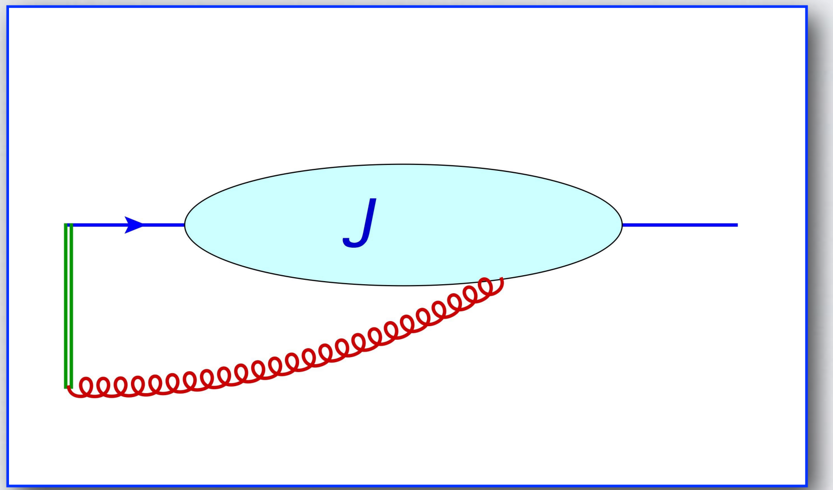
Collinear cross sections: pictorial

Consider next collinear final state divergences. They are associated with individual partons.

Collinear cross sections: pictorial

Consider next collinear final state divergences. They are associated with individual partons.

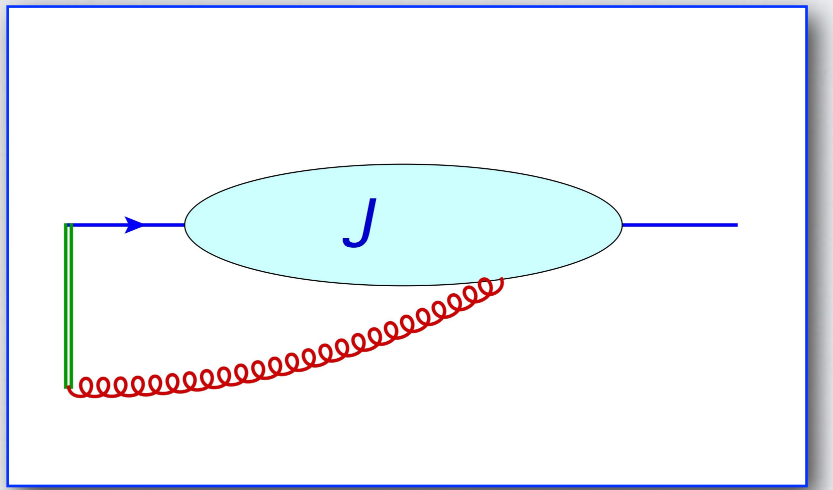
At amplitude level poles factorise and exponentiate.



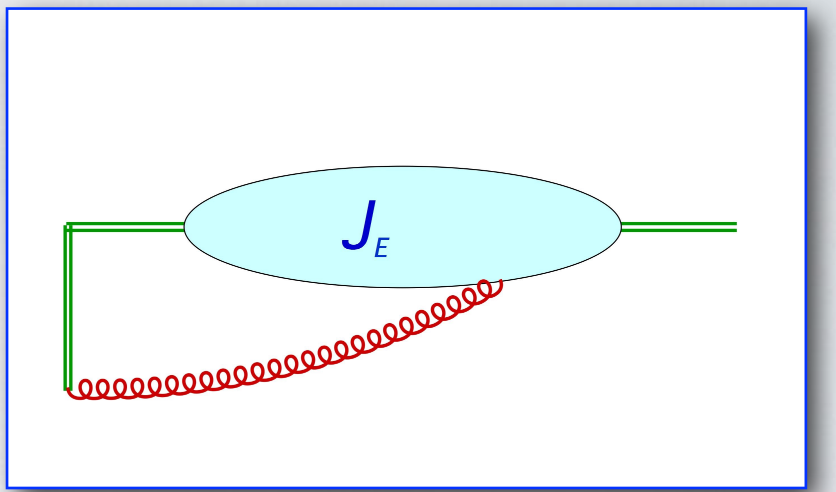
Collinear cross sections: pictorial

Consider next collinear final state divergences. They are associated with individual partons.

At amplitude level poles factorise and exponentiate.



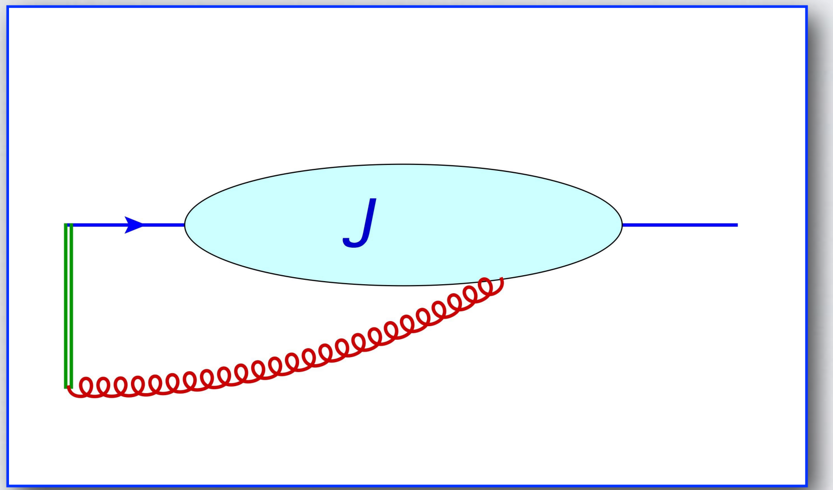
Soft-collinear poles can be subtracted



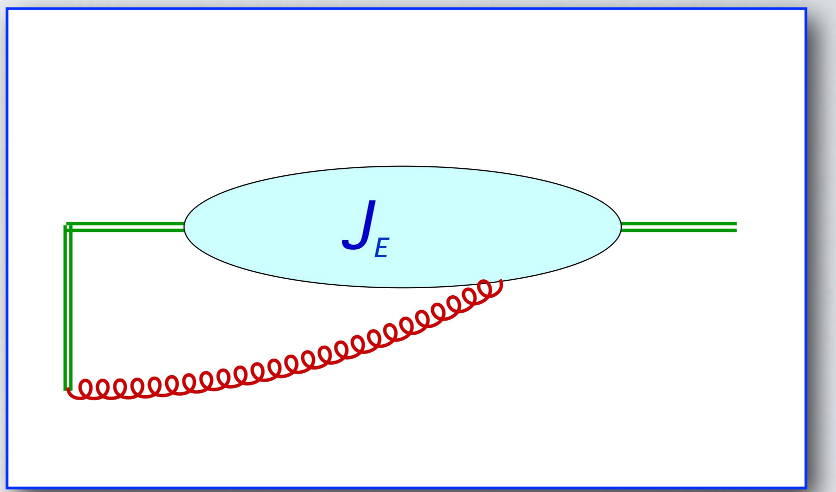
Collinear cross sections: pictorial

Consider next collinear final state divergences. They are associated with individual partons.

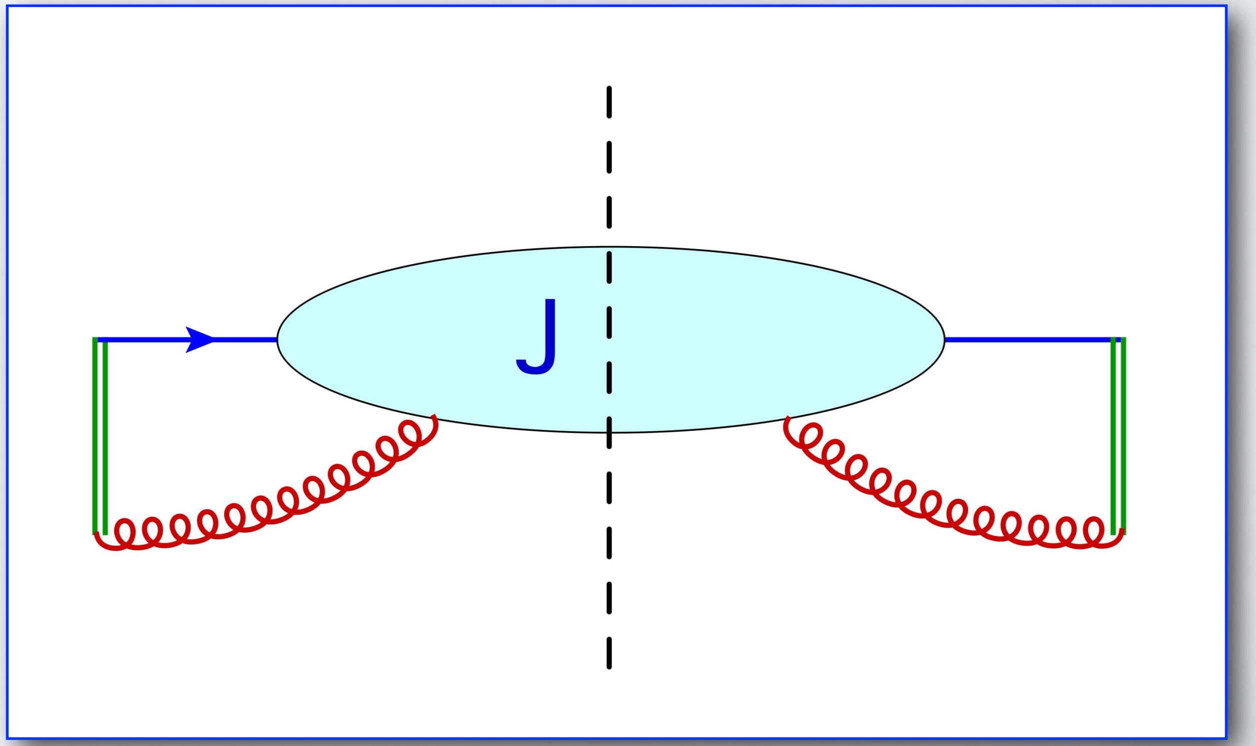
At amplitude level poles factorise and exponentiate.



Soft-collinear poles can be subtracted



- Inclusive `jet cross sections' are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local collinear counterterms.
- Eikonal jet cross sections subtract the soft-collinear double counting.



See also Feige, Schwartz 2014

Soft counterterms: all orders

Introduce eikonal form factors for the emission of m soft partons from n hard ones.

$$\mathcal{S}_{n,m}\left(k_1,\ldots,k_m;\beta_i\right) \equiv \langle k_1,\lambda_1;\ldots;k_m,\lambda_m| \prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\;|0\rangle$$
$$\equiv \epsilon_{\mu_1}^{*\,(\lambda_1)}(k_1)\ldots\epsilon_{\mu_m}^{*\,(\lambda_m)}(k_m)\;J_{\mathcal{S}}^{\mu_1\ldots\mu_m}\left(k_1,\ldots,k_m;\beta_i\right)$$
$$\equiv \sum_{p=0}^{\infty}\mathcal{S}_{n,m}^{(p)}\left(k_1,\ldots,k_m;\beta_i\right)$$

These matrix elements define soft gluon multiple emission currents. They are gauge invariant and they contain loop corrections to all orders.

Existing finite order calculations and all-order arguments are consistent with the factorisation

$$\mathcal{A}_{n,m}\left(k_1,\ldots,k_m;p_i\right) = \mathcal{S}_{n,m}\left(k_1,\ldots,k_m;\beta_i\right)\mathcal{H}_n(p_i) + \mathcal{R}_{n,m}\left(k_1,\ldots,k_m;p_i\right)$$

with corrections that are finite in dimensional regularisation, and integrable in the soft gluon phase space. It is a working assumption: a formal all-order proof is still lacking.

Soft counterterms: all orders

The factorisation is reflected at cross-section level, for fixed final state quantum numbers.

$$\sum_{\lambda_i} |\mathcal{A}_{n,m}(k_1,\ldots,k_m;p_i)|^2 \simeq \mathcal{H}_n^\dagger(p_i)\, S_{n,m}(k_1,\ldots,k_m;\beta_i)\, \mathcal{H}_n(p_i)$$

The *cross-section level* "radiative soft functions" are Wilson-line squared matrix elements

$$S_{n,m}\Big(\{k_m\},\{\beta_i\}\Big) \equiv \sum_{p=0}^{\infty} S_{n,m}^{(p)}\Big(\{k_m\},\{\beta_i\}\Big)$$
$$\equiv \sum_{\{\lambda_i\}} \langle 0|\,\overline{T}\left[\prod_{i=1}^{n}\Phi_{\beta_i}(0,\infty)\right]|k_1,\lambda_1;\ldots;k_m,\lambda_m\rangle\,\langle k_1,\lambda_1;\ldots;k_m,\lambda_m|\,T\left[\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\right]|0\rangle\,,$$

These functions provide a complete list of local soft subtraction counterterms, to all orders. Indeed, summing over particle numbers and integrating over the soft phase space one finds

$$\sum_{m=0}^{\infty}\int d\Phi_m\, S_{n,m}\Big(\{k_m\};\{\beta_i\}\Big) = \langle 0|\,\overline{T}\left[\prod_{i=1}^{n}\Phi_{\beta_i}(0,\infty)\right] T\left[\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\right]|0\rangle\,.$$

"Completeness relation"

This is a finite fully inclusive soft cross section, order by order in perturbation theory.

Collinear counterterms: all orders

For collinear poles, introduce jet matrix elements for the emission of m partons. For quarks

$$\overline{u}_s(p)\,\mathcal{J}_{q,\,m}\,(k_1,\ldots,k_m;p,n)\;\equiv\;\langle p,s;k_1,\lambda_1;\ldots;k_m,\lambda_m|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle$$

At *cross-section level*, "radiative jet functions" can be defined as Fourier transforms of squared matrix elements, to account for the non-trivial momentum flow. We propose

$$J_{q,\,m}(\{k_m\};l,p,n)\;\equiv\;\sum_{p=0}^{\infty}J_{q,\,m}^{(p)}(\{k_m\};l,p,n)$$
$$\equiv\int d^dx\,\mathrm{e}^{\mathrm{i}l\cdot x}\sum_{\{\lambda_m\}}\langle 0|\,\overline{T}\Big[\Phi_n(\infty,x)\,\psi(x)\Big]\,|p,s;\{k_m,\lambda_m\}\rangle\,\langle p,s;\{k_m,\lambda_m\}|\,T\Big[\overline{\psi}(0)\,\Phi_n(0,\infty)\Big]\,|0\rangle\;,$$

These functions provide a complete list of local collinear counterterms, to all orders. Summing over particle numbers and integrating over the collinear phase space one finds

$$\sum_{m=0}^{\infty}\int d\Phi_{m+1}\,J_{q,\,m}(\{k_m\};l,p,n)\;=\;\mathrm{Disc}\Big[\int d^dx\,\mathrm{e}^{\mathrm{i}l\cdot x}\,\langle 0|\,T\Big[\Phi_n(\infty,x)\psi(x)\overline{\psi}(0)\Phi_n(0,\infty)\Big]\,|0\rangle\Big]\,.$$

"Completeness relation"

A "two-point function", finite order by order in perturbation theory. Note however

- The collinear limit must still be taken (as $l^2 \to 0$), unlike the case of radiative soft functions.
- $n^2 \neq 0$ avoids spurious collinear poles, but is cumbersome ➔ use SCET-like anti-collinear n^μ.

NLO subtraction

A "top-down" approach $\mathcal{A}_n(p_i) = \prod_{i=1}^{n} \left[\frac{\mathcal{J}_i(p_i, n_i)}{\mathcal{J}_{E,i}(\beta_i, n_i)} \right] \mathcal{S}_n(\beta_j)\, \mathcal{H}_n(p_i)$

The outlines of a subtraction procedure emerge. Begin by expanding the virtual matrix element

$$\mathcal{A}_n(p_i) = \Bigg[\mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) + \sum_{i=1}^{n} \left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i) \right) \mathcal{S}_n^{(0)}(\beta_i)\, \mathcal{H}_n^{(0)}(p_i) \Bigg] \left(1 + \mathcal{O}\left(\alpha_s^2\right) \right)$$

From the master formula, get the virtual poles of the cross section in terms of virtual kernels

$$V_n \equiv 2\,\mathbf{Re}\Big[\mathcal{A}_n^{(0)*} \mathcal{A}_n^{(1)} \Big] \simeq \mathcal{H}_n^{(0)\,\dagger}(p_i) S_{n,\,0}^{(1)}(\beta_i) \mathcal{H}_n^{(0)}(p_i) + \sum_i \left(J_{i,\,0}^{(1)}(p_i) - J_{\mathrm{E},\,i,\,0}^{(1)}(\beta_i) \right) \Big| \mathcal{A}_n^{(0)}(p_i) \Big|^2$$

Go through the list of proposed soft and collinear counterterms to collect the relevant ones

$$S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1\, S_{n,\,1}^{(0)}(k, \beta_i) = \text{finite}$$

$$J_{i,\,0}^{(1)}(l, p, n) + \int d\Phi_1\, J_{i,\,1}^{(0)}(k; l, p, n) = \text{finite}$$

Construct the appropriate local functions.

$$K_{n+1}^{\mathrm{NLO,\,s}} = \mathcal{H}_n^{(0)\,\dagger}(p_i)\, S_{n,\,1}^{(0)}(k, \beta_i)\, \mathcal{H}_n^{(0)}(p_i)$$

$$K_{n+1}^{\mathrm{NLO,\,c}} = \sum_{i=1}^{n} J_{i,\,1}^{(0)}(k_i; l, p_i, n_i) \left| \mathcal{A}_n^{(0)}\left(p_1, \ldots, p_{i-1}, l, p_{i+1}, \ldots, p_n\right) \right|^2$$

with a similar expression for the anti-subtraction of the soft-collinear region in terms of $J_{\mathcal{E}}$.

NLO subtraction

A "top-down" approach $\mathcal{A}_n(p_i) = \prod_{i=1}^{n} \left[\frac{\mathcal{J}_i(p_i, n_i)}{\mathcal{J}_{E,i}(\beta_i, n_i)} \right] \mathcal{S}_n(\beta_j)\, \mathcal{H}_n(p_i)$

The outlines of a subtraction procedure emerge. Begin by expanding the virtual matrix element

$$\mathcal{A}_n(p_i) = \Big[\mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) + \sum_{i=1}^{n} \left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i) \right) \mathcal{S}_n^{(0)}(\beta_i)\, \mathcal{H}_n^{(0)}(p_i) \Big] \left(1 + \mathcal{O}\left(\alpha_s^2\right) \right)$$

From the master formula, get the virtual poles of the cross section in terms of virtual kernels

$$V_n \equiv 2\,\mathbf{Re}\Big[\mathcal{A}_n^{(0)*} \mathcal{A}_n^{(1)} \Big] \simeq \mathcal{H}_n^{(0)\,\dagger}(p_i) S_{n,\,0}^{(1)}(\beta_i) \mathcal{H}_n^{(0)}(p_i) + \sum_i \left(J_{i,\,0}^{(1)}(p_i) - J_{\mathrm{E},\,i,\,0}^{(1)}(\beta_i) \right) \left| \mathcal{A}_n^{(0)}(p_i) \right|^2$$

Go through the list of proposed soft and collinear counterterms to collect the relevant ones

$$S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1\, S_{n,\,1}^{(0)}(k, \beta_i) = \text{finite}$$

$$J_{i,\,0}^{(1)}(l, p, n) + \int d\Phi_1\, J_{i,\,1}^{(0)}(k; l, p, n) = \text{finite}$$

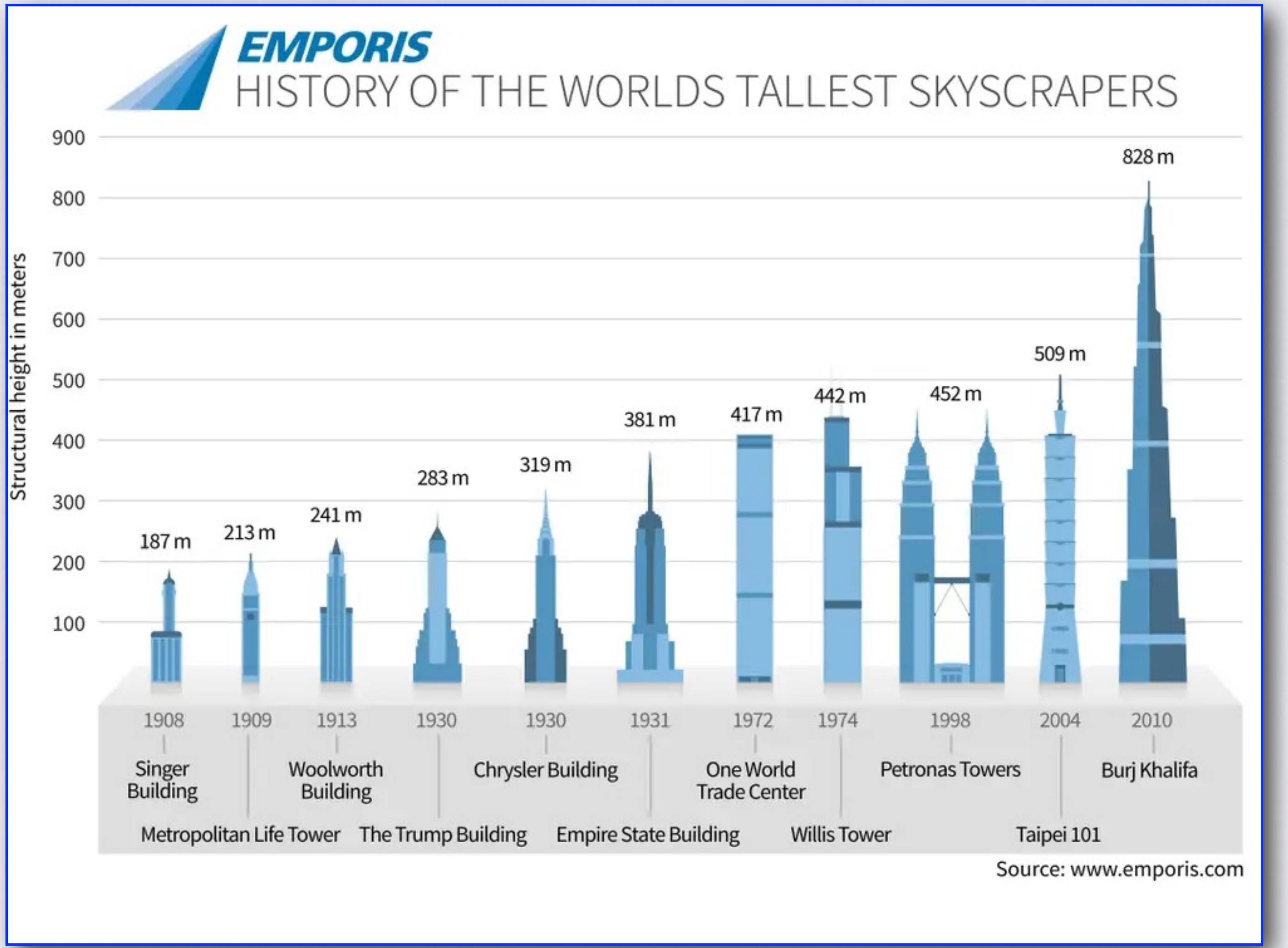
Construct the appropriate local functions.

$$K_{n+1}^{\mathrm{NLO,\,s}} = \mathcal{H}_n^{(0)\,\dagger}(p_i)\, S_{n,\,1}^{(0)}(k, \beta_i)\, \mathcal{H}_n^{(0)}(p_i)$$

$$K_{n+1}^{\mathrm{NLO,\,c}} = \sum_{i=1}^{n} J_{i,\,1}^{(0)}(k_i; l, p_i, n_i) \left| \mathcal{A}_n^{(0)}(p_1, \ldots, p_{i-1}, l, p_{i+1}, \ldots, p_n) \right|^2$$

with a similar expression for the anti-subtraction of the soft-collinear region in terms of $J_{\mathcal{E}}$.

BUILDING STRONG ORDERING



S. Catani, M. Ciafaloni 1984
S. Catani, M. Ciafaloni, G. Marchesini 1985

Soft refactorisation: tree level

The tree-level double soft-gluon current simplifies considerably in the strong-ordering limit

$$\left[J_{\rm CG}^{(0),\,{\rm s.o.}}\right]^{a_1 a_2}_{\mu_1 \mu_2}(k_1, k_2; \beta_i) \, = \, \left(J^{(0)\, a_2}_{\mu_2}(k_2)\, \delta^{a_1 a} + {\rm i} g_s \, f^{a_1 a_2 a} \, \frac{k_{1,\,\mu_2}}{k_1 \cdot k_2} \right) J^{(0)}_{\mu_1,\, a}(k_1)\,,$$

$$J^{(0)\, a}_{\mu}(k) \, = \, g_s \sum_{i=1}^{n} \frac{\beta_{i,\,\mu}}{\beta_i \cdot k} \, T_i^a$$

One may define a strongly-ordered soft form factor by contracting with physical polarisations

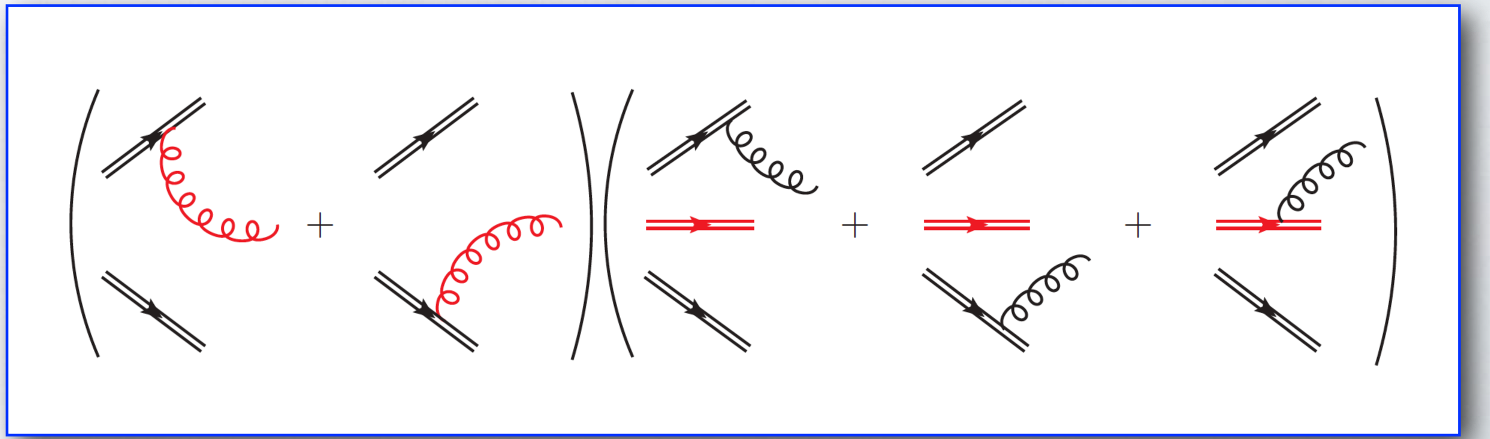
$$\left[\mathcal{S}^{(0)}_{n;\,1,\,1}\right]^{a_1 a_2}(k_1, k_2; \beta_i) = \epsilon^{*\,\mu_1}(k_1)\, \epsilon^{*\,\mu_2}(k_2) \, \left[J_{\rm CG}^{(0),\,{\rm s.o.}}\right]^{a_1 a_2}_{\mu_1 \mu_2}(k_1, k_2; \beta_i)\,.$$

The form factor is given by an interesting "re-factorisation" of the double-radiative soft function

$$\begin{aligned} \left[\mathcal{S}^{(0)}_{n;\,1,\,1}\right]^{a_1 a_2}_{\{d_i e_i\}}(k_1, k_2; \beta_i) \;&\equiv\; \langle k_2, a_2 | \, \Phi_{\beta_{k_1}}^{\ a_1 b}(0, \infty) \prod_{i=1}^{n} \Phi_{\beta_i,\, d_i}^{\quad\ c_i}(0, \infty) \, | 0 \rangle \\ &\quad \times \langle k_1, b | \prod_{i=1}^{n} \Phi_{\beta_i,\, c_i e_i}(0, \infty) \, | 0 \rangle \Big|_{\rm tree} \\ &= \left[\mathcal{S}^{(0)}_{n+1,1}\right]^{a_2,\, a_1 b}_{\{d_i c_i\}}(k_2; \beta_{k_1}, \beta_i) \, \left[\mathcal{S}^{(0)}_{n,1}\right]_{b,\, \{c_i e_i\}}(k_1; \beta_i)\,, \end{aligned}$$

Notice the non-trivial colour structure: the product is ordered.

The original system of n Wilson lines radiates the harder gluon, which then "Wilsonises". The augmented system of (n+1) Wilson lines radiates the softer gluon



Soft refactorisation: tree level

This framework generalises to arbitrary patterns of strong ordering for multiple soft radiation at tree level. For example for strongly-ordered triple radiation one can define

$$
\begin{aligned}
\left[\mathcal{S}^{(0)}_{n;1,1,1}\right]^{a_1 a_2 a_3}_{\{f_i e_i\}}(k_1,k_2,k_3;\beta_i) \;\equiv\; & \left[\mathcal{S}^{(0)}_{n+2,1}\right]^{a_3}_{\{f_i d_i\},\,a_1 b_1,\,a_2 b_2}\left[\mathcal{S}^{(0)}_{n+1,1}\right]^{b_2}_{\{d_i c_i\},\,b_1 g_1}\left[\mathcal{S}^{(0)}_{n,1}\right]^{g_1}_{\{c_i e_i\}} \\
= \; & \langle k_3, a_3|\,\Phi^{a_1 b_1}_{\beta_{k_1}}(0,\infty)\,\Phi^{a_2 b_2}_{\beta_{k_2}}(0,\infty)\prod_{i=1}^{n}\Phi^{f_i d_i}_{\beta_i}(0,\infty)|0\rangle \\
& \times \langle k_2, b_2|\,\Phi^{b_1 g_1}_{\beta_{k_1}}(0,\infty)\prod_{i=1}^{n}\Phi^{d_i c_i}_{\beta_i}(0,\infty)|0\rangle \\
& \times \langle k_1, g_1|\prod_{i=1}^{n}\Phi^{c_i e_i}_{\beta_i}(0,\infty)|0\rangle\Big|_{\text{tree}}\,,
\end{aligned}
$$

Computing the form factors, one reproduces the strongly-ordered limit of (Catani et al. 2019).

$$
\begin{aligned}
\left[\mathcal{S}^{(0)}_{n;1,1,1}\right]^{a_1 a_2 a_3} \; = \; & \epsilon^*_{\mu_3}(k_3)\,\epsilon^*_{\mu_2}(k_2)\,\epsilon^*_{\mu_1}(k_1) \\
& \times\left[J^{\mu_3}_{a_3}(k_3)\,\delta^{a_1 b_1}\,\delta^{a_2 b_2} + {\rm i} g_s\, f^{a_1 a_3 b_1}\,\delta^{a_2 b_2}\,\frac{k_1^{\mu_3}}{k_1\cdot k_3} + {\rm i} g_s\, f^{a_2 a_3 b_2}\,\delta^{a_1 b_1}\,\frac{k_2^{\mu_3}}{k_2\cdot k_3}\right] \\
& \times\left[J^{\mu_2}_{b_2}(k_2)\,\delta^{b_1 c_1} + {\rm i} g_s\, f^{b_1 b_2 c_1}\,\frac{k_1^{\mu_2}}{k_1\cdot k_2}\right] J^{\mu_1}_{c_1}(k_1)\,,
\end{aligned}
$$

- Generalising to strongly-ordered soft radiation of m gluons is natural (and tested for m=3).
- Similar definitions hold for soft form factors for multiple ordered subsets of several gluons.
- Preliminary evidence suggests that similar soft re-factorisations may hold to higher orders.

Strongly-ordered soft counterterms

The top-down approach suggests an expression for the soft real-virtual counterterm

$$K^{(\mathbf{RV}),\,\mathrm{s}}_{n+1} \;=\; \mathcal{H}^{(0)\,\dagger}_{n}\, S^{(1)}_{n,\,1}\, \mathcal{H}^{(0)}_{n} \;+\; \text{finite}$$

Collinear poles?

The refactorisation of strongly-ordered soft radiation suggests an expression for the soft $K^{(12)}$

$$\begin{aligned}
K^{(\mathbf{12}),\,\mathrm{s}}_{n+2} \;&=\; \mathcal{H}^{(0)\,\dagger}_{n}\, S^{(0)}_{n,\,1,1}\, \mathcal{H}^{(0)}_{n} \\
&=\; \mathcal{H}^{(0)\,\dagger}_{n} \left[\mathcal{S}^{b,\,(0)}_{n,\,1}(\beta_i;k_1)\right]^{\dagger} \left[\mathcal{S}^{a_2,\,a_1 b\,(0)}_{n+1,\,1}(\beta_i,\beta_{k_1};k_2)\right]^{\dagger} \mathcal{S}^{a_2,\,a_1 c,(0)}_{n+1,\,1}(\beta_i,\beta_{k_1};k_2)\, \mathcal{S}^{c,\,(0)}_{n,\,1}(\beta_i;k_1)\, \mathcal{H}^{(0)}_{n} \\
&\equiv\; \mathcal{H}^{(0)\,\dagger}_{n} \left[\mathcal{S}^{b}_{n,\,1}(\beta_i;k_1)\right]^{\dagger} S^{bc,(0)}_{n+1,\,1}(\beta_i,\beta_{k_1};k_2)\, \mathcal{S}^{c}_{n,\,1}(\beta_i;k_1)\, \mathcal{H}^{(0)}_{n}
\end{aligned}$$

One can now use the finiteness of inclusive soft cross sections to cancel soft poles arising from the phase-space integration of $K^{(12)}$, using

$$S^{bc,(1)}_{n+1,\,0}(\beta_i,\beta_{k_1}) \;+\; \int d\Phi_1(k_2)\, S^{bc,(0)}_{n+1,\,1}(\beta_i,\beta_{k_1};k_2) \;=\; \text{finite}$$

"Completeness relation"

This gives a new expression for the real-virtual soft counterterm

$$K^{(\mathbf{RV}),\,\mathrm{s}}_{n+1} \;=\; \mathcal{H}^{(0)\,\dagger}_{n} \left[\mathcal{S}^{b,\,(0)}_{n,\,1}(\beta_i;k_1)\right]^{\dagger} S^{bc,(1)}_{n+1,\,0}(\beta_i,\beta_{k_1})\, \mathcal{S}^{c,\,(0)}_{n,\,1}(\beta_i;k_1)\, \mathcal{H}^{(0)}_{n} \;+\; \text{finite}$$

A "bottom-up" approach

The two definitions have identical soft poles, which was checked with a non-trivial calculation.

A top-down approach

This result is better understood by taking more seriously the idea of refactorisation

- The radiative soft function is not a pure counterterm: it has IR poles and finite contributions.
- It can be considered as an amplitude in the presence of sources: virtual IR poles will factorise.

Applying the standard soft-jet-hard factorisation for scattering amplitudes we write

$$\mathcal{A}_n(p_i) = \prod_{i=1}^{n} \left[\frac{\mathcal{J}_i(p_i, n_i)}{\mathcal{J}_{E,i}(\beta_i, n_i)} \right] \mathcal{S}_n(\beta_j)\, \mathcal{H}_n(p_i) \quad \to \quad \mathcal{S}_{n,1}(k;\beta_i) = \frac{\mathcal{J}_g(k,n)}{\mathcal{J}_{E,g}(\beta_k, n)} \mathcal{S}_{n+1,0}(\beta_k, \beta_i)\, \mathcal{S}^{\rm fin}_{n,1}(k;\beta_i)$$

Expanding to one-loop order, the terms containing IR poles are

$$\mathcal{S}^{(1)}_{n,1}(k;\beta_i) = \underline{\mathcal{S}^{(1)}_{n+1,0}(\beta_k, \beta_i)\, \mathcal{S}^{(0)}_{n,1}(k;\beta_i)} + \underline{\left(\mathcal{J}^{(1)}_g(k,n) - \mathcal{J}^{(1)}_{E,g}(\beta_k, n) \right)} \mathcal{S}^{(0)}_{n,1}(k;\beta_i)$$

We recognise (upon squaring) the soft contribution to $K^{(RV)}$, plus hard collinear corrections.

This can be explicitly checked against the general expression for the soft limit of RV

$$S^{(1)}_{n,1}(k;\beta_i) = \mathbf{S}_k RV - \frac{\alpha_S^2 \mu^{2\epsilon}}{S_\epsilon} \sum_{i>j}^{n} \frac{\beta_i \cdot \beta_j}{\beta_i \cdot k\, \beta_j \cdot k} \mathbf{T}_i \cdot \mathbf{T}_j \left[\sum_{m=1}^{n} \frac{\gamma_m^{(1)}}{\epsilon} + \frac{b_0}{2\epsilon} \right]$$

To match the two calculations, one must subtract the hard-collinear poles of the virtual part.

Collinear refactorisation

The top-down approach suggests an expression for the collinear real-virtual counterterm

$$K^{(\mathbf{RV}),\,\mathrm{c},\,i}_{n+1} \;=\; \mathcal{H}^{(0)\,\dagger}_n \, J^{(1)}_{i,\,1} \, \mathcal{H}^{(0)}_n \;+\; \text{finite}$$

Soft poles?

In the bottom-up approach one starts with strongly-ordered collinear kernels, for example

$$\lim_{\theta_{12}\ll\theta_{13}\ll 1} RR_{n+2} \;=\; \frac{\mathcal{N}^2}{s_{12}\, s_{[12]3}} \, P^{\alpha\beta}_{gq}\big(z_{[12]}, q_\perp\big)\, d_{\alpha\mu}\big(k_{[12]}, n\big)\, P^{\mu\nu}_{q\bar{q}}\left(\frac{z_1}{z_{[12]}}, k_\perp\right) d_{\nu\beta}\big(k_{[12]}, n\big)\,,$$

$q \to q\, q\, \bar{q}$

This can be directly translated in the language of jet functions. At cross-section level

$$J^{(0)}_{q,1,1}\,(k_1, k_2; k_3, n)\Big|_{gg,\,\mathrm{ab.}} \;=\; J^{(0)}_{q,1}\,\big(k_1; k_{[23]}, n\big)\; J^{(0)}_{q,1}\,(k_2; k_3, n)\;\;,$$

$q \to q\, g\, g$, abelian

One can now use the finiteness of inclusive collinear cross sections to cancel collinear poles arising from the phase-space integration of $K^{(12)}$, using

$$K^{(\mathbf{12}),\,\mathrm{c},\,q}_{n+2} \;=\; \mathcal{H}^{(0)\,\dagger}_n \, J^{(0)}_{q,1,1} \, \mathcal{H}^{(0)}_n\,; \qquad J^{(1)}_{q,0}\big(k_{[23]}, n\big) + \int d\Phi_1(k_3)\, J^{(0)}_{q,1}(k_2; k_3, n) \;=\; \text{finite}$$

"Completeness relation"

This gives a new expression for the real-virtual collinear counterterm

$$K^{(\mathbf{RV}),\,\mathrm{c},\,q}_{n+1} \;=\; \mathcal{H}^{(0)\,\dagger}_n \, J^{(0)}_{q,1}\big(k_1; k_{[23]}, n\big)\, J^{(1)}_{q,0}\big(k_{[23]}, n\big)\, \mathcal{H}^{(0)}_n \;+\; \text{finite}$$

Bottom-up approach

The two definitions have identical collinear poles, which again calls for an explanation.

Top-down collinear

Once again, the result is better understood by means of a refactorisation of the radiative jet

- The radiative jet function has both UV and IR poles, as well as phase-space singularities.
- As before, it is an amplitude in the presence of sources: virtual IR poles will factorise.

Applying the standard soft-jet-hard factorisation for amplitudes we write

$$\mathcal{J}_{f,1}(k;p,n) = \left[\frac{\mathcal{J}(k,n_k)}{\mathcal{J}_E(\beta_k,n_k)}\,\frac{\mathcal{J}(p,n_p)}{\mathcal{J}_E(\beta_p,n_p)}\right] \mathcal{S}_3(\beta_k,\beta_p,\beta_n)\,\mathcal{J}_{f,1}^{\rm fin}(k,p,n)$$

Expanding to one-loop order, the terms containing IR poles are

$$\mathcal{J}_{f,1}^{(1)}(k;p,n) = \Big[\mathcal{J}^{(1)}(k,n_k) - \mathcal{J}_E^{(1)}(\beta_k,n_k) + \mathcal{J}^{(1)}(p,n_p) - \mathcal{J}_E^{(1)}(\beta_p,n_p) + \mathcal{S}_3^{(1)}(\beta_k,\beta_p,\beta_n)\Big]\,\mathcal{J}_{f,1}^{{\rm fin},\,0}(k,p,n)$$

One reconstructs (upon squaring) the collinear contribution to $K^{(RV)}$, plus soft corrections.

- For gg radiation, hard collinear terms are identical and phase space provides a factor 1/2.
- The three-point soft function does not affect collinear factorisation: it simplifies to a singlet quantity when the collinear limit is taken.

The cancellation of poles between $K^{(RV)}$ and the integral of $K^{(12)}$ is now built in the definitions.

OUTLOOK

Outlook

- Infrared subtraction beyond NLO requires understanding all strongly-ordered IR limits.
- Factorisation provides definitions for local soft and collinear kernels to all orders.
- Soft and collinear kernels are expressed by matrix elements of fields and Wilson lines.
- In strongly ordered limits the kernels re-factorise into lower-order matrix elements.
- Known strongly ordered IR limits at NNLO and N3LO are reproduced by factorisation.
- "Completeness relations" link strongly-ordered kernels and real-virtual counterterms.
- Upon implementing phase-space mappings, the cancellation of RV poles can be checked.
- The refactorisation approach to strong-ordering generalises smoothly to higher orders.
- The architecture of infrared subtraction is becoming clear to all orders.

THANK YOU