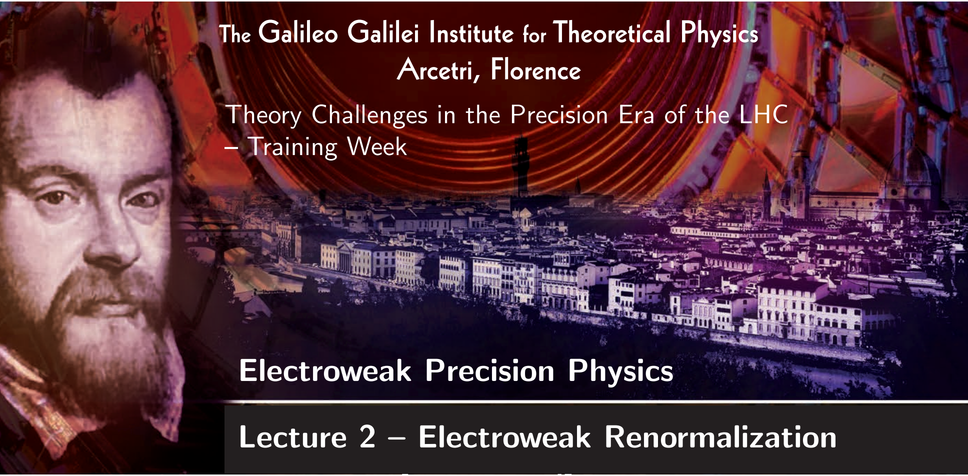


The Galileo Galilei Institute for Theoretical Physics
Arcetri, Florence

Theory Challenges in the Precision Era of the LHC
– Training Week

Electroweak Precision Physics

Lecture 2 – Electroweak Renormalization

Stefan Dittmaier

universität freiburg

Table of contents

Table of contents

Structure of loop corrections and the concept of renormalization

General procedure in higher-order calculations

Formulate theory: Lagrangian

$\Downarrow$

quantization $\rightarrow$ gauge fixing, Faddeev–Popov ghosts

$\Downarrow$

Perturbative evaluation: Feynman rules

$\Downarrow$

Feynman graphs

$\Downarrow$

loop integrals $\rightarrow$ technical problem: divergences (UV, IR)

$\Downarrow$

regularization $\rightarrow$ divergences mathematically meaningful

$\Downarrow$

Define input parameters: renormalization $\rightarrow$ eliminates UV divergences

$\Downarrow$

Theoretical predictions: calculation of observables (cross sections, decay widths, etc.)

$\hookrightarrow$ IR divergences cancel for sufficiently inclusive quantities by virtue of KLN theorem
(e.g. inclusion of photon bremsstrahlung)

Green functions, transition amplitudes, and observables

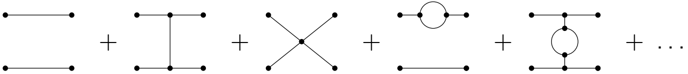
("Reducible") Green functions $G^{\phi_1\cdots\phi_n}$:

Time-ordered vacuum expectation values of field correlators:

$$G^{\phi_1\cdots\phi_n}(x_1,\dots,x_n) = \langle 0|\, T\,\phi_1(x_1)\cdots\phi_n(x_n)\,|0\rangle$$

$\hookrightarrow$ central objects of QFT, directly derivable from functional integral, systematic perturbative expansion via Feynman diagrams

Example:

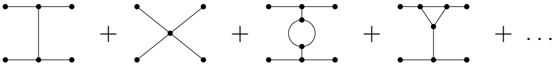
$G^{\phi\phi\phi\phi} =$ 

"Connected" Green functions:

$$G_c^{\phi_1\cdots\phi_n} = G^{\phi_1\cdots\phi_n}\Big|_{\text{only connected graphs}}$$

$\hookrightarrow$ relevant for genuine scattering processes

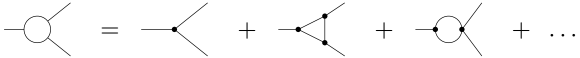
Example:

$G_c^{\phi\phi\phi\phi} =$ 

“Amputated” Green functions $G_{\mathrm{amp}}^{\phi_1\ldots\phi_n}$:

calculated as sum of all connected Feynman diagrams with n external legs $\phi_1,\ldots,\phi_n$ with external propagators (and propagator corrections) discarded:

$G_{\mathrm{amp}}^{\phi_1\phi_2\phi_3}$ =



Transition amplitude $\mathcal{M}_{fi}$ for $|i\rangle \rightarrow |f\rangle$:

calculated from amputated Green functions $G_{\mathrm{amp}}^{\phi_1\ldots\phi_n}$ by “LSZ reduction”:

- put external momenta to their mass shell, $p_j^2 = m_j^2$
- contract with wave functions of external particles
  (Dirac spinors, polarization vectors)

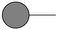
  Note: fields must be normalized: $R_{\phi_j} = 1$ (= residue of propagator pole), otherwise multiply by $\sqrt{R_{\phi_j}}$ for each external leg

Cross section for transition $|i\rangle \rightarrow |f\rangle$:

$$\sigma \;=\; \text{flux} \times \int \underbrace{\mathrm{dLIPS}}_{\text{Lorentz-invariant phase space}} |\mathcal{M}_{fi}|^2$$

"Vertex functions" $\Gamma^{\phi_1 \cdots \phi_n}$ as irreducible building blocks:

- Convenient choice for 1-point functions ("tadpoles"):

 $= \Gamma^{\phi} \stackrel{!}{=} 0$

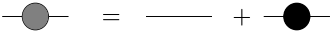
  - automatically fulfilled if ϕ carries a conserved quantum number (spin, charge, colour)
  - condition can be enforced by "tadpole renormalization"
  - $\Gamma^{\phi} = 0$ assumed below (otherwise relation between $\Gamma^{\phi_1\phi_2}$ and $G^{\phi_1\phi_2}$ modified)

- 2-point functions and propagators:

$$\mathrm{i}\Gamma^{\phi_1\phi_2} \equiv -(G^{\phi_1\phi_2})^{-1} = -(\text{inverse propagator})$$

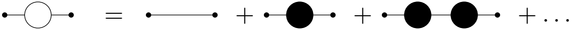
Example: scalar 2-point function (momentum transfer p)

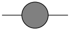
$$\mathrm{i}\Gamma^{\phi\phi}(p) = \mathrm{i}(p^2 - m^2) + \mathrm{i}\Sigma(p^2),$$



$\Sigma =$ self-energy $=$ sum of 1PI graphs
1PI $=$ 1-particle-irreducible
(graph cannot be disconnected by cutting *one* line)

$$G^{\phi\phi}(p) = \frac{\mathrm{i}}{p^2 - m^2} + \frac{\mathrm{i}}{p^2 - m^2}\,\mathrm{i}\Sigma(p^2)\,\frac{\mathrm{i}}{p^2 - m^2} + \dots \qquad \text{(Dyson series)}$$

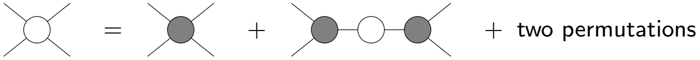


$$= \frac{\mathrm{i}}{p^2 - m^2 + \Sigma(p^2)} = -\left(\mathrm{i}\Gamma^{\phi\phi}(p)\right)^{-1} = -\left(\right.$$  $$\left.\right)^{-1}$$

- n-point functions with $n > 2$:

  $\mathrm{i}\Gamma^{\phi_1\cdots\phi_n} \equiv G_{\mathrm{amp}}^{\phi_1\cdots\phi_n}\Big|_{\text{only 1PI graphs}}$

  Example:



  $G_{\mathrm{amp}}^{\phi\phi\phi\phi}$ = $\mathrm{i}\Gamma^{\phi\phi\phi\phi}$ + $\mathrm{i}\Gamma^{\phi\phi\phi}\; G^{\phi\phi}\; \mathrm{i}\Gamma^{\phi\phi\phi}$ + two permutations

General idea:

Control/cancel UV divergences in building blocks $\Gamma^{\phi_1\cdots\phi_n}$

$\hookrightarrow$ $G_{\mathrm{amp}}^{\phi_1\cdots\phi_n}$ and $\mathcal{M}_{fi}$ become UV finite

Loop integrals and regularization

Regularization of divergences

Observation: loop integrals involve divergences

- UV divergences for $q \to \infty$, e.g.:

$$\int \mathrm{d}^4 q \, \frac{1}{(q^2 - m_0^2)(q^2 - m_1^2)} \sim \int \frac{\mathrm{d}q}{q} \text{ for } q \to \infty \quad \to \text{logarithmic divergence}$$

- IR divergences for $q \to q_0$, e.g.:

$$\int \mathrm{d}^4 q \, \frac{1}{q^2(q^2 + 2qp_1)(q^2 + 2qp_2)} \sim \int \frac{\mathrm{d}q}{q} \text{ for } q \to 0 \quad \to \text{log. divergence}$$

"Regularization": extension of theory by free parameter δ such that

- integrals (and thus the theory) become finite, i.e. well defined
- original theory is obtained as limiting case $\delta \to \delta_0$

  $\hookrightarrow$ fix input parameters $x_{i,0}$ of regularized theory ($\delta \neq \delta_0$) by experiment

  But: Limit $\lim_{\delta \to \delta_0} x_{i,0}(\delta)$ might not exist!

  $\hookrightarrow$ Split $x_{i,0}$ into divergent and finite parts, $x_{i,0} = x_i + \delta x_0$,
  and reparametrize theory by finite part x_i (="renormalization", discussed later)

Convenient regularization schemes:

- Dimensional regularization: switch to $D \neq 4$ space-time dimensions
  - regularizes UV (and IR) divergences, gauge invariant, easy to use
  - prescription: (μ = arbitrary reference mass, drops out in observables)

    $$\int \mathrm{d}^4 q \;\to\; (2\pi\mu)^{4-D} \int \mathrm{d}^D q$$

    and D-dim. momenta, metric, Dirac algebra
    and analytic continuation to complex $D \equiv 4 - 2\epsilon$!
  - divergences appear as poles $\frac{1}{\epsilon}$ in results

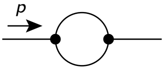
    $\hookrightarrow$ define $\Delta \equiv \frac{1}{\epsilon} - \gamma_{\mathrm{E}} + \ln(4\pi) = \frac{1}{\epsilon} + \mathrm{const.}$
- Mass regularization for IR singularities:
  infinitesimal photon mass m_γ and (if relevant) by small fermion masses m_f
  - photon propagator pole $\frac{1}{q^2} \to \frac{1}{q^2 - m_\gamma^2}$

    $\hookrightarrow$ $\ln(m_\gamma)$ terms at one-loop (appearing as $\Delta + \ln\mu$ in dim. reg.)
  - small fermion masses kept only in propagator denominators
    and asymptotic limit $m_f \to 0$ taken in integrals

    $\hookrightarrow$ $\ln(m_f)$ and $\ln^2(m_f)$ terms at one loop

Standard one-loop integrals:

- 2-point integrals:

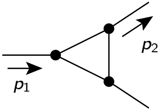


$$B_{0,\mu,\mu\nu,\dots}(p,m_0,m_1) \;=\; \frac{(2\pi\mu)^{2\epsilon}}{i\pi^2}\int \mathrm{d}^D q\, \frac{1,\, q_\mu,\, q_\mu q_\nu,\, \dots}{(q^2-m_0^2+\mathrm{i}0)[(q+p)^2-m_1^2+\mathrm{i}0]}$$

scalar integral B_0 = logarithmically UV divergent $= \Delta +$ finite,

vector integral $B_\mu = -\frac{1}{2}p_\mu\Delta +$ finite, etc.

- 3-point integrals:



$$C_{0,\mu,\mu\nu,\dots}(p_1,p_2,m_0,m_1,m_2)$$

$$= \frac{(2\pi\mu)^{2\epsilon}}{i\pi^2}\int \mathrm{d}^D q\, \frac{1,\, q_\mu,\, q_\mu q_\nu,\, \dots}{(q^2-m_0^2+\mathrm{i}0)[(q+p_1)^2-m_1^2+\mathrm{i}0][(q+p_2)^2-m_2^2+\mathrm{i}0]}$$

$C_0,\ C_\mu$ = UV finite,

$C_{\mu\nu}$ = logarithmically UV divergent $= \frac{1}{4}g_{\mu\nu}\Delta +$ finite, etc.

- 4-point integrals: $D_{\dots}$, already cumbersome integrals for multiple scales
- n-point integrals with $n > 4$: algebraically reducible to $D_{\dots}$ functions for $\epsilon \to 0$

Features of one-loop integrals:

- sign of infinitesimally small imaginary part i0 in mass terms reflects causality
- general results for one-loop integrals known
  (complicated but straightforward calculation)
  - momentum integrals can be carried out after "Feynman parametrization"
    $\hookrightarrow$ $(n-1)$-dimensional integrals for n-point functions
  - B functions $\rightarrow$ can be expressed in terms of log's
  - C, D, etc. $\rightarrow$ involve dilogarithms $\mathrm{Li}_2(x) = -\int_0^x \frac{\mathrm{d}t}{t} \ln(1-t)$
- tensor integrals can be decomposed into Lorentz covariants:

$$B^\mu = p^\mu B_1, \qquad B^{\mu\nu} = g^{\mu\nu} B_{00} + p^\mu p^\nu B_{11},$$

$$C^\mu = p_1^\mu C_1 + p_2^\mu C_2, \quad C^{\mu\nu} = p_1^\mu p_1^\nu C_{11} + p_2^\mu p_2^\nu C_{22} + (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu) + g^{\mu\nu} C_{00},$$

$\hookrightarrow$ tensor coefficients B_1, B_{ij}, C_i, etc. can be obtained as
linear combinations of scalar integrals B_0, C_0, etc.
(e.g. by "Passarino–Veltman reduction")

Renormalization at the one-loop level ("next-to-leading order", NLO)

Propagators and 2-point functions:

Structure of one-loop self-energies (scalar case as example):

$$\Sigma(p^2) \;=\; C_1\, p^2\, \Delta \;+\; C_2\, \Delta \;+\; \Sigma_{\rm finite}(p^2) \;=\; \text{UV divergent}$$

Behaviour of propagator near pole for free propagation at NLO:

$$G^{\phi\phi}(p^2) \;=\; \frac{\mathrm{i}}{p^2 - m^2 + \Sigma(p^2)} \;\underset{p^2\to m^2}{\widetilde{\quad\;}}\; \frac{1}{1+\Sigma'(m^2)}\,\frac{\mathrm{i}}{p^2 - m^2 + \Sigma(m^2)}$$

$\hookrightarrow$ higher-order corrections change location and residue of propagator pole

Interaction vertices:

Example: scalar 4-point interaction $\mathcal{L}_{\phi^4} = \lambda\phi^4/4!$

$$\mathrm{i}\Gamma^{\phi\phi\phi\phi}(p_1,p_2,p_3) \;=\; \mathrm{i}\lambda \;+\; \mathrm{i}\Lambda\,(p_1,p_2,p_3)$$

momentum-dependent one-loop correction:

$$\Lambda\,(p_1,p_2,p_3) \;=\; C_3\Delta \;+\; \Lambda_{\rm finite}\,(p_1,p_2,p_3) \;=\; \text{UV divergent}$$

$\hookrightarrow$ higher-order corrections change coupling strengths

Structure of UV divergences:

- Renormalizable field theories:

  UV divergences in vertex functions have analytical form of elementary vertex structures (directly related to $\mathcal{L}$)

  $\hookrightarrow$ idea: absorb divergences in free parameters

  $\Rightarrow$ Reparametrization of theory (=renormalization)

  Different types of renormalizable theories:

  - theories with unrelated couplings of non-negative mass dimensions
    $\hookrightarrow$ renormalizability proven by power counting and "BPHZ procedure"
  - gauge theories (couplings unified by gauge invariance)
    $\hookrightarrow$ renormalizability non-trivial consequence of gauge symmetry
    't Hooft '71

- Non-renormalizable field theories:

  typically theories with couplings of negative mass dimensions (e.g. Fermi model)

  Operators of higher and higher mass dimensions needed to absorb UV divergences in higher loop orders

  - infinitely many free parameters, i.e. less predictive power
  - only finitely many counterterms and free parameters per loop order
    $\hookrightarrow$ basis of *effective field theories*

Practical procedure for renormalization:

Consider original ("bare") parameters and fields as preliminary
(denoted with subscripts "0" in the following)

$\hookrightarrow$ Switch to new "renormalized" parameters and fields obeying certain conditions

Propagators and 2-point functions:

- mass renormalization: $m_0^2 = m^2 + \delta m^2$,
  $m^2 \stackrel{!}{=}$ location of propagator pole = "physical mass"

- wave-function ren.: rescale fields $\phi_0 = \sqrt{Z_\phi}\phi$, $G_{\rm R}^{\phi\phi} = Z_\phi^{-1} G^{\phi_0\phi_0}$
  fix $Z_\phi = 1 + \delta Z_\phi$ such that residue of $G_{\rm R}^{\phi\phi}$ at $p^2 = m^2$ equals 1

$\Rightarrow$ Renormalized propagator $G_{\rm R}^{\phi\phi}$: $G_{\rm R}^{\phi\phi}(p^2) = \dfrac{\rm i}{p^2 - m^2 + \Sigma_{\rm R}(p^2)}$

$$\Sigma_{\rm R}(p^2) = \Sigma(p^2) - \delta m^2 + \delta Z_\phi (p^2 - m^2) \ = \ \text{ren. self-energy}$$

$$\Sigma_{\rm R}(m^2) \stackrel{!}{=} 0 \quad \Rightarrow \quad \delta m^2 = \Sigma(m^2)$$

$$\Sigma'_{\rm R}(m^2) \stackrel{!}{=} 0 \quad \Rightarrow \quad \delta Z_\phi = -\Sigma'(m^2)$$

$$\Rightarrow \Sigma_{\rm R}(p^2) = \Sigma(p^2) - \Sigma(m^2) - (p^2 - m^2)\Sigma'(m^2)$$

$$= \Sigma_{\rm finite}(p^2) - \Sigma_{\rm finite}(m^2) - (p^2 - m^2)\Sigma'_{\rm finite}(m^2) \ = \ \text{UV finite}$$

Vertex functions for interactions:

Unrenormalized vertex function:

$$\mathrm{i}\Gamma^{\phi_0\phi_0\phi_0\phi_0}(p_1,p_2,p_3) = \mathrm{i}\lambda_0 + \underbrace{\mathrm{i}\Lambda(p_1,p_2,p_3)}$$

= unrenormalized vertex correction, contains UV-divergent constant

Coupling renormalization: $\lambda_0 \;=\; \lambda + \delta\lambda$

Fix $\delta\lambda$, e.g., by "momentum subtraction"
such that λ assumes a measured value for special kinematics p_i^{exp}

$\Rightarrow$ Renormalized vertex function $\Gamma_{\mathrm{R}}^{\phi\phi\phi\phi}$:

$$\begin{aligned}\mathrm{i}\Gamma_{\mathrm{R}}^{\phi\phi\phi\phi}(p_1,p_2,p_3) &= Z_\phi^2\mathrm{i}\Gamma^{\phi_0\phi_0\phi_0\phi_0}(p_1,p_2,p_3)\\ &= \mathrm{i}\lambda + \underbrace{\mathrm{i}\delta\lambda + 2\mathrm{i}\delta Z_\phi\lambda + \mathrm{i}\Lambda(p_1,p_2,p_3)}_{\equiv\, \mathrm{i}\Lambda_{\mathrm{R}}(p_1,p_2,p_3)} + \dots\end{aligned}$$

$$\mathrm{i}\Gamma_{\mathrm{R}}^{\phi\phi\phi\phi}(p_1^{\mathrm{exp}},p_2^{\mathrm{exp}},p_3^{\mathrm{exp}}) \overset{!}{=} \mathrm{i}\lambda \qquad \Rightarrow\; \delta\lambda = -2\delta Z_\phi\lambda - \Lambda(p_1^{\mathrm{exp}},p_2^{\mathrm{exp}},p_3^{\mathrm{exp}})$$

$$\Lambda_{\mathrm{R}}^{\phi\phi\phi\phi}(p_1,p_2,p_3) = \Lambda_{\mathrm{finite}}(p_1,p_2,p_3) - \Lambda_{\mathrm{finite}}(p_1^{\mathrm{exp}},p_2^{\mathrm{exp}},p_3^{\mathrm{exp}}) \;=\; \text{UV finite}$$

An alternative: "$\overline{\mathrm{MS}}$ prescription", which keeps only contributions $\propto \Delta$ at NLO and interprets μ as "renormalization scale"

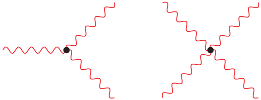
Table of contents

Electroweak renormalization at NLO

Loop corrections

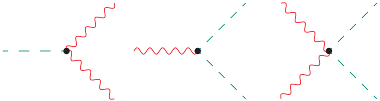
Recapitulation of elementary SM couplings (vertices)

gauge-boson self-couplings:

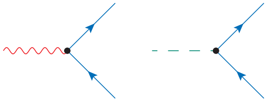


Higgs self-couplings:

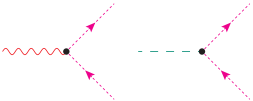
gauge-boson–Higgs couplings:



fermion couplings:



Faddeev–Popov couplings:

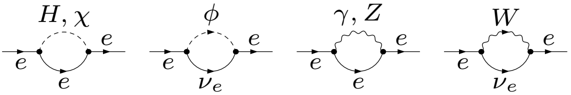


$\Rightarrow$ Large variety of loop diagrams !

Examples for 2-point functions at one loop: ('t Hooft–Feynman gauge)

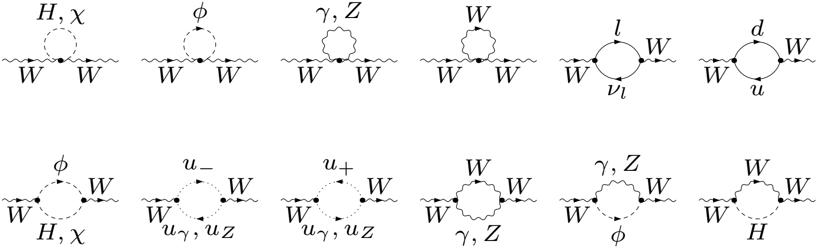
Electron self-energy:

$$i\Gamma^{\bar{e}e}(p) \;=\; i(\not{p} - m_e) + i\not{p}\omega_+\Sigma^{e,R}(p^2) + i\not{p}\omega_-\Sigma^{e,L}(p^2) + i m_e \Sigma^{e,S}(p^2)$$



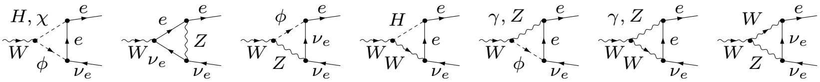
W-boson self-energy:

$$i\Gamma^{W^+W^-}_{\mu\nu}(k) \;=\; -ig_{\mu\nu}(k^2 - M_W^2) - i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Sigma_T^W(k^2) - i\frac{k_\mu k_\nu}{k^2}\Sigma_L^W(k^2)$$

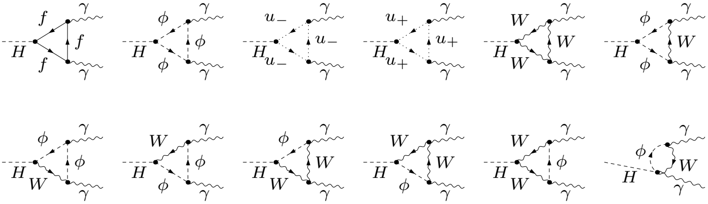


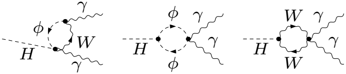
Examples for 3-point functions at one loop:

$\mathrm{We}\nu_\mathrm{e}$ vertex correction:



$\mathrm{H}\gamma\gamma$ vertex (loop induced):





$+$ diagrams (if different) with photons interchanged

Renormalization

Bare input parameters: $e_0,\, M_{\mathrm{W},0},\, M_{\mathrm{Z},0},\, M_{\mathrm{H},0},\, m_{f,0},\, V_{ij,0}$

Renormalization transformation:

- Parameter renormalization:

$$
\begin{aligned}
e_0 &= (1+\delta Z_e)e, \\
M_{\mathrm{W},0}^2 &= M_{\mathrm{W}}^2 + \delta M_{\mathrm{W}}^2, \quad M_{\mathrm{Z},0}^2 = M_{\mathrm{Z}}^2 + \delta M_{\mathrm{Z}}^2, \quad M_{\mathrm{H},0}^2 = M_{\mathrm{H}}^2 + \delta M_{\mathrm{H}}^2, \\
m_{f,0} &= m_f + \delta m_f, \quad V_{ij,0} = V_{ij} + \delta V_{ij}, \quad (\text{both } V_{ij,0}, V_{ij} \text{ unitary})
\end{aligned}
$$

  Note:
  renormalization of $c_{\mathrm{W}}, s_{\mathrm{W}}$ fixed by on-shell condition $c_{\mathrm{W}} = \frac{M_{\mathrm{W}}}{M_{\mathrm{Z}}}$
  (s_{W} is *not* a free parameter if M_{W} and M_{Z} are used as input parameters)

- Field renormalization

$$
W_0^{\pm} = \sqrt{Z_W}\, W^{\pm}, \quad \begin{pmatrix} Z_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} \sqrt{Z_{ZZ}} & \sqrt{Z_{ZA}} \\ \sqrt{Z_{AZ}} & \sqrt{Z_{AA}} \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix}, \quad H_0 = \sqrt{Z_H}\, H,
$$

$$
\psi_{f,0}^{\mathrm{L}} = \sqrt{Z_{ff'}^{\mathrm{L}}}\, \psi_{f'}^{\mathrm{L}}, \qquad \psi_{f,0}^{\mathrm{R}} = \sqrt{Z_{ff'}^{\mathrm{R}}}\, \psi_{f'}^{\mathrm{R}}
$$

  Note:
  matrix renormalization well suited to account for loop-induced mixing

On-shell (OS) renormalization conditions:

- **Mass renormalization:**

  on-shell definition: mass^2 tied to location of pole in propagator,
  i.e. to the zero of the 2-point vertex function $\Gamma^{\cdots}$

$$\left[\mathrm{Re}\,\Gamma^{V^\dagger V}_{\mathrm{R},\mu\nu}(-k,k)\right]\varepsilon^\nu(k)\Big|_{k^2=M_V^2} = 0, \quad V = W, Z, A$$

$+$ similarly for the Higgs bosons and fermions

$\Rightarrow$ Mass renormalization constants:

$$\delta M_V^2 = \mathrm{Re}\{\Sigma_{\mathrm{T}}^{V^\dagger V}(M_V^2)\}, \quad V = W, Z$$ (similarly for δM_{H}^2 and δm_f)

Comments:

- "Re" in the renormalization condition defines the "real OS scheme"
  $\hookrightarrow$ "running widths" in propagators, mostly used in exp. analyses
  (but problem with gauge dependences at two loops)
- Condition without "Re" defines the "(complex) pole scheme"
  $\hookrightarrow$ "constant widths" in propagators,
  (gauge-independent) complex masses for unstable particles
- other definitions of quark masses often more appropriate
  (running masses, masses in effective field theories)

- On-shell field renormalization:

"Diagonal parts of propagators":

residues of propagators (physical parts) normalized to 1

$$\lim_{k^2\to M_V^2} \frac{1}{k^2 - M_V^2} \left[\mathrm{Re}\,\Gamma^{V^\dagger V}_{\mathrm{R},\mu\nu}(-k,k)\right] \varepsilon^\nu(k) = -\,\varepsilon_\mu(k), \quad V = W, Z, A$$

$\Rightarrow$ Diagonal field renormalization constants:

$\delta Z_W = -\,\mathrm{Re}\{\Sigma_{\mathrm{T}}^{W\,\prime}(M_{\mathrm{W}}^2)\}$ (similarly for $\delta Z_{AA}, \delta Z_{ZZ}, \delta Z_H, \delta Z_{ff}^{\mathrm{L/R}}$)

"Non-diagonal parts of propagators":

suppression of mixing propagators on particle poles

$$\left[\mathrm{Re}\,\Gamma^{V'V}_{\mathrm{R},\mu\nu}(-k,k)\right] \varepsilon^\nu(k)\Big|_{k^2=M_V^2} = 0, \quad V'V = AZ, ZA$$

$+$ similarly for fermions

$\Rightarrow$ Non-diagonal field renormalization constants:

$\delta Z_{AZ} = -2\,\mathrm{Re}\,\frac{\Sigma_{\mathrm{T}}^{AZ}(M_{\mathrm{Z}}^2)}{M_{\mathrm{Z}}^2}, \quad \delta Z_{ZA} = 2\frac{\Sigma_{\mathrm{T}}^{AZ}(0)}{M_{\mathrm{Z}}^2}$ (similarly for $\delta Z_{ff'}^{\mathrm{L/R}}$, $f \neq f'$)

Note: problems for unstables particles beyond one loop
(field-renormalization constants become complex)

- Mixing-angle renormalization:

Recall: If M_{W} and M_{Z} are taken as free parameters, then c_{W} is fixed!

Relation between $M_{\mathrm{W}}, M_{\mathrm{Z}}, c_{\mathrm{W}}$ holds for bare and renormalized quantities:
(otherwise problems with self-consistency/gauge invariance)

$$c_{\mathrm{W},0} = \frac{M_{\mathrm{W},0}}{M_{\mathrm{Z},0}}, \qquad c_{\mathrm{W}} = \frac{M_{\mathrm{W}}}{M_{\mathrm{Z}}}$$

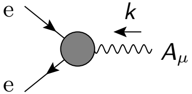
$\Rightarrow$ Renormalization constant δc_{W}^2 in $c_{\mathrm{W},0}^2 = c_{\mathrm{W}}^2 + \delta c_{\mathrm{W}}^2$ fixed by δM_{W}^2, δM_{Z}^2:

$$\frac{\delta c_{\mathrm{W}}^2}{c_{\mathrm{W}}^2} = \frac{\delta M_{\mathrm{W}}^2}{M_{\mathrm{W}}^2} - \frac{\delta M_{\mathrm{Z}}^2}{M_{\mathrm{Z}}^2} = \frac{\mathrm{Re}\{\Sigma_{\mathrm{T}}^{WW}(M_{\mathrm{W}}^2)\}}{M_{\mathrm{W}}^2} - \frac{\mathrm{Re}\{\Sigma_{\mathrm{T}}^{ZZ}(M_{\mathrm{Z}}^2)\}}{M_{\mathrm{Z}}^2}$$

Note also:
$s_{\mathrm{W},0}^2 + c_{\mathrm{W},0}^2 = s_{\mathrm{W}}^2 + c_{\mathrm{W}}^2 = 1$ with $s_{\mathrm{W},0}^2 = s_{\mathrm{W}}^2 + \delta s_{\mathrm{W}}^2$ implies

$$\delta s_{\mathrm{W}}^2 = -\delta c_{\mathrm{W}}^2, \qquad \delta s_{\mathrm{W}} = \frac{\delta s_{\mathrm{W}}^2}{2 s_{\mathrm{W}}} = -\frac{\delta c_{\mathrm{W}}^2}{2 s_{\mathrm{W}}}$$

- Charge renormalization: define e in Thomson limit



$\xrightarrow{k\to 0}$ $\mathrm{i}e\gamma_\mu$

low-energy interaction of photons with physical (on-shell!) $\mathrm{e}^\pm$

$\Rightarrow$ e = elementary charge of classical electrodynamics

fine-structure constant $\alpha(0) = \dfrac{e^2}{4\pi} = 1/137.03599976$

Gauge invariance relates δZ_e to photon wave-function renormalization:

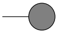
$$\delta Z_e \;=\; -\frac{1}{2}\delta Z_{AA} - \frac{s_\mathrm{W}}{2c_\mathrm{W}}\delta Z_{ZA}$$

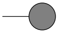
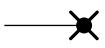
- Quark-field and CKM-matrix renormalization $\to$ fixes $\delta Z^{\mathrm{L/R}}_{qq'}, \delta V_{ij}$

  $\hookrightarrow$ rotation to mass eigenstates

  CKM part requires a careful (non-trivial) investigation of mixing self-energies, mass eigenstates, LSZ reduction, etc.

- Tadpole renormalization:

  tadpole diagrams unavoidable: $\Gamma^H = H$  $\neq 0$

  Very convenient:
  cancel tadpoles against counterterm: $\Gamma^H_{\mathrm{R}} = H$  $+ \; H$  $\stackrel{!}{=} 0$

$$\Rightarrow \delta t = -\Gamma^H$$

  Comments:

  - δt can be introduced in the course of parameter and/or field renormalization $\rightarrow$ different "tadpole schemes" (TS)
  - predictions for observables
    - ◇ do not depend on the TS in OS renormalization schemes
    - ◇ depend on the TS in $\overline{\mathrm{MS}}$ renormalization schemes
  - several vertex counterterms receive δt terms, depending on the TS
  - δt is gauge dependent

    $\hookrightarrow$ care required to avoid gauge dependences in $\overline{\mathrm{MS}}$ renormalization

- **Tadpole renormalization:** (continued)

  Bare Lagrangian contains $t_0 H_0$ term with $t_0 = \frac{1}{4} v_0 \left(4\mu_{2,0}^2 - \lambda_{2,0} v_0^2\right)$.

  Schemes for introducing δt:

  Parameter-Renormalized Tadpole Scheme (PRTS): e.g. Böhm/Hollik/Spiesberger '86, Denner '93

  $\delta t = t_0$

  $\hookrightarrow$ expansion of H field about corrected minimum of effective potential

  $\oplus$ moderate corrections in $\overline{\mathrm{MS}}$ schemes

  $\ominus$ gauge-dependent terms $\propto \delta t$ enter relations between renormalized parameters and predicted observables

  Fleischer–Jegerlehner Tadpole Scheme (FJTS): Fleischer/Jegerlehner '80, Actis et al. '06

  $t_0 = 0, \quad \delta t$ generated via field shift $H_0 \;\to\; H_0 + \Delta v$ with $\Delta v = -\delta t / M_\mathrm{H}^2$

  $\oplus$ no gauge dependences (field shift just redistributes terms)

  $\ominus$ $\overline{\mathrm{MS}}$ predictions prone to very large corrections

  "Gauge-invariant tadpole scheme" (GIVS): S.D., Rzehak, arXiv:2203.07236

  hybrid scheme of PRTS and FJTS

  $\oplus$ no gauge dependences

  $\ominus$ perturbative stability in $\overline{\mathrm{MS}}$ schemes

Renormalization of the unphysical sector:

$\hookrightarrow$ refers to unphysical fields (Goldstone fields, ghosts) and gauge parameters

- irrelevant for S-matrix elements
- required to render all Green functions UV finite
- can be adjusted to maintain form of Slavnov–Taylor identities after renormalization

Overall result:

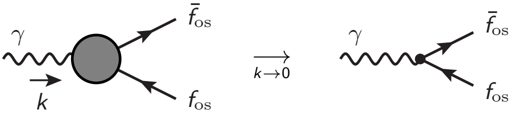
All renormalization constants obtained from self-energies (non-trivial for δZ_e!).

Table of contents

Charge renormalization to all orders

Renormalization of e:

- renormalization condition in Thomson limit
  $\hookrightarrow$ no corrections in the low-energy limit for on-shell (OS) electrons



$$\bar{u}(p')\,\Gamma^{A\bar{f}f}_{\mathrm{R},\mu}(k,-p',p)\,u(p) \quad \underset{k\to 0}{\longrightarrow} \quad -\,Q_f\,e\,\bar{u}(p)\gamma_\mu u(p)$$

- renormalization transformation: $\underset{\text{bare}}{e_0} = Z_e \underset{\text{ren.}}{e} = (1+\delta Z_e)e$

  $\hookrightarrow$ charge ren. constant Z_e fixed by vertex correction $\Gamma^{A\bar{f}f}_{\mu}(0,-p,p)$
- QED: Z_e derived from $\gamma\gamma$ self-energy Σ^{AA} via QED Ward identity
- SM(+beyond): Z_e calculable from $\Sigma^{AA/AZ}$, but underlying gauge-invariance arguments much more complicated!

Relevance of relating Z_e to (f-independent) self-energies $\Sigma^{AA/AZ}$?
$\hookrightarrow$ General understanding, proving theorems, technical simplifications, ...

On the history of charge renormalization in the SM

Charge renormalization at NLO:

- pioneering works on electroweak renormalization:
  - δZ_e from explicit one-loop calculations Ross, Taylor '73; Sirlin '80; Bardin et al. '80; Aoki et al. '82; Fleischer, Jegerlehner '81; Böhm et al. '86; Hollik '90; Denner '93
- more recently: derivation of δZ_e via Lee identities Denner, S.D. '19

Charge renormalization at NNLO:

- δZ_e from explicit 2-loop calculations Degrassi, Vicini '03; Actis et al. '06

Charge renormalization to all orders:

- Background-Field Method (BFM): $\rightarrow$ backup slides
  generalization of QED-like result to SM Denner, S.D., Weiglein '94
  $\hookrightarrow$ in particular proves "charge universality"
  $\Rightarrow$ renormalization of e does not depend on charged particle,
  so that any charged particle can be used for renormalization of e
- conventional R_ξ gauge:
  - correct conjecture, but incorrect proofs Bauberger '97; Freitas et al. '02; Awramik et al. '02
  - new approach via charge universality S.D. '21
    $\hookrightarrow$ confirmation of previous conjecture $\rightarrow$ explained below!

Charge renormalization in QED

Ward identity for the unrenormalized $Af\bar{f}$ vertex function:

$$k^\mu \Gamma^{A\bar{f}f}_\mu(k,\bar{p},p) = -Q_f e_0 \left[\Gamma^{\bar{f}f}(\bar{p},-\bar{p}) - \Gamma^{\bar{f}f}(-p,p)\right]$$

Renormalization transformation: $\psi_{0,f}(x) = Z_f^{1/2}\,\psi_f(x),\; A_0^\mu(x) = Z_{AA}^{1/2}\,A^\mu(x)$

$$G^{Af\bar{f}}_\mu(k,\bar{p},p) = Z_{AA}^{1/2}\,Z_f\,G^{Af\bar{f}}_{\mathrm{R},\mu}(k,\bar{p},p) \qquad \text{(full, reducible Green function)}$$

$$\Gamma^{A\bar{f}f}_{\mathrm{R},\mu}(k,\bar{p},p) = Z_{AA}^{1/2}\,Z_f\,\Gamma^{A\bar{f}f}_\mu(k,\bar{p},p) \qquad \text{(amputated, 1PI vertex function)}$$

$\Rightarrow$ Ward identity for the renormalized $Af\bar{f}$ vertex function:

$$k^\mu \Gamma^{A\bar{f}f}_{\mathrm{R},\mu}(k,\bar{p},p) = -Q_f e\, Z_e\, Z_{AA}^{1/2} \left[\Gamma^{\bar{f}f}_{\mathrm{R}}(\bar{p},-\bar{p}) - \Gamma^{\bar{f}f}_{\mathrm{R}}(-p,p)\right]$$

$\hookrightarrow$ Expansion for $k \to 0$ yields

$$\underbrace{\bar{u}(p)\,\Gamma^{A\bar{f}f}_{\mathrm{R},\mu}(0,-p,p)\,u(p)}_{\overset{!}{=}\,-Q_f e\,\bar{u}(p)\gamma_\mu u(p)} = -Q_f e\, Z_e\, Z_{AA}^{1/2}\, \underbrace{\bar{u}(p)\,\frac{\partial \Gamma^{\bar{f}f}_{\mathrm{R}}(-p,p)}{\partial p^\mu}\,u(p)}_{=\,\bar{u}(p)\,\gamma_\mu\,u(p)}$$

(wave-fct. ren. of ψ_f)

$\Rightarrow$ $Z_e = Z_{AA}^{-1/2}$ = independent of f $\Rightarrow$ Charge universality of QED!

From QED to the SM

Charge renormalization in the SM @ NLO

QED	SM
Gauge symmetry:	
$\mathrm{U(1)_{em}}$ exact	$\mathrm{U(1)}_Y$ spontaneously broken, $\mathrm{U(1)_{em}}$ mixes $\mathrm{SU(2)}_I$ and $\mathrm{U(1)}_Y$ trafos
Neutral gauge bosons:	
photon field A_μ	2 neutral gauge fields W^3_μ, B_μ, mass basis via Weinberg rotation: $\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} c_\mathrm{W} & -s_\mathrm{W} \\ s_\mathrm{W} & c_\mathrm{W} \end{pmatrix} \begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix}$ $\hookrightarrow$ mixing of A_μ, Z_μ in higher orders
BRS symmetry:	
fully decoupling FP ghost fields $\hookrightarrow$ ghost-free Ward identities	FP ghosts non-decoupling $\hookrightarrow$ ghost contributions in Slavnov–Taylor (ST) ids. for Green fcts. $G^{\Psi_1\Psi_2\cdots}$ and Lee ids. for 1PI vertex fcts. $\Gamma^{\Psi_1\Psi_2\cdots}$

Charge renormalization in the SM @ NLO in R_ξ gauge

One-loop Ward identity: derived via Lee identities or ST identities

(Denner, S.D., 1912.06823)

$$\bar{u}(p)\,\Gamma_\mu^{A\bar{f}f}(0,-p,p)\,u(p) = -\,Q_f e_0\,\bar{u}(p)\,\frac{\partial\Gamma^{\bar{f}f}(-p,p)}{\partial p^\mu}\,u(p) - \frac{I^3_{\mathrm{w},f} e_0}{s_{\mathrm{W},0}c_{\mathrm{W},0}}\,\frac{\Sigma_\mathrm{T}^{AZ}(0)}{M_\mathrm{Z}^2}\,\bar{u}(p)\,\gamma_\mu\omega_-\,u(p), \qquad (*)$$

$$\omega_\pm = \tfrac{1}{2}(1\pm\gamma_5)$$

Generalization beyond one-loop level not available!

Renormalization:

$$\begin{pmatrix} Z_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} Z_{ZZ}^{1/2} & Z_{ZA}^{1/2} \\ Z_{AZ}^{1/2} & Z_{AA}^{1/2} \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix}, \qquad \omega_\sigma f_0 = f_0^\sigma = \left(Z^{f,\sigma}\right)^{1/2} f^\sigma$$

$$\underbrace{\bar{u}(p)\,\Gamma_{\mathrm{R},\mu}^{A\bar{f}f}(0,-p,p)\,u(p)}_{\stackrel{!}{=}\,-Q_f e\,\bar{u}(p)\,\gamma_\mu\,u(p)} = \bar{u}(p)\,\Gamma_\mu^{A\bar{f}f}(0,-p,p)\,u(p) + \tfrac{1}{2}\delta Z_{ZA}\,\bar{u}(p)\,\Gamma_{\mu,0}^{Z\bar{f}f}\,u(p) + \tfrac{1}{2}\left(\delta Z_{AA} + \delta Z^{f,+} + \delta Z^{f,-}\right)\bar{u}(p)\,\Gamma_{\mu,0}^{A\bar{f}f}\,u(p)$$

$$\underset{(*)}{=} -\,Q_f e\left(1 + \delta Z_e + \frac{1}{2}\delta Z_{AA} + \frac{s_\mathrm{W}}{2c_\mathrm{W}}\delta Z_{ZA}\right)\bar{u}(p)\,\gamma_\mu\,u(p) + \frac{I^3_{\mathrm{w},f} e}{s_\mathrm{W} c_\mathrm{W}}\underbrace{\left(\frac{1}{2}\delta Z_{ZA} - \frac{\Sigma_\mathrm{T}^{AZ}(0)}{M_\mathrm{Z}^2}\right)}_{=0\ \text{(OS renormalization)}}\bar{u}(p)\,\gamma_\mu\omega_-\,u(p)$$

$$\Rightarrow\quad \delta Z_e = -\frac{1}{2}\delta Z_{AA} - \frac{s_\mathrm{W}}{2c_\mathrm{W}}\delta Z_{ZA}$$

Charge renormalization in the SM to all orders in R_ξ gauge

S.D., arXiv:2101.05154

Idea: exploit charge universality and
introduce fake fermion η with infinitesimal weak hypercharge:

$$\tfrac{1}{2} Y_{\mathrm{w},\eta} = Q_\eta \to 0, \qquad I_{\mathrm{w},\eta} = 0, \qquad m_\eta = \text{arbitrary}$$

Lagrangian:

$$\mathcal{L}_\eta = \bar\eta\Big(\mathrm{i}\not\partial - \tfrac{1}{2} g_1 Y_{\mathrm{w},\eta} \not B - m_\eta\Big)\eta = \bar\eta\left[\mathrm{i}\not\partial - Q_\eta e\left(\not A + \frac{s_\mathrm{W}}{c_\mathrm{W}} \not Z\right) - m_\eta\right]\eta$$

Charge renormalization condition for η:

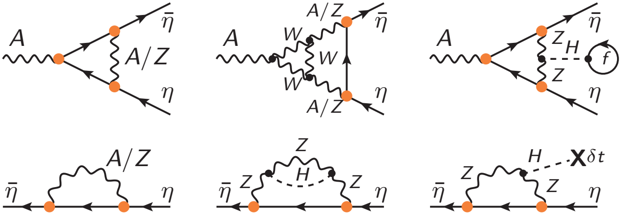
$$\bar u(p)\,\Gamma^{A\bar\eta\eta}_{\mathrm{R},\mu}(0,-p,p)\,u(p)\Big|_{p^2=m_\eta^2} = -Q_\eta e\,\bar u(p)\gamma_\mu u(p)$$

Renormalization transformation: $\eta_0 = Z_\eta^{1/2}\,\eta$

$$\Gamma^{A\bar\eta\eta}_{\mathrm{R},\mu}(k,\bar p,p) = Z_\eta Z_{AA}^{1/2}\,\underbrace{\Gamma^{A\bar\eta\eta}_{\mu}(k,\bar p,p)}_{=-Q_\eta e_0\gamma_\mu+\text{h.o.}} + Z_\eta Z_{ZA}^{1/2}\,\underbrace{\Gamma^{Z\bar\eta\eta}_{\mu}(k,\bar p,p)}_{=-Q_\eta e_0\frac{s_{\mathrm{W},0}}{c_{\mathrm{W},0}}\gamma_\mu+\text{h.o.}}$$

But: higher-order contributions to Z_η and $\Gamma^{V\bar\eta\eta}_\mu$ are of $\mathcal{O}(Q_\eta^2)$!

Some sample diagrams for corrections to Z_η and $\Gamma_\mu^{V\bar\eta\eta}$:



$\mathcal{O}(Q_\eta^2)$

$\Rightarrow$ Direct calculation of $\Gamma^{A\bar\eta\eta}_{\mathrm{R},\mu}(0,-p,p)$ (without Ward identities, etc.):

$$\Gamma^{A\bar\eta\eta}_{\mathrm{R},\mu}(0,-p,p) = -Q_\eta e\gamma_\mu Z_e \left[Z_{AA}^{1/2} + Z_{ZA}^{1/2} \frac{s_{\mathrm{W},0}}{c_{\mathrm{W},0}} \right] + \mathcal{O}(Q_\eta^2)$$
$$\stackrel{!}{=} -Q_\eta e\gamma_\mu$$

$$\Rightarrow Z_e = \left[Z_{AA}^{1/2} + Z_{ZA}^{1/2} \frac{s_{\mathrm{W},0}}{c_{\mathrm{W},0}} \right]^{-1} = \left[Z_{AA}^{1/2} + Z_{ZA}^{1/2} \sqrt{\frac{s_{\mathrm{W}}^2 - \delta c_{\mathrm{W}}^2}{c_{\mathrm{W}}^2 + \delta c_{\mathrm{W}}^2}} \right]^{-1}$$

= function of gauge-boson self-energies only

and in agreement with previously "conjectured" results of Bauberger '97; Freitas et al. '02; Awramik et al. '02

Literature

$\hookrightarrow$ See Lecture 1 !

Backup slides

Basics of the Background-Field Method (BFM)

DeWitt '67,'80; 't Hooft '75; Boulware '81; Abbott '81 SM: Denner, S.D., Weiglein '94

Renormalization in the background-field method

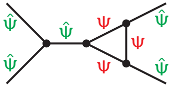
Fields Ψ split into background and quantum parts: $\Psi \to \hat{\Psi} + \Psi$

- Background fields $\hat{\Psi}$:
  - sources of the BFM effective action $\hat{\Gamma}[\hat{\Psi}]$
  - external and tree lines in Feynman diagrams
  - gauge of background gauge fields $\hat{A}^a_\mu$ not fixed in $\hat{\Gamma}[\hat{\Psi}]$
- Quantum fields Ψ:
  - integration variables of the functional integral $\int \mathcal{D}\Psi \, \exp\left\{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}\right\}$
  - loop lines of Feynman diagrams
  - R_ξ-type gauge-fixing to support gauge invariance of $\hat{\Gamma}[\hat{\Psi}]$

BFM gauge invariance and gauge fixing of $\hat{\Psi}$:

- $\hat{\Gamma}[\hat{\Psi}]$ fully invariant under "ordinary" gauge transformations of $\hat{\Psi}$

  $\hookrightarrow$ "ghost-free" QED-like Ward identities for BFM vertex fcts. $\hat{\Gamma}^{\hat{\Psi}\cdots}$
- Reducible Green fcts. $\hat{G}^{\hat{\Psi}\cdots}$ and S-matrix elements:
  - gauge-fixing of $\hat{\Psi}$ required for bkg. propagators $\hat{G}^{\hat{\Psi}\hat{\Psi}}$
  - formed from trees with vertex fcts. $\hat{\Gamma}^{\hat{\Psi}\cdots}$



BFM version of charge renormalization in the SM

Denner, S.D., Weiglein '94

BFM Ward identity for the unrenormalized $\hat{A}\bar{f}f$ vertex function:

$$k^\mu \hat{\Gamma}^{\hat{A}\hat{V}}_{\mu\nu}(k,-k) = 0, \qquad \hat{V} = \hat{A}, \hat{Z},$$

$$k^\mu \hat{\Gamma}^{\hat{A}\bar{f}_i f_j}_{\mu}(k,\bar{p},p) = -\,Q_f e_0 \left[\hat{\Gamma}^{\bar{f}_i f_j}(\bar{p},-\bar{p}) - \hat{\Gamma}^{\bar{f}_i f_j}(-p,p)\right]$$

Renormalization transformation: $\begin{pmatrix}\hat{Z}_0\\ \hat{A}_0\end{pmatrix} = \begin{pmatrix} Z^{1/2}_{\hat{Z}\hat{Z}} & Z^{1/2}_{\hat{Z}\hat{A}} \\ Z^{1/2}_{\hat{A}\hat{Z}} & Z^{1/2}_{\hat{A}\hat{A}}\end{pmatrix} \begin{pmatrix}\hat{Z}\\ \hat{A}\end{pmatrix}, \quad f^\sigma_{0,n} = \sum_j \left(Z^{f,\sigma}_{nj}\right)^{1/2} f^\sigma_j$

$$\hat{\Gamma}^{\bar{f}_i f_j}_{\mathrm{R}}(-p,p) = \sum_{l,n} \left(Z^{f,\sigma *}_{li}\right)^{1/2} \left(Z^{f,\sigma}_{nj}\right)^{1/2} \hat{\Gamma}^{\bar{f}_l f_n}(-p,p),$$

$$\hat{\Gamma}^{\hat{A}\bar{f}_i f_j}_{\mathrm{R},\mu}(k,\bar{p},p) = \sum_{\hat{V}=\hat{A},\hat{Z}} \sum_{l,n} \underbrace{Z^{1/2}_{\hat{V}\hat{A}}}_{Z_{\hat{Z}\hat{A}} = 0 \text{ due to BFM gauge invariance}} \left(Z^{f,\sigma *}_{li}\right)^{1/2} \left(Z^{f,\sigma}_{nj}\right)^{1/2} \hat{\Gamma}^{\hat{V}\bar{f}_l f_n}_{\mu}(k,\bar{p},p)$$

BFM Ward identity for the renormalized $\hat{A}\bar{f}f$ vertex function:

$$k^\mu \hat{\Gamma}^{\hat{A}\bar{f}_i f_j}_{\mathrm{R},\mu}(k,\bar{p},p) = -\,Q_f e\, Z_e\, Z^{1/2}_{\hat{A}\hat{A}} \left[\hat{\Gamma}^{\bar{f}_i f_j}_{\mathrm{R}}(\bar{p},-\bar{p}) - \hat{\Gamma}^{\bar{f}_i f_j}_{\mathrm{R}}(-p,p)\right]$$

$\Rightarrow\ Z_e = Z^{-1/2}_{\hat{A}\hat{A}}$ analogously to QED $\Rightarrow$ Charge universality of the SM!