UNIVERSITY OF COPENHAGEN

DANMARKS FRIE FORSKNINGSFOND

Co-funded by the European Union

Collider Physics Tools for Precision Gravitational Wave Physics

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Based on work with C. Dlapa, G. Kälin, R. Porto, Z. Yang, J. Neef, R. Jinno, H. Rubira

2304.01275 2210.05541 2209.01091 2112.11296

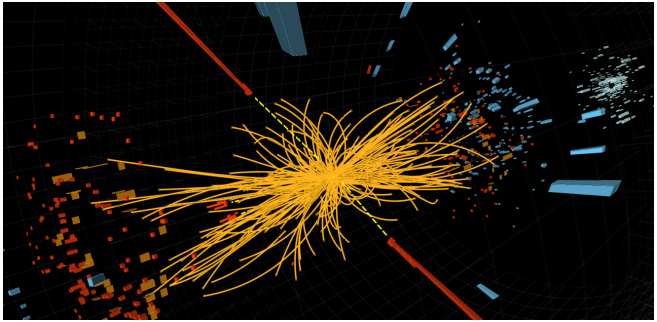
2106.08276 2007.04977 2102.10059 2008.06047

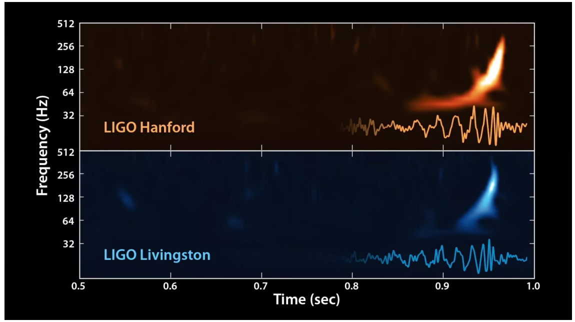
Theory Challenges in the Precision Era of the Large Hadron Collider

Galileo Galilei Institute, Firenze

September 1, 2023

Precision Era of Fundamental Physics

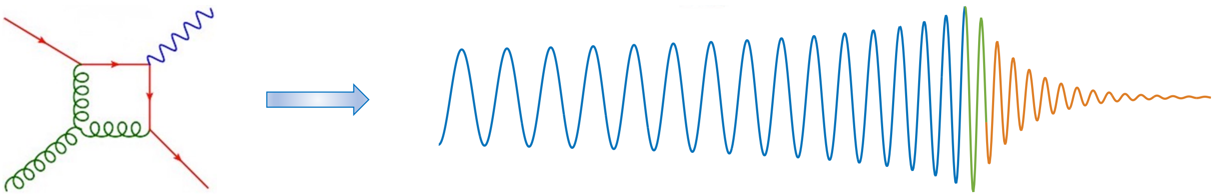




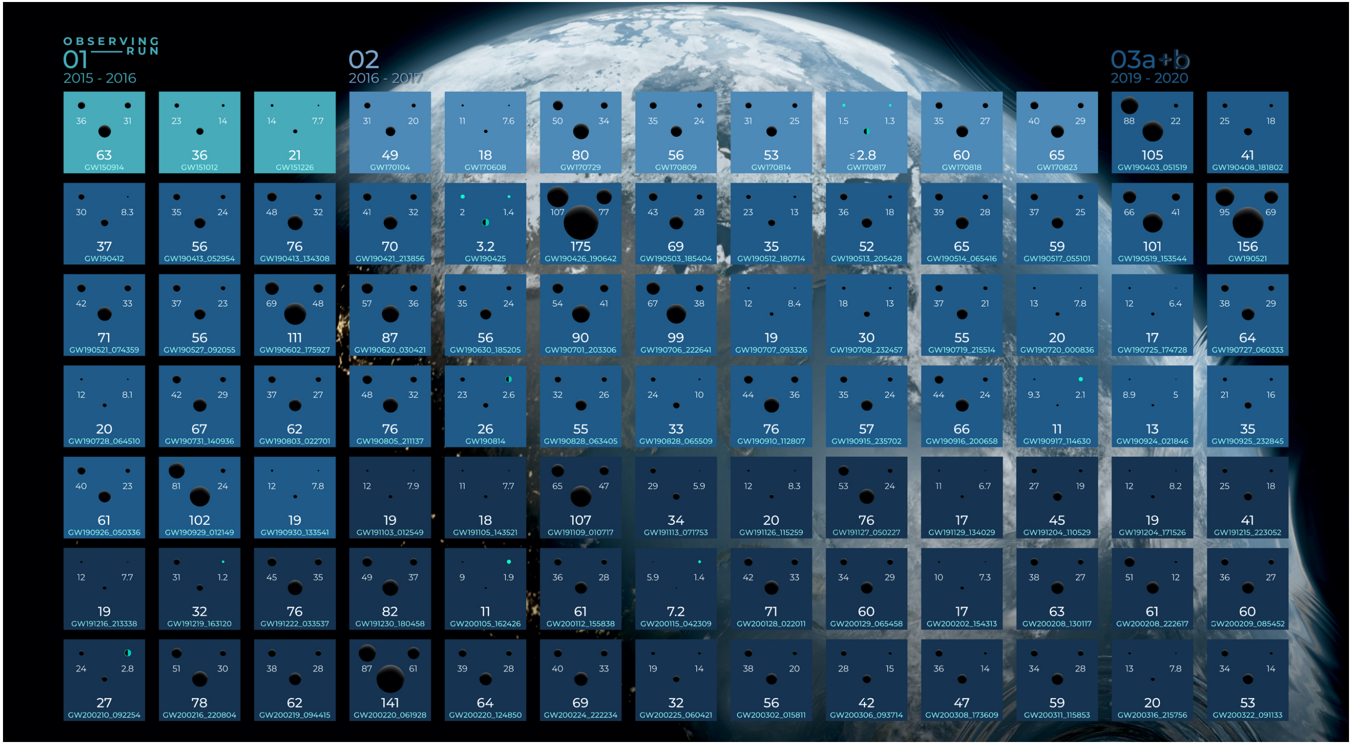
Two historic breakthroughs in science:

- Higgs bosons from the LHC (2012)
- Gravitational waves from the LIGO (2016)
- High-energy and gravitational physics entered a precision era!

Modern techniques from collider physics are playing a crucial role in precision GW physics!



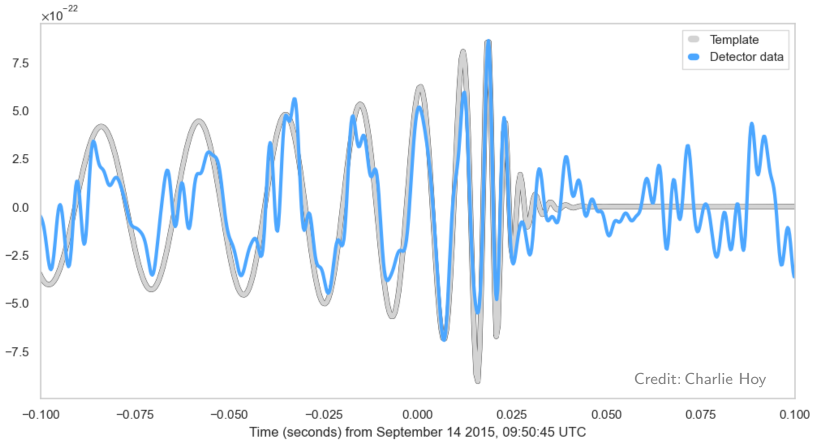
Gravitational waves



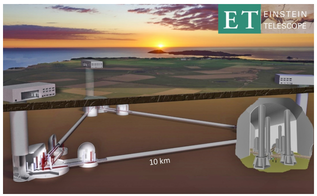
Credit: Carl Knox (OzGrav, Swinburne)

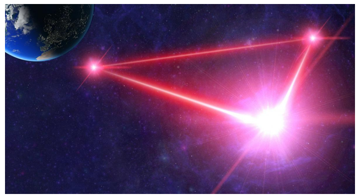
GWTC-3: 90 GW events – the majority are binary black holes (BH), but also several binary neutron stars (NS) and mixed NS-BHs.

Gravitational-wave science



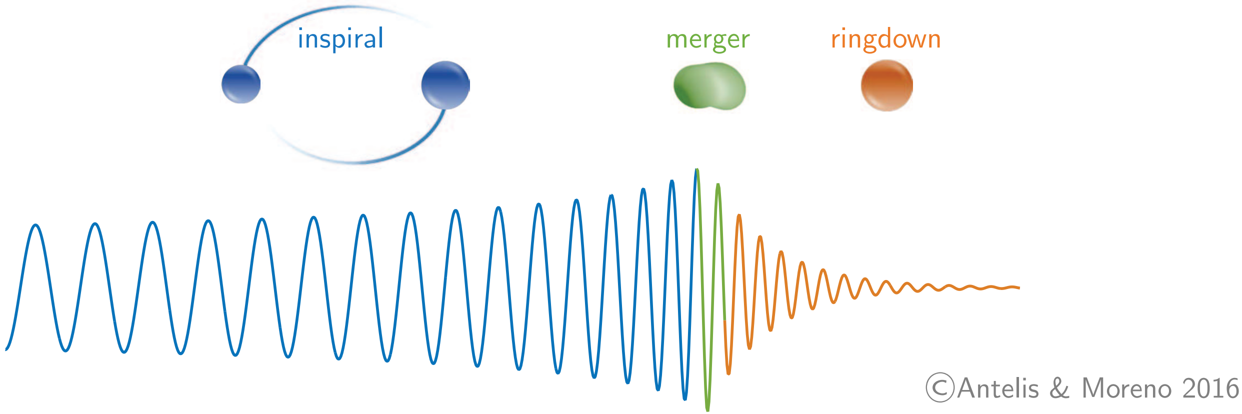
Waveforms will be far more complex and carry more information than expected. Improved modeling will be needed for extracting the GW's information. Kip Thorne 1993





Precise theoretical predictions for the motion of GW sources are crucial in interpreting data and maximizing discovery potential for present and future observations.

Gravitational waves from binary coalescences



Merger: Numerical Relativity

Ringdown: black hole perturbation theory

Inspiral: the relative velocity v is small

$$v^2 \sim \frac{GM}{r} \ll 1$$

- Numerical Relativity: accurately, but computationally expensive
- Analytic methods: corrections in v or G are perturbatively calculable

Post-Newtonian/post-Minkowskian expansion

Inspiralling dynamics

	0PN		1PN		2PN		3PN		4PN		5PN		6PN		$\cdots$	
$G($	1	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	v^{10}	$+$	v^{12}	$+$	$\cdots)$	1PM
	$G^2($		1	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	v^{10}	$+$	$\cdots)$	2PM
			$G^3($		1	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	$\cdots)$	3PM
					$G^4($		1	$+$	v^2	$+$	v^4	$+$	v^6	$+$	$\cdots)$	4PM
							$G^5($		1	$+$	v^2	$+$	v^4	$+$	$\cdots)$	5PM
									$G^6($		1	$+$	v^2	$+$	$\cdots)$	6PM

STATE OF THE ART

1PM: Bertotti 1956

2PM: Westpfahl 1985

Collider physics tools have transformed this field!

3PM: Bern, Cheung, Roiban, Shen, Solon, Zeng 2019; Kälin, **ZL**, Porto 2020
Di Vecchia, Heissenberg, Russo, Veneziano 2021; Bjerrum-Bohr, Damgaard, Planté, Vanhove 2021
Brandhuber, Chen, Travaglini, Wen 2021,...

4PM: Dlapa, Kälin, **ZL**, Neef, Porto 2021, 2022; Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng 2021
Damgaard, Hansen, Planté, Vanhove 2023; Jakobsen, Mogull, Plefka, Sauer, Xu 2023

Effective Field Theory

- In the inspiral phase

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}$$

- Effective action for gravitational binary systems   Goldberger-Rothstein 2004

$$e^{i\mathcal{S}_{\text{eff}}[x_a(\tau)]} = \int \mathcal{D}h_{\mu\nu}\, e^{i\mathcal{S}_{\text{WL}}+i\mathcal{S}_{\text{GR}}}$$

with

$$\mathcal{S}_{\text{WL}} = \sum_{i=1,2}\left[-\frac{m_i}{2}\int dt\, g_{\mu\nu}\dot{x}_i^{\mu}\dot{x}_i^{\nu} + \cdots\right], \quad \mathcal{S}_{\text{GR}} = \frac{-1}{16\pi G}\int d^4x\,\sqrt{-g}\,R + \cdots$$

- Post-Minkowskian expand in powers of G

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + G\mathcal{L}_1 + G^2\mathcal{L}_2 + \cdots \qquad \mathcal{L}_0 = -\sum_i \frac{m_i}{2}\eta_{\mu\nu}\dot{x}_i^{\mu}\dot{x}_i^{\nu}$$

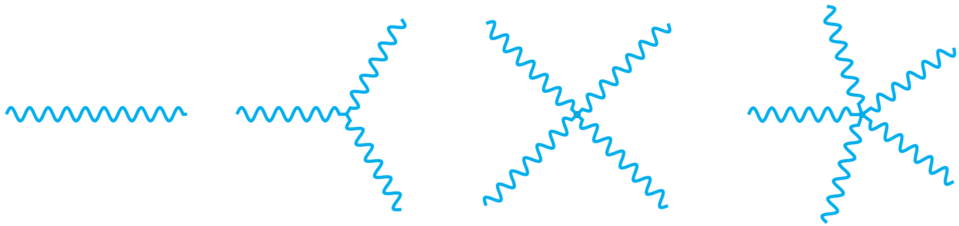
The equations of motion for trajectories:

$$m_i\ddot{x}_i^{\mu} = -\eta^{\mu\nu}\sum_{n=1}^{\infty}\left(\frac{\partial\mathcal{L}_n}{\partial x_i^{\nu}} - \frac{d}{d\tau_i}\frac{\partial\mathcal{L}_n}{\partial\dot{x}_i^{\nu}}\right) \qquad x_i^{\mu} = b_i^{\mu} + u_i^{\mu}\tau + \delta x_i^{\mu}(\tau) + \cdots$$

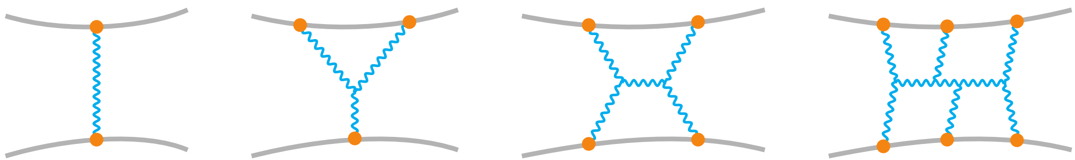
Effective Field Theory

- Worldlines as sources in path integral:

- Hilbert-Einstein: $\mathcal{L}_{\text{HE}} = \mathcal{L}_{hh} + \mathcal{L}_{hhh} + \mathcal{L}_{hhhh} + \cdots$



- Classical physics: we use the saddle-point approximation in path integrals.



- Enjoy the advantages of quantum methods and classical physics

  powerful and systematic & purely classical at all steps (simplicity)

Effective Field Theory

- Observables at $\mathcal{O}(G^N)$ 2007.04977 2008.06047 2102.10059 2304.01275

$$\Delta p_i^\mu \sim \int d^D q \, \frac{e^{i q\cdot b}\,\delta(q\cdot u_1)\delta(q\cdot u_2)}{|q^2|^\sharp} \int \left(\prod_{i=1}^{N-1} d^D\ell_i \, \frac{\delta(\ell_j\cdot u_a)}{(\ell_i\cdot u_b - i0)^{\nu_i}} \right) \frac{\mathcal{N}^\mu(q,u_a)}{D_1 D_2 D_3\cdots}$$

  - Graviton propagators:

$$\frac{1}{D_i} \longrightarrow \frac{1}{(\ell^0 \pm i0)^2 - \vec{\ell}^{\,2}} \quad \text{or} \quad \frac{1}{\ell^2 + i0}$$

  - Cut: always one delta function $\delta(\ell_i\cdot u_a)$ for each loop
  - Kinematics: $q\cdot u_a = 0,\ u_a^2 = 1,\ u_1\cdot u_2 = \gamma \implies$ single scale γ to all orders

- Multi-loop technology from collider physics can be used to solve gravitational problems!

Collider Physics Toolbox

Post-Minkowskian Loop Integrals at $\mathcal{O}(G^N)$

$$\int \left(\prod_{i=1}^{N-1} d^D\ell_i \, \frac{\delta(\ell_j \cdot u_{a_i})}{(\ell_i \cdot u_{b_i} - i0)^{\alpha_i}} \right) \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots}$$

- Reverse Unitarity: replace the delta-function by the cut-propagator Anastasiou-Melnikov 2002

$$\delta(k_i \cdot u_a) \to \frac{1}{2\pi i} \left(\frac{1}{k_i \cdot u_a - i0} - \frac{1}{k_i \cdot u_a + i0} \right)$$

Then standard loop-integral techniques can be applied straightforwardly!

- IBP reduction: any integral = a linear combination of a small number of basis integrals

$$\vec{f} = \{I_1, I_2, \ldots\}$$

Publicly-available programs: `Reduze`, `FIRE`, `LiteRed`, `Kira`, `FiniteFlow`, `NeatIBP`

See Andreas von Manteuffel's talk

Collider Physics Toolbox

- Differential equations:

$$\frac{d\vec{f}(x,\epsilon)}{dx} = M(x,\epsilon)\,\vec{f}(x,\epsilon) \qquad D = 4 - 2\epsilon \qquad \gamma = \frac{x^2+1}{2x}$$

- Canonical form

Henn 2013 Lee 2014

$$\frac{d\vec{g}(x,\epsilon)}{dx} = \epsilon\, Q(x)\,\vec{g}(x,\epsilon) \qquad \vec{g} = T \cdot \vec{f}$$

- Then the solution can be written in terms of multiple polylogarithms to any order in ϵ.

Chen 1977 Goncharov 2001

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1}\, G(a_2,\ldots,a_n;t)$$

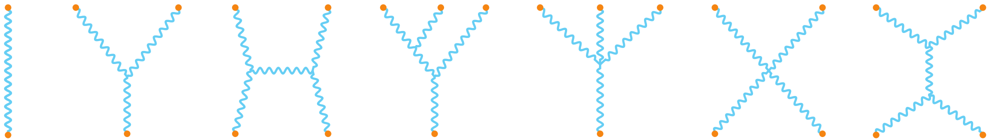
- From 3-loop level, elliptic integrals appear and differential equations remain powerful.

- Boundary constants can be computed in Post-Newtonian limit using the method of regions.

potential: $\ell^\mu \sim (v,1)$ radiation: $\ell^\mu \sim (v,v)$ Beneke-Smirnov 1997

- Post-Minkowskian physics can be bootstrapped from Post-Newtonian data!

Inspiralling dynamics at NNLO



- $\mathcal{O}(G^3)$: two-loop intgerals 2007.04977 2008.06047

$$\int \frac{d^D\ell_1 d^D\ell_2\, \delta(\ell_1\cdot u_1)\,\delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2]^{a_1}\,[\pm\ell_2\cdot u_1]^{a_2}}\,\frac{1}{[\ell_1^2]^{a_3}\,[\ell_2^2]^{a_4}\,[(\ell_1+\ell_2-q)^2]^{a_5}\,[(\ell_1-q)^2]^{a_6}\,[(\ell_2-q)^2]^{a_7}}$$

- The reduction and evaluation of integrals can be performed in standard techniques.

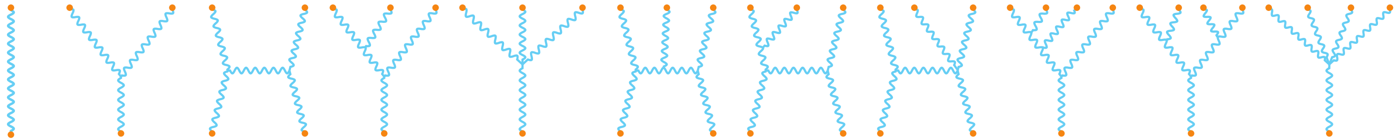
- Conservative dynamics at $\mathcal{O}(G^3)$: 2007.04977

$$\Delta p_1^\mu = \frac{G^3 b^\mu}{|b^2|^2}\left(\frac{8m_1^2m_2^2(4\gamma^4-12\gamma^2-3)}{(\gamma^2-1)}\,\mathrm{Arccosh}(\gamma) - \frac{2m_1m_2(m_1^2+m_2^2)(16\gamma^6-32\gamma^4+16\gamma^2-1)}{(\gamma^2-1)^{5/2}}\right.$$
$$\left. - \frac{4m_1^2m_2^2\gamma(20\gamma^6-90\gamma^4+120\gamma^2-53)}{3(\gamma^2-1)^{5/2}}\right) + \frac{3\pi}{2}\frac{(2\gamma^2-1)\,(5\gamma^2-1)}{(\gamma^2-1)^2}\frac{G^3m_1m_2(m_1+m_2)}{|b^2|^{3/2}}\Big((m_1+\gamma m_2)u_2^\mu - (m_2+\gamma m_1)u_1^\mu\Big)$$

- We provided the first confirmation for the BCRSSZ's derivation.
  Bern-Cheung-Roiban-Shen-Solon-Zeng (BCRSSZ) 2019

- We also obtained the quadrupolar and octupolar tidal corrections at $\mathcal{O}(G^3)$.
  2008.06047

Inspiralling dynamics at NNNLO



$\mathcal{O}(G^4)$: three-loop integrals 2106.08276 2112.11296 2210.05541 2304.01275

$$\int d^D\ell_1 d^D\ell_2 d^D\ell_3 \frac{\delta(\ell_1\cdot u_1)\,\delta(\ell_2\cdot u_2)\,\delta(\ell_3\cdot u_2)}{[\ell_1\cdot u_2]^{\alpha_1}\,[\ell_2\cdot u_1]^{\alpha_2}\,[\ell_3\cdot u_1]^{\alpha_3}} \frac{D_8^{-\nu_8} D_9^{-\nu_9}}{D_1^{\nu_1} D_2^{\nu_2}\cdots D_7^{\nu_7}}$$

$\{\ell_1^2, \ell_2^2, (\ell_1-q)^2, (\ell_2-q)^2, (\ell_3-q)^2,$
$\ell_3^2, (\ell_1-\ell_2)^2, (\ell_2-\ell_3)^2, (\ell_3-\ell_1)^2\}$

IBP reduction: LiteRed $\oplus$ FIRE6

conservative: $\mathcal{O}(10^2)$ master integrals full: $\mathcal{O}(10^3)$ master integrals

Differential Equations

$$\frac{d\vec{f}(x,\epsilon)}{dx} = M(x,\epsilon)\,\vec{f}(x,\epsilon)$$

- The majority can be solved in terms of multiple polylogarithms.
- Elliptic integrals appear in post-Minkwskian gravity for the first time.

Elliptic differential equations

Elliptic differential equations 2106.08276 2304.01275

$$\frac{d}{dx}\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} \frac{1-x^2}{2x(1+x^2)} & \frac{1+x^2}{4x(1-x^2)} & \frac{3x}{(1-x^2)(1+x^2)} \\ -\frac{1-x^2}{x(1+x^2)} & \frac{3(1+x^2)}{2x(1-x^2)} & -\frac{6x}{(1-x^2)(1+x^2)} \\ \frac{1-x^2}{x(1+x^2)} & -\frac{1+x^2}{2x(1-x^2)} & -\frac{1-4x^2+x^4}{x(1-x^2)(1+x^2)} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} + \mathcal{O}(\epsilon)$$

It can then be written as a third-order differential equation:

$$\left[\frac{d^3}{dx^3} - \frac{6x}{1-x^2}\frac{d^2}{dx^2} - \frac{1-4x^2+7x^4}{x^2(1-x^2)^2}\frac{d}{dx} - \frac{1+x^2}{x^3(1-x^2)} + \mathcal{O}(\epsilon)\right] f_1(x) = 0$$

It is easy to find the three solutions:

$$x\,\mathrm{K}^2(1-x^2), \qquad x\,\mathrm{K}(1-x^2)\mathrm{K}(x^2), \qquad x\,\mathrm{K}^2(x^2)$$

Complete elliptic integrals:

$$\mathrm{K}(x) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}} \qquad \mathrm{E}(x) \equiv \int_0^1 \frac{\sqrt{1-xt^2}}{\sqrt{1-t^2}}\,dt$$

Inspiralling dynamics at NNNLO

The full impulse at $\mathcal{O}(G^4)$: 2106.08276 2112.11296 2210.05541 2304.01275

$$\Delta p_1^\mu\Big|_{\mathrm{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b\,\frac{b^\mu}{|b|} + c_1\,\frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2\,\frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}\right)$$

$$\begin{aligned}
\frac{c_b}{\pi} = &-\frac{3h_{34}m_2m_1(m_1^3+m_2^3)}{64v_\infty^5} + \frac{m_1^2m_{12}m_2^2}{4}\Bigg[\frac{3h_6\mathrm{K}^2(w_2)}{4v_\infty^3} - \frac{3h_8\mathrm{K}(w_2)\mathrm{E}(w_2)}{4v_\infty^3} + \frac{21h_5w_3\mathrm{E}^2(w_2)}{8v_\infty^3} - \frac{\pi^2h_{16}v_\infty}{4(\gamma+1)} + \frac{3\gamma h_{10}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{w_3v_\infty^2}\\
&+\log(v_\infty)\left(\frac{h_{32}}{2v_\infty^3} - \frac{3h_{14}\log(\frac{w_3}{2})}{v_\infty} - \frac{3\gamma h_{22}\log(w_1)}{2v_\infty^4}\right)\Bigg] + m_2^2m_1^3\Bigg[\frac{h_{52}}{48v_\infty^6} - \frac{h_{63}}{768\gamma^9w_3v_\infty^5} - \frac{3v_\infty(h_{40}\mathrm{Li}_2(w_2)+2w_3h_{33}\mathrm{Li}_2(-w_2))}{64w_3}\\
&+\frac{3h_{14}\log(\frac{w_3}{2})\log(w_3)}{4v_\infty} + \frac{\gamma h_{39}\log(w_1)}{8w_3^3v_\infty^2} + \frac{3\gamma h_{22}\log(w_3)\log(w_1) - h_{35}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{h_{56}\log(2) - h_{57}\log(w_3) + 2\gamma h_{55}\log(\gamma)}{32v_\infty^5} - \frac{\gamma h_{51}\log(w_1)}{16v_\infty^7}\Bigg]\\
&+m_1^2m_2^3\Bigg[\frac{h_{58}}{192\gamma^7v_\infty^5} + \frac{h_{53}}{48v_\infty^6} + \frac{\gamma h_{49}\log(w_1)}{16v_\infty^6} - \frac{2\gamma h_{50}\log(w_1) + 3\gamma^2h_{13}\log^2(w_1)}{32v_\infty^7} - \frac{h_{41}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{3\gamma\log(w_1)(5h_{26}\log(2) + 8h_{12}\log(w_3))}{8v_\infty^4}\\
&-\frac{h_{36}\log(w_3)}{4v_\infty^3} + \frac{\gamma h_{30}\log(\gamma)}{2v_\infty^3} + \frac{h_{37}\log(2)}{8v_\infty^3} + \frac{3(h_{17}w_3\mathrm{Li}_2(w_2) - 2h_{23}\mathrm{Li}_2(-w_2) + h_{15}\log^2(w_3) - h_9\log^2(2))}{8v_\infty} - \frac{3h_7\log(2)\log(w_3)}{v_\infty}\Bigg]
\end{aligned}$$

$$c_1 = m_1m_2^2\left(\frac{2h_{46}m_{12s}}{v_\infty^6} + \frac{9\pi^2h_1m_{12}^2}{32v_\infty^2}\right) + m_1^2m_2^3\left(\frac{4\gamma h_{47}}{3v_\infty^6} - \frac{8\gamma h_2\log(w_1)}{v_\infty^6} + \frac{16h_{25}\log(w_1)}{v_\infty^3} - \frac{8h_3}{3v_\infty^5}\right)$$

$$\begin{aligned}
c_2 = &-m_1^4m_2\left(\frac{9\pi^2h_1}{32v_\infty^2} + \frac{2h_{46}}{v_\infty^6}\right) + m_2^2m_1^3\Bigg[+\frac{h_{60}}{705600\gamma^8v_\infty^5} - \frac{4\gamma h_{48}}{3v_\infty^6} + \frac{3h_{38}(\mathrm{Li}_2(w_2) - 4\mathrm{Li}_2(\sqrt{w_2})) - \gamma h_{21}(\mathrm{Li}_2(-w_1^2) + 2\log(\gamma)\log(w_1))}{16v_\infty^4}\\
&+\frac{3\gamma h_{31}(2\mathrm{Li}_2(-w_1) + \log(w_1)\log(w_3))}{8v_\infty^4} + \frac{h_{62}\log(w_1)}{6720\gamma^9v_\infty^6} + \frac{32\gamma^2h_{44}\log^2(w_1)}{v_\infty^7} + \frac{8\gamma(2h_4\log(2) - h_{27}\log(w_1))\log(w_1)}{v_\infty^4} - \frac{32h_{29}\log(w_1)}{3v_\infty^3} + \frac{\pi^2h_{42}}{192v_\infty^4}\Bigg]\\
&+m_2^3m_1^2\Bigg[\frac{h_{59}}{1440\gamma^7v_\infty^5} - \frac{h_{19}(\mathrm{Li}_2(-w_1^2) + 2\log(\gamma)\log(w_1))}{8v_\infty^4} + \frac{h_{43}(\mathrm{Li}_2(w_2) - 4\mathrm{Li}_2(\sqrt{w_2}))}{32v_\infty^4} - \frac{h_{20}(2\mathrm{Li}_2(-w_1) + \log(w_1)\log(w_3))}{4v_\infty^4}\\
&-\frac{h_{61}\log(w_1)}{480\gamma^8v_\infty^6} - \frac{16\gamma^2h_{11}\log^2(w_1)}{v_\infty^4} - \frac{32\gamma h_{45}\log^2(w_1)}{v_\infty^7} + \frac{16\gamma h_{28}\log(w_1)}{5v_\infty^3} - \frac{32h_{24}\log(2)\log(w_1)}{v_\infty^4} - \frac{\pi^2h_{18}}{48v_\infty^4} - \frac{2h_{54}}{45v_\infty^6}\Bigg]
\end{aligned}$$

with $\gamma \equiv u_1\cdot u_2$, $v_\infty = \sqrt{\gamma^2-1}$, $w_1 = \gamma - v_\infty$, $w_2 = \frac{\gamma-1}{\gamma+1}$, $w_3 = \gamma+1$, h_i = polynomial in γ.

$$\mathrm{Li}_2(z) \equiv \int_0^z dx\,\frac{\log(1-x)}{-x}$$

$$K(z) \equiv \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}}$$

$$E(k) \equiv \int_0^1 dx\,\frac{\sqrt{1-zx^2}}{\sqrt{1-x^2}}$$

Inspiralling dynamics at NNNLO

The full impulse at $\mathcal{O}(G^4)$: 2106.08276 2112.11296 2210.05541

$$\Delta p_1^\mu\big|_{\text{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b\,\frac{b^\mu}{|b|} + c_1\,\frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2\,\frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}\right)$$

- We obtained the full dynamics of binary inspirals at $\mathcal{O}(G^4)$ for the first time.
- Perfect agreement with the state-of-the-art PN computations

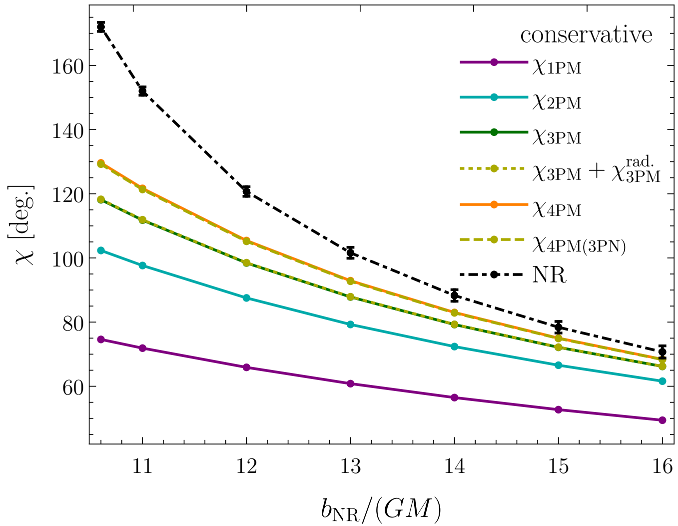
  Cho-Dandapat-Gopakumar 2021 Cho 2022 Bini-Geralico 2021 2022 Bini-Damour 2022
- Conservative part agrees perfectly with Amplitudes' derivations.

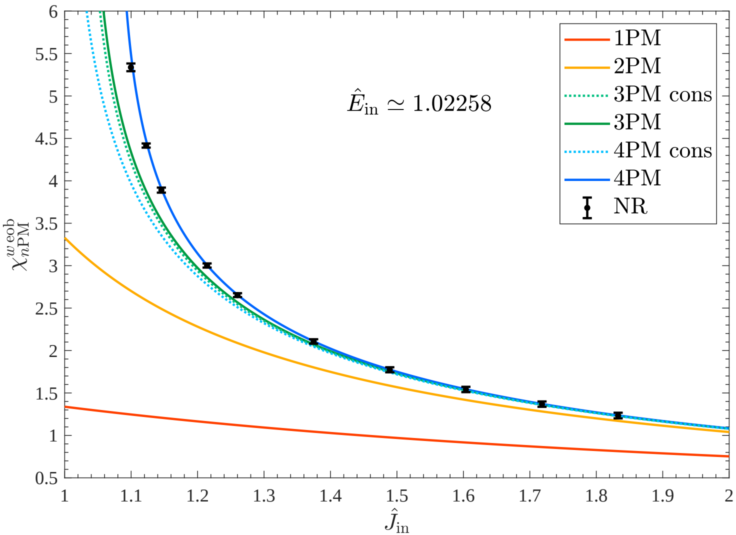
  Bern-Parra-Martinez-Roiban-Ruf-Shen-Solon-Zeng 2021
- Very recently two new calculations confirmed our results.

  Damgaard-Hansen-Planté-Vanhove 2023 (exponentiation of amplitudes)

  Jakobsen-Mogull-Plefka-Sauer-Xu 2023 (worldline)

Comparison with Numerical Relativity





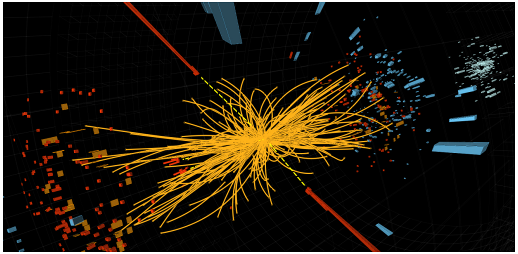
Khalil-Buonanno-Steinhoff-Vines 2204.05047

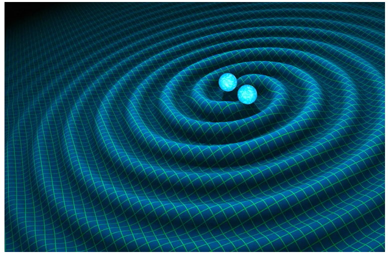
Damour-Rettegno 2211.01399

Rettegno-Pratten-Thomas-Schmidt-Damour 2307.06999

Conclusion

Modern techniques from collider physics have already proven useful to improve theoretical predictions for gravitational-wave observables.





We have developed an efficient framework and made breakthroughs to NNNLO.

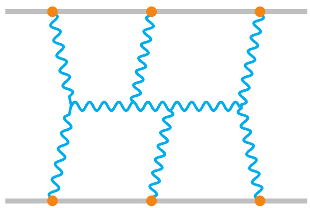
- Conservative spin effects to NLO JHEP 06 (2021) 012
- Non-spinning conservative dynamics to NNLO PRL 125 (2020) 261103
- Quadrupolar and octupolar tidal corrections to NLO PRD 102 (2020) 124025
- Non-spinning conservative dynamics to NNNLO PLB 822 (2021) 136698 PRL 128 (2022) 161104
- Non-spinning full dynamics to NNNLO PRL 130 (2023) 101401
- Novel techniques to evaluate loop integrals in gravity JHEP 07 (2023) 181 JHEP 08 (2023) 109

Outlook

- The era of gravitational-wave science is just starting!
- New discoveries rely highly on the precision of theoretical predictions.

Discovery Potential = Precise Theoretical Predictions

- $\mathcal{O}(G^5)$ is highly desirable for next-generation observations (ET, LISA...)



$$= \int \frac{d^D\ell_1 d^D\ell_2 d^D\ell_3 d^D\ell_4\, \delta(\ell_1\cdot u_1)\delta(\ell_2\cdot u_1)\delta(\ell_3\cdot u_2)\delta(\ell_4\cdot u_2)}{D_1^{\nu_1} D_2^{\nu_2}\cdots D_{18}^{\nu_{18}}}$$

$$D_i \in \{\ell_1^2, \ell_2^2, \ell_4^2, (\ell_3-q)^2, (\ell_2-\ell_3)^2, (\ell_3-\ell_4)^2, (\ell_1+\ell_2-q)^2, (\ell_1+\ell_3-q)^2, (\ell_2-\ell_3+\ell_4)^2, \ldots\}$$

  - Four loops: 22 indices = 4 (cuts) + 9 (propagators) + 9 (irreducible scalar products)
  - Numerators: $8 = 2\cdot\mathrm{degree}(\ell^2) + \mathrm{degree}(\ell\cdot u)$
  - Beyond polylogarithms: elliptics, Calabi-Yau...

- I am happy to learn new technology from the participants in this workshop!

Grazie!

HORIZON 2020

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 847523.