Neutrinos in cosmology

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June 25 – 28, 2024

The grand lecture plan…

Part 1: Neutrinos in homogeneous cosmology

1. The homogeneous and isotropic universe
2. The hot universe and the cosmic neutrino background
3. Precision CνB

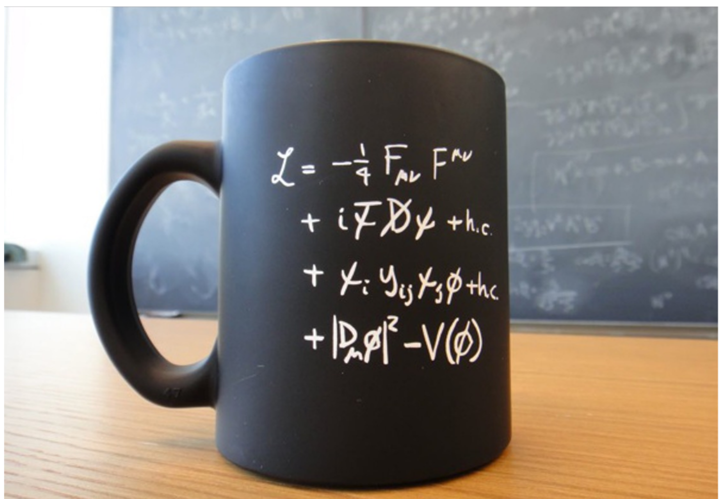
Part 2: Neutrinos in inhomogeneous cosmology

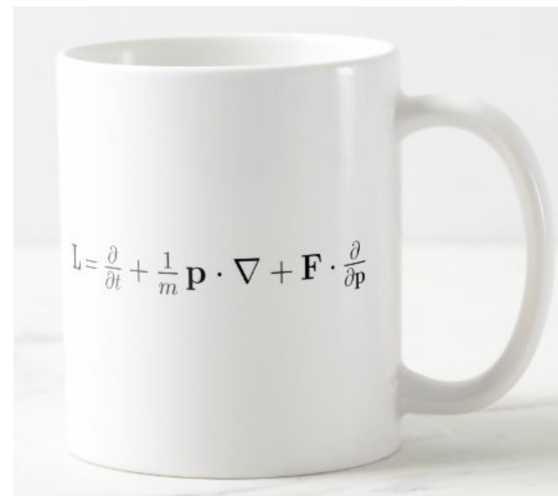
1. Theory of inhomogeneities
2. Neutrinos and structure formation
3. Relativistic neutrino free-streaming and non-standard interactions

Part 1: Neutrinos in homogeneous cosmology

1. The homogeneous and isotropic universe
2. The hot universe and the cosmic neutrino background
3. Precision CνB

3. Precision CνB...

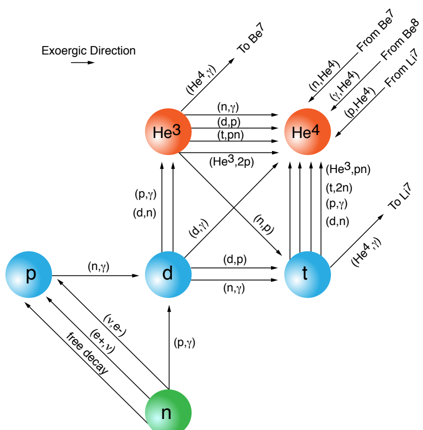




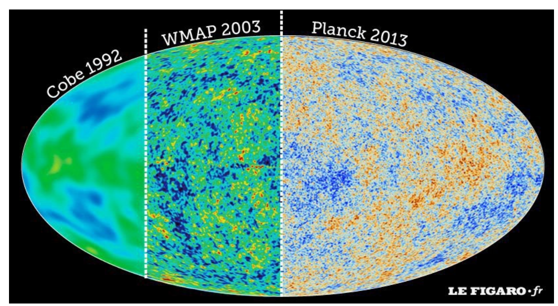
Observable CνB...

We **cannot detect the CνB** in the lab. But we can discern its presence from its impact on the events that take place after its formation.

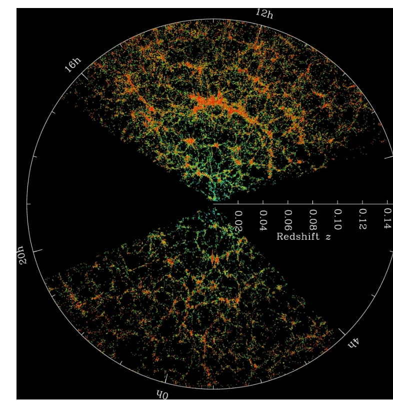
Light element abundances



CMB anisotropies



Large-scale matter distribution



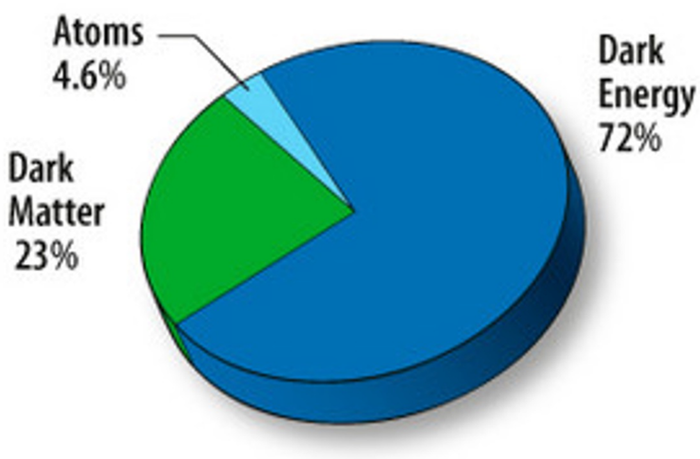
Properties of the CνB probed:

- Light element abundances: **N_{eff} (expansion rate)** ← Part 1
- CMB anisotropies: **N_{eff} (expansion rate)** ← Part 1
  - $\sum m_\nu$ (perturbation growth)
  - Interactions (free-streaming)
  - Lifetime (free-streaming)
  - } Part 2
- Large-scale matter distribution: $\sum m_\nu$ (perturbation growth)

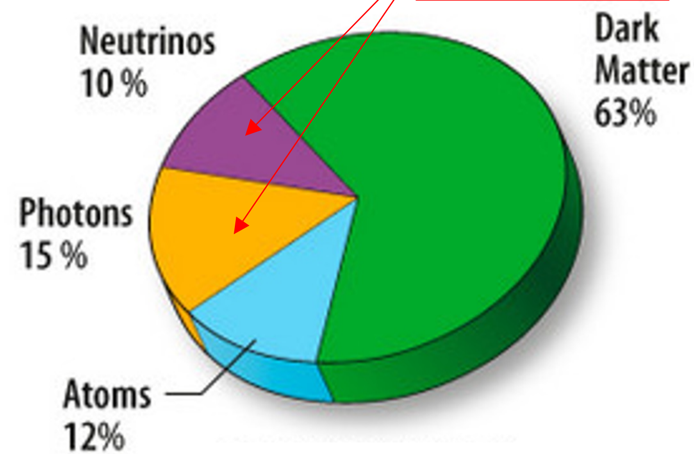
The concordance flat ΛCDM model...

The simplest model consistent with **present observations.**

$$\frac{\rho_\nu}{\rho_\gamma} \approx 0.67$$



Composition today



13.4 billion years ago
(at photon decoupling)

Plus flat spatial geometry+initial conditions
from single-field inflation

Neutrino-to-photon energy density ratio…

Recall the standard hot big bang prediction for the **CνB temperature**:

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

→ At early times when $T_\nu \gg m_\nu$ (i.e., relativistic neutrinos), the energy density in one family of $\nu + \bar{\nu}$ is:

$\rho_{\text{relativistic}} \sim T^4$

$$\rho_{\nu_\alpha} \simeq \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}} \rho_\gamma \simeq 0.227\rho_\gamma$$

Fermion vs boson

- **Summing** over all three families: $\sum_{\nu_e,\nu_\mu,\nu_\tau,} \rho_{\nu_\alpha} \approx 3 \times 0.227\, \rho_\gamma \approx 0.68\, \rho_\gamma$

Effective number of neutrinos…

A common practice is to express the neutrino-to-photon energy density ratio in terms of the **effective number of neutrino $N_{\mathbf{eff}}$** parameter.

$$\sum_{\nu_e,\nu_\mu,\nu_\tau,} \rho_{\nu_\alpha} = N_{\mathrm{eff}} \times \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

Energy density in one thermalised species of massless fermions with 2 internal d.o.f. and temperature $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$.

The SM value is $N_{\mathrm{eff}}^{\mathrm{SM}} = 3.0440 \pm 0.0002$, for

- 3 families of neutrinos + antineutrinos
- A variety of %-level SM effects that alter **both** ρ_{ν_α} and ρ_γ from their naïve expectations.

More on this later on.

Extending N_{eff} to light BSM thermal relics…

Any light (~sub-eV mass), feebly-interacting particle species produced by scattering in the early universe will **look sort of like a neutrino** as far as cosmology is concerned.

- E.g., light sterile neutrinos, thermal axions, …

- At leading order, these **light thermal relics** add to the SM neutrino energy density as if $N_{\text{eff}} \gtrsim 3$.

→ Re-interpret N_{eff} as the early-time non-photon radiation content:

$$\sum_{\nu_e,\nu_\mu,\nu_\tau,} \rho_{\nu_\alpha} + \rho_{\text{other}} = N_{\text{eff}} \times \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

$$N_{\text{eff}} = N_{\text{eff}}^{\text{SM}} + \Delta N_{\text{eff}}$$

N_{eff} and the expansion rate…

The primary impact of N_{eff} is on the expansion rate during & shortly after radiation domination.

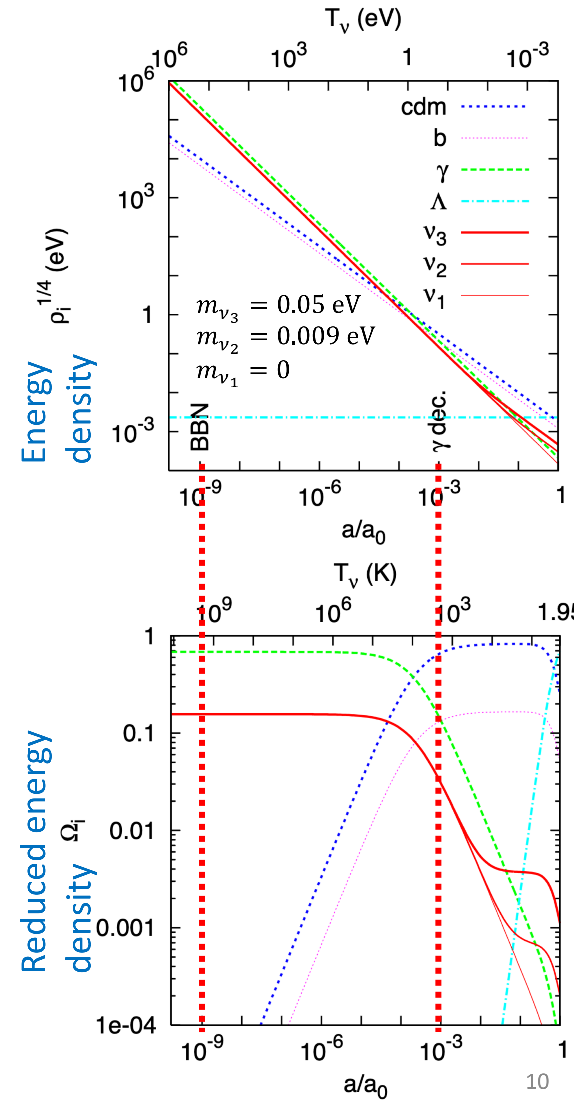
Hubble expansion rate (Friedmann equation)

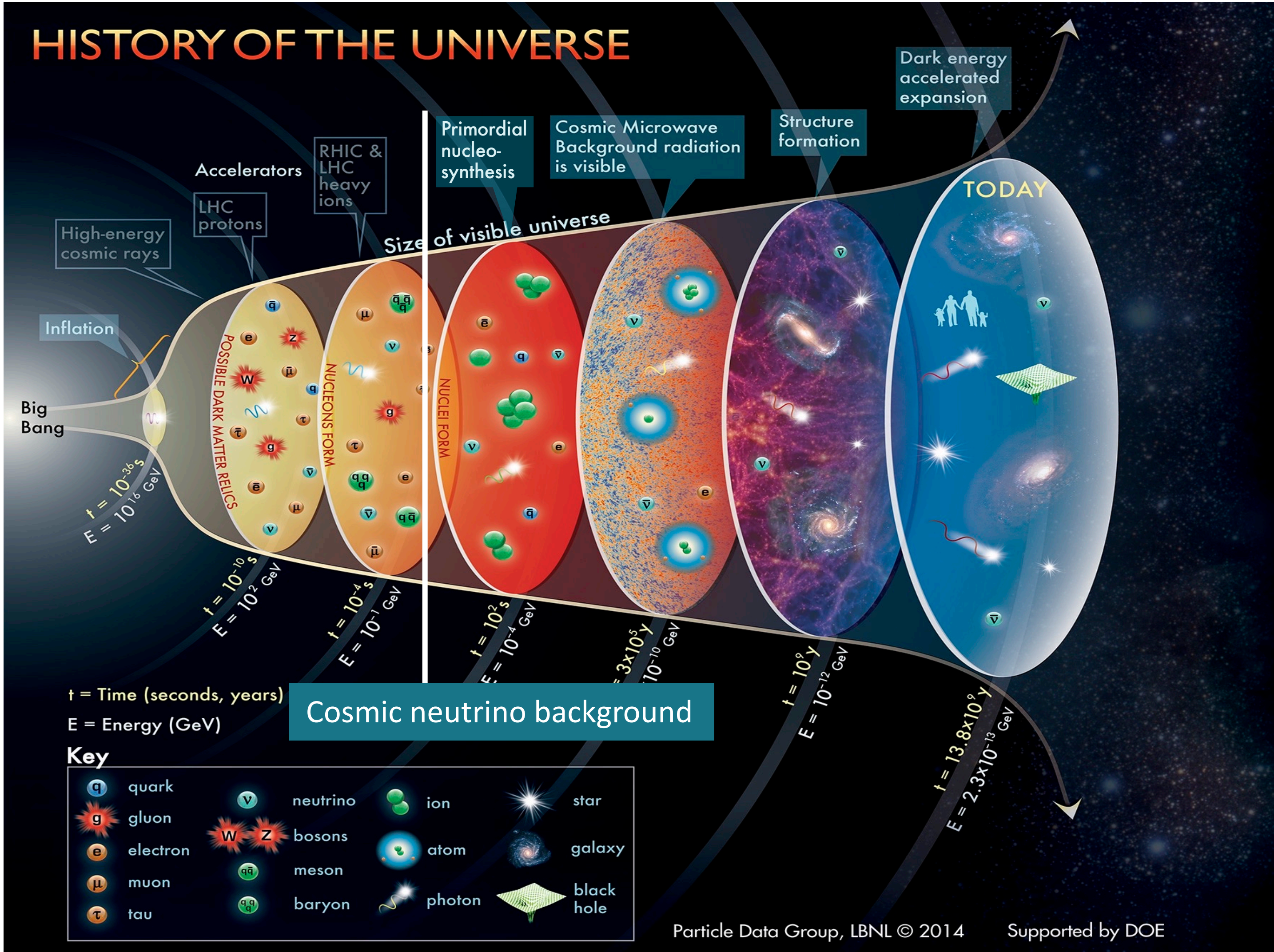
$$H^2(t) \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_\alpha \rho_\alpha - \frac{K}{a^2}$$

At $a \lesssim O(10^{-3})$ $$\approx \frac{8\pi G}{3} \sum_{\mathrm{relativistic}} \rho_\alpha$$

N_{eff} goes in here

- Light element abundances (particularly Helium-4) from **primordial nucleosynthesis** and the **CMB anisotropies** are sensitive to this parameter.





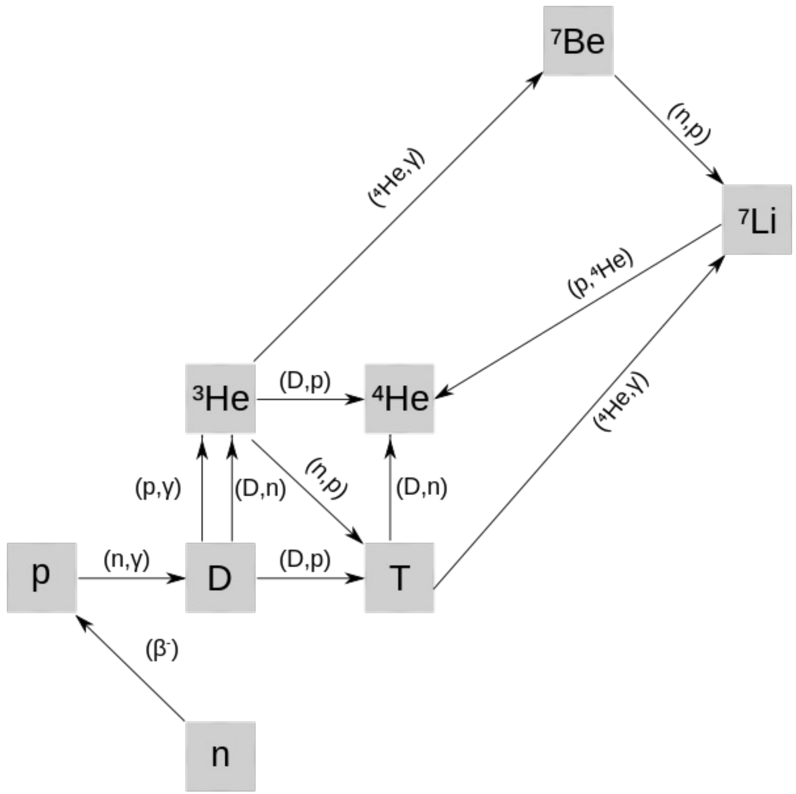
Cosmic neutrino background

N_{eff} and nucleosynthesis...

Primordial nucleosynthesis takes place at $T{\sim}O(100) - O(10)$ keV, shortly after neutrino decoupling.

- Changing the expansion rate affects the production of **all** light elements.
- The largest effect is on He4, because
  - Almost all neutrons end up in He4.
  - The **neutron-to-proton ratio** depends strongly on how expansion affects the β-processes:

$$\nu_e + n \leftrightarrow p + e^-$$
$$\bar{\nu}_e + p \leftrightarrow n + e^+$$



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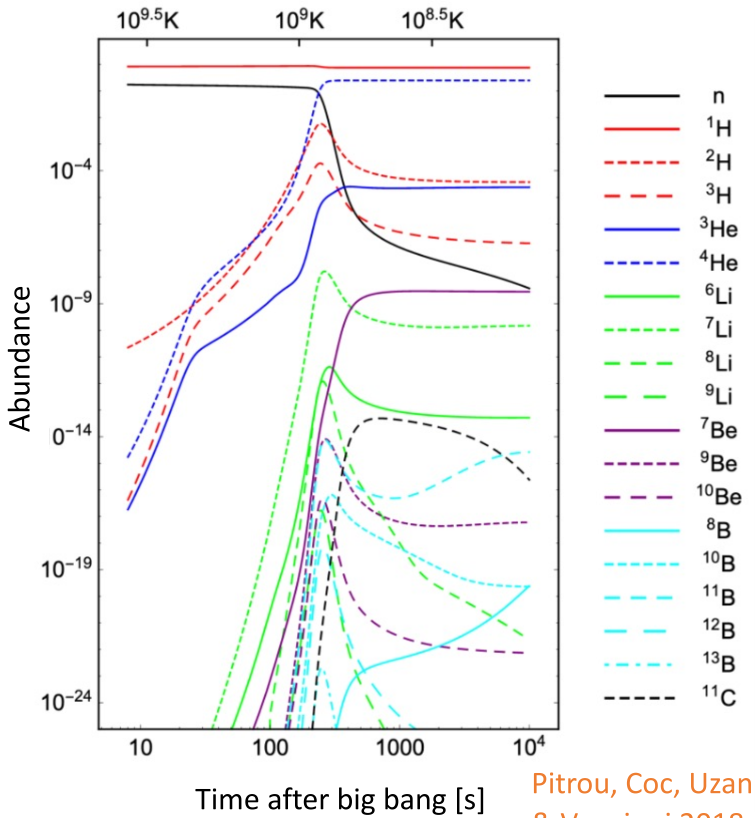

Pitrou, Coc, Uzan & Vangioni 2018

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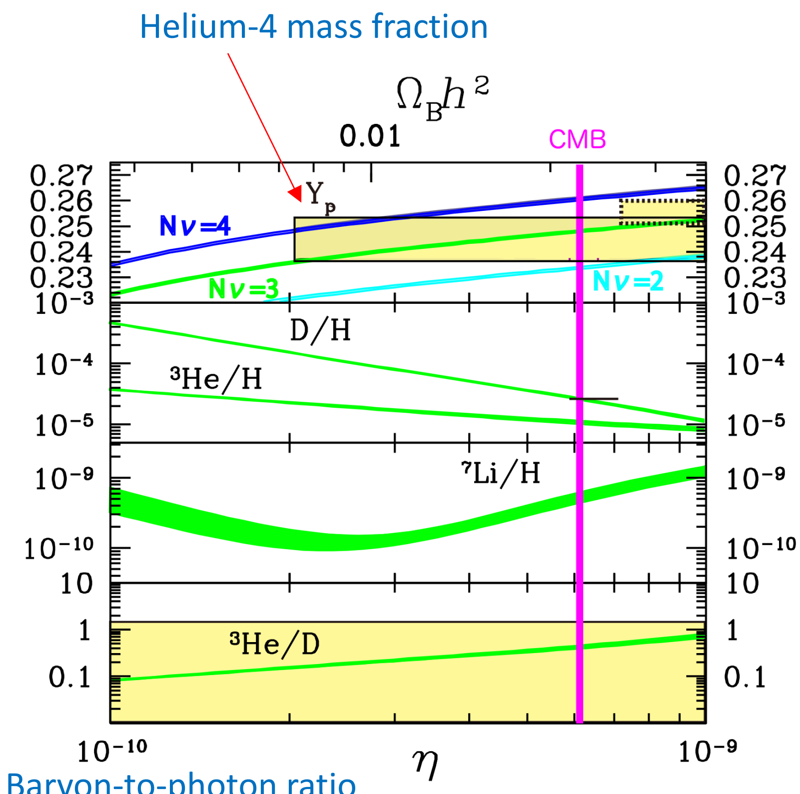

Kawasaki, Kohri, Moroi & Takaesu 2018

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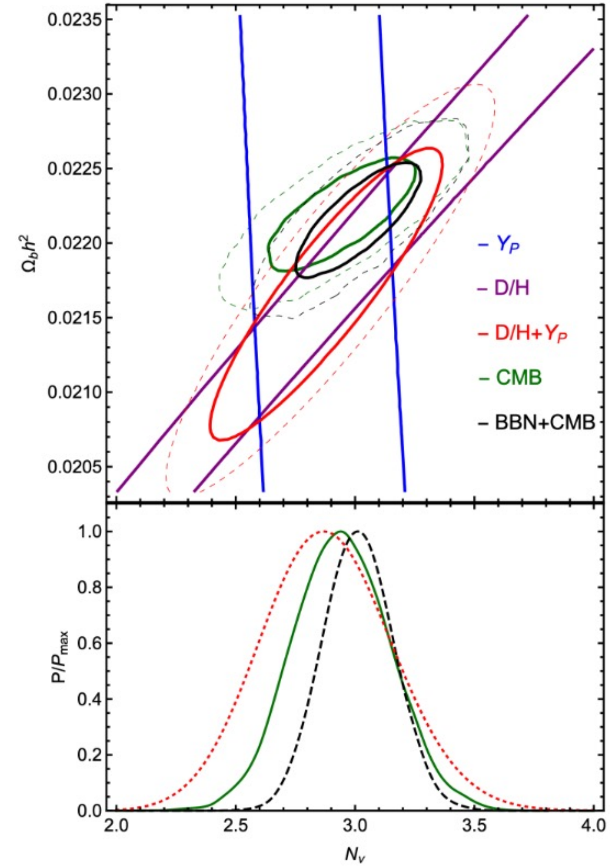


$$N_{\text{eff}} = 2.88 \pm 0.27 \text{ (68\% CL)}$$

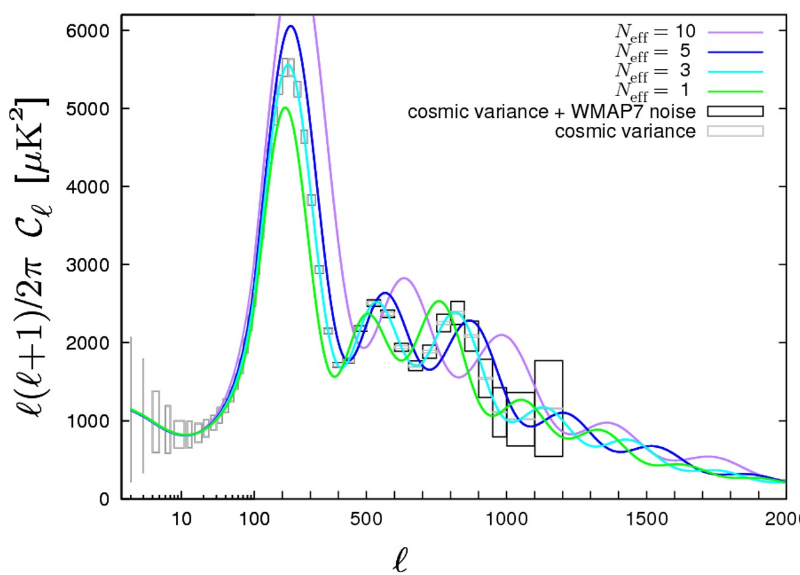
Pitrou, Coc, Uzan & Vangioni 2018

N_{eff} and the CMB anisotropies…

N_{eff} also affects the **expansion rate at recombination** ($T\sim 0.2\ eV$), observable in the CMB temperature power spectrum

- If you plug different values of N_{eff} into CAMB or CLASS, this is what you'll get.
- But this is **not** the "real" effect of N_{eff}, because degeneracy with, e.g., the matter density ω_{m}, the Hubble parameter h, etc., can largely offset it.

"Naïve" signature



N_{eff} and the CMB anisotropies…

N_{eff} also affects the **expansion rate at recombination** ($T\sim 0.2$ eV), observable in the CMB temperature power spectrum

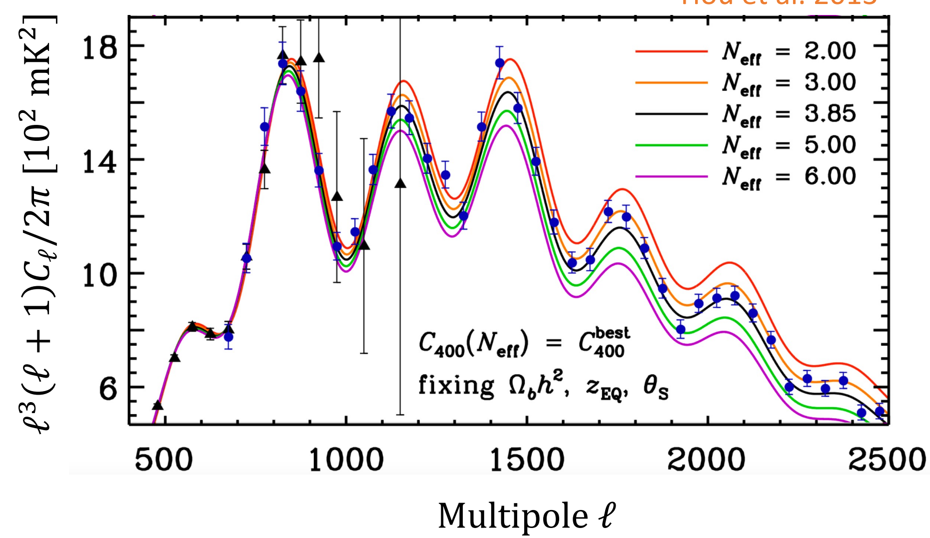
- Adjusting ω_{m} and h to match the first peak height and location, the irreducible signature of N_{eff} is in the damping tail.

Diffusion damping scale

$$r_d^2 \simeq (2\pi)^2 \int_0^{a_*} \frac{da}{a^3 \sigma_T n_e H} \left[\frac{R^2 + (16/15)(1+R)}{6(1+R)^2} \right]$$

Thomson cross section; Free electron density; Hubble expansion; Baryon-to-photon density ratio

Hou et al. 2013



$$N_{\text{eff}} = 2.99 \pm 0.34\ (95\%\ \text{CL})$$

Planck TTTEEE +lowE+lensing+BAO; 7-parameters

Aghanim et al. [Planck] 2021

Precision $N_{\text{eff}}^{\text{SM}}$ …

The **SM value of N_{eff}** can be calculated very precisely:

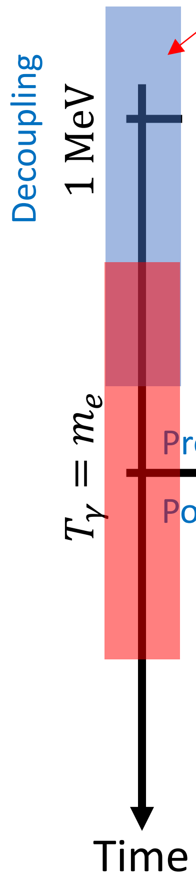
$$\sum_{\nu_e,\nu_\mu,\nu_\tau,} \rho_{\nu_\alpha} = N_{\text{eff}}^{\text{SM}} \times \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

with $N_{\text{eff}}^{\text{SM}} = 3.0440 \pm 0.0002$, where the %-level corrections come from Bennett et al. 2019, 2020; Akita and Yamaguchi 2020; Froustey, Pitrou & Volpe 2020; Drewes et al. 2024

- Non-instantaneous neutrino decoupling
- Neutrino flavour oscillations
- Non-relativistic electron gas across neutrino decoupling
- Finite-temperature QED effects in the photon/electron plasma

(first three items: ρ_ν; last two items: ρ_γ)

Deviations, or what's wrong with this picture?



Neither neutrino decoupling nor annihilation is instantaneous.

QED plasma

$e^\pm, \gamma$

At $T \sim 1$ MeV, $e^\pm$ are not ultra-relativistic, nor is the QED plasma an ideal gas to justify this expression.

$$S_{\rm QED} = \frac{11}{45}\pi^2 T_\gamma^3 a^3$$

3 families of neutrinos + antineutrinos

$\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

$$S_\nu = \frac{7}{30}\pi^2 T_\nu^3 a^3$$

Pre-e^+e^--annihilation

Post-e^+e^--annihilation

γ

$$S_{\rm QED} = \frac{4}{45}\pi^2 T_\gamma^3 a^3$$

Conservation: $S_{\rm QED}^{\rm post} = S_{\rm QED}^{\rm pre}$

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

The neutrino temperature+ FD distribution does not completely capture the ν-to-γ energy density ratio.

Tracking non-instantaneous decoupling…

The effect of an out-of-equilibrium interaction on a particle species can be tracked using the Boltzmann equation.

f_1 = Phase space density of the particle species of interest

$$\frac{\partial f_1}{\partial t} = -\{f_1, H\} + \frac{1}{E_1} C[f_1]$$

Collision term

Hamiltonian for particle propagation

- **The collision term** for e.g., $1 + 2 \rightarrow 3 + 4$

9D phase space integral

Energy-momentum conservation

Matrix element

$$C[f_1] = \frac{1}{2} \int \prod_{i=2}^{4} \frac{d^3 p_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P_4) |M|^2$$
$$\times [f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4)]$$

Quantum statistical factors

Tracking decoupling including oscillations...

Tracking neutrino decoupling is complicated by **neutrino oscillations**.

- We promote the classical Boltzmann equation for the phase space density to a quantum kinetic equation (QKE) for the density matrix of the neutrino ensemble.

Boltzmann

$$\frac{\partial f_1}{\partial t} = -\{f_1, H\} + \frac{1}{E_1} C[f_1]$$

Quantum kinetic equation

$$\frac{\partial \hat{\rho}_1}{\partial t} = -\frac{1}{i\hbar}[\hat{\rho}_1, \hat{H}] + \frac{\hat{C}[\hat{\rho}]}{E_1}$$

Collision term

e.g., Sigl & Raffelt 1993

Density matrix (momentum-dependent)

$$\hat{\rho} = \begin{pmatrix} \rho_{ee} & \rho_{e\mu} & \rho_{e\tau} \\ \rho_{\mu e} & \rho_{\mu\mu} & \rho_{\mu\tau} \\ \rho_{\tau e} & \rho_{\tau\mu} & \rho_{\tau\tau} \end{pmatrix}$$

Diagonal ~ occupation numbers
Off-diagonal ~ oscillation phases

Hamiltonian

$$\hat{H} = \frac{1}{2p} U \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} U^\dagger + \hat{V}_{\mathrm{matter}}$$

Vacuum oscillations + matter effects

Interactions at $0.1 < T < 10$ MeV…

The particle content and interactions at $0.1 < T < 10$ MeV determine the properties of the CνB.

- **QED plasma**: $e^{\pm}, \gamma$
- **3 families of $\boldsymbol{\nu + \bar{\nu}}$**: $\nu_e, \bar{\nu}_e,$ $\nu_\mu, \bar{\nu}_\mu,$ $\nu_\tau, \bar{\nu}_\tau$

EM interactions (always in equilibrium @ $0.1 < T < 10$ MeV):

$$e^+e^- \leftrightarrow \gamma\gamma$$
$$e^+e^- \leftrightarrow e^+e^-$$
$$e^{\pm}e^{\mp} \leftrightarrow e^{\pm}e^{\mp}$$
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Coupled @ $T > O(1)$MeV

$$\nu_\alpha e^{\pm} \leftrightarrow \nu_\alpha e^{\pm}$$
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$$\nu_\alpha \nu_\beta \leftrightarrow \nu_\alpha \nu_\beta$$
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$$\bar{\nu}_\alpha \bar{\nu}_\beta \leftrightarrow \bar{\nu}_\alpha \bar{\nu}_\beta$$

These processes go into the collision integral and matter effects.

Collision integrals @ next-to-leading order…

Weak annihilation and scattering rates are currently computed to $O(G_F^2)$.

- Recent interest in computing QED corrections (quantum + finite-temperature) to these rates

  Significant effect claimed in

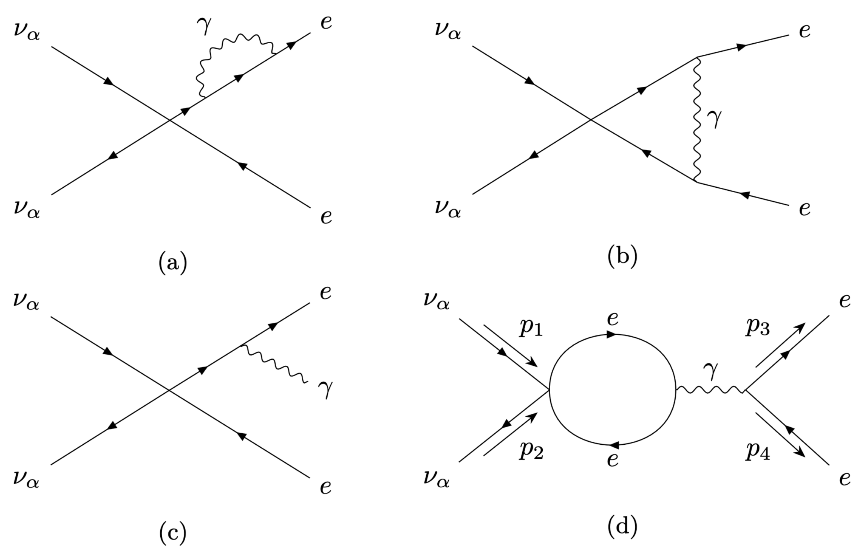
  Cielo, Escudero, Mangano & Pisanti 2023

  Versus negligible effect

  Jackson & Laine 2023

  Drewes, Georis, Klasen, Wiggering & Y^{3}W 2024

- No complete picture yet, but there's work in progress. So stay tuned!



Aside: sterile neutrinos…

See Physics Reports by
Dasgupta & Kopp 2021
Abazajian 2017

The QKE formalism can also be used to compute **sterile neutrino production**.

- Light (sub-eV to eV mass) sterile states motivated by the short-baseline anomalies
- keV sterile neutrino dark matter

Differences:

- Sterile states have **no** matter effects or collisions.
- Sub-eV to eV sterile states are produced most efficiently at $T \sim O(1)$ MeV, while keV states are produced much earlier at $T \sim O(100)\mathrm{MeV}$, i.e., subject to different background expansion and SM neutrino interactions.

Aside: non-standard neutrino interactions…

You can also use the QKE formalism to compute the effects of non-standard neutrino interactions (NSI) on the N_{eff}.

- NSI can change neutrino scattering and annihilation through the collision integral.
- NSI can add to the matter effects through the oscillation Hamiltonian.

De Salas, Gariazzo, Martinez-Mirave, Pastor & Tortola 2021

ε^L_{ee}	$\varepsilon^L_{\tau\tau}$	N_{eff}	N_{eff} - $N_{\text{eff}}^{\text{no NSI}}$	$N_{\text{eff}}^{\text{osc}}$	$N_{\text{eff}}^{\text{osc}}$ - $N_{\text{eff}}^{\text{no NSI}}$	$N_{\text{eff}}^{\text{coll}}$	$N_{\text{eff}}^{\text{coll}}$ - $N_{\text{eff}}^{\text{no NSI}}$
0.2	-0.3	3.05714	1.4 $\times 10^{-2}$	3.04357	-7 $\times 10^{-5}$	3.05714	1.4 $\times 10^{-2}$
-0.3	0.2	3.03199	-1.2 $\times 10^{-2}$	3.04367	3×10^{-5}	3.03198	-1.2 $\times 10^{-2}$

Table 3: Comparison between the value of N_{eff} obtained for two sets of NSI parameters considering only its impact on oscillations through equation (16) ($N_{\text{eff}}^{\text{osc}}$) or in the collisional integrals through the G^X matrices in equation (11) ($N_{\text{eff}}^{\text{coll}}$). The deviation from the value of N_{eff} expected in the absence of NSI under the same assumptions is presented as a reference, where $N_{\text{eff}}^{\text{no NSI}} = 3.04364$ (muons are not included).

DIY: neutrino QKE codes…

Two **publicly available neutrino QKE codes** for fully momentum-dependent decoupling/sterile neutrino production calculations.

- **FortEPiaNO**:
  - 3 active +3 sterile
  - CP symmetric only
  - Precision mode includes corrections to the QED equation of state to $O(e^3)$.
  - https://bitbucket.org/ahep_cosmo/fortepiano_public

- **LASAGNA**:
  - 1 active + 1 sterile
  - Can handle CP asymmetry
  - https://github.com/ThomasTram/LASAGNA_public

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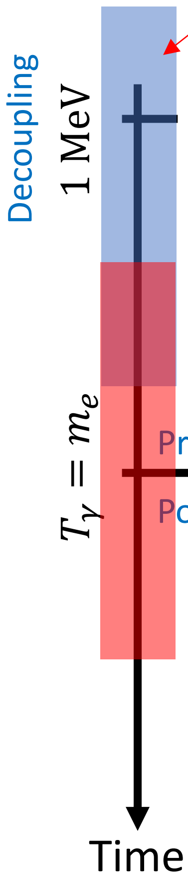

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Deviations from an ideal gas described by thermal QED

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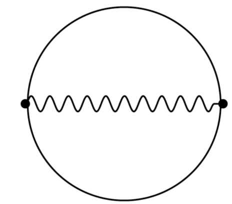
Finite-temperature QED…

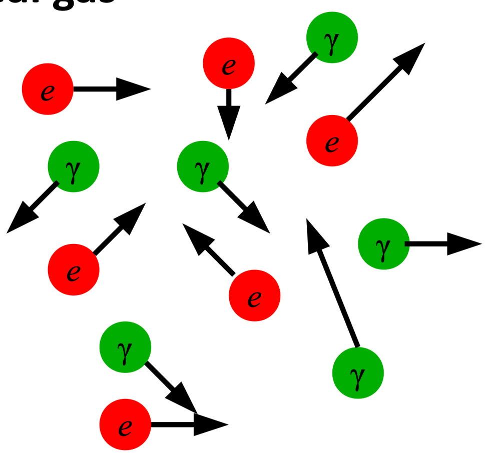
Lowest-order correction of
the QED partition function

$$\ln Z^{(2)} = -\frac{1}{2}$$

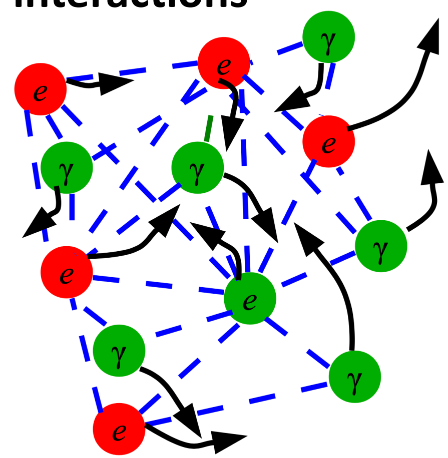


Interactions of $e^{\pm}, \gamma$ **modify the QED plasma** away from an ideal gas.

Ideal gas



+ EM interactions



T-dependent
dispersion relation
+ Forces

Energy = kinetic energy + rest mass

Pressure = from kinetic energy

Energy = **modified** kinetic energy + **T-dependent masses** + **interaction potential energy**

Pressure = from **modified** kinetic energy + **EM forces**

Modified QED equation of state

Summary of precision $N_{\rm eff}^{\rm SM}$ ...

Accounting for all corrections, the current SM benchmark for $N_{\rm eff}$ is:

$$N_{\rm eff}^{\rm SM} = 3.0440 \pm 0.0002$$

Bennett et al. 2019, 2020;
Akita and Yamaguchi 2020;
Froustey, Pitrou & Volpe 2020;
Drewes et al. 2024

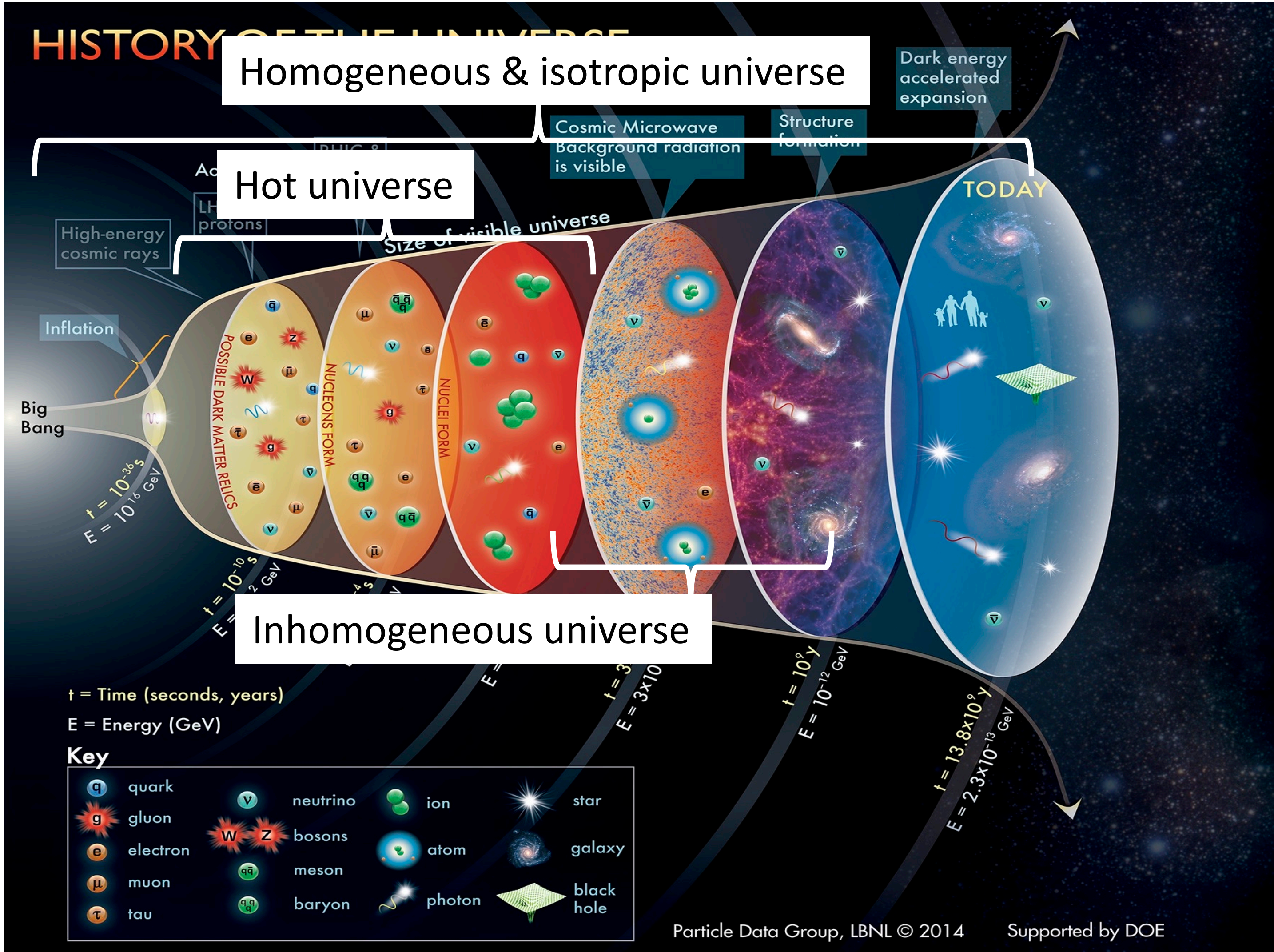
Standard-model corrections to $N_{\rm eff}^{\rm SM}$	Leading-digit contribution
m_e/T_d correction	$+0.04$
$\mathcal{O}(e^2)$ FTQED correction to the QED EoS	$+0.01$
Non-instantaneous decoupling+spectral distortion	-0.006
$\mathcal{O}(e^3)$ FTQED correction to the QED EoS	-0.001
Flavour oscillations	$+0.0005$
Type (a) FTQED corrections to the weak rates	$\lesssim 10^{-4}$
Sources of uncertainty	
Numerical solution by `FortEPiaNO`	± 0.0001
Input solar neutrino mixing angle θ_{12}	± 0.0001

Summary: Part 1...

- SM+FLRW cosmology predicts a **cosmic neutrino background**.
- The homogeneous properties of this background can be computed very precisely.
- A central technique in this computation are the **Quantum Kinetic Equations (QKEs)**, which track how neutrinos go out of equilibrium in an expanding space.
- The QKEs also have uses beyond SM neutrino decoupling calculations.

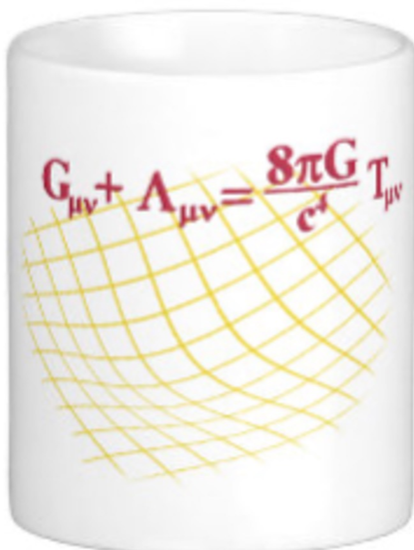
Part 2: Neutrinos in inhomogeneous universe

1. Theory of inhomogeneities
2. Neutrinos and structure formation
3. Relativistic neutrino free-streaming and non-standard interactions



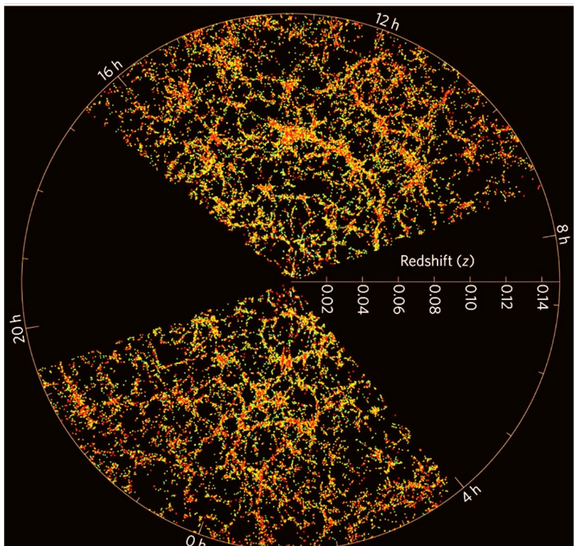
Homogeneous & isotropic universe
Hot universe
Inhomogeneous universe
Dark energy accelerated expansion
Cosmic Microwave Background radiation is visible
Structure
TODAY
High-energy cosmic rays
Inflation
Big Bang
POSSIBLE DARK MATTER RELICS
NUCLEONS FORM
NUCLEI FORM
$t = 10^{-36}$ s
$E = 10^{16}$ GeV
$t = 10^{-10}$ s
$t = 10^{9}$ y
$E = 10^{-12}$ GeV
$t = 13.8 \times 10^{9}$ y
$E = 2.3 \times 10^{-13}$ GeV
t = Time (seconds, years)
E = Energy (GeV)
Key
quark
gluon
electron
muon
tau
neutrino
bosons
meson
baryon
ion
atom
photon
star
galaxy
black hole
Particle Data Group, LBNL © 2014
Supported by DOE

1. Theory of inhomogeneities...

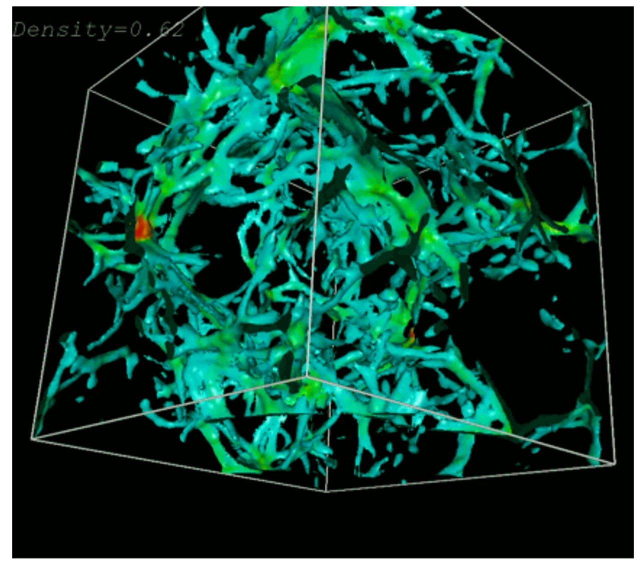


Overview…

The distribution of matter in the universe, even on large scales, is **not** homogeneous and isotropic!



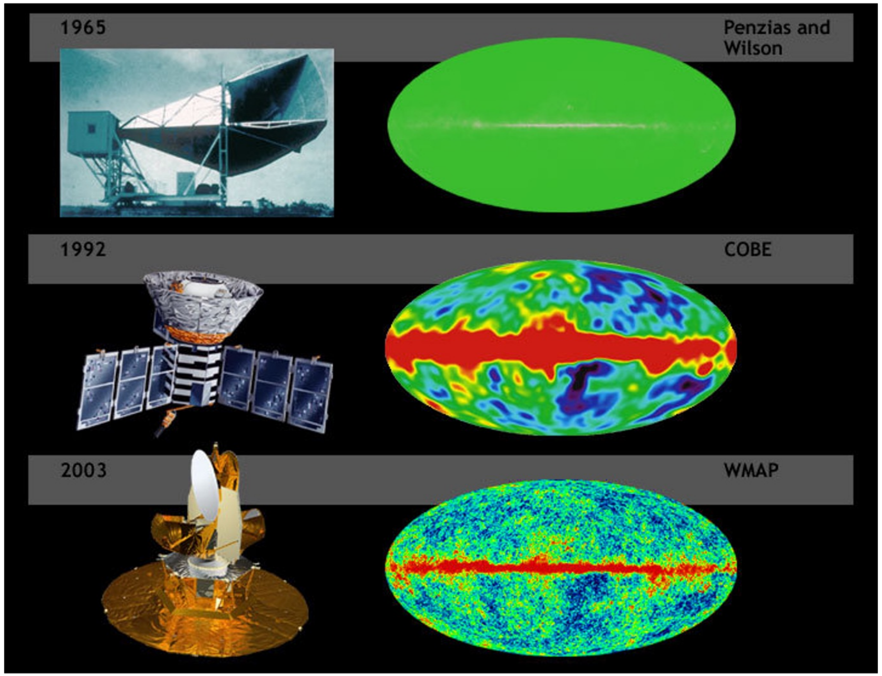
Red galaxies observed by the Sloan Digital Sky Survey



Intergalactic hydrogen clouds (simulations)

Overview...

The cosmic microwave background radiation is anisotropic.



2.73 K background

~ 10^{-5} temperature fluctuations (~ 7 deg resolution)

~ 0.2 deg resolution

Theory of inhomogeneities…

Our current understanding of the inhomogeneous universe:

Inflation: Quantum fluctuations on a scalar field
→ initial perturbations on spacetime metric
→ initial conditions for CMB & large-scale structure

Neutrino decoupling

Matter-radiation equality

Photon decoupling

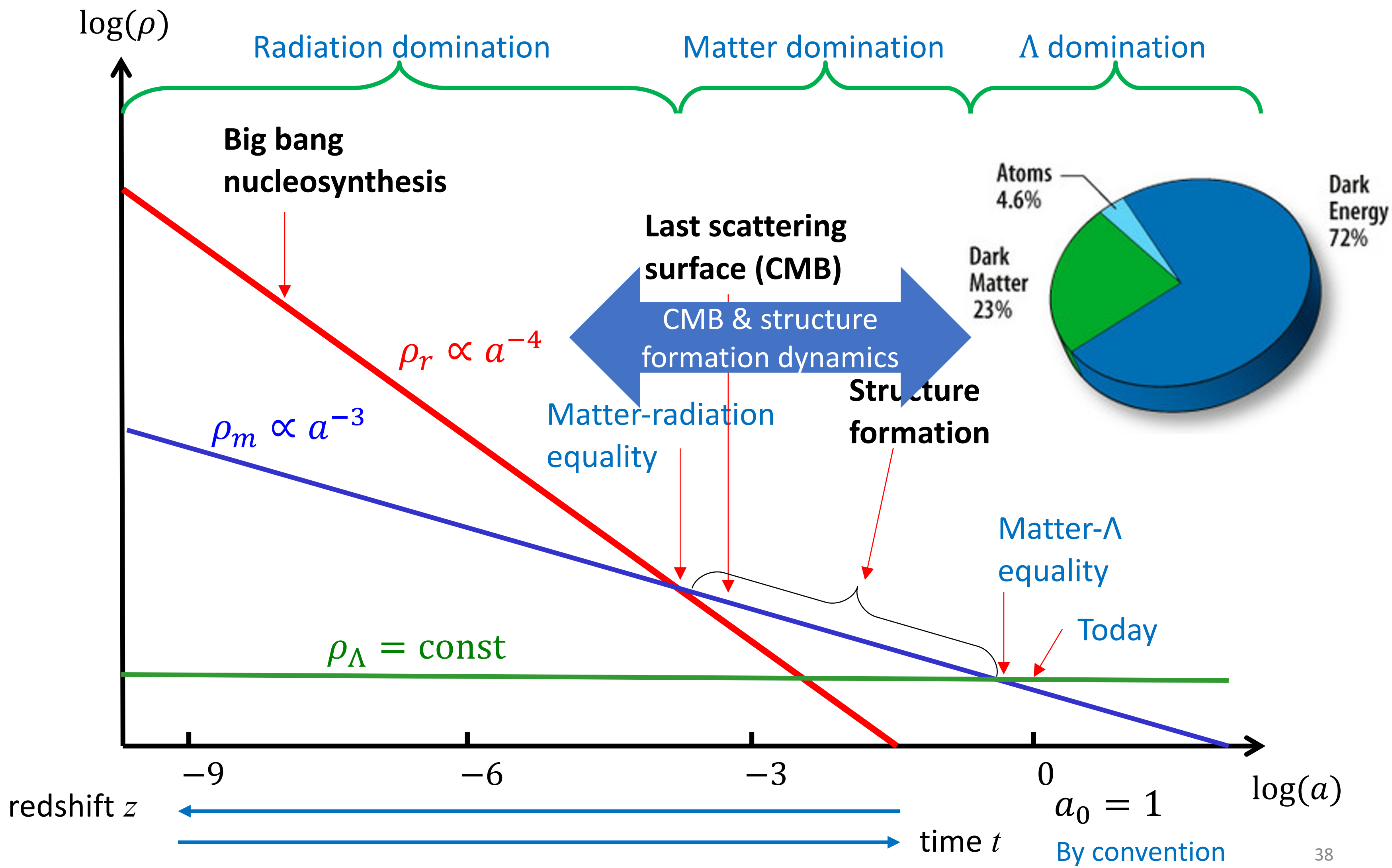
time

Gravity competes with pressure to amplify primordial inhomogeneities

CMB anisotropies: Perturbations in photon energy density and polarisation

Structure formation: Perturbations in nonrelativistic matter

Perturbative GR + Boltzmann


log(ρ)
Radiation domination
Matter domination
Λ domination
Big bang nucleosynthesis
Last scattering surface (CMB)
CMB & structure formation dynamics
Atoms 4.6%
Dark Energy 72%
Dark Matter 23%
$\rho_r \propto a^{-4}$
$\rho_m \propto a^{-3}$
Matter-radiation equality
Structure formation
Matter-Λ equality
Today
$\rho_\Lambda = \mathrm{const}$
−9
−6
−3
0
log(a)
redshift z
$a_0 = 1$
time t
By convention

Theory of inhomogeneities…

We study large-scale inhomogeneities by perturbing around the FLRW spacetime geometry and stress-energy tensor:

$\bar{g}_{\mu\nu}$ = Unperturbed FLRW metric	$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu}$	$\lvert h_{\mu\nu} \rvert \ll 1$	Gravity
$\bar{T}_{\mu\nu}$ = Homogeneous and isotropic	$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$	$\lvert \delta T_{\mu\nu} \rvert \ll \bar{\rho}$	Energy-momentum of the “stuff” in the universe

- Linear perturbations suffice for large length scales (e.g., CMB):
  - **Einstein’s equation** → How $h_{\mu\nu}$ evolves due to $\delta T_{\mu\nu}$
  - **Boltzmann equation** → How $\delta T_{\mu\nu}$ evolves due to $h_{\mu\nu}$ and the properties of the “stuff”

Theory of inhomogeneities…

Like any space metric, the perturbed part of the metric has **10 degrees of freedom**

$\bar{g}_{\mu\nu}$ = Unperturbed FLRW metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

- 4 are gauge degrees of freedom
- Of the 6 physical degrees of freedom:
  - **2 are relevant for structure formation**: scalar modes
  - 2 represent gravitational waves: tensor modes
  - 2 are not present in standard inflationary ΛCDM: vector modes

Theory: scalar perturbations…

A physically intuitive way to represent and understand the behaviours of the two scalar modes is to use the **conformal Newtonian gauge**:

$$ds^2 = a^2(\eta)[-(1+2\Psi)d\eta^2 + (1-2\Phi)\delta_{ij}dx^i dx^j]$$

Cosmic time → $dt = a\, d\eta$ ← Conformal time

- Cf the weak-field metric: $ds^2 = -(1+2\Psi)dt^2 + (1-2\Psi)\delta_{ij}dx^i dx^j$
- In the non-relativistic limit, $\Psi = \Phi$ corresponds to the Newtonian gravitational potential.
- I use the conformal Newtonian gauge in the following.

Theory: perturbations in $T_{\mu\nu}$…

There is one set of perturbations for each matter/energy component, e.g., one for cold dark matter, one for photons, etc.

$$T^{\mu}{}_{\nu} = \begin{bmatrix} -\bar\rho & 0 & 0 & 0 \\ 0 & \bar P & 0 & 0 \\ 0 & 0 & \bar P & 0 \\ 0 & 0 & 0 & \bar P \end{bmatrix}$$

Unperturbed part

$$+\begin{bmatrix} -\delta\rho & (\bar\rho+\bar P)v_\parallel & (\bar\rho+\bar P)v_\parallel & (\bar\rho+\bar P)v_\parallel \\ -(\bar\rho+\bar P)v_\parallel & \delta P & \Sigma^1_2 & \Sigma^1_3 \\ -(\bar\rho+\bar P)v_\parallel & \Sigma^2_1 & \delta P & \Sigma^2_3 \\ -(\bar\rho+\bar P)v_\parallel & \Sigma^3_1 & \Sigma^3_2 & \delta P \end{bmatrix}$$

Perturbed part: showing only scalar perturbations; but vector velocity and vector/tensor anisotropic stresses are also possible.

Density perturbations

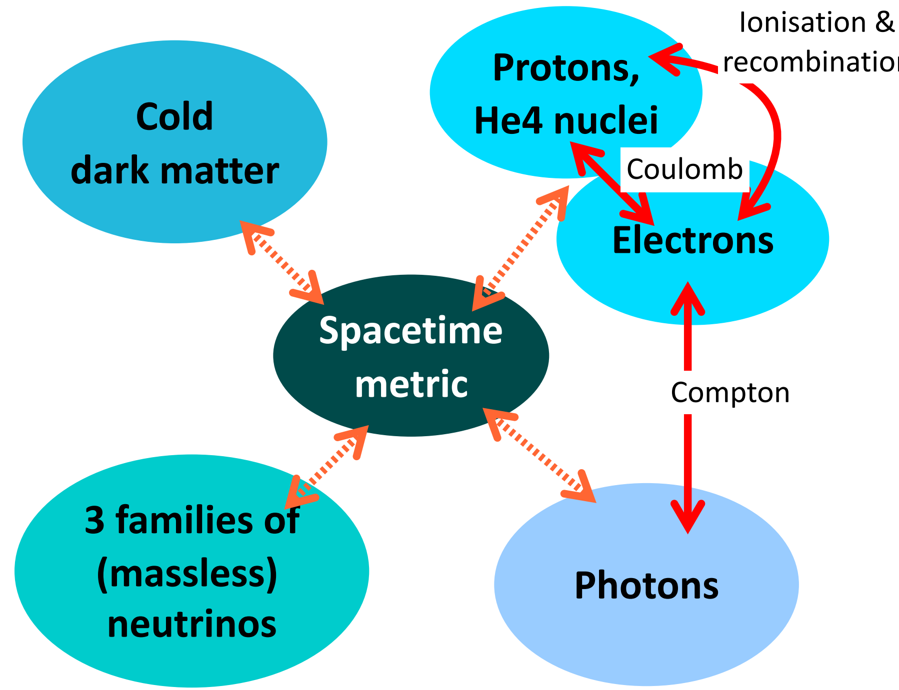
Velocity perturbations

Anisotropic stress

Pressure perturbations

Theory: perturbations in $T_{\mu\nu}$...

In standard inflationary ΛCDM, we track 4 forms of matter/energy.



Theory: perturbations in $T_{\mu\nu}$...

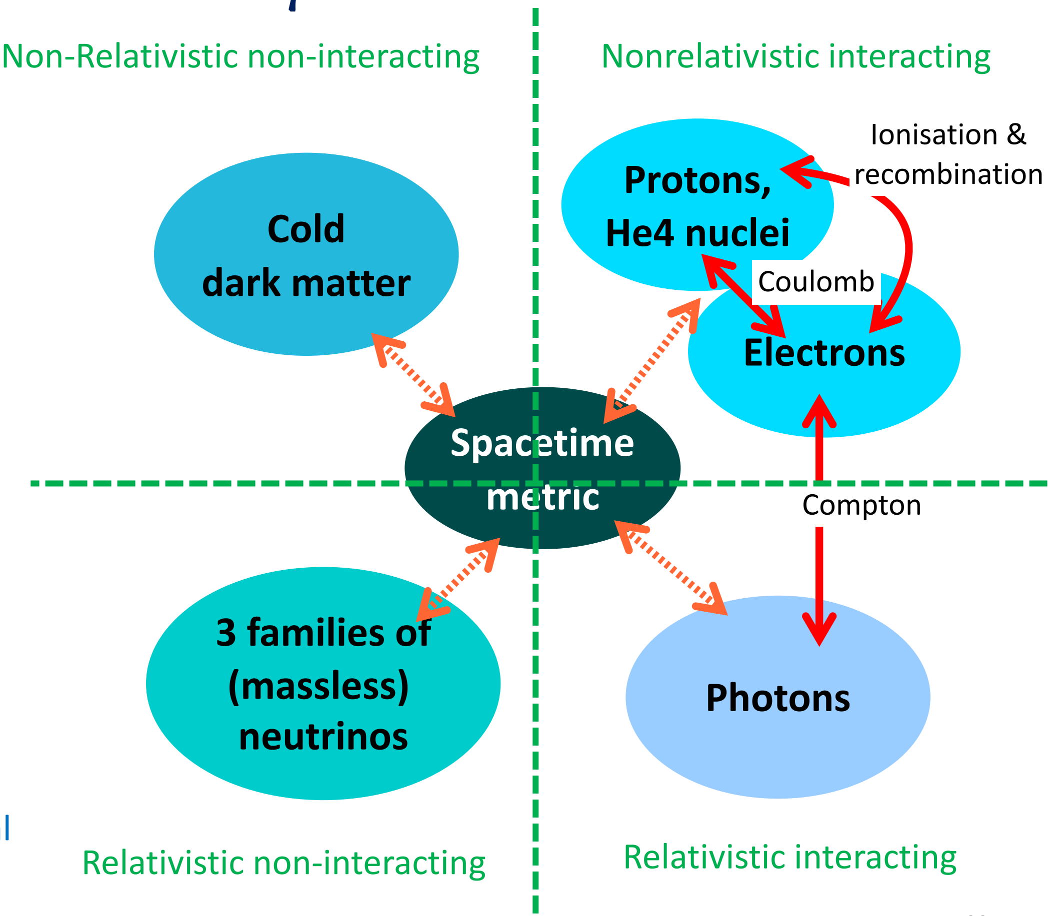
In standard inflationary ΛCDM, we track 4 forms of matter/energy.

- Each matter/energy form develops its own perturbations, tracked by the **Boltzmann equation**:

$$P^{\mu}\frac{\partial f_{\alpha}}{\partial x^{\mu}} - \Gamma^{i}_{\mu\nu}P^{\mu}P^{\nu}\frac{\partial f_{\alpha}}{\partial P^{i}} = C[f_{\alpha}]$$

Gravitational effects

Non-gravitational interactions



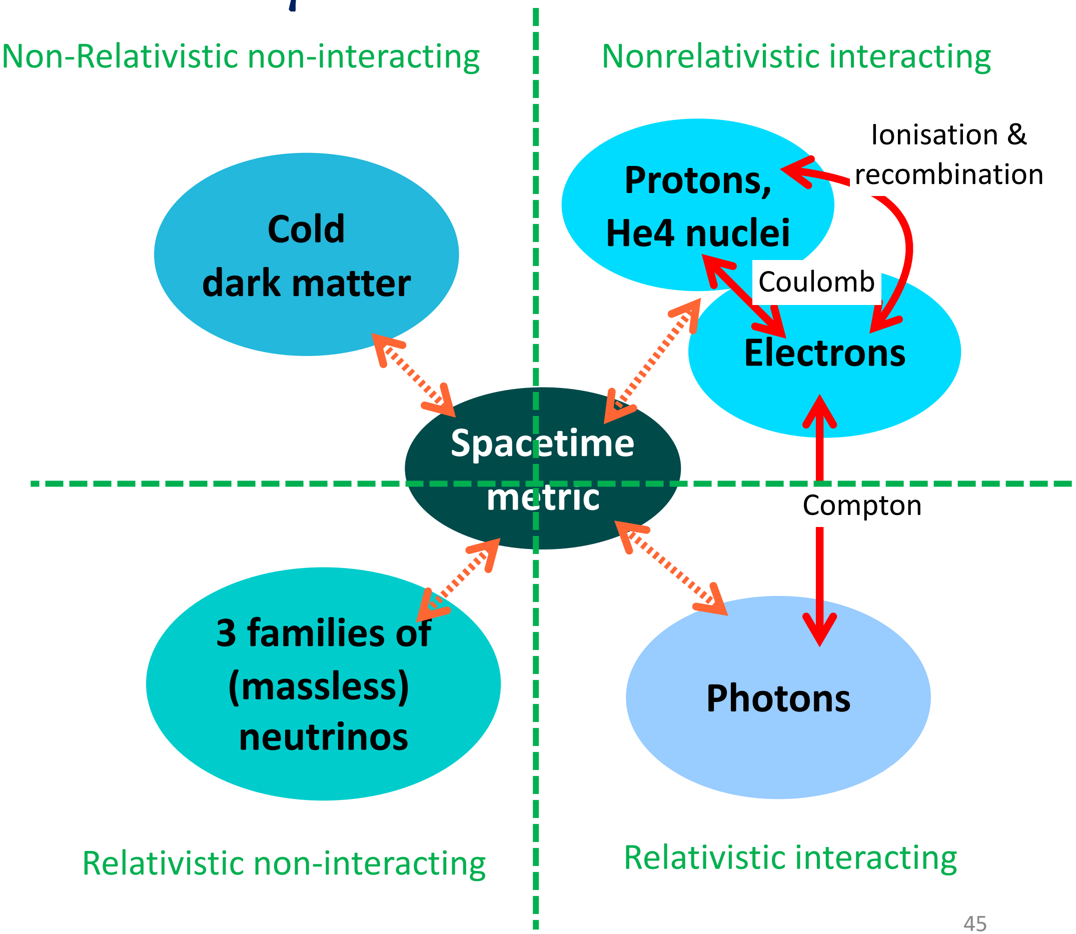
Theory: perturbations in $T_{\mu\nu}$...

In standard inflationary ΛCDM, we track 4 forms of matter/energy.

- But **all** forms of matter/energy "interact" with the spacetime metric (i.e., gravity), whose evolution is governed by **Einstein's equation**:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

Metric goes in here   Matter/energy



Linear Einstein-Boltzmann system...

A selection of the relevant equations

Metric

$$ds^2 = a^2(\tau)\left\{-(1+2\psi)d\tau^2 + (1-2\phi)dx^i dx_i\right\} .$$

Stress-energy tensor

$$\begin{aligned} T^0{}_0 &= -(\bar\rho + \delta\rho)\,, \\ T^0{}_i &= (\bar\rho + \bar P)v_i = -T^i{}_0\,, \\ T^i{}_j &= (\bar P + \delta P)\delta^i{}_j + \Sigma^i{}_j\,, \qquad \Sigma^i{}_i = 0\,, \end{aligned}$$

Einstein's equation

$$\begin{aligned} k^2\phi + 3\frac{\dot a}{a}\left(\dot\phi + \frac{\dot a}{a}\psi\right) &= 4\pi G a^2 \delta T^0{}_0(\mathrm{Con})\,, \\ k^2\left(\dot\phi + \frac{\dot a}{a}\psi\right) &= 4\pi G a^2(\bar\rho + \bar P)\theta(\mathrm{Con})\,, \\ \ddot\phi + \frac{\dot a}{a}(\dot\psi + 2\dot\phi) + \left(2\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right)\psi + \frac{k^2}{3}(\phi - \psi) &= \frac{4\pi}{3} G a^2 \delta T^i{}_i(\mathrm{Con})\,, \\ k^2(\phi - \psi) &= 12\pi G a^2(\bar\rho + \bar P)\sigma(\mathrm{Con})\,, \end{aligned}$$

(Derived from) Boltzmann equation

CDM

$$\dot\delta_c = -\theta_c + 3\dot\phi\,, \quad \dot\theta_c = -\frac{\dot a}{a}\theta_c + k^2\psi\,.$$

Baryons

$$\begin{aligned} \dot\delta_b &= -\theta_b + 3\dot\phi\,, \\ \dot\theta_b &= -\frac{\dot a}{a}\theta_b + c_s^2 k^2\delta_b + \frac{4\bar\rho_\gamma}{3\bar\rho_b} a n_e \sigma_T(\theta_\gamma - \theta_b) + k^2\psi\,. \end{aligned}$$

Photons

$$\begin{aligned} \dot\delta_\gamma &= -\frac{4}{3}\theta_\gamma + 4\dot\phi\,, \\ \dot\theta_\gamma &= k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + k^2\psi + a n_e\sigma_T(\theta_b - \theta_\gamma)\,, \\ \dot F_{\gamma 2} &= 2\dot\sigma_\gamma = \frac{8}{15}\theta_\gamma - \frac{3}{5}kF_{\gamma 3} - \frac{9}{5}a n_e\sigma_T\sigma_\gamma + \frac{1}{10}a n_e\sigma_T\left(G_{\gamma 0} + G_{\gamma 2}\right)\,, \\ \dot F_{\gamma l} &= \frac{k}{2l+1}\left[lF_{\gamma(l-1)} - (l+1)F_{\gamma(l+1)}\right] - a n_e\sigma_T F_{\gamma l}\,, \quad l \ge 3 \\ \dot G_{\gamma l} &= \frac{k}{2l+1}\left[lG_{\gamma(l-1)} - (l+1)G_{\gamma(l+1)}\right] + a n_e\sigma_T\left[-G_{\gamma l} + \frac{1}{2}\left(F_{\gamma 2} + G_{\gamma 0} + G_{\gamma 2}\right)\left(\delta_{l0} + \frac{\delta_{l2}}{5}\right)\right] \end{aligned}$$

Massless neutrinos

$$\begin{aligned} \dot\delta_\nu &= -\frac{4}{3}\theta_\nu + 4\dot\phi\,, \\ \dot\theta_\nu &= k^2\left(\frac{1}{4}\delta_\nu - \sigma_\nu\right) + k^2\psi\,, \\ \dot F_{\nu l} &= \frac{k}{2l+1}\left[lF_{\nu(l-1)} - (l+1)F_{\nu(l+1)}\right]\,, \quad l \ge 2\,. \end{aligned}$$

e.g., Ma & Bertschinger 1995

Linear Einstein-Boltzmann system…

There are several publicly available numerical codes that solve the full linear Einstein-Boltzmann system:

- CAMB: https://camb.info
- CLASS: http://class-code.net/

- Mostly optimised for for standard inflationary ΛCDM, but can be fairly easily modified to accommodate "exotic" models.
- More on **cosmological perturbation theory**, e.g.,
  - Hu, Covariant linear perturbation formalism, astro-ph/0402060
  - Seljak, Lectures on dark matter, *ICTP Lect.Notes Ser.* 4 (2001) 33-77

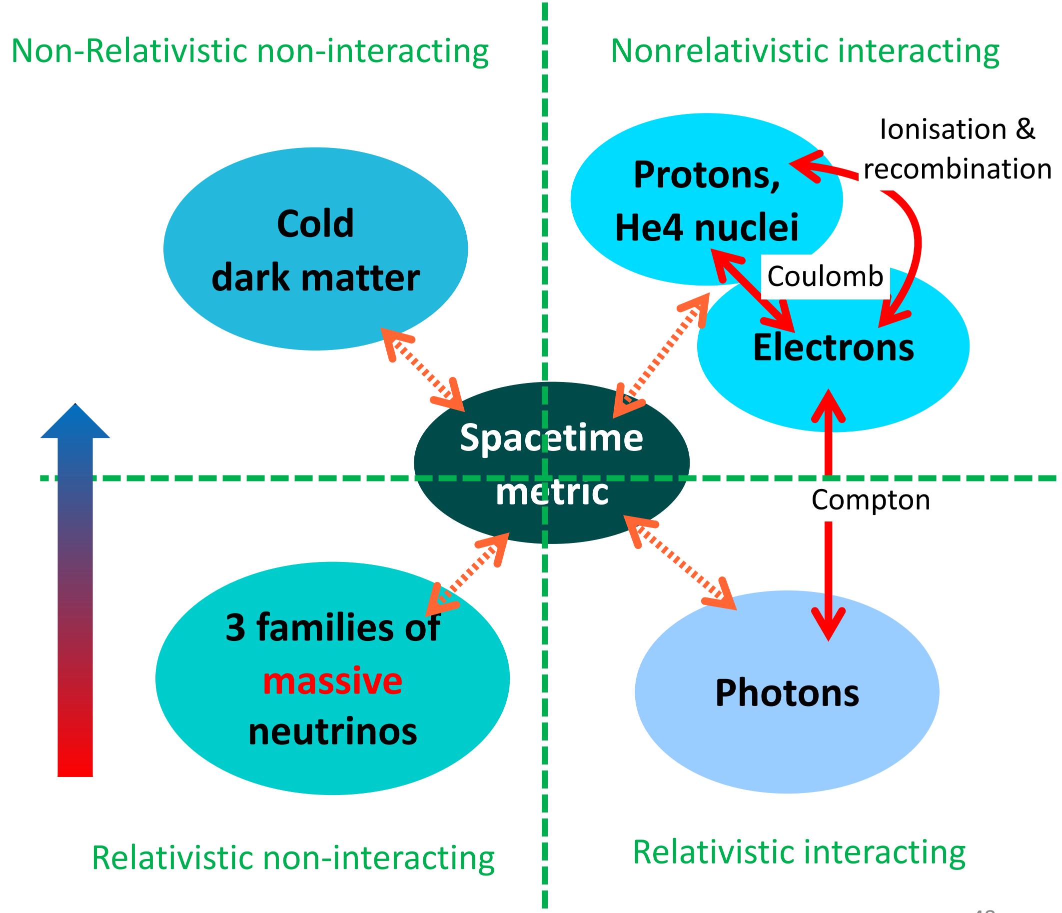
How real neutrinos fit into this picture…

Real neutrinos are of course **massive**.

- For sub-eV masses, relativistic-to-NR transition happens at redshifts $z = O(100) - O(1000)$.

→ Technically NR today **but may not be totally "cold"**.

→ Spend a substantial amount of time in the CMB/structure formation epoch as relativistic particles.



Massive neutrino Boltzmann equation…

We also use the linearised Boltzmann equation to track the massive neutrino phase space density $f_\nu(p, x, t)$ and how inhomogeneities evolve in their presence.

$$P^\mu \frac{\partial f_\alpha}{\partial x^\mu} - \Gamma^i_{\mu\nu} P^\mu P^\nu \frac{\partial f_\alpha}{\partial P^i} = 0$$

No non-gravitational Interactions for neutrinos

Gravity

- Fundamentally **no different** from the massless case.
- However, adding masses leads to new energy/time scales in the problem.
  - Need to track how each momentum mode goes NR and responds to gravity (in contrast with the massless case, where every mode has speed c and hence evolves in the same way).
  - $\sim \mathbf{10-20}$ **times more equations** to solve numerically for every new mass.

Massive neutrino Boltzmann equation…

Split into background + perturbed part

$$f(x^i, P_j, \tau) = f_0(q)\left[1 + \Psi(x^i, q, n_j, \tau)\right] \; .$$

Legendre decomposition

$$\Psi(\vec{k}, \hat{n}, q, \tau) = \sum_{l=0}^{\infty} (-i)^l (2l+1) \Psi_l(\vec{k}, q, \tau) P_l(\hat{k}\cdot\hat{n}) \, .$$

Legendre-decomposed Boltzmann equation aka Boltzmann hierarchy

$$\begin{aligned}
\dot{\Psi}_0 &= -\frac{qk}{\epsilon}\Psi_1 - \dot{\phi}\frac{d\ln f_0}{d\ln q} \, , \\
\dot{\Psi}_1 &= \frac{qk}{3\epsilon}\left(\Psi_0 - 2\Psi_2\right) - \frac{\epsilon\, k}{3q}\psi\frac{d\ln f_0}{d\ln q} \, , \\
\dot{\Psi}_l &= \frac{qk}{(2l+1)\epsilon}\left[l\Psi_{l-1} - (l+1)\Psi_{l+1}\right] \, , \quad l \geq 2 \, .
\end{aligned}$$

Linear Boltzmann equation for perturbed part

$$\frac{\partial \Psi}{\partial \tau} + i\frac{q}{\epsilon}(\vec{k}\cdot\hat{n})\,\Psi + \frac{d\ln f_0}{d\ln q}\left[\dot{\phi} - i\frac{\epsilon}{q}(\vec{k}\cdot\hat{n})\,\psi\right] = 0$$

Density, pressure, velocity & anistropic stress from Legendre moments

$$\begin{aligned}
\delta\rho_h &= 4\pi a^{-4}\int q^2 dq\, \epsilon f_0(q)\Psi_0 \, , \\
\delta P_h &= \frac{4\pi}{3}a^{-4}\int q^2 dq\, \frac{q^2}{\epsilon} f_0(q)\Psi_0 \, , \\
(\bar{\rho}_h + \bar{P}_h)\theta_h &= 4\pi k a^{-4}\int q^2 dq\, q f_0(q)\Psi_1 \, , \\
(\bar{\rho}_h + \bar{P}_h)\sigma_h &= \frac{8\pi}{3}a^{-4}\int q^2 dq\, \frac{q^2}{\epsilon} f_0(q)\Psi_2 \, .
\end{aligned}$$

e.g., Ma & Bertschinger 1995

These are all coded up in public Boltzmann codes **CAMB** and **CLASS**.

Fun things with cosmological neutrinos…

I'm **not** going to talk about technical details here. Rather, I want to give you a physical picture of what's happening.

- **Non-relativistic neutrinos** in the low-redshift ($z \lesssim 1000$) universe in structure formation
  - Neutrino mass constraints
- **Free-streaming relativistic neutrinos** around CMB times ($z \sim 1000$)
  - As a probe of non-standard neutrino interactions
  - Neutrino decay