Hard Scattering Beyond the Leading Power

(EFT, Background field interpretation, Gravity)

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Theory Challenges in the Precision Era of the Large Hadron Collider
GGI, Firenze, Aug 28, 2023 - Oct 13, 2023
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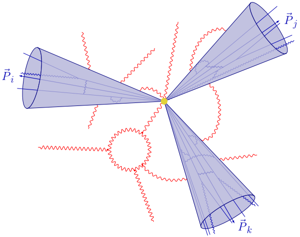

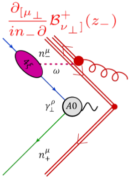

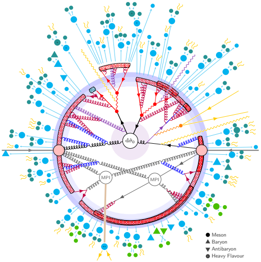

[Figure credit: 2203.11601]

Based on 1912.01585 (MB, Broggio, Jaskiewicz, Vernazza), 2110.02969, 2112.04983, 2210.09336 (MB, Hager, Szafron)

Hard scattering beyond the leading power

"Higher-twist"

$$F(Q, \Lambda; \alpha_s) = f_0(\alpha_s) + \frac{\Lambda}{Q} f_1(\alpha_s) + \ldots$$
$$Q \gg \Lambda, \; \lambda = \sqrt{\Lambda/Q}$$

Double expansion in α_s and powers.
Mathematically, one constructs an asymptotic / resurgent expansion of an observable in powers and logarithms of the ratio of Λ/Q.

"Perturbative resummation problem"

$$F(Q, E_s, \Lambda; \alpha_s) = \left[f_0(\alpha_s, \ln \frac{E_s}{Q}) + \frac{E_s}{Q} f_1(\alpha_s, \ln \frac{E_s}{Q}) + \ldots \right] + \mathcal{O}\left(\frac{\Lambda}{Q}\right)$$
$$Q \gg E_s \gg \Lambda, \; \lambda = \sqrt{E_s/Q}$$

Compared to "old" higher twist: Soft physics does not cancel, No local OPE.

Effective Field Theory

Scattering amplitudes, phase-space integrals have infrared singularities, when small scales are set to zero, which obstruct the Taylor expansion in $1/Q$

Effective field theory turns the analysis of IR singularities of diagrams into an ultraviolet renormalization calculations of EFT operators. Resummation becomes a RGE problem. Essential change of point of view for the analysis of power corrections.

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Every object (fields, derivatives, momenta, ...) in the EFT should have a unique scaling with λ.

collinear mode momentum

$$p_c \sim Q(1, \lambda, \lambda^2), p_c^2 \sim Q^2\lambda^2$$

soft mode momentum

$$p_s \sim Q(\lambda^2, \lambda^2, \lambda^2), p_s^2 \sim Q^2\lambda^4$$

Light-like reference vectors $n_\pm, n_\pm^2 = 0, n_+ \cdot n_- = 2$

$$\begin{aligned} p^\mu &= (n_+p)\frac{n_-^\mu}{2} + p_\perp^\mu + (n_-p)\frac{n_+^\mu}{2} \\ &= (n_+p, p_\perp, n_-p) \end{aligned}$$

Need separate fields for collinear and soft modes

$$\text{QCD}[A, \psi] \quad \longrightarrow \quad \text{SCET}[A_{c_i}, A_s, \xi_{c_i}, q_s]$$

Several collinear directions $n_{i\pm} \rightarrow$ several copies of collinear fields.

SCET at leading power

$$\mathcal{L}_{\rm SCET}^{(0)} = \sum_{i=1}^{N} \mathcal{L}_{c_i}^{(0)} + \mathcal{L}_{\rm soft}$$

$$\mathcal{L}_c^{(0)}(x) = \bar{\xi}_c \left(in_- D_c + g_s n_- A_s(x_-) + i\not{D}_{\perp c} \frac{1}{in_+ D_c} i\not{D}_{\perp c} \right) \frac{\not{n}_+}{2} \xi_c + \mathcal{L}_{c,\rm YM}^{(0)}$$

$$iD_c = i\partial + g_s A_c, \qquad x_-^\mu = \frac{1}{2} n_+ \cdot x \, n_-^\mu$$

- Non-local along light-cone sn_+^μ, since $n_+ p_c \sim Q$.
- Introduce collinear-gauge invariant "jet" fields

$$\chi \equiv W_c^\dagger \xi, \qquad \mathcal{A}_{c\perp}^\mu \equiv W_c^\dagger [iD_{c\perp}^\mu W_c]$$

SCET at leading power

$$\mathcal{L}^{(0)}_{\text{SCET}} = \sum_{i=1}^{N} \mathcal{L}^{(0)}_{c_i} + \mathcal{L}_{\text{soft}}$$

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$$iD_c = i\partial + g_s A_c\,, \qquad x^\mu_- = \frac{1}{2} n_+ \cdot x\, n^\mu_-$$

- Non-local along light-cone sn^μ_+, since $n_+ p_c \sim Q$.
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- In soft-collinear interactions, soft fields are multipole-expanded around x^μ_- in collinear interactions. [BCDF, 2002]
- Only $n_- A_s$ appears, with eikonal vertex $ig_s n^\mu_-$.
- Introduce the soft-decoupling transformation [BPS, 2001]

$$\xi(x) \to Y_{n_-}(x_-)\xi(x) \qquad A^\mu_c(x) \to Y_{n_-}(x_-) A^\mu_c(x) Y^\dagger_{n_-}(x_-)$$

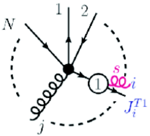
LP soft physics $\leftrightarrow$ Matrix element of light-like Wilson line correlators

SCET Lagrangian beyond LP

Lagrangian describes collinear splittings and interactions of (separate) collinear and soft fields

$$\mathcal{L}_{\text{SCET}} = \sum_{i=1}^{N} \mathcal{L}_{c_i} + \mathcal{L}_{\text{soft}}$$

$$\mathcal{L}_{c_i} = \mathcal{L}_{c_i}^{(0)} + \mathcal{L}_{c_i}^{(1)} + \mathcal{L}_{c_i}^{(2)} + \ldots$$



Soft-collinear interactions

$$\mathcal{L}_{sc}^{\text{gluon}}(x) = \bar{\xi}\Big[\underbrace{g_s n_- A_s(x_-)}_{\text{LP, eikonal}} + \underbrace{x_\perp^\mu n_-^\nu\, W_c\, g_s F^{s}_{\mu\nu}(x_-) W_c^\dagger}_{\text{NLP}}\Big]\frac{\not{n}_+}{2}\,\xi + \mathcal{O}(\lambda^2)$$

$$\mathcal{L}_{sc}^{\text{quark}}(x) = \underbrace{\bar{q}(x_-)\, W_c^\dagger\, i\not{D}_{\perp c}\,\xi}_{\text{starts at NLP}} + \text{h.c.} + \mathcal{O}(\lambda^2)$$

- No purely collinear subleading interactions. At least one soft field in every vertex.
- Execpt for LP eikonal $n_- A_s$ only covariant field-strength interactions.
- NLP needs $\mathcal{O}(\lambda^2)$, i.e. $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$.
  Lagrangian known to all orders (tree-level exact).

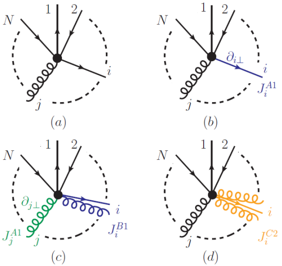
NLP hard process

Hard processes are represented by N-jet light-ray operators of the collinear “jet” fields

- The standard soft anomalous dimension is the anomalous dimension of the leading power N-jet operator

$$\chi_{c1}(t_1 n_{1+})\mathcal{A}^{\mu}_{\perp c2}(t_2 n_{2+})\ldots\chi_{cN}(t_N n_{N+})$$

- Subleading power: transverse derivatives and more jet fields in the <u>same</u> collinear direction.



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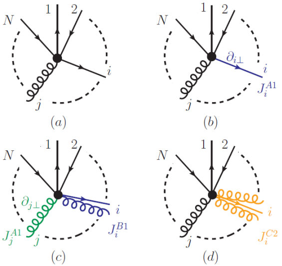


- Automatic factorization into **single scale** objects, which have **gauge-invariant operator definitions: hard, jet/collinear and soft functions**

$$\sigma \sim H \otimes J_1 \otimes \ldots \otimes J_n \otimes S$$

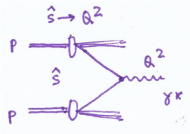
Hard matching coefficient

Jet functions - matrix elements of collinear fields

Soft function - matrix element of soft fields

Drell-Yan process, partonic threshold

$pp \to \gamma^*(Q) + X \qquad \text{for } \hat{s} \to Q^2: \qquad q\bar{q} \to \gamma^*(Q) + X_{\text{soft}}$



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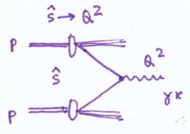
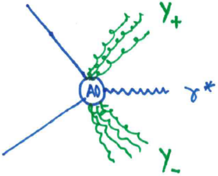


$$\langle 0|\bar{\psi}\,\gamma^\rho \psi(0)|q\bar{q}\rangle = \int dt\, d\bar{t}\, C^{A0,A0}(t,\bar{t}\,)\,\langle 0|\bar{\chi}_{\bar{c}}\,(\bar{t}n_-)|\bar{q}\rangle\,\gamma_\perp^\rho\,\langle 0|\chi_c(tn_+)|q\rangle$$

$$\times\,\langle X_s|\,\underbrace{\mathbf{T}\left(\left[Y_-^\dagger(0)Y_+(0)\right]\right)}_{\text{Exponentiated LP eikonal}}|0\rangle$$

$$Y_\pm(x) = \mathbf{P}\exp\left[ig_s\int_{-\infty}^0 ds\, n_\mp A_s\,(x+sn_\mp)\right]$$



[Korchemsky, Marchesini, 1993]

NLP with SCET

Field redefinition

$$\xi_c(z) \to Y_+(z_-)\,\xi_c^{(0)}(z) \qquad A_c^\mu(z) \to Y_+(z_-)A_c^{(0)\mu}(z)Y_+^\dagger(z_-)$$

$$\mathcal{L}_{\text{soft}}^{\text{gluon}}(x) = \bar\xi\Big[\underbrace{gn_-A_s(x_-)}_{\text{LP, eikonal}} + \underbrace{x_\perp^\mu n_-^\nu\, W_c\, gF^{\text{s}}_{\mu\nu}(x_-)W_c^\dagger}_{\text{NLP, }\mathcal{O}(\lambda)}\Big]\frac{\not n_+}{2}\,\xi + \mathcal{O}(\lambda^2)$$

$$\to \bar\chi^{(0)}\Big[\underbrace{0}_{\text{LP decoupling}} + i\underbrace{x_\perp^\mu\,[in_-\partial\mathcal{B}_\mu^+(x_-)]}_{\text{NLP, }\mathcal{O}(\lambda)}\Big]\frac{\not n_+}{2}\,\chi^{(0)} + \mathcal{O}(\lambda^2)$$

$$\bar\chi_{\bar c}(\bar t n_-)\,\gamma_\perp^\rho\,\chi_c(tn_+) \to \bar\chi_{\bar c}^{(0)}(\bar t n_-)\,\gamma_\perp^\rho\,\chi_c^{(0)}(tn_+) \times \Big[Y_-^\dagger(0)Y_+(0)\Big]$$

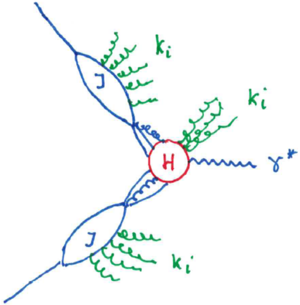
- Gauge-invariant soft fields

$$\mathcal{B}_\pm^\mu = Y_\pm^\dagger\,[i\,D_s^\mu\,Y_\pm]\,, \qquad q^\pm = Y_\pm^\dagger\,q_s$$

- No soft-collinear decoupling at subleading power

General form of the radiative (DY) amplitude

- Collinear functions appear from NLP and "dress" the soft emissions
  [Del Duca, 1991; Bonocore et al. 2015]
- Gauge-invariant definition through matching of gauge-invariant operators
  [MB, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, Wang, 2018]



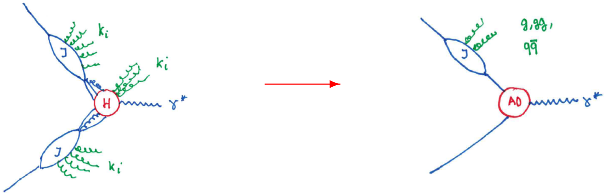
$$i^m \int \{d^d z_j\}\, \mathbf{T}\left[\{\chi_c(t_k n_+)\} \times \left\{\mathcal{L}^{(l)}(z_j)\right\}\right]$$
$$= 2\pi \sum_i \int du \int \{dz_{j-}\}\, \tilde{J}_i\left(\{t_k\}, u; \{z_{j-}\}\right) \chi_c^{\text{ext}}(u n_+)\, \mathfrak{s}_i(\{z_{j-}\})$$

$$\mathfrak{s}_i(\{z_{j-}\}) \in \left\{\mathcal{B}^+_{\mu_\perp}(z_{1-}), \mathcal{B}^+_{\mu_\perp}(z_{1-})\mathcal{B}^+_{\nu_\perp}(z_{2-}), q_{+\sigma}(z_{1-})\bar{q}_{+\lambda}(z_{2-}), \ldots\right\}$$

Simplifications at next-to-leading power

- Only one collinear line to hard vertex, no soft from vertex

  $\hookrightarrow$ universal hard vertex (no collinear radiation into final state!)
- Jet function only at one leg at a time
- But also two soft gluon and soft quark-antiquark emission



$$\langle X_s|\mathbf{T}\left(\left[Y_-^\dagger(0)Y_+(0)\right]\right)|0\rangle \to \sum_i J_i(\omega_k)\otimes\langle X_s|\mathbf{T}\left(\left[Y_-^\dagger(0)Y_+(0)\right]\,\mathfrak{s}_i(\omega_k)\right)|0\rangle$$

NLP factorization formula (leading-log accurate)

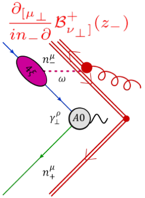
$$\hat{\sigma}_{q\bar{q}}^{\rm NLP}(z) = H^{\rm LP}(Q^2)\,Q\,J(\omega)\otimes_\omega S_{2\xi}(Q(1-z);\omega) + \text{h.c.}$$

Generalized soft function

$$S_{2\xi}(\Omega,\omega) = \mathbf{FT}_{\{x^0,z_-\}}\, \frac{1}{N_c}\, \mathrm{Tr}\, \langle 0|\overline{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x)\right]\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\,\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\perp\nu}(z_-)\right])|0\rangle$$

$$= \frac{\alpha_s C_F}{2\pi}\left\{\theta(\Omega)\delta(\omega)\left(-\frac{1}{\epsilon}+\ln\frac{\Omega^2}{\mu^2}\right)+\left[\frac{1}{\omega}\right]_+\theta(\omega)\theta(\Omega-\omega)\right\}$$

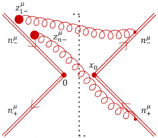
Two-loop available 1912.01585 and 2107.07353 (Broggio, Jaskiewicz, Vernazza)



Renormalization group equation involves mixing with

$$S_{x_0}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega/2}\, \frac{-2i}{x^0-i\varepsilon}\, \frac{1}{N_c}\, \mathrm{Tr}\, \langle 0|\overline{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x^0)\right]\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\right]|0\rangle$$

$$\frac{d}{d\ln\mu}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega)\end{pmatrix} = \frac{\alpha_s}{\pi}\begin{pmatrix} 4C_F\ln\frac{\mu}{\mu_s} & -C_F\delta(\omega) \\ 0 & 4C_F\ln\frac{\mu}{\mu_s}\end{pmatrix}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x0}(\Omega)\end{pmatrix}$$

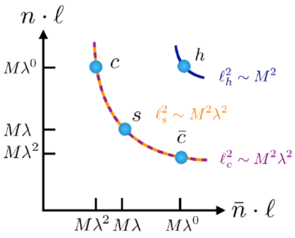


↪ leading log resummation
not understood beyond leading log

Subtleties

- Glauber modes violate LP factorization
  [Rothstein, Stewart, 2016]

- Dim r… izes modes with different virt… odes with equal virtuality bu… ngitudinal c… lmann, 2003]
  ↪ rapi… F
  [Chiu et al., 2…



[Figure T. Cohen, 1903.03622]

CANCELLED

- Violation of KSZ theorem … renormalization of sub-leading power light-ray operators due to … rny, Szafron, Wang, 2019]

$$\lim_{p^2\to 0}\left[\frac{\partial}{\partial p_\perp^\mu}\frac{1}{\epsilon^2}\,(p^2)^{-\epsilon} \ldots\right]$$

- Endpoint-divergent convolutions in $H \otimes J \otimes S$ when $\epsilon \to 0$ be…

$$\int_0^\Omega d\omega\;\underbrace{(n_+p\omega)^{-\epsilon}}_{\text{collinear function}}\;\underbrace{\frac{1}{\omega}^{1+\epsilon}\;\frac{1}{(\Omega-\omega)^\epsilon}}_{\text{soft function}}$$

Collinear and soft gauge symmetries of SCET

Separate soft gauge symmetry and collinear gauge symmetry for every collinear / jet direction
Transformation of the soft fields

$$\text{collinear:} \quad A_s \to A_s\,, \qquad \phi_s \to \phi_s\,,$$

$$\text{soft:} \quad A_s \to U_s A_s U_s^\dagger + \frac{i}{g} U_s \left[\partial\,, U_s^\dagger\right] \qquad \phi_s \to U_s \phi_s$$

Transformation of the collinear fields

$$\text{collinear:} \quad \hat{A}_c \to U_c \hat{A}_c U_c^\dagger + \frac{i}{g} U_c \left[D_s(x_-)\,, U_c^\dagger\right]\,, \qquad \hat{\phi}_c \to U_c \hat{\phi}_c$$

$$\text{soft:} \quad \hat{A}_c \to U_s(x_-) \hat{A}_c U_s^\dagger(x_-)\,, \qquad \hat{\phi}_c \to U_s(x_-) \hat{\phi}_c$$

with a λ-homogeneous soft-background field covariant derivative

$$D_{s\mu} = \partial_\mu - ign_- A_s(x_-)\,\frac{n_{+\mu}}{2}$$

- $n_- A_s(x_-)$ (and only this) appears as a background gauge field, which lives on the light-cone / classical trajectory of the collinear particle.

Background field interpretation

$$\mathcal{L}_{\text{SCET}} = \frac{1}{2}\left[n_+ D_c \hat{\phi}_c\right]^\dagger n_- D \hat{\phi}_c + \frac{1}{2}\left[n_- D \hat{\phi}\right]^\dagger n_+ D_c \hat{\phi}_c + \left[D_{c\mu_\perp} \hat{\phi}_c\right]^\dagger D_c^{\mu_\perp} \hat{\phi}_c$$

$$+ \frac{1}{2} n_-^\nu g x_\perp^\mu F^a_{s\mu\nu} n_+ j^a + \ldots \qquad \text{(scalar “matter”)}$$

$$j^a_\mu = i \hat{\chi}_c^\dagger t^a \overleftrightarrow{\mathcal{D}}_{c\mu} \hat{\chi}_c \,, \qquad \mathcal{D}_{c\mu} = D_{s\mu}(x_-) - ig \hat{\mathcal{A}}_{c\mu}$$

$n_- A_s(x_-)$ (and only this) appears as a background gauge field, which lives on the light-cone / classical trajectory of the collinear particle. Soft fields appear only in $n_- D(x_-)$ and the covariant field-strength tensor.

$$D_{s\mu} = \partial_\mu - ig n_- A_s(x_-) \, \frac{n_{+\mu}}{2}$$

Background field interpretation

$$\mathcal{L}_{\rm SCET} = \frac{1}{2}\left[n_+ D_c \hat{\phi}_c\right]^\dagger n_- D \hat{\phi}_c + \frac{1}{2}\left[n_- D \hat{\phi}\right]^\dagger n_+ D_c \hat{\phi}_c + \left[D_{c\mu_\perp}\hat{\phi}_c\right]^\dagger D_c^{\mu_\perp}\hat{\phi}_c$$

$$+ \frac{1}{2} n_-^\nu g x_\perp^\mu F^a_{s\mu\nu} n_+ j^a + \ldots \qquad \text{(scalar “matter”)}$$

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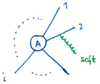
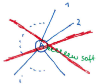
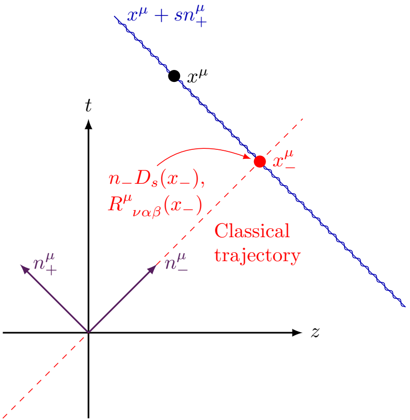
$$D_{s\mu} = \partial_\mu - ig n_- A_s(x_-)\,\frac{n_{+\mu}}{2}$$

$$\mathcal{A}_{\rm rad} = -g \sum_{i=1}^{n} t_i^a\, \overline{u}(p_i) \Bigg(\underbrace{\frac{\varepsilon^a_\mu(k) p_i^\mu}{p_i \cdot k}}_{\text{soft cov. derivative}} + \underbrace{\frac{k_\nu \varepsilon^a_\mu(k) J_i^{\mu\nu}}{p_i \cdot k}}_{\text{multipole exp}} \Bigg) \mathcal{A}$$





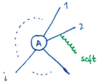
Space-time picture

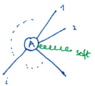


Why gravity?

- Internal vs. space-time symmetry but intriguing relations between scattering amplitudes.
  $\hookrightarrow$ "double copy", Gravity $\sim$ [Gauge]^2
- Collinear divergences very different at LP, soft similiar
- Soft theorem structure
  Colour-kinematics dual structure $g t_i^a \to -\frac{\kappa}{2} p_i^\nu$, but ...

$$\mathcal{A}_{\rm rad} = -g \sum_{i=1}^{n} t_i^a \, \bar{u}(p_i) \left(\frac{\varepsilon_\mu^a(k) p_i^\mu}{p_i \cdot k} + \frac{k_\nu \varepsilon_\mu^a(k) J_i^{\mu\nu}}{p_i \cdot k} \right) \mathcal{A}$$





$$\mathcal{A}_{\rm rad} = \frac{\kappa}{2} \sum_i \bar{u}(p_i) \left(\frac{\varepsilon_{\mu\nu}(k) p_i^\mu p_i^\nu}{p_i \cdot k} + \frac{k_\rho \varepsilon_{\mu\nu}(k) p_i^\nu J_i^{\mu\rho}}{p_i \cdot k} + \frac{1}{2} \frac{\varepsilon_{\mu\nu}(k) k_\rho k_\sigma J_i^{\rho\mu} J_i^{\sigma\nu}}{p_i \cdot k} \right) \mathcal{A}\,.$$

soft (eikonal): Bloch, Nordsieck 1930s / Weinberg 1965
(next-to-)next-to-soft: Low 1958; Burnett, Kroll, 1968 / Cachazo, Strominger 2014

Leading collinear & soft interactions

- No IR divergence in collinear graviton emission due to spin-2 [Weinberg 1965; Akhoury, Saotome, Sterman 2011; MB, Kirilin 2012]

$$\mathcal{L}_c^{(0)} = \bar{\xi}\left(in_-D_c + i\not{D}_{\perp c}\frac{1}{in_+D_c}\,i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi$$

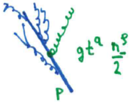
$$\mathcal{L}_c^{(0)} = \bar{\xi}\left(in_-\partial + i\not{\partial}_{\perp}\frac{1}{in_+\partial}\,i\not{\partial}_{\perp}\right)\frac{\not{n}_+}{2}\,\xi$$

Leading collinear & soft interactions

- No IR divergence in collinear graviton emission due to spin-2 [Weinberg 1965; Akhoury, Saotome, Sterman 2011; MB, Kirilin 2012]

$$\mathcal{L}_c^{(0)} = \bar{\xi}\left(in_-D_c + i\not{D}_{\perp c}\frac{1}{in_+D_c}\,i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi$$

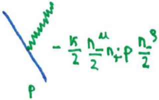
$$\mathcal{L}_c^{(0)} = \bar{\xi}\left(in_-\partial + i\not{\partial}_{\perp}\frac{1}{in_+\partial}\,i\not{\partial}_{\perp}\right)\frac{\not{n}_+}{2}\,\xi$$



Add soft:

$$\mathcal{L}^{(0)} = \bar{\xi}\left(in_-D_c + g n_-^\nu A_{s\nu}(x_-) + i\not{D}_{\perp c}\frac{1}{in_+D_c}\,i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi$$

$$\mathcal{L}^{(0)} = \bar{\xi}\left(in_-\partial - \frac{\kappa}{4} n_-^\mu n_-^\nu s_{\mu\nu}(x_-) n_+\partial + i\not{\partial}_{\perp}\frac{1}{in_+\partial}\,i\not{\partial}_{\perp}\right)\frac{\not{n}_+}{2}\,\xi$$



- Leading (eikonal) soft coupling very similar,
$gt^a \to -\frac{\kappa}{2}\frac{n_-^\mu}{2} n_+\partial \simeq -\frac{\kappa}{2}\,p_c^\mu$

Beyond leading power? Soft-collinear gravity

Collinear gravitons

$$h_{\mu\nu} \sim \frac{1}{\lambda} \times p_\mu p_\nu \qquad \text{[while } A_{c\mu} \sim p_\mu\text{]}$$

$h_{++} \sim 1/\lambda$ and $h_{\perp +} \sim 1$ must be controlled to all orders by an analogue of the Wilson line.

- Wilson lines are elements of the gauge group.
  Construct a local translation $W_c^{-1} = T_{\theta_c[h]}$ such that the gauge-transformed field $\mathfrak{h}_{\mu\nu}$ satisfies collinear light-cone gauge $\mathfrak{h}_{+\mu} = 0$

$$\eta_{\mu\nu} + \mathfrak{h}_{\mu\nu}(x) = W_c{}^\alpha{}_\mu W_c{}^\beta{}_\nu [W_c^{-1}(\eta_{\alpha\beta} + h_{\alpha\beta}(x))]$$

- $\mathfrak{h}_{\mu\nu}$ and $\chi_c = \left[W_c^{-1}\varphi\right]$ are the collinear gauge-invariant fields

- Note similarity at linear level

$$\mathfrak{h}_{\mu\nu} = h_{\mu\nu} - \frac{\partial_\mu}{n_+\partial}\left(h_{\nu+} - \frac{1}{2}\frac{\partial_\nu}{n_+\partial}h_{++}\right) - \frac{\partial_\nu}{n_+\partial}\left(h_{\mu+} - \frac{1}{2}\frac{\partial_\mu}{n_+\partial}h_{++}\right) + \mathcal{O}(\lambda h_{\mu\nu})$$

$$\mathcal{A}_{c\mu} = W_c^\dagger\,[iD_{c\mu}W_c] = A_{c\mu} - \frac{\partial_\mu}{n_+\partial}A_{c+} + \mathcal{O}(gA_{c\mu})$$

Soft metric background field and emergent gauge symmetry

- Define generalization of Riemann normal coordinates: fixed-line normal coordinates by

$$(x - x_-)^\alpha (x - x_-)^\beta \Gamma^\mu{}_{\alpha\beta}(x) = 0$$

Does not fix the gauge completely. Residual gauge invariance on the light cone including first derivative of $s_{\mu\nu}$.

- Construct the corresponding coordinate transformation (translation by $\theta^\mu_{\text{FLNC}}[s_{\rho\sigma(x)}]$) and then $R^{-1}_{\text{FLNC}}(x) = T_{\theta_{\text{FLNC}}(x)}$ and

$$\check{g}_{s\mu\nu}(x) \equiv R^\alpha{}_\mu(x) R^\beta{}_\nu(x) \left[R^{-1}_{\text{FLNC}}(x) g_{s\alpha\beta}(x)\right] \equiv \underbrace{\hat{g}_{s\mu\nu}(x)}_{\text{background metric}} + \mathfrak{g}_{s\mu\nu}(x)$$

- $\mathfrak{g}_{s\mu\nu}(x)$ is covariant and expressed in terms of the Riemann tensor at x_- after multipole expansion

$$\mathfrak{g}_{s\mu\nu}(x) = x_\perp^\alpha x_\perp^\beta \left(-\frac{n_{+\mu} n_{+\nu}}{4} R_{\alpha-\beta-}(x_-) + \ldots\right) + \mathcal{O}(\lambda^3)$$

Soft background and emergent gauge symmetry in gravity

Soft background vierbein ($\to$ metric)

$$\hat{e}_s{}^a{}_\mu(x) = \left(\delta^a_{\perp\mu} + \frac{n_{-\mu}}{2} n^a_+ + \frac{n_{+\mu}}{2}\left(e_s{}^a{}_-(x_-) + (x-x_-)^\beta [\Omega_-]^a{}_\beta(x_-)\right)\right)$$

- Two independent gauge symmetries and gauge fields, "vierbein" $e_-{}^\alpha(x_-)$ and "spin connection" $[\Omega_-]_{\alpha\beta}(x_-)$ on the light-cone, corresponding to local translations and local Lorentz transformations

$$e_-{}^\alpha = \delta^\alpha_- + \frac{1}{2} s^\alpha_- - \frac{1}{8} s_{-\beta} s^{\beta\alpha} + \mathcal{O}(s^3)$$

$$[\Omega_-]_{\alpha\beta} = -\frac{1}{2}\left([\partial_\alpha s_{\beta -}] - [\partial_\beta s_{\alpha -}]\right) + \mathcal{O}(s^2)$$

- Covariant derivative

$$n_- D_s \equiv \hat{E}^\mu_{s-}\partial_\mu = \partial_- - \frac{\kappa}{2} s_{-\mu}\partial^\mu + \frac{\kappa}{2}[\Omega_-]_{\mu\nu} J^{\mu\nu} + \mathcal{O}(s^2)$$

where $J^{\mu\nu} = (x-x_-)^\mu \partial^\nu - (x-x_-)^\nu \partial^\mu \equiv (x-x_-)^{[\mu}\partial^{\nu]}$ denotes the angular momentum (Lorentz generator) operator.

- Same structure as in gauge theory [review: 2210.09336]

Soft-collinear gravity and soft theorem

$$\mathcal{L}_{\rm SCETgr} = \frac{1}{2}\partial_+\hat\varphi_c D_-\hat\varphi_c + \frac{1}{2}\partial_{\alpha\perp}\hat\varphi_c\partial^{\alpha\perp}\hat\varphi_c \underbrace{-\frac{1}{2}\hat h^{\mu\nu}\partial_\mu\hat\varphi_c\partial_\nu\hat\varphi_c}_{\mathcal{L}^{(1)}}$$

$$\underbrace{-\frac{1}{8}x_\perp^\alpha x_\perp^\beta R^s_{\alpha-\beta-}(\partial_+ W_c^{-1}\hat\varphi_c)^2 + \ldots}_{\mathcal{L}^{(2)}} + \ldots$$

Soft-collinear gravity and soft theorem

$$\mathcal{L}_{\text{SCETgr}} = \frac{1}{2}\partial_+\hat{\varphi}_c D_-\hat{\varphi}_c + \frac{1}{2}\partial_{\alpha\perp}\hat{\varphi}_c\partial^{\alpha\perp}\hat{\varphi}_c \underbrace{-\frac{1}{2}\hat{h}^{\mu\nu}\partial_\mu\hat{\varphi}_c\partial_\nu\hat{\varphi}_c}_{\mathcal{L}^{(1)}}$$

$$\underbrace{-\frac{1}{8}x_\perp^\alpha x_\perp^\beta R^s_{\alpha-\beta-}(\partial_+ W_c^{-1}\hat{\varphi}_c)^2 + \ldots}_{\mathcal{L}^{(2)}} + \ldots$$

$$\mathcal{A}_{\text{rad}} = \frac{\kappa}{2}\sum_i \bar{u}(p_i)\left(\underbrace{\left[\varepsilon_{\mu\nu}(k)p_i^\nu + k_\rho\varepsilon_{\mu\nu}(k)J_i^{\nu\rho}\right]\frac{p_i^\mu}{p_i\cdot k}}_{\text{soft cov. derivative}} + \underbrace{\frac{1}{2}\frac{\varepsilon_{\mu\nu}(k)k_\rho k_\sigma J_i^{\rho\mu}J_i^{\sigma\nu}}{p_i\cdot k}}_{\text{multipole exp}}\right)\mathcal{A}$$

- Universal terms = Lagrangian insertions on external legs ONLY
  No emission from hard vertex up to (including) $\mathcal{O}(\lambda^2)$ = next-to-soft in gauge theory and $\mathcal{O}(\lambda^4)$ = next-to-next-to-soft in gravity

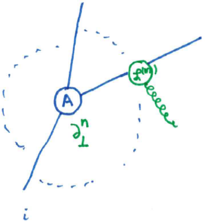
- Consequence of soft gauge invariance, no calculation needed

Soft theorem from the SCET Lagrangian

Goal — Operatorial form of the soft theorem by manipulating the Lagrangian, assuming that it is evaluated at tree level in a matrix element with one soft graviton and one energetic particle per direction.

$$\mathcal{A}_{\text{rad}} \sim \langle p_i, k| \sum_{n,m=0} C^{(An)}(p_i p_j) \int d^4x\, T(\partial_\perp^n \chi_{c_i}^\dagger, \mathcal{L}_\chi^{(m)}(x))|0\rangle$$

with $n + m \leq 4$ for next-to-next-to-soft (gravity).
Sum over all directions $i = 1 \ldots N$.



- Wlog choose a frame where external p_i is aligned with reference vector n_{-i}, i.e. $p_{\perp i} = 0$.
- Lorentz-invariance / reparameterization invariance relates $C^{(An)}$ to $C^{(A0)}$ for all n

$\hookrightarrow$ $\mathcal{A}_{\text{non}-\text{rad}}$ factors

Organizing the Lagrangian

- Many terms already have a suggestive form due to the multipole expansion

$$J^{\mu\nu} = x^\mu \partial_\nu - x^\nu \partial^\mu + \Sigma^{\mu\nu} = L^{\mu\nu} + \Sigma^{\mu\nu}$$

- Matter field is scalar

$$\mathcal{L}^{(0)}_{\rm eik} = \frac{\kappa}{4} \chi_c^\dagger \, s_{--} n_+\partial \, n_+\partial \chi_c$$

$$\mathcal{L}^{(1)}_{\rm eik} = \frac{\kappa}{2} [\partial_\mu s_{\nu -}] \chi_c^\dagger \overleftarrow{L}^{\mu\nu}_{+\perp} n_+\partial \chi_c$$

$$\mathcal{L}^{(2)}_{\rm eik} = \frac{\kappa}{2} [\partial_\mu s_{\nu -}] \chi_c^\dagger \overleftarrow{L}^{\mu\nu}_{+-} n_+\partial \chi_c$$

$$\mathcal{L}^{(2)}_{\rm R} = \frac{1}{4} R^s_{\mu\alpha\nu\beta} \, \chi_c^\dagger \overleftarrow{L}^{\mu\alpha}_{+\perp} \overrightarrow{L}^{\nu\beta}_{+\perp} \chi_c$$

$$\mathcal{L}^{(3)}_{\rm R} = \frac{1}{4} R^s_{\mu\alpha\nu\beta} \, \chi_c^\dagger (\overleftarrow{L}^{\mu\alpha}_{+-} \overrightarrow{L}^{\nu\beta}_{+\perp} + \overleftarrow{L}^{\mu\alpha}_{+\perp} \overrightarrow{L}^{\nu\beta}_{+-}) \chi_c$$

$$\mathcal{L}^{(4)}_{\rm R} = \frac{1}{4} R^s_{\mu\alpha\nu\beta} \, \chi_c^\dagger \overleftarrow{L}^{\mu\alpha}_{+-} \overrightarrow{L}^{\nu\beta}_{+-} \chi_c$$

For gauge theory, also considered spin-1/2 and spin-1 matter, and recover the spin term analogously [SCET gravity for fermions, 2212.02525]

Lagrangian derivation of the soft theorem provides explanation of angular momentum in terms of gauge symmetry and multipole expansion.

Back-up

All-order construction of SCET [MB, Feldmann, 2002]
Step 1: Field split and soft background gauge symmetry

$$A_\mu = A_{c\mu} + A_{s\mu}\,,$$

$$\phi = \phi_c + WZ^\dagger \phi_s\,,$$

collinear:

$$A_c \to U_c A_c U_c^\dagger + \frac{i}{g} U_c \left[D_s\,, U_c^\dagger \right]\,, \qquad \phi_c \to U_c \phi_c\,,$$

$$A_s \to A_s\,, \qquad \phi_s \to \phi_s\,,$$

soft:

$$A_c \to U_s A_c U_s^\dagger\,, \qquad \phi_c \to U_s \phi_c\,,$$

$$A_s \to U_s A_s U_s^\dagger + \frac{i}{g} U_s \left[\partial\,, U_s^\dagger \right] \qquad \phi_s \to U_s \phi_s$$

- $A_\mu = A_{c\mu} + A_{s\mu}$ and ϕ transform in the usual way under both gauge symmetries.
- Background field A_s for collinear trafo
- Collinear trafo of A_c mixes different orders in λ, because

$$\int d^4x\, \phi_c(x)\phi_s(x) \qquad \text{is \underline{not} homogeneous.}$$

Step 2: Multipole expansion

Recall interactions of light with atoms

$$\langle n|\vec{A}_k(\hat{X})\cdot\hat{P}|m\rangle \sim \langle n|e^{i\vec{k}\cdot\hat{X}}\hat{P}|m\rangle \approx \langle n|(1+i\vec{k}\cdot\hat{X}+\ldots)\hat{P}|m\rangle$$

On fields

$$\int d^4x\,\psi^\dagger(x)i\vec{\partial}\psi(x)\vec{A}(x) \approx \int d^4x\,\psi^\dagger(x)i\vec{\partial}\psi(x)\left[\vec{A}(t,0)+[(\vec{x}\vec{\partial})\vec{A}](t,0)+\ldots\right]$$

Subleading term gives $\psi^\dagger\vec{x}\cdot\vec{E}\psi$.

For any soft field in a product with collinear expand in $n_-x, x_\perp$,

$$\psi_s(x) = \psi_s(x_-) + \left[x_\perp^\alpha\partial\psi_s\right](x_-) + \frac{1}{2}\left[n_-x n_+\partial\psi_s\right](x_-) + \frac{1}{2}\left[x_\perp^\alpha x_\perp^\beta\partial_\alpha\partial_\beta\psi_s\right](x_-) + \mathcal{O}(\lambda^3\psi_s)\,,$$

i.e. around the light-cone $x_-^\mu = \frac{1}{2}n_+\cdot x\, n_-^\mu$ of the collinear field .

- Soft fields must always be evaluated at x_- in products with collinear.
- Must adapt the gauge symmetry to this expansion.

Step 3: Homogeneous gauge symmetry and collinear field redefinition

Redefine collinear fields such that

$$\text{collinear:} \quad \hat{A}_c \to U_c \hat{A}_c U_c^\dagger + \frac{i}{g} U_c \left[D_s(x_-)\,, U_c^\dagger \right]\,, \qquad \hat{\phi}_c \to U_c \hat{\phi}_c$$

$$\text{soft:} \quad A_c \to U_s(x_-) \hat{A}_c U_s^\dagger(x_-)\,, \qquad \hat{\phi}_c \to U_s(x_-) \hat{\phi}_c$$

with homogeneous soft-background field covariant derivative

$$D_{s\mu} = \partial_\mu - ign_- A_s(x_-)\, \frac{n_{+\mu}}{2}$$

Pull back gauge trafo from x to x_- by redefining collinear fields with

$$R(x) = P \exp\left(ig \int_C dy_\mu A_s^\mu(y) \right)$$

(C a straight path from x_- to x.)

Then

$$\phi_c = R \underbrace{W_c^\dagger \hat{\phi}_c}_{\hat{\chi}_c}, \qquad A_c = R\left(W_c^\dagger \hat{A}_c W_c + \frac{i}{g} W_c^\dagger \left[D_s(x_-)\,, W_c \right] \right) R^\dagger$$

- Background gauge field is now only $n_- A_s(x_-)$ and lives on the light cone. This is the emergent soft gauge symmetry of the effective Lagrangian.

Step 4: Collinear gauge-invariant and soft gauge-covariant fields

Collinear fields

$$\hat{\mathcal{A}}_{c\perp\mu} = W_c^\dagger \left[i\hat{D}_{c\perp\mu} W_c \right]$$

$$\hat{\chi}_c = W_c^\dagger \hat{\phi}_c$$

$$W_c(x) = P \exp\left(ig \int_{-\infty}^{0} ds\, n_+ \hat{A}_c(x + s n_+) \right)$$

Collinear gauge-invariant
Soft covariant
Covariant version of collinear light-cone gauge, $n_+ \hat{\mathcal{A}}_c = 0$. Controls the $\mathcal{O}(1)$ field.

Soft fields

$n_- A_s(x_-)$ only in the background covariant derivative $in_- D_s(x_-)$

$$\mathcal{A}_s(x) \equiv R^\dagger A_s(x) R + \frac{i}{g} R^\dagger \left[D_s\,, R \right] ,$$

satisfying fixed-line gauge

$$(x - x_-) \cdot \mathcal{A}_s(x) = 0$$

Leaves $n_- A_s(x_-)$ uncontrained.

Gauge fields in fixed-line gauge can be expressed in terms of the field-strength and multipole-expanded, e.g.

$$\begin{aligned} n_- \mathcal{A}_s(x) &= \int_0^1 ds\, (x - x_-)^\mu n_-^\nu R^\dagger(y(s)) F_{s\mu\nu}(y(s)) R(y(s)) \\ &= x_\perp^\mu n_-^\nu F_{s\mu\nu}(x_-) + \frac{1}{2} n_- x n_+^\mu n_-^\nu F_{s\mu\nu}(x_-) + \frac{1}{2} x_\perp^\mu x_{\perp\rho} n_-^\nu \left[D_s^\rho\,, F_{s\mu\nu} \right](x_-) + \mathcal{O}(\lambda^5) \end{aligned}$$

Minimally coupled scalar + Einstein Hilbert theory

$$S = -\frac{2}{\kappa^2}\int d^4x\,\sqrt{-g}R + \int d^4x\sqrt{-g}\,\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$$

Diffeomorphisms as local translations $x^\mu \to y^\mu(x) = x^\mu + \varepsilon^\mu(x)$.
Implement as active transformation

$$\varphi(x) \to \varphi'(x) = U(x)\varphi(x) = T_\varepsilon^{-1}\varphi(x)$$

$$T_\varepsilon f(x) = f(x) + \varepsilon^\alpha(x)\partial_\alpha f(x) + \frac{1}{2}\varepsilon^\alpha(x)\varepsilon^\beta(x)\partial_\alpha\partial_\beta f(x) + \mathcal{O}(\varepsilon^3)\,.$$

Minimally coupled scalar + Einstein Hilbert theory

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Weak-field expansion around Minkowski space
Similar to (abelian) gauge symmetry at linear level, but inherently non-linear.

$$g_{\mu\nu}(x) \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}(x), \qquad h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu + \mathcal{O}(\varepsilon^2)$$

$$S = \sum_{k=0}^{\infty} \kappa^k S^{(k)}[h_{\mu\nu}]$$

Loop corrections to the gravitational soft theorem [2210.09336]

- Follow from λ-power counting of SCET gravity and flat-space Lorentz invariance [$\rightarrow$ scaleless integrals]

The leading soft factor is never modified by loops, the subleading factor is only corrected by one-loop, and the sub-subleading factor is only modified by one- and two-loop contributions. Higher loop-corrections cannot affect the terms of the gravitational soft theorem.

[Diagrammatically: Bern, Davies, Nohle, 1405.1015]

i) In the purely-collinear sector, that is in the Lagrangian terms containing only collinear but no soft fields, there are no leading power interactions. The λ expansion corresponds to the weak-field expansion, and the first collinear interaction appears in $\mathcal{O}(\lambda)$.

A collinear loop can only be connected by purely-collinear vertices, which are power-suppressed in gravity. Thus, adding a collinear loop always brings suppression of at least $\mathcal{O}(\lambda^2)$.

ii) In the purely-soft sector, that is in the Lagrangian terms containing only soft but no collinear fields, there are also no leading power interactions. Here, the weak-field expansion agrees with the λ^2 expansion, corresponding to an expansion in soft momenta $k \sim \lambda^2$. Purely-soft interaction vertices thus start at $\mathcal{O}(\lambda^2)$.

A soft loop is scaleless, unless it is directly connected to the external soft graviton by a *purely-soft interaction vertex*, due to the multipole expansion in soft-collinear vertices. Since purely-soft interactions are power-suppressed in gravity by a factor of λ^2, adding a soft loop yields a suppression of at least $\mathcal{O}(\lambda^2)$.