Neutrino Mixing and Leptonic CP-Violation – Theory and Tests in Future Experiments

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The Framework: the Reference 3-ν Mixing Scheme

$$\nu_{l\mathrm{L}} = \sum_{j=1}^{3} U_{lj}\,\nu_{j\mathrm{L}} \qquad l = e, \mu, \tau.$$

The PMNS matrix U - 3×3 unitary.

ν_j, $m_j \neq 0$: Dirac or Majorana particles.

Data: 3 νs are light: $\nu_{1,2,3}$, $m_{1,2,3} \lesssim 0.5$ eV.

3-ν mixing: 3-flavour neutrino oscillations possible.

ν_μ, E; at distance L: $P(\nu_\mu \to \nu_{\tau(e)}) \neq 0$, $P(\nu_\mu \to \nu_\mu) < 1$

$P(\nu_l \to \nu_{l'}) = P(\nu_l \to \nu_{l'}; E, L; U; m_2^2 - m_1^2, m_3^2 - m_1^2)$

Reference 3-ν Mixing Scheme

$$\nu_{l\mathrm{L}} = \sum_{j=1}^{3} U_{lj}\, \nu_{j\mathrm{L}} \qquad l = e, \mu, \tau.$$

Data: 3 νs are light: $\nu_{1,2,3}$, $m_{1,2,3} \lesssim 0.5$ eV;
KATRIN: $m_{\bar{\nu}_e} < 0.45$ eV;
Cosmology: $\sum_j m_j <$ 0.12 - 0.77 eV (95% CL; 2107.00532).

The value of $\min(m_j)$ and "mass ordering" unknown.

Δm^2_{21}, $|\Delta m^2_{31}|$ - **known** ($\mathrm{sgn}(\Delta m^2_{31})$ - **unknown**).

ν_j, $m_j \neq 0$: **nature - Dirac or Majorana - unknown.**

The PMNS matrix U - 3×3 unitary: θ_{12}, θ_{13}, θ_{23} - known; CPV phases δ, α_{21}, α_{31} - unknown.

Thus, 5 known + 4 unknown parameters + MO.
"Known" = measured; "unknown" = not measured.

m_e, m_μ, m_τ **also known - used as input.**

PMNS Matrix: Standard Parametrization

$$U = V\,P\,, \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix},$$

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

- $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$, $\theta_{ij} = [0, \frac{\pi}{2}]$,
- δ - Dirac CPV phase, $\delta = [0, 2\pi]$; CP inv.: $\delta = 0, \pi, 2\pi$;
- α_{21}, α_{31} - Majorana CPV phases; CP inv.: $\alpha_{21(31)} = k(k')\pi$, $k(k') = 0, 1, 2...$

S.M. Bilenky et al., 1980

- $\Delta m^2_{\odot} \equiv \Delta m^2_{21} \cong 7.34 \times 10^{-5}$ eV$^2 > 0$, $\sin^2\theta_{12} \cong 0.305$, $\cos 2\theta_{12} \gtrsim 0.306$ (3σ),
- $|\Delta m^2_{31(32)}| \cong 2.448\ (2.502) \times 10^{-3}$ eV^2, $\sin^2\theta_{23} \cong 0.545\ (0.551)$, NO (IO) ,
- θ_{13} - the CHOOZ angle: $\sin^2\theta_{13} = 0.0222\ (0.0223)$

F. Capozzi et al. (Bari Group), arXiv:2003.08511.

• $\text{sgn}(\Delta m^2_{\text{atm}}) = \text{sgn}(\Delta m^2_{31(32)})$ not determined

$$\Delta m^2_{\text{atm}} \equiv \Delta m^2_{31} > 0, \quad \text{normal mass ordering (NO)}$$

$$\Delta m^2_{\text{atm}} \equiv \Delta m^2_{32} < 0, \quad \text{inverted mass ordering (IO)}$$

Convention: $m_1 < m_2 < m_3$ - NO, $m_3 < m_1 < m_2$ - IO

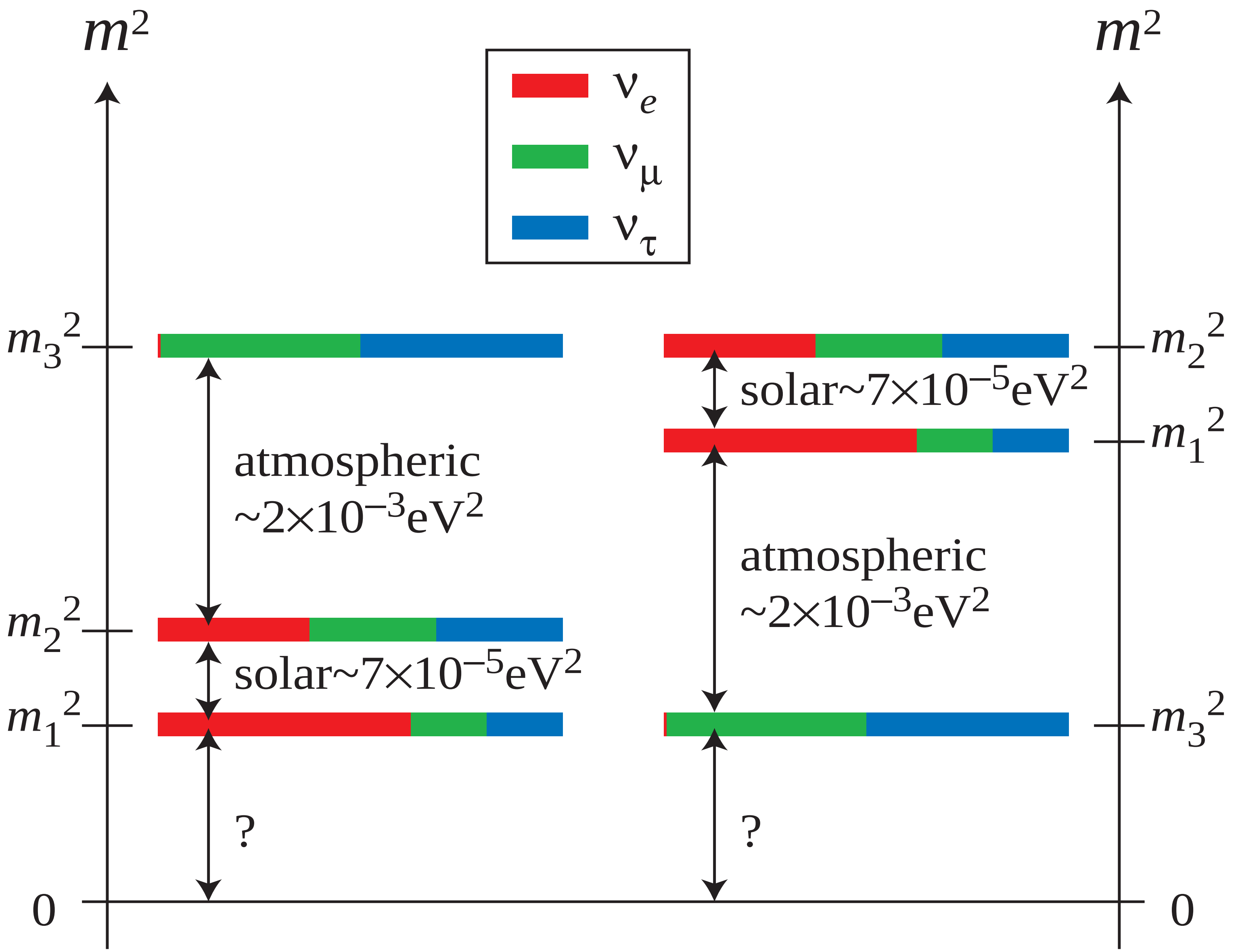
$$m_1 \ll m_2 < m_3, \qquad \text{NH},$$

$$m_3 \ll m_1 < m_2, \qquad \text{IH},$$

$$m_1 \cong m_2 \cong m_3,\ m^2_{1,2,3} >> |\Delta m^2_{31(32)}|,\ \text{QD};\ m_j \gtrsim 0.10\ \text{eV}.$$

• $m_2 = \sqrt{m_1^2 + \Delta m^2_{21}}$, $m_3 = \sqrt{m_1^2 + \Delta m^2_{31}}$ - NO;

• $m_1 = \sqrt{m_3^2 + \Delta m^2_{23} - \Delta m^2_{21}}$, $m_2 = \sqrt{m_3^2 + \Delta m^2_{23}}$ - IO;



m^2
m^2
ν_e
ν_μ
ν_τ
m_3^2
m_2^2
m_1^2
atmospheric
$\sim 2\times10^{-3}$eV^2
solar$\sim 7\times10^{-5}$eV^2
?
0
m_2^2
m_1^2
m_3^2
solar$\sim 7\times10^{-5}$eV^2
atmospheric
$\sim 2\times10^{-3}$eV^2
?
0

• **Dirac phase** δ: $\nu_l \leftrightarrow \nu_{l'}$, $\bar{\nu}_l \leftrightarrow \bar{\nu}_{l'}$, $l \neq l'$; $A_{\rm CP}^{(l,l')} \propto J_{\rm CP} \propto \sin\theta_{13} \sin\delta$:

3ν—mixing:

P.I. Krastev, S.T.P., 1988

$$A_{\rm CP}^{(l,l')} \equiv P(\nu_l \to \nu_{l'}) - P(\bar{\nu}_l \to \bar{\nu}_{l'}) \ , \quad l \neq l' = e, \mu, \tau$$

$$A_{\rm T}^{(l,l')} \equiv P(\nu_l \to \nu_{l'}) - P(\nu_{l'} \to \nu_l), \ l \neq l'$$

$$A_{\rm CP(T)}^{(e,\mu)} = A_{\rm CP(T)}^{(\mu,\tau)} = -A_{\rm CP(T)}^{(e,\tau)}$$

In vacuum: $A^{(e,\mu)}_{\mathrm{CP(T)}} = 4\, J_{\mathrm{CP}} F^{vac}_{osc}$ $\;(A^{(e,\mu)}_{\mathrm{CP(T)}} = A^{(\mu,\tau)}_{\mathrm{CP(T)}} = -\, A^{(e,\tau)}_{\mathrm{CP(T)}})$

$$J_{CP} = \mathrm{Im}\left\{U_{e1}\, U_{\mu 2}\, U^{*}_{e2}\, U^{*}_{\mu 1}\right\} = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos\theta_{13} \sin\delta$$

$$F^{vac}_{osc} = \sin(\frac{\Delta m^2_{21}}{2E}L) + \sin(\frac{\Delta m^2_{32}}{2E}L) + \sin(\frac{\Delta m^2_{13}}{2E}L)$$

P.I. Krastev, S.T.P., 1988

In matter: **Matter effects violate**

$$\mathrm{CP}: \qquad P(\nu_l \rightarrow \nu_{l'}) \neq P(\bar{\nu}_l \rightarrow \bar{\nu}_{l'})$$

$$\mathrm{CPT}: \qquad P(\nu_l \rightarrow \nu_{l'}) \neq P(\bar{\nu}_{l'} \rightarrow \bar{\nu}_l)$$

P. Langacker et al., 1987

Can conserve the T-invariance (constant density or density profile symmetric relative to the middle point, e.g., Earth)

$$P(\nu_l \rightarrow \nu_{l'}) = P(\nu_{l'} \rightarrow \nu_l),\; l \neq l'$$

In matter with constant density (T2K, NOνA, T2HK, DUNE):

$$\mathrm{J^{mat}_{CP} = J^{vac}_{CP}\; R_{CP}},\quad \mathrm{A^{(e,\mu)}_{T} = J^{mat}_{CP} F^{mat}_{osc}}$$

R_{CP}- real, does not depend on θ_{23} and δ; $\;|R_{CP}| \lesssim 2.5$

P.I. Krastev, S.T.P., 1988

2018: $R_{CP} > 0$, $R_{CP} \leq 1.2$; **numerically** R_{CP}**=Naumov-HS factor (from 1991).**

S.T.P., Y.-L. Zhou, 1806.09112

Current data: $|J_{CP}| \lesssim 0.040$ (can be relatively large!); b.f.v. with $\delta = 3\pi/2$: $J_{CP} \cong -0.035$.

- **Majorana phases** α_{21}, α_{31}:
  - $\nu_l \leftrightarrow \nu_{l'}$, $\bar{\nu}_l \leftrightarrow \bar{\nu}_{l'}$ not sensitive;

S.M. Bilenky, J. Hosek, S.T.P.,1980;
P. Langacker, S.T.P., G. Steigman, S. Toshev, 1987

  - $|<m>|$ in $(\beta\beta)_{0\nu}$−decay depends on α_{21}, α_{31};
  - $\Gamma(\mu \to e + \gamma)$ etc. in SUSY theories depend on $\alpha_{21,31}$;
  - BAU, leptogenesis scenario: $\delta, \alpha_{21,31}$!

Parameter	Ordering	Best fit	1σ range	2σ range	3σ range	"1σ" (%)
$\delta m^2/10^{-5}$ eV^2	NO, IO	7.36	7.21 − 7.52	7.06 − 7.71	6.93 − 7.93	2.3
$\sin^2\theta_{12}/10^{-1}$	NO, IO	3.03	2.90 − 3.16	2.77 − 3.30	2.63 − 3.45	4.5
$\lvert\Delta m^2\rvert/10^{-3}$ eV^2	NO	2.485	2.454 − 2.508	2.427 − 2.537	2.401 − 2.565	1.1
	IO	2.455	2.430 − 2.485	2.403 − 2.513	2.376 − 2.541	1.1
$\sin^2\theta_{13}/10^{-2}$	NO	2.23	2.17 − 2.30	2.11 − 2.37	2.04 − 2.44	3.0
	IO	2.23	2.17 − 2.29	2.10 − 2.38	2.03 − 2.45	3.1
$\sin^2\theta_{23}/10^{-1}$	NO	4.55	4.40 − 4.73	4.27 − 5.81	**4.16 − 5.99**	6.7
	IO	5.69	5.48 − 5.82	4.30 − 5.94	**4.17 − 6.06**	5.5
δ/π	NO	1.24	1.11 − 1.42	0.94 − 1.74	0.77 − 1.97	16
	IO	1.52	1.37 − 1.66	1.22 − 1.78	**1.07 − 1.90**	9
$\Delta\chi^2_{\mathrm{IO-NO}}$	IO−NO	+6.5 (2.5σ)				

Global 3ν analysis of oscillation parameters: best-fit values and allowed ranges at $N_\sigma = 1$, 2 and 3, for either NO or IO, including all data. The latter column shows the formal "1σ fractional accuracy" for each parameter, defined as 1/6 of the 3σ range, divided by the best-fit value and expressed in percent. We recall that $\Delta m^2 = m_3^2 - (m_1^2 + m_2^2)/2$ and that $\delta \in [0, 2\pi]$ (cyclic). The last row reports the difference between the χ^2 minima in IO and NO.

F. Capozzi et al. (Bari Group), arXiv:2107.00532.

θ_{12}, θ_{23} - **large,** θ_{13} - **small (very different from the quark mixing angles).** $\sin^2\theta_{23}$ - **relatively large uncertainty.**

$$\Delta m^2_{21}/|\Delta m^2_{31}| \cong 1/30.$$

The new T2K and NOνA data presented at Neutrino 2024 (June 17-22, 2024, Milano) essentially did not improve on NMO, δ and $\sin^2_{23}$.

The Problem

Understanding the origin of the peculiar pattern of neutrino mixing of two large and one small mixing angles is one of the major problems in neutrino physics.

It is a part of the more general highly challenging and still unresolved fundamental problem in particle physics of understanding the origins of the patterns of the charged lepton and neutrino masses and of neutrino mixing, of quark masses and mixing, and of CP violation in the quark and lepton sector, i.e., understanding the origins of the lepton and quark flavours.

"Asked what single mystery, if he could choose, he would like to see solved in his lifetime, Weinberg doesnt have to think for long: he wants to be able to explain the observed pattern of quark and lepton masses."
From Model Physicist, CERN Courier, 13 October 2017.

The renewed attempts to seek new better solutions of the flavour problem than those already proposed were stimulated primarily by the remarkable progress made in the studies of neutrino oscillations, which began 24 years ago with the discovery of oscillations of atmospheric ν_μ and $\bar{\nu}_\mu$ by SuperKamiokande experiment. This lead, in particular, to the determination of the pattern of the 3-neutrino mixing, which turn out to consist of two large and one small mixing angles.

The Lepton Flavour Problem

Consists of three basic elements (sub-problems), namely, understanding:

• **Why** $m_{\nu_j} <<< m_{e,\mu,\tau}, m_q,\ q = u, c, t, d, s, b$ ($m_{\nu_j} \lesssim 0.5$ **eV**, $m_l \geq 0.511$ **MeV**, $m_q \gtrsim 2$ **MeV**);

• **The origins of the patterns of**
i) neutrino mixing of 2 large and 1 small angles ($\theta^l_{12} = 33.4^\circ$, $\theta^l_{23} = 42.4^\circ$ (49.0°), $\theta^l_{13} = 8.59^\circ$), **and of ii)** Δm^2_{ij}, **i.e., of** $\Delta m^2_{21} \ll |\Delta m^2_{31}|$, $\Delta m^2_{21}/|\Delta m^2_{31}| \cong 1/30$.

• **The origin of the hierarchical pattern of charged lepton masses:**
$m_e \ll m_\mu \ll m_\tau$, $m_e/m_\mu \cong 1/200$, $m_\mu/m_\tau \cong 1/17$.

The first two added new important aspects to the flavour problem.

$m_{\nu_j} <<< m_{e,\mu,\tau}, m_q,\ q = u, c, t, d, s, b$:

seesaw mechanism(s), Weinberg operator, radiative ν mass generation, extra dimensions.
However, additional input (symmetries) needed to explain the pattern of lepton mixing and to get specific testable predictions.

The Neutrino Mixing Problem

The most elegant, simple and testable solution of the neutrino mixing problem is arguably provided by the non-Abelian discrete symmetry approach.

In what concerns the lepton flavour problem, in the last 5 years a very successful and attractive approach based on Modular Invariance has been and continues to be developed.

The Non-Abelian Discrete Symmetry Approach

With the observed pattern of neutrino mixing Nature is sending us a Message. The Message is encoded in the values of the neutrino mixing angles, leptonic CP violation phases and neutrino masses. In my opinion, Nature gave us also a hint what the content of Nature's Message is.

Neutrino Mixing: New Symmetry?

- $\theta_{12} = \theta_{\odot} \cong \frac{\pi}{5.4}$, $\quad \theta_{23} = \theta_{\rm atm} \cong \frac{\pi}{4}(?)$, $\quad \theta_{13} \cong \frac{\pi}{20}$

$$U_{\rm PMNS} \cong \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & \epsilon \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}}(?) \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}}(?) \end{pmatrix};$$

Very different from the CKM-matrix!

- $\theta_{12} \cong \sin^{-1}\frac{1}{\sqrt{3}}(= \frac{\pi}{5}) - 0.020$; $\theta_{12} \cong \pi/4 - 0.20$, $\theta_{13} \cong 0 + \pi/20$, $\quad \theta_{23} \cong \pi/4 \mp 0.10$.
- $U_{\rm PMNS}$ **due to new approximate symmetry?**

A Natural Possibility: $U_{\mathsf{PMNS}} = U_{\mathsf{l}}^{\dagger} U_{\nu}$

$$\mathbf{U}_{\mathsf{PMNS}} = \mathbf{U}_{\mathsf{l}}^{\dagger}(\theta^{\ell}_{\mathbf{ij}}, \delta^{\ell})\ \mathbf{Q}(\psi,\omega)\mathbf{U}_{\mathsf{TBM,BM,LC,...}}\ \bar{\mathbf{P}}(\xi_1,\xi_2)\,,$$

with

$$U_{\mathsf{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix};\quad U_{\mathsf{BM}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \pm\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \mp\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

- $U_{\mathsf{lep}}^{\dagger}(\theta^{\ell}_{ij}, \delta^{\ell})$ - from diagonalization of the l^- mass matrix;
- $U_{\mathsf{TBM,BM,LC,...}}\ \bar{P}(\xi_1,\xi_2)$ - from diagonalization of the ν mass matrix;
- $Q(\psi,\omega)$, - from diagonalization of the l^- and/or ν mass matrices.

P. Frampton, STP, W. Rodejohann, 2003

$U_{\rm LC}$, $U_{\rm GRAM}$, $U_{\rm GRBM}$, $U_{\rm HGM}$:

$$U_{\rm LC} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{c^{\nu}_{23}}{\sqrt{2}} & \frac{c^{\nu}_{23}}{\sqrt{2}} & s^{\nu}_{23} \\ \frac{s^{\nu}_{23}}{\sqrt{2}} & -\frac{s^{\nu}_{23}}{\sqrt{2}} & c^{\nu}_{23} \end{pmatrix}; \quad \mu - \tau \ \text{symmetry}: \ \ \theta^{\nu}_{23} = \mp\pi/4\,;$$

$$U_{\rm GR} = \begin{pmatrix} c^{\nu}_{12} & s^{\nu}_{12} & 0 \\ -\frac{s^{\nu}_{12}}{\sqrt{2}} & \frac{c^{\nu}_{12}}{\sqrt{2}} & -\sqrt{\frac{1}{2}} \\ -\frac{s^{\nu}_{12}}{\sqrt{2}} & \frac{c^{\nu}_{12}}{\sqrt{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}; \quad U_{\rm HGM} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \theta^{\nu}_{12} = \pi/6\,.$$

$U_{\rm GRAM}$: $\sin^2\theta^{\nu}_{12} = (2+r)^{-1} \cong 0.276$, $r = (1+\sqrt{5})/2$
(GR: $r/1$; $a/b = a + b/a$, $a > b$**)**

$U_{\rm GRBM}$: $\sin^2\theta^{\nu}_{12} = (3-r)/4 \cong 0.345$.

GRB and HG mixing: W. Rodejohann et al., 2009.

$U_{\rm TBM(BM)}$...: Groups A_4, T', S_4 (S_4),...(vast literature)

E. Ma, G. Rajasekaran, hap-ph/0106291; K. Babu, E. Ma, J.F.W. Valle, hep-ph/0206292; G. Altarelli, F. Feruglio, hep-ph/0512103; C.S. Lam, 0708.3665 and 0804.2622; W. Grimus, L.Lavoura, 0809.0226; Z.-Z. Xing, 1106.3244; S. Zhou, 1205.0761; F. Feruglio, C. Hagedorn, R. Ziegler, 1211.5560; M. Holthausen, M. Lindner, M.A. Schmidt, 1211.6953; A. Meroni, S.T.P., M. Spinrath, 1312:1966; S.T.P., 1405.6006; ...
(Reviews: G. Altarelli, F. Feruglio, arXiv:1002.0211; M. Tanimoto et al., arXiv:1003.3552; S. King, Ch. Luhn, arXiv:1301.1340;...)

- $U_{\rm GRA}$: Group A_5,...; $s^2_{13} = 0$ and possibly $s^2_{12} = 0.276$ and $s^2_{23} = 1/2$ must be corrected.

L. Everett, A. Stuart, arXiv:0812.1057;...

- $U_{\rm LC}$: alternatively $U(1)$, $L' = L_e - L_\mu - L_\tau$

S.T.P., 1982

- $U_{\rm LC}$: $s^2_{12} = 1/2$, $s^2_{13} = 0$, s^ν_{23} - free parameter; $s^2_{13} = 0$ and $s^2_{12} = 1/2$ must be corrected.

• $U_{\rm GRB}$: Group D_{10},...; $s^2_{13} = 0$ and possibly $s^2_{12} = 0.345$ and $s^2_{23} = 1/2$ must be corrected.

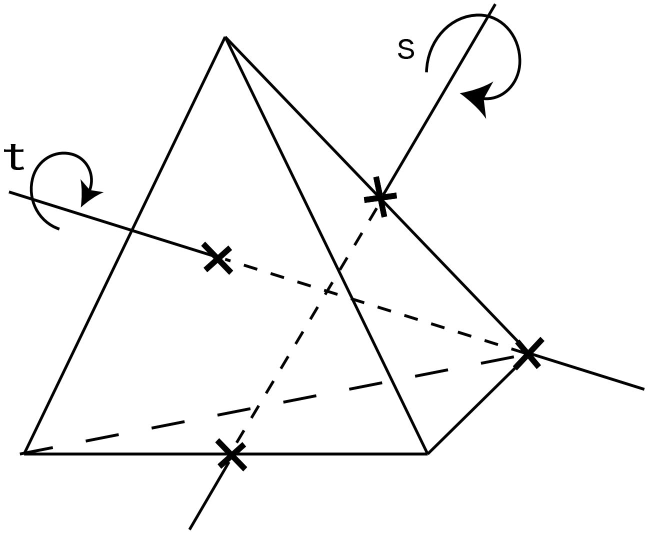
• $U_{\rm HG}$: Group D_{12},...; $s^2_{13} = 0$, $s^2_{12} = 0.25$ and possibly $s^2_{23} = 1/2$ must be corrected.

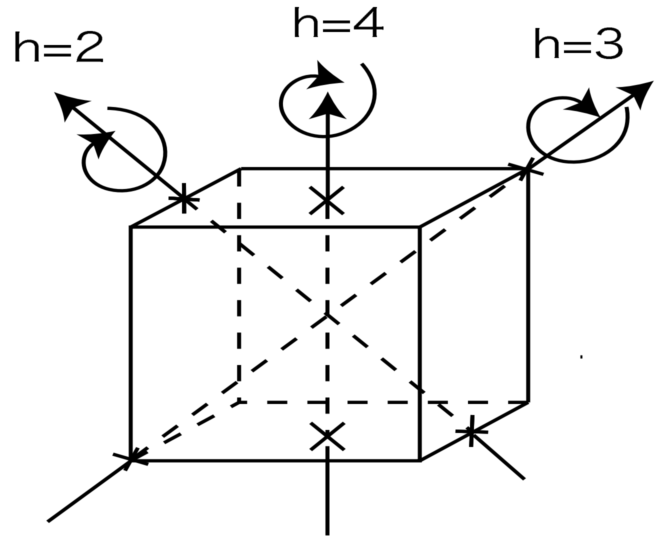
For all symmetry forms considered we have: $\theta^{\nu}_{13} = 0$, $\theta^{\nu}_{23} = \mp\pi/4$.
They differ by the value of θ^{ν}_{12}:
TBM, BM, GRA, GRB and HG forms correspond to $\sin^2\theta^{\nu}_{12} = 1/3;\ 0.5;\ 0.276;\ 0.345;\ 0.25$.

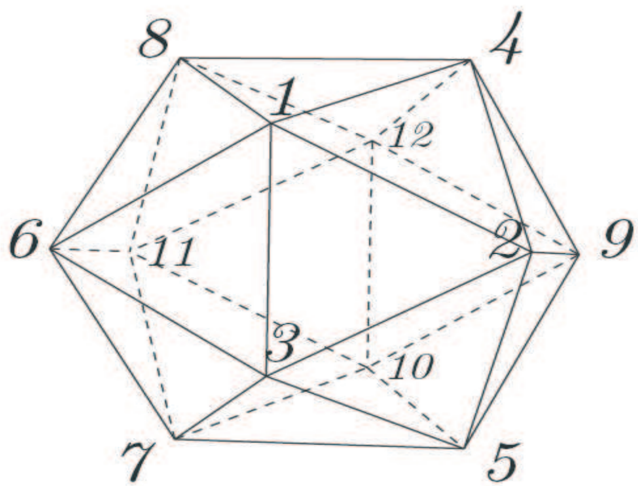
The observed pattern of 3-ν mixing, two large and one small mixing angles,
$\theta_{12} \cong 33^\circ$, $\theta_{23} \cong 45^\circ \pm 6^\circ$ and $\theta_{13} \cong 8.4^\circ$,
can most naturally be explained by extending the Standard Model (SM) with a flavour symmetry corresponding to a non-Abelian discrete (finite) group G_f.

$G_f = A_4$, T', S_4, A_5, D_{10}, D_{12},...

Vast literature; reviews: G. Altarelli, F. Feruglio, 1002.0211; H. Ishimori et al., 1003.3552; M. Tanimoto, AIP Conf.Proc. 1666 (2015) 120002; S. King , Ch. Luhn, 1301.1340; D. Meloni, 1709.02662; STP, 1711.10806; F. Feruglio, A. Romanino, 1912.06028







Examples of symmetries: A_4, S_4, A_5

From M. Tanimoto et al., arXiv:1003.3552

Group	Number of elements	Generators	Irreducible representations
S_4	24	S, T (U)	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3'}$
S'_4	48	S, T (R)	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3'}$, $\hat{\mathbf{1}}$, $\hat{\mathbf{1}}'$, $\hat{\mathbf{2}}$, $\hat{\mathbf{3}}$, $\hat{\mathbf{3}}'$
A_4	12	S, T	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{1''}$, $\mathbf{3}$
T'	24	S, T (R)	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{1''}$, $\mathbf{2}$, $\mathbf{2'}$, $\mathbf{2''}$, $\mathbf{3}$
A_5	60	$\tilde{S}$, $\tilde{T}$	$\mathbf{1}$, $\mathbf{3}$, $\mathbf{3'}$, $\mathbf{4}$, $\mathbf{5}$
A'_5	120	$\tilde{S}$, $\tilde{T}$	$\mathbf{1}$, $\mathbf{3}$, $\mathbf{3'}$, $\mathbf{4}$, $\mathbf{5}$, $\hat{\mathbf{2}}$, $\hat{\mathbf{2}}'$, $\hat{\mathbf{4}}$, $\hat{\mathbf{6}}$.

Number of elements, generators and irreducible representations of S_4, S'_4, A_4, $A'_4 \equiv T'$, A_5 and A'_5 discrete groups.

Predictions and Correlations

$U_\nu = U_{\text{TBM,BM,GRA,GRB,HG}}\,\bar{P}(\xi_1, \xi_2)$; θ^ν_{12};

$U_\ell^\dagger = R_{12}(\theta^\ell_{12})\,Q$, $Q = \text{diag}(e^{i\varphi}, 1, 1)$: $\theta^\ell_{12}, \varphi$
(the "minimal"= simplest case ($SU(5) \times T'$,...)

$U_\ell^\dagger = R_{12}(\theta^\ell_{12})\,R_{23}(\theta^\ell_{23})Q$, $Q = \text{diag}(1, e^{-i\psi}, e^{-i\omega})$,
(next-to-minimal case): $\theta^\ell_{12}, \hat{\theta}^\ell_{23}, \phi$

$\theta_{12}, \theta_{23}, \theta_{13}, \delta$ **in terms of** $\theta^\ell_{12}, \hat{\theta}^\ell_{23}, \phi$ + θ^ν_{12}

$\cos\delta = \cos\delta(\theta_{12}, \theta_{23}, \theta_{13}; \theta^\nu_{12}, ...)$,

$J_{CP} = J_{CP}(\theta_{12}, \theta_{23}, \theta_{13}, \delta) = J_{CP}(\theta_{12}, \theta_{23}, \theta_{13}; \theta^\nu_{12}, ...)$,

θ^ν_{12},... - **known (fixed) parameters, depend on the underlying symmetry.**

For arbitrary fixed θ^{ν}_{12} and any θ_{23} ("minimal" and "next-to-minimal" cases):

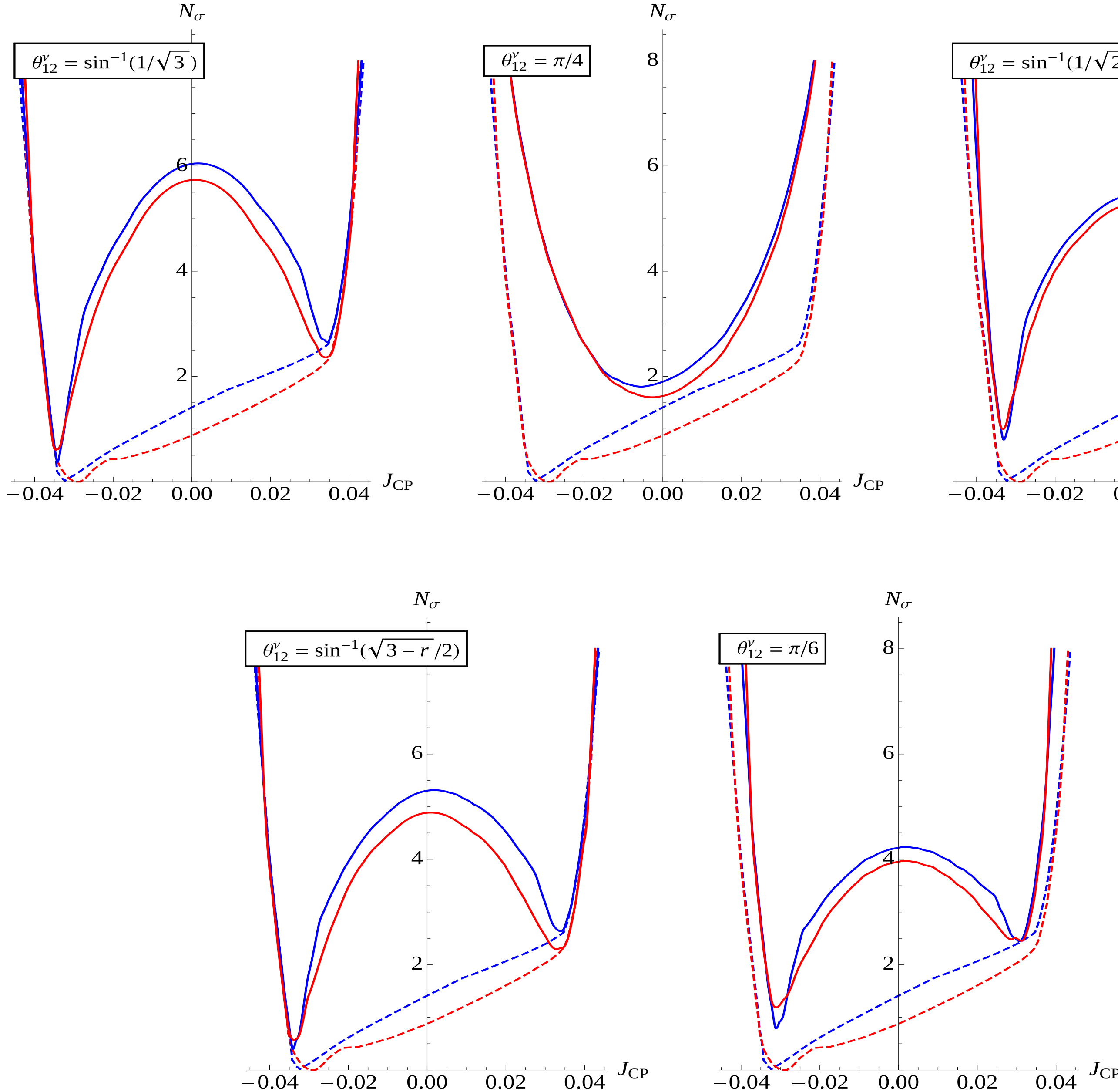
$$\cos\delta = \frac{\tan\theta_{23}}{\sin 2\theta_{12}\sin\theta_{13}}\Big[\cos 2\theta^{\nu}_{12}$$

$$+\left(\sin^2\theta_{12} - \cos^2\theta^{\nu}_{12}\right)\left(1 - \cot^2\theta_{23}\,\sin^2\theta_{13}\right)\Big]\,.$$

S.T.P., arXiv:1405.6006

This results is exact.

"Minimal" case: $\sin^2\theta_{23} = \frac{1}{2}\,\frac{1-2\sin^2\theta_{13}}{1-\sin^2\theta_{13}}$.



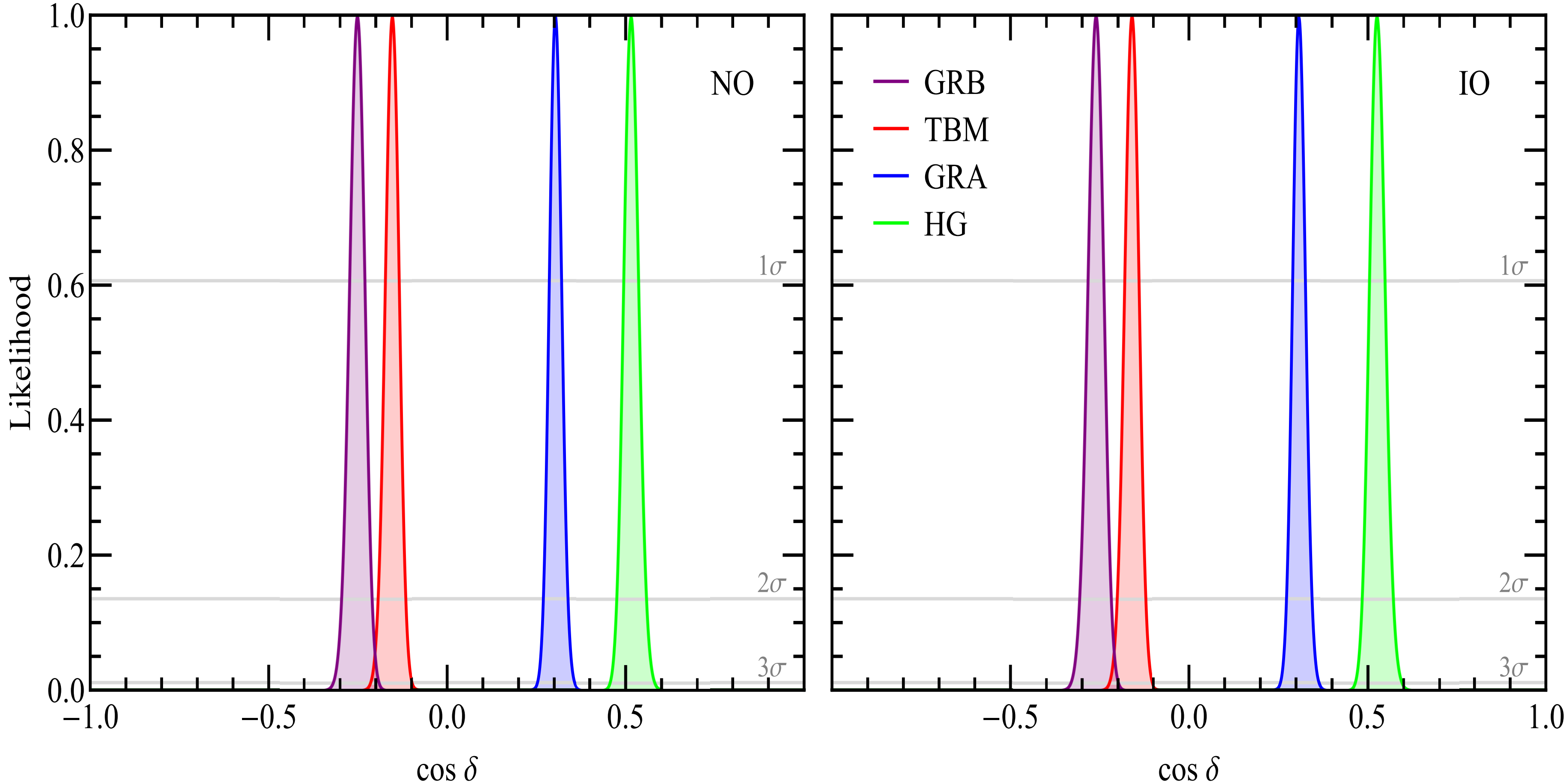
I. Girardi, S.T.P., A. Titov

Prospective precision:

$\delta(\sin^2\theta_{12}) = 0.7\%$ **(JUNO),**

$\delta(\sin^2\theta_{13}) = 3\%$ **(Daya Bay),**

$\delta(\sin^2\theta_{23}) = 3\%$ **(T2K, NOνA combined).**

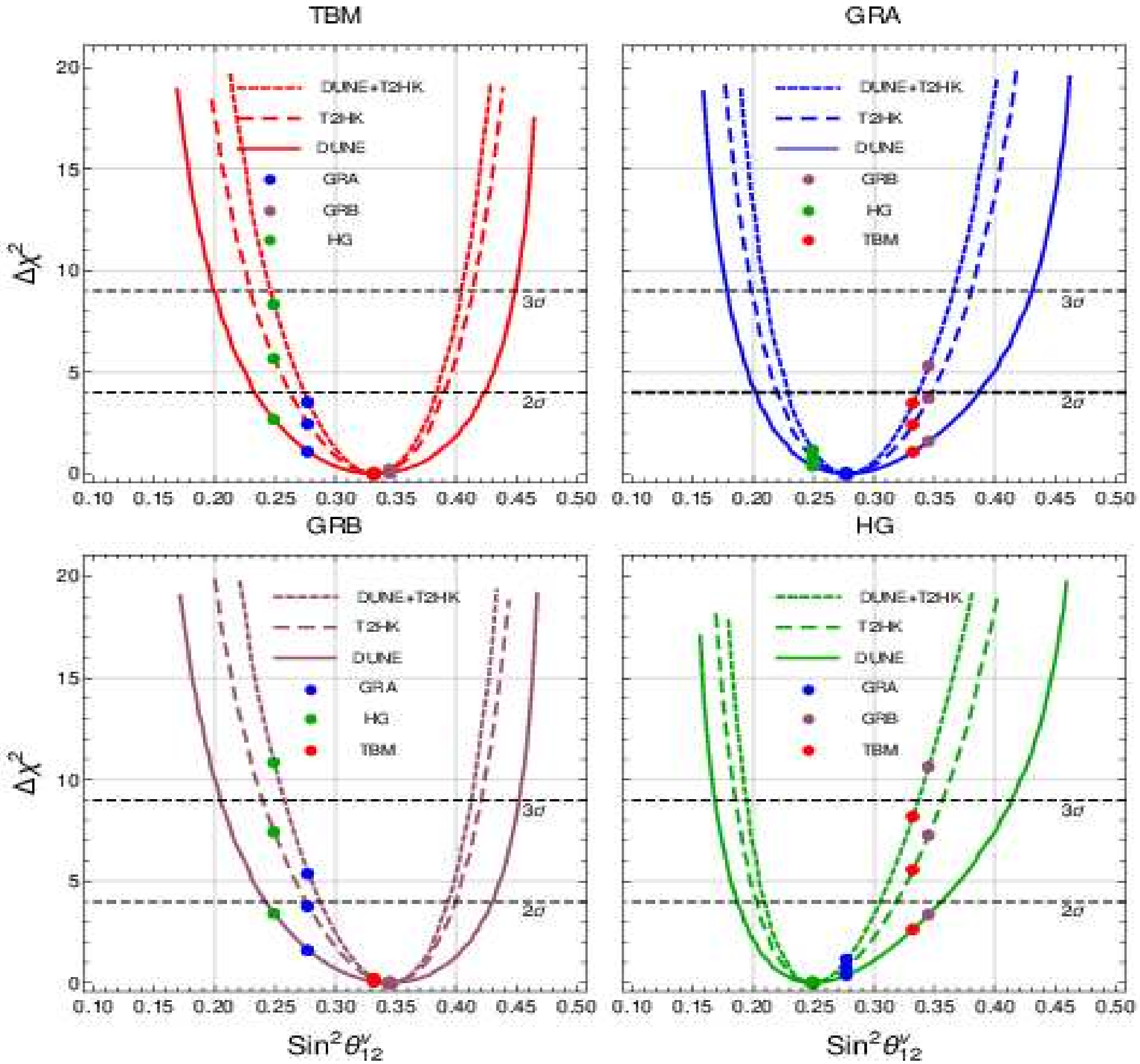


I. Girardi, S.T.P., A. Titov

b.f.v. of $\sin^2\theta_{ij}$ **(Esteban et al., Jan., 2018) + the prospective precision used.**

$$\cos\delta = \frac{\tan\theta_{23}}{\sin 2\theta_{12}\sin\theta_{13}}\left[\cos 2\theta^{\nu}_{12} + \left(\sin^2\theta_{12} - \cos^2\theta^{\nu}_{12}\right)\left(1 - \cot^2\theta_{23}\,\sin^2\theta_{13}\right)\right]\,.$$

$\delta(\sin^2\theta_{23}) = 3\%$ **(T2HK, DUNE).**



Agarwalla, Chatterjee, STP, Titov, arXiv:1711.02107

GRB - HG $> 3\sigma$; GRA - GRB $\geq 2\sigma$; TMB - HG $\cong 3\sigma$; TMB - GRA $\cong 2\sigma$. DUNE + T2HK prospective data used.

How does it Work.

Choose G_f.

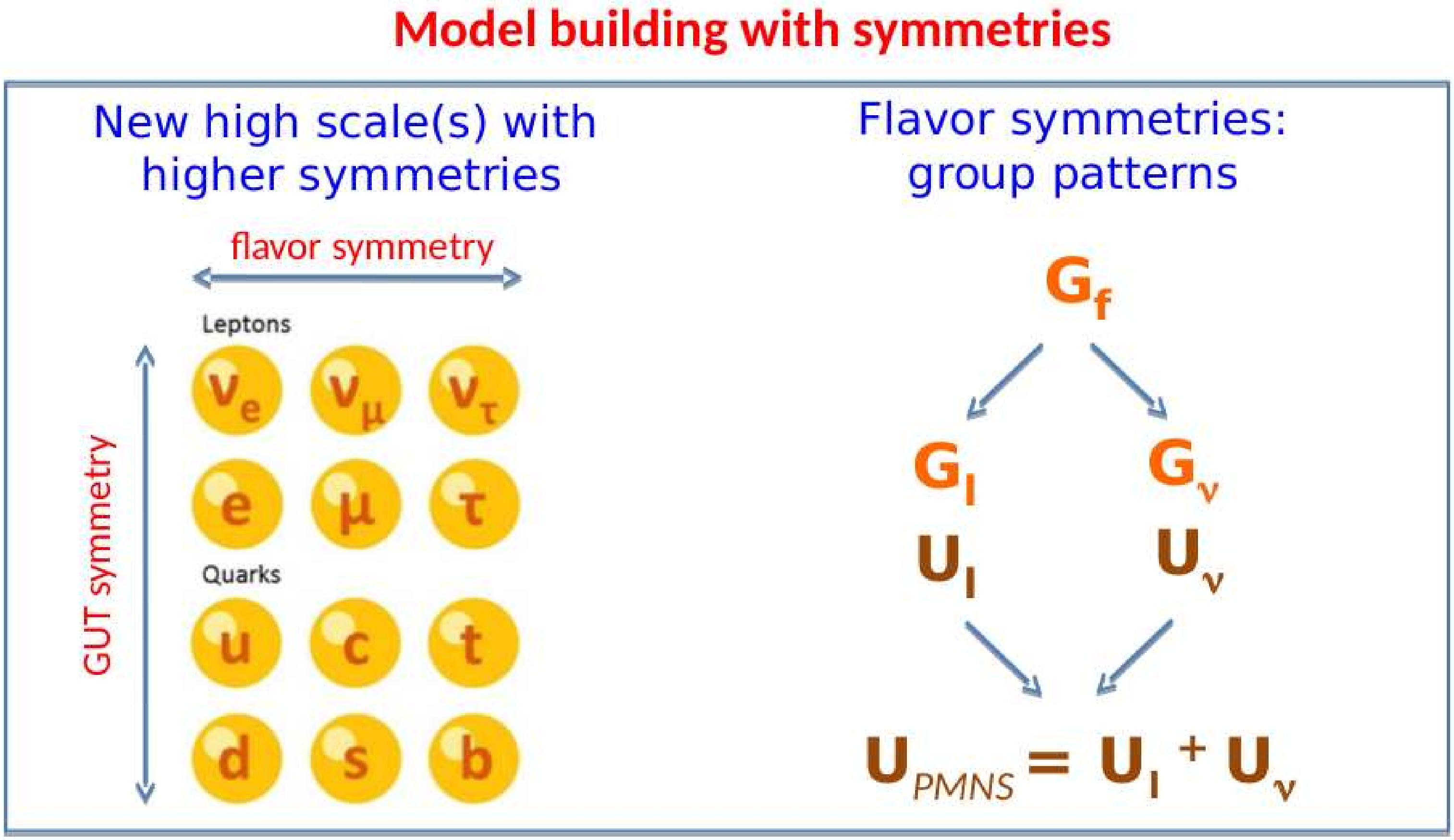
$\nu_{eL}(x), \nu_{\mu L}(x), \nu_{\tau L}(x)$: **assigned to $\rho^{(\nu)}(g_f)$ - irreducible representation of G_f, where g_f is an element of G_f.**

$e_L(x), \mu_L(x), \tau_L(x)$: **assigned to $\rho^{(e)}(g_f)$ - IRREP of G_f.**

$G_f = S_4$, A_4, T', A_5: $\rho^{(\nu)}(g_f)$, $\rho^{(e)}(g_f)$ - **triplet IRREP.**

$e_R(x), \mu_R(x), \tau_R(x)$: **singlets of G_f.**

How Does it Work



E. Lisi, TAUP 2019

ν_j, **Majorana mass term,** $m_j \neq m_k$, $j \neq k = 1, 2, 3$: $G_\nu = Z_2 \times Z_2$, Z_2

$G_e = Z_2$; Z_n, $n > 2$; $Z_n \times Z_m$, $n, m \geq 2$

M_e - charged lepton mass matrix (L-R convention).

U_e: $U_e^\dagger M_e M_e^\dagger U_e = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2)$.

G_e - residual symmetry group of $M_e M_e^\dagger$:

$$\rho^{(e)}(g_e)^\dagger M_e M_e^\dagger \rho(g_e) = M_e M_e^\dagger,$$

$\rho^{(e)}(g_e)$ generator(s) of G_e in the triplet rep.

$\rho^{(e)}(g_e)$ and $M_e M_e^\dagger$ commute: both are diagonalised by U_e.
$\rho^{(e)}(g_e)$ - known! Thus, U_e - fixed!

M_ν - **neutrino Majorana mass matrix (R-L convention).**

U_ν: $U_\nu^T M_\nu U_\nu = \mathrm{diag}(m_1, m_2, m_3)$.

G_ν - **residual symmetry group of** M_ν:

$\rho(g_\nu)^T M_\nu \rho(g_\nu) = M_\nu$,

g_ν: **an element of** G_ν,
$\rho(g_\nu)$ **generator of** G_ν **in the triplet repr.**

$\rho(g_\nu)$ **and** $M_\nu^\dagger M_\nu$ **commute: both are diagonalised by** U_ν.
$\rho(g_\nu)$ - **known! Thus,** U_ν**-fixed.**

$$U_{\text{PMNS}} = U_e^\dagger U_\nu$$

A_4: $G_e = Z_3^T = \{1, T, T^2\}$, $G_\nu = Z_2^S = \{1, S\}$

$(S^2 = T^3 = (ST)^3 = \mathbf{I})$

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \omega = e^{i2\pi\tau/3} \text{ (A − F)}.$$

$U_e = \mathbf{I}$, $U_{\rm PMNS} = U_e^\dagger U_\nu = U_{\rm TBM} U_{13}(\theta^\nu_{13}, \alpha)$, θ^ν_{13}, α - free.

W. Grimus, L. Lavoura, 2008

$\sin^2\theta_{12} = \frac{1}{3(1-\sin^2\theta_{13})} \cong 0.34$;

$\cos\delta = \frac{\cos 2\theta_{23} \cos 2\theta_{13}}{\sin 2\theta_{23} \sin\theta_{13} (2-3\sin^2\theta_{13})^{\frac{1}{2}}}$; if $\theta_{23} = \frac{\pi}{4}$, $\delta = \pm\frac{\pi}{2}$.

Examples of Predictions and Correlations II.

- $\sin^2\theta_{23} = \frac{1}{2}$.
- $\sin^2\theta_{23} \cong \frac{1}{2}(1 \mp \sin^2\theta_{13}) + O(\sin^4\theta_{13}) \cong \frac{1}{2}(1 \mp 0.022)$.
- $\sin^2\theta_{23} = 0.455;\ 0.463;\ 0.537;\ 0.545;\ 0.604$.
- $\sin^2\theta_{12} \cong \frac{1}{3}(1 + \sin^2\theta_{13}) + O(\sin^4\theta_{13}) \cong 0.340$.
- $\sin^2\theta_{12} \cong \frac{1}{3}(1 - 2\sin^2\theta_{13}) + O(\sin^4\theta_{13}) \cong 0.319$.
- **and/or** $\cos\delta = \cos\delta(\theta_{12}, \theta_{23}, \theta_{13}; \theta^{\nu}_{12}, ...)$,

$J_{CP} = J_{CP}(\theta_{12}, \theta_{23}, \theta_{13}, \delta) = J_{CP}(\theta_{12}, \theta_{23}, \theta_{13}; \theta^{\nu}_{12}, ...)$,

$\theta^{\nu}_{12}, ...$ - **known (fixed) parameters, depend on the underlying symmetry**.

The Approach is testable/falsifyable experimentally!

The measurement of the Dirac phase in the PMNS mixing matrix, together with an improvement of the precision on the mixing angles θ_{12}, θ_{13} and θ_{23}, can provide unique information about the possible existence of new fundamental symmetry in the lepton sector.

Prospective (useful/requested) precision:

$\delta(\sin^2\theta_{12}) = 0.7\%$ **(JUNO),**

$\delta(\sin^2\theta_{13}) = 3\%$ **(Daya Bay),**

$\delta(\sin^2\theta_{23}) = 3\%$ **(T2HK, DUNE; T2K+NOνA(?)).**

$\delta(\delta) \leq 14^\circ$ **at** $\delta = 3\pi/2$ **($\delta(\delta) = 10^\circ$)**
(ESSνSB: ~8%, A. Alekou et al., EPJ ST 231 (2022) 379; THKK?; DUNE: accounting for both the 1st and 2nd probability maxima, Jogesh Rout, Poonam Mehta et al., PRD 2021, S. Goswami et al., 2012.04958)

The Power of Data

Systematic analysis (I. Girardi *et al.*):
all possible combinations of residual symmetries G_e and G_ν of the lepton flavour symmetry groups $G_f = S_4$, A_4, T' and A_5, leading to correlations between some of the three neutrino mixing angles and/or between the neutrino mixing angles and the Dirac CPV phase δ, were considered.

(A) $G_e = Z_2$ **and** $G_\nu = Z_k$, $k > 2$ **or** $Z_m \times Z_n$, $m, n \geq 2$;

(B) $G_e = Z_k$, $k > 2$ **or** $Z_m \times Z_n$, $m, n \geq 2$ **and** $G_\nu = Z_2$;

(C) $G_e = Z_2$ **and** $G_\nu = Z_2$.

In these cases $U_e^\dagger$ **and/or** U_ν **of** $U = U_e^\dagger U_\nu = (\tilde{U}_e)^\dagger \Psi \tilde{U}_\nu Q_0$, **are partially (or fully) determined by residual discrete symmetries of** $G_f = S_4$, A_4, T', A_5.

More specifically:

A. $G_e = Z_2$, $G_\nu = Z_n$, $n > 2$ **or** $Z_n \times Z_m$, $n, m \geq 2$;
U_ν **fixed;** $U_e = U_{ij}(\theta^e_{ij}, \delta_{ij})$, $ij = 12(A1); 13(A2), 23(A3)$;
θ_{23}, $\cos\delta$ **(**θ_{12}, θ_{13}**) predicted.**

B. $G_e = Z_n$, $n > 2$ **or** $G_e = Z_n \times Z_m$, $n, m \geq 2$, $G_\nu = Z_2$;
$U_\nu = U_{\rm sym} U_{ij}(\theta^\nu_{ij}, \delta_{ij}) Q_0$, $ij = 13, 12, 23$**(B1,B2,B3);**
U_e **fixed:** θ_{12}, $\cos\delta$ **(**θ_{23}, θ_{13}**) predicted.**

C. $G_e = Z_2$ **and** $G_\nu = Z_2$: U_e **- up to** $U_{ij}(\theta^e_{ij}, \delta^\varepsilon_{ij})$,
U_ν **- up to** $U_{ij}(\theta^\nu_{ij}, \delta^\nu_{ij})$, $ij = 12; 13, 23$ **(C1 - C9);**
θ_{12} **or** θ_{23} **or** $\cos\delta$ **predicted.**

For A_4, S_4 and A_5 the total number of models to be analysed is extremely large. However, a total of only 14 models survive the 3σ constraints on $\sin^2\theta_{ij}$ from the current data and the requirement $|\cos\delta| \leq 1$.

$G_f = A_4$, S_4, T', A_5.

A_4: **3** Z_2, **4** Z_3, **1** $Z_2 \times Z_2$ **subgroups (total 8).**

T': **similar to** A_4.

S_4: **9** Z_2, **4** Z_3, **3** Z_4, **4** $Z_2 \times Z_2$ **subgroups (total 20).**

A_5: **has 15** Z_2, **10** Z_3, **6** Z_5, **5** $Z_2 \times Z_2$ **subgroups (36).**

In the case of A_4 (T') symmetry only there are 64 models (up to permutation of rows and columns).

A_4:
$(G_e, G_\nu) = (Z_2, Z_3)$, **A1 - A3;**
$(G_e, G_\nu) = (Z_2, Z_2)$, **A1 - A3;**
$(G_e, G_\nu) = (Z_3, Z_2)$, **B1 - B3;**
$(G_e, G_\nu) = (Z_2 \times Z_2, Z_2)$, **B1 - B3;**
$(G_e, G_\nu) = (Z_2, Z_2)$, **C1 - C9.**

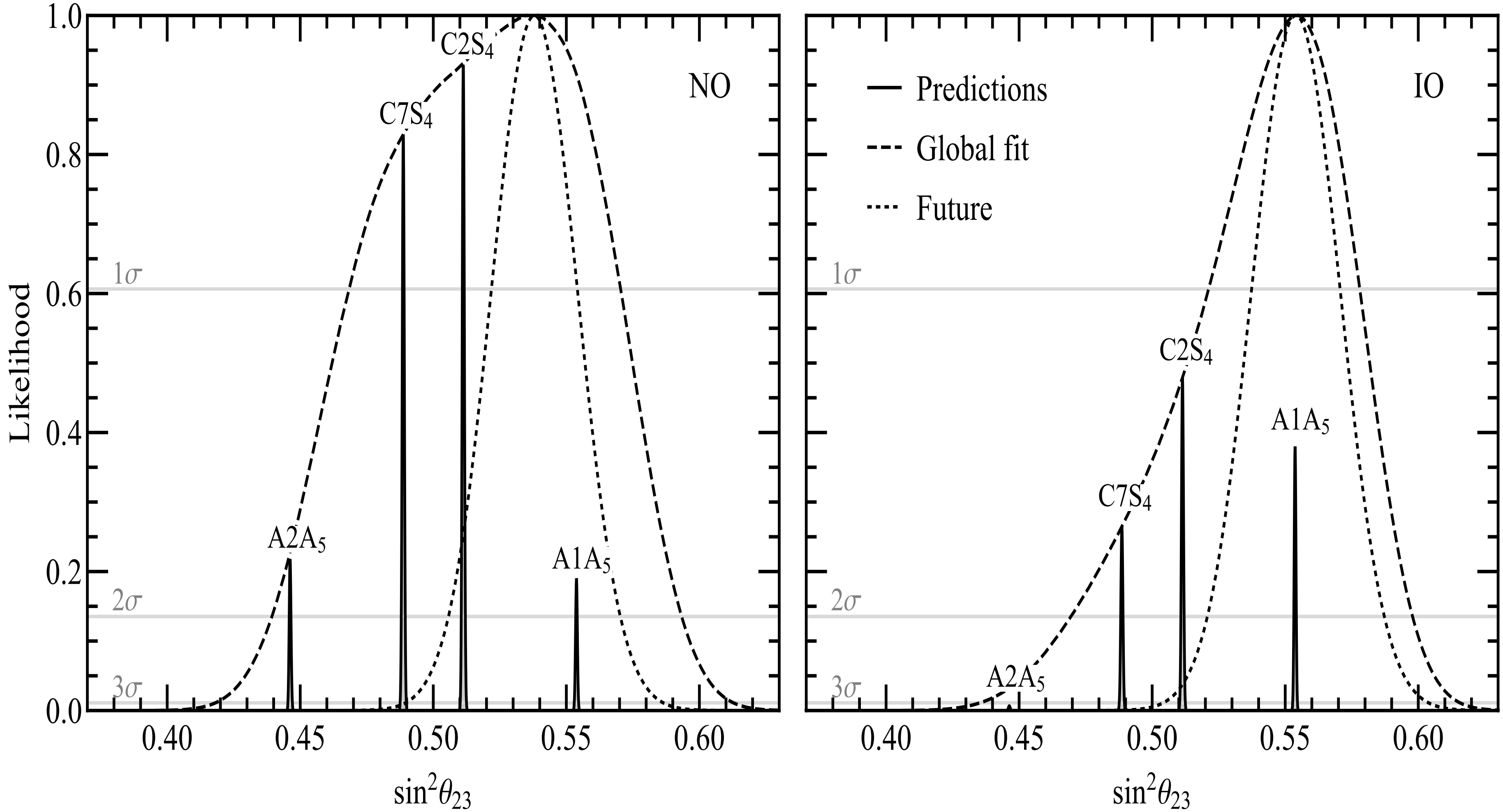
For A_4, S_4 and A_5 the total number of models to be analysed is extremely large. However, a total of only 14 models survive the 3σ constraints on $\sin^2\theta_{ij}$ from the current data and the requirement $|\cos\delta| \leq 1$.

Phenomenologically Viable Predictions

A1 (A2), A_5 **(**$G_e = Z_2$, $G_\nu = Z_3$ **(Dirac** ν_j**)):**
$\sin^2\theta_{23} \cong 0.553$ (0.447); $\cos\delta \cong 0.716$ (-0.716).

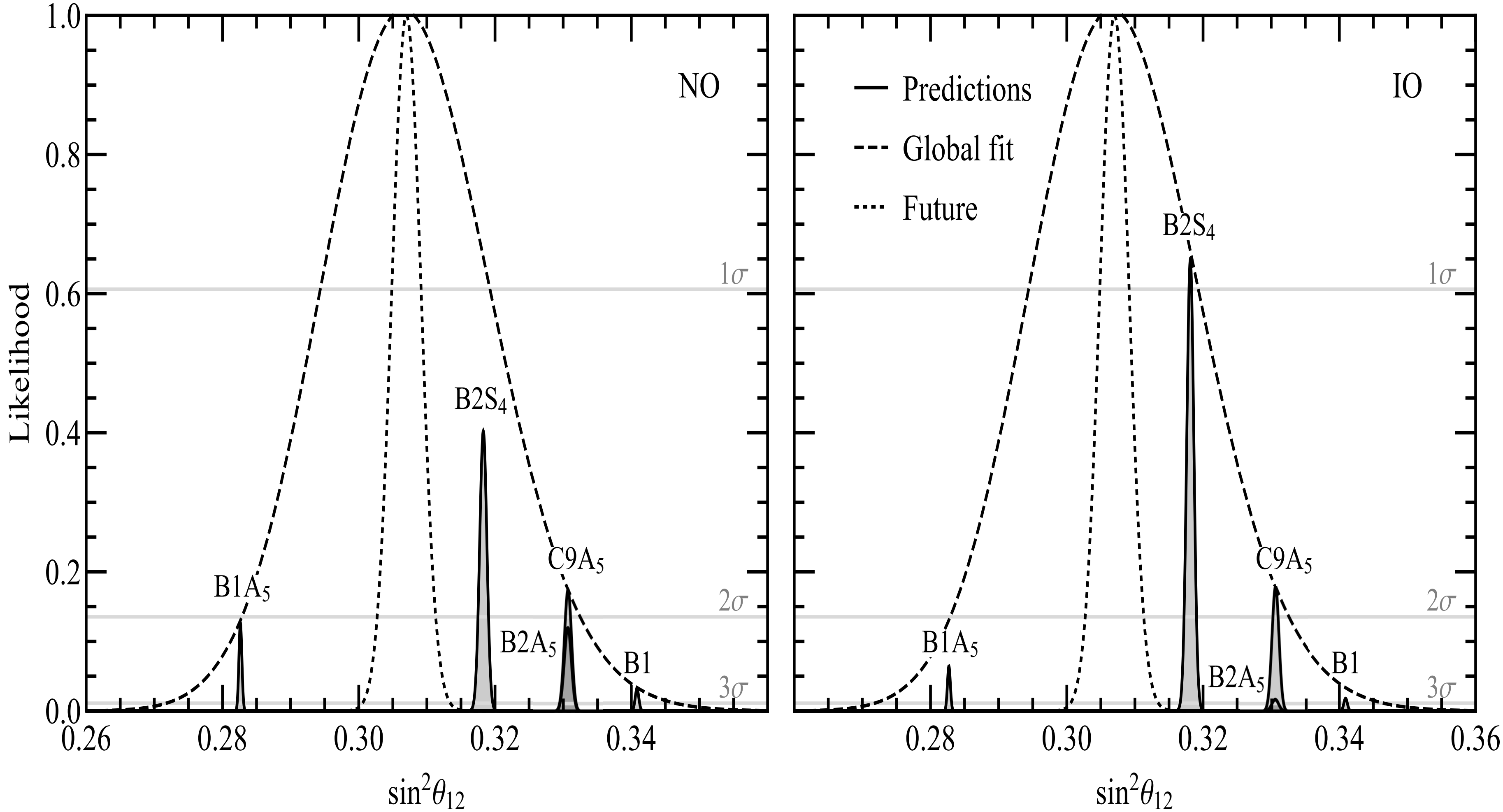
B1, A_4 **(**T', S_4, A_5**)** **(**$G_e = Z_3^T$, $G_\nu = Z_2^S$**):**
$U_{\rm PMNS} = U_{\rm TBM} U_{13}(\theta^\nu_{13}, \delta_{13}) Q_0$;
$\sin^2\theta_{12} = 1/(3\cos^2\theta_{13}) \cong 0.340$; $\cos\delta \cong 0.570$.

B2, S_4 **(**$G_e = Z_3^T$, $G_\nu = Z_2^{SU}$**):**
$\sin^2\theta_{12} \cong (1 - 2\sin^2\theta_{13})/3 = 0.319$; $\cos\delta \cong -0.269$.



S.T.P., A. Titov, arXiv:1804.00182

Future: $\delta(\sin^2\theta_{23}) = 3\%$ **(T2HK, DUNE).**



S.T.P., A. Titov, arXiv:1804.00182

Future: $\delta(\sin^2\theta_{12}) = 0.7\%$ **(JUNO).**

A total of 6 models would survive out of the currently viable 14 (of the extremely large number) considered if $\delta(\sin^2\theta_{23}) = 3\%$, $\delta(\sin^2\theta_{12}) = 0.7\%$ and the current b.f.v. would not change:

A1A_5, C3, C3A_5, C4A_5, C8, C2S_4.

Will be constrained further by the data on δ.

The Symmtry Breaking

The correct lepton mixing pattern in a model with non-Abelian discrete symmetry G_f is determined by the appropriate choice of residual symmetries G_e and G_ν and is not directly related to the charged lepton and neutrino mass generation.
The breaking of G_f has to ensure the correct generation of the fermion masses and keep G_e and G_ν intact.

The symmetry breaking in the lepton and quark flavour models based on non-Abelian discrete symmetries is impressively complicated: it requires the introduction of a plethora of “flavon” scalar fields having elaborate potentials, which in turn require large shaping symmetries to ensure the requisite breaking of the symmetry leading to correct mass and mixing patterns.

The Flavour Problem: Modular Invariance Approach

Modular invariance approach to the flavour problem was proposed in F. Feruglio, arXiv:1706.08749 and has been intensively developed in the last five years.

In this approach the flavour (modular) symmetry is broken by the vacuum expectation value (VEV) of a single scalar field - the modulus τ. The VEV of τ can also be the only source of violation of the CP symmetry.

Many (if not all) of the drawbacks of the widely studied alternative approaches are absent in the modular invariance approach to the flavour problem.

The first phenomenologically viable "minimal" (in terms of fields, i.e., without flavons) lepton flavour model based on modular symmetry appeared in June of 2018 (J.T. Penedo, STP, arXiv:1806.11040). Since then various aspects of this approach were and continue to be extensively studied – the number of publications on the topic exceeds 190.

Matter Fields and Modular Forms

The matter(super)fields (charged lepton, neutrino, quark) transform under $\overline{\Gamma} \simeq PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$, $\mathbb{Z}_2 = \{I, -I\}$ $\quad$ ($\Gamma \simeq SL(2,\mathbb{Z})$) as "weighted" multiplets:

$$\psi_i \xrightarrow{\gamma} (c\tau + d)^{-k_\psi} \rho_{ij}(\tilde{\gamma})\psi_j \,, \gamma \in \overline{\Gamma} \ (\gamma \in \Gamma) \,,$$

$$\left(\gamma\tau = \frac{a\tau+b}{c\tau+d} \,, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,, a, b, c, d \in \mathbb{Z} \,, ad - bc = 1 \,, \mathrm{Im}\tau > 0\right)$$

k_ψ **is the weight of** ψ; $k_\psi \in \mathbb{Z}$ **(or rational number).**
$\Gamma(N)$ **- principal congruence (normal) subgroup of** $SL(2,\mathbb{Z})$.
$\rho(\tilde{\gamma})$ **is a unitary representation of the** *inhomogeneous (homogeneous)* **finite modular group** $\Gamma_N = \overline{\Gamma}/\overline{\Gamma}(N)$ **(**$\Gamma'_N = \Gamma/\Gamma(N)$**),** $\tilde{\gamma}$ **− representation of** γ **in** Γ_N **(**Γ'_N**)**

F. Feruglio, arXiv:1706.08749; S. Ferrara et al., Phys.Lett. B233 (1989) 147, B225 (1989) 363

As we have indicated in brackets, one can consider also the case of Γ and $\gamma \in \Gamma(N)$. Then $\rho(\gamma)$ will be a unitary representation of the *homogeneous* **finite modular group Γ'_N.**

Remarkably, for $N \leq 5$, the inhomogeneous finite modular groups Γ_N are isomorphic to non-Abelian discrete groups widely used in flavour model building:
$\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$.
Γ_N is presented by two generators S and T satisfying:

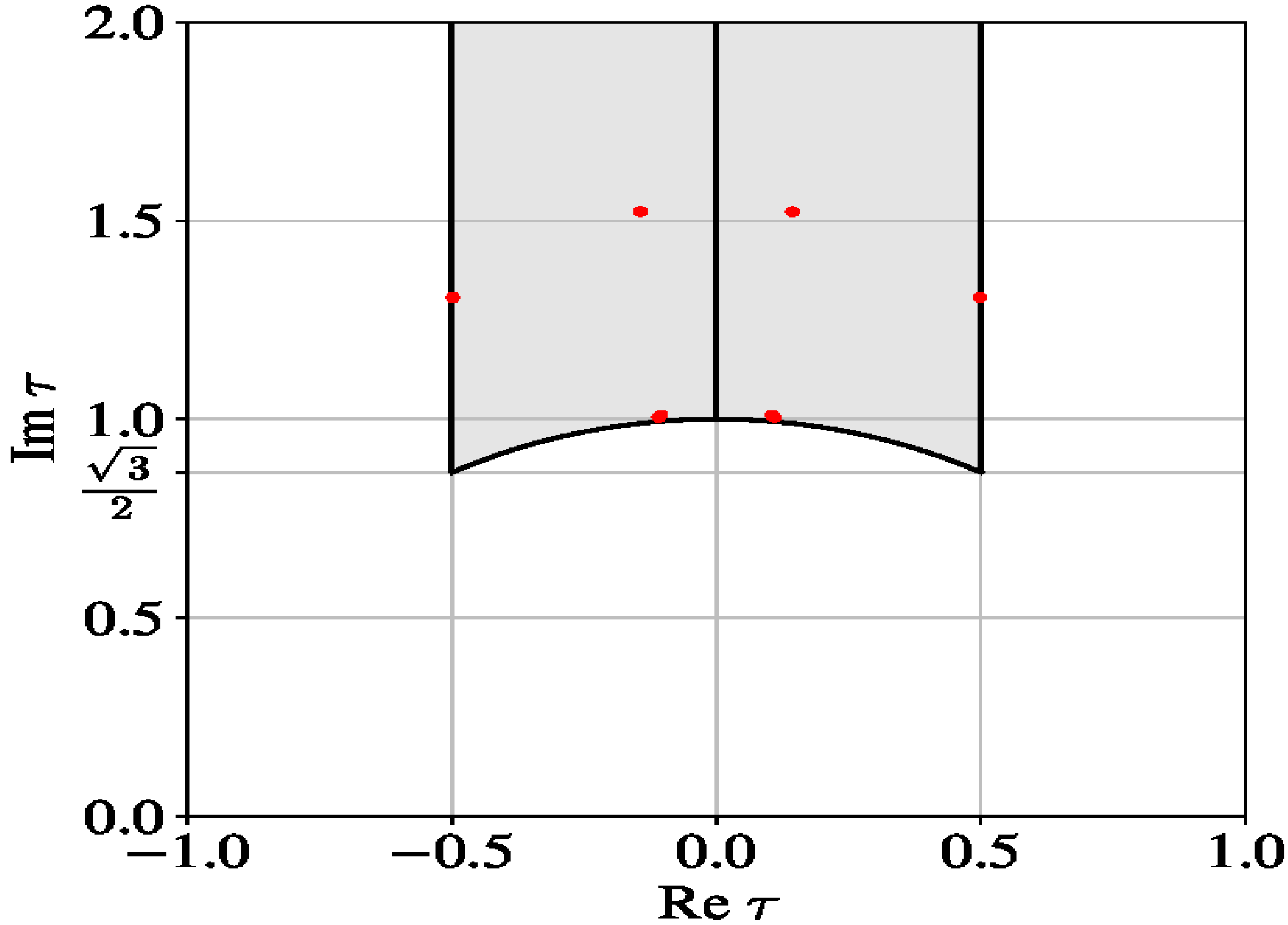
$$S^2 = (ST)^3 = T^N = I\,.$$

The group theory of $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$ is summarized, e.g., in P.P. Novichkov *et al.*, JHEP 07 (2019) 165, arXiv:1905.11970.

$\Gamma \simeq SL(2,\mathbb{Z})$ – *homogeneous* modular group, $\Gamma(N)$ and the quotient groups $\Gamma'_N \equiv \Gamma/\Gamma(N)$ – homogeneous finite modular groups. For $N = 3,4,5$, Γ'_N are isomorphic to the double covers of the corresponding non-Abelian discrete groups:
$\Gamma'_3 \simeq A'_4 \equiv T'$, $\Gamma'_4 \simeq S'_4$ and $\Gamma'_5 \simeq A'_5$.
Γ'_N is presented by two generators S and T satisfying:

$$S^4 = (ST)^3 = T^N = I\,,\;\; S^2\,T = T\,S^2 \quad (S^2 = R)\,.$$

The group theory of $\Gamma'_3 \simeq A'_4$, $\Gamma'_4 \simeq S'_4$ and $\Gamma'_5 \simeq A'_5$ for flavour model building was developed in X.-G. Liu, G.-J. Ding, arXiv:1907.01488 (A'_4); P.P. Novichkov et al., arXiv:2006.03058 (S'_4); C.-Y. Yao et al., arXiv:2011.03501 (A'_5).



The Fundamental Domain of $\overline{\Gamma}$ (Γ) shown for $\mathrm{Im}\tau \leq 2$ (the red dots correspond to solutions of the lepton flavour problem, see further).

P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933.

Mass Matrices

Consider the bilinear (i.e., mass term)

$$\psi_i^c \, M(\tau)_{ij} \, \psi_j \,,$$

where the fields ψ and ψ^c transform as

$$\begin{aligned} \psi &\xrightarrow{\gamma} (c\tau+d)^{-k}\rho_r(\gamma)\,\psi \quad (\rho(\gamma)\,,\ \Gamma_N^{(\prime)}\,,\ N=2,3,4,5)\,, \\ \psi^c &\xrightarrow{\gamma} (c\tau+d)^{-k^c}\rho^c_{r^c}(\gamma)\,\psi^c\,, \quad (\rho^c(\gamma)\,,\ \Gamma_N^{(\prime)})\,. \end{aligned}$$

Modular invariance: $M(\tau)_{ij}$ must be modular form of level N and weight $K \equiv k + k^c$,

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau+d)^K \, \rho_{r_Y}(\gamma) \, M(\gamma\tau)\,,$$

where $\rho(\gamma)_{r_Y}$ - irrep of $M(\gamma\tau)$:

$$K = k + k^c\,,$$

$$\mathbf{r}_Y \otimes \mathbf{r} \otimes \mathbf{r^c} \supset \mathbf{1}\,.$$

Modular Forms

Within the considered framework the elements of the Yukawa coupling and fermion mass matrices in the Lagrangian of the theory are expressed in terms of modular forms of a certain level N and weight k_f.
The modular forms are functions of a single complex scalar field – the modulus τ – and have specific transformation properties under the action of the modular group.
Both the modular forms of given level N and weight k_f and the matter fields (supermultiplets) are assumed to transform in representations of an inhomogeneous (homogeneous) finite modular group $\Gamma_N^{(\prime)}$.
Once τ acquires a VEV, the modular forms and thus the Yukawa couplings and the form of the mass matrices get fixed, and a certain flavour structure arises.
Quantitatively and barring fine-tuning, the magnitude of the values of the non-zero elements of the fermion mass matrices and therefore the fermion mass ratios are determined by the modular form values (which in turn are functions of the τ's VEV).

Modular Forms (contd.)

The key elements of the considered framework are modular forms $f(\tau)$ of *weight k_f and level N* **– holomorphic functions of τ, which transform under $\overline{\Gamma}$ (Γ) as follows:**

$$F(\gamma\tau) = (c\tau + d)^{k_F}\,\rho_{\mathbf{r}}(\tilde{\gamma})F(\tau)\,, \quad \gamma \in \overline{\Gamma} \quad (\gamma \in \Gamma)\,,$$

F. Feruglio, arXiv:1706.08749

ρ_{r} is a unitary representation of the finite modular group Γ_N (Γ'_N).
In the case of $\overline{\Gamma}$ (Γ) *non-trivial* **modular forms exist only for positive even integer (positive integer) weight k_F.**
For given k, N (N is a natural number), the modular forms span a linear space of finite dimension:
of weight k and level 3, $\mathcal{M}_k(\Gamma_3^{(\prime)} \simeq A_4^{(\prime)})$, is $k+1$;
of weight k and level 4, $\mathcal{M}_k(\Gamma_4^{(\prime)} \simeq S_4^{(\prime)})$, is $2k+1$;
of weight k and level 5, $\mathcal{M}_k(\Gamma_5^{(\prime)} \simeq A_5^{(\prime)})$, is $5k+1$.
Thus, $\dim\mathcal{M}_1(\Gamma'_3 \simeq A'_4) = 2$, $\dim\mathcal{M}_1(\Gamma'_4 \simeq S'_4) = 3$, $\dim\mathcal{M}_1(\Gamma'_5 \simeq A'_5) = 6$.

Multiplets of Γ_N (Γ'_N) of higher weight modular forms can be constructed from tensor products of the lowest weight 2 (weigh 1) multiplets (they represent homogeneous polynomials of the lowest weight modular forms).

Following arXiv:1706.08749, it was of highest priority and of crucial importance for model building to find the basis of modular forms of the lowest weight 2 (weight 1) transforming in irreps of Γ_N (Γ'_N).

It took about two years to find the requisite bases for Γ_N (Γ'_N), $N = 2, 3, 4, 5$.

F. Feruglio, 1706.08749 ($\Gamma_3 \simeq A_4$, $k_f = 2$: the 3 mod.forms form a 3 of A_4);
T. Kobayashi et al., 1803.10391 ($\Gamma_2 \simeq S_3$, $k_f = 2$: the 2 mod. forms form a **2** of S_3);
J. Penedo, STP, 1806.11040 ($\Gamma_4 \simeq S_4$, $k_f = 2$: the 5 mod. forms form a **2** and $3'$ of S_4);
P.P. Novichkov et al., 1812.02158; G.-J. Ding et al.: the 11 basis modular forms were shown to form a **3**, a $3'$ and a **5** of A_5).
More elegant constuction: modular forms for A'_4, S'_4, A'_5 (and A_4, S_4, A_5).

The weight 1 modular forms
i) of A'_4 form a 2 of A'_4, ii) of S'_4 form a $\hat{3}$ of S'_4, iii) of A'_5 form a 5 of A'_5, as was proven respectively in X.-G. Liu, G.-J. Ding, 1907.01488, P.P. Novichkov et al., 2006.03058 and C.-Y. Yao et al., 2011.03501.

In each of the cases of A_4', S_4' and A_5' the lowest weight 1 modular forms, and thus all higher weight modular forms, icluding those (of even weight) associated with A_4, S_4 and A_5, constructed from tensor products of the weight 1 multiplets, were shown (respectively in X.-G. Liu, G.-J. Ding, 1907.01488, P.P. Novichkov et al., 2006.03058 and C.-Y. Yao et al., 2011.03501) to be expressed in terms of only two independent functions of τ.

These pairs of functions are different for the three different groups; but they all are related (in different ways) to the Dedekind η-function (in the case of A_5' (A_5) - to two Jacobi theta constants also) and have similar (fastly converging) $q-$expansions, i.e., power series expansions in $q = e^{2\pi i\tau}$.

Thus, in the case of a flavour symmetry described by a finite modular group $\Gamma_N^{(\prime)}$, $N = 2, 3, 4, 5$, the elements of the matices of the Yukawa couplings in the considered approach represent homogeneous polynomials of various degree of only two (holomorphic) functions of τ. They include also a limited (relatively small) number of constant parameters.

The modular forms of level $N = 2, 3, 4, 5$ for $\Gamma^{(\prime)}_{2,3,4,5} \simeq S_3, A^{(\prime)}_4, S^{(\prime)}_4, A^{(\prime)}_5$ have been constructed by use of the of Dedekind eta function, $\eta(\tau)$:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) = q^{\frac{1}{24}} \sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{n(3n-1)}{2}} \, , q = e^{i2\pi\tau} \, .$$

In the cases of $\Gamma^{(\prime)}_5 \simeq A^{(\prime)}_5$ two "Jacobi theta constants" are also used.
Modular forms of level $N = 4$ for $\Gamma'_4 \simeq S'_4$ ($\Gamma_4 \simeq S_4$) − in terms of $\theta(\tau)$, $\varepsilon(\tau)$:

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau) \, , \; \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau) \, .$$

$\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants, $\eta(a\tau)$, $a = 1, 2, 4$, is the Dedekind eta.
Modular forms of level $N = 3$ for $\Gamma'_3 \simeq A'_4$ ($\Gamma_3 \simeq A_4$) − in terms of $\hat{e}_1$ and $\hat{e}_2$:

$$\hat{e}_1 = \frac{\eta^3(3\tau)}{\eta(\tau)} \, , \quad \hat{e}_2 = \frac{\eta^3(\tau/3)}{\eta(\tau)} \, .$$

Modular forms of level $N = 5$ for $\Gamma'_3 \simeq A'_5$ ($\Gamma_3 \simeq A_4$) − in terms of $\theta_5(\tau)$ and $\varepsilon_5(\tau)$: $\theta_5(\tau) = \exp(-\,i\pi/10)\,\Theta_{\frac{1}{10},\frac{1}{2}}(5\tau)\,\eta^{-3/5}(\tau)$,
$\varepsilon_5(\tau) = \exp(-\,i\,3\pi/10)\,\Theta_{\frac{3}{10},\frac{1}{2}}(5\tau)\,\eta^{-3/5}(\tau)$.

Example: S_4'

P.P. Novichkov, J.T. Penedo. S.T.P., arXiv:2006.03058

Weight 1 modular forms furnishing a $\hat{3}$ of S_4':

$$Y^{(1)}_{\hat{\mathbf{3}}}(\tau) = \begin{pmatrix} \sqrt{2}\,\varepsilon\,\theta \\ \varepsilon^2 \\ -\theta^2 \end{pmatrix}$$

Modular S_4 lowest-weight 2 multiplets furnish a 2 and a $3'$ irreducible representations of S_4 (S_4') and are given by: :

$$Y^{(2)}_{\mathbf{2}}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4 + \varepsilon^4\right) \\ -\sqrt{6}\,\varepsilon^2\,\theta^2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad Y^{(2)}_{\mathbf{3'}}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4 - \varepsilon^4\right) \\ -2\,\varepsilon\,\theta^3 \\ -2\,\varepsilon^3\,\theta \end{pmatrix} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}.$$

At weight $k=3$, a non-trivial singlet and two triplets exclusive to S_4' arise:

$$Y^{(3)}_{\hat{\mathbf{1}}'}(\tau) = \sqrt{3}\left(\varepsilon\,\theta^5 - \varepsilon^5\,\theta\right),$$

$$Y^{(3)}_{\hat{\mathbf{3}}}(\tau) = \begin{pmatrix} \varepsilon^5\,\theta + \varepsilon\,\theta^5 \\ \frac{1}{2\sqrt{2}}\left(5\,\varepsilon^2\,\theta^4 - \varepsilon^6\right) \\ \frac{1}{2\sqrt{2}}\left(\theta^6 - 5\,\varepsilon^4\,\theta^2\right) \end{pmatrix}, \quad Y^{(3)}_{\hat{\mathbf{3}}'}(\tau) = \frac{1}{2}\begin{pmatrix} -4\sqrt{2}\,\varepsilon^3\,\theta^3 \\ \theta^6 + 3\,\varepsilon^4\,\theta^2 \\ -3\,\varepsilon^2\,\theta^4 - \varepsilon^6 \end{pmatrix}.$$

The functions $\theta(\tau)$ and $\varepsilon(\tau)$ are given by:

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau)\,, \qquad \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau)\,.$$

$\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants, $\eta(a\tau)$, $a = 1, 2, 4$, is the Dedekind eta function.

The functions $\theta(\tau)$ and $\varepsilon(\tau)$ admit the following q-expansions - power series expansions in $q_4 \equiv \exp(i\pi\tau/2)$ ($\mathrm{Im}(\tau) \geq \sqrt{3}/2$, $|q_4| \lesssim 0.26$) :

$$\theta(\tau) = 1 + 2\sum_{k=1}^{\infty} q_4^{(2k)^2} = 1 + 2\,q_4^4 + 2\,q_4^{16} + \ldots\,,$$

$$\varepsilon(\tau) = 2\sum_{k=1}^{\infty} q_4^{(2k-1)^2} = 2\,q_4 + 2\,q_4^9 + 2\,q_4^{25} + \ldots\,.$$

In the "large volume" limit $\mathrm{Im}\,\tau \to \infty$, $\theta \to 1$, $\varepsilon \to 0$.
In this limit $\varepsilon \sim 2\,q_4$ and ε can be used as an expansion parameter instead of q_4.
Due to quadratic dependence in the exponents of q_4, the $q-$expansion series converge rapidly in the fundamental domain of the modular group, where $\mathrm{Im}(\tau) \geq \sqrt{3}/2$ and $|q_4| \leq \exp(-\pi\sqrt{3}/4) \simeq 0.26$.

Similar conclusions are valid for the pair of functions in terms of which the lowest weight 1 modular forms, and thus all higher weight modular forms of A_4' and A_5' are expressed.

Lepton sector: reference 3-ν mixing scheme

$$\nu_{l\mathrm{L}} = \sum_{j=1}^{3} U_{lj}\, \nu_{j\mathrm{L}} \qquad l = e, \mu, \tau.$$

Data: 3 νs are light: $\nu_{1,2,3}$, $m_{1,2,3} \lesssim 0.5$ eV;
KATRIN: $m_{\bar{\nu}_e} < 0.45$ eV (06/2024);
Cosmology: $\sum_j m_j <$ 0.12 - 0.77 eV (95% CL; 2107.00532).

The value of min(m_j) and "mass ordering" unknown.

Δm^2_{21}, $|\Delta m^2_{31}|$ - **known** (sgn(Δm^2_{31}) - **unknown**).

ν_j, $m_j \neq 0$: **nature - Dirac or Majorana - unknown.**

The PMNS matrix U - 3×3 unitary: θ_{12}, θ_{13}, θ_{23} - known; CPV phases δ, α_{21}, α_{31} - unknown.

Thus, 5 known + 4 unknown parameters + MO.
"Known" = measured; "unknown" = not measured.

m_e, m_μ, m_τ **also known - used as input.**

Example: Lepton Flavour Models Based on S_4 (Seesaw Models without Flavons)

P.P. Novichkov et al., arXiv:1811.04933

We assume that neutrino masses originate from the (supersymmetric) type I seesaw mechanism.

The fields involved:

- two Higgs doublets H_u and H_d; transform trivially under Γ_4, $\rho_u = \rho_d \sim \mathbf{1}$, $k_u = k_d = 0$;
- three lepton $SU(2)$ doublets L_1, L_2, L_3; furnish a 3-dim. irrep of S_4, i.e., $\rho_L \sim \mathbf{3}$ or $\mathbf{3'}$, and carry weight $k_L = 2$;
- three neutral lepton gauge singlets N_1^c, N_2^c, N_3^c; transform as a triplet of Γ_4, $\rho_N \sim \mathbf{3}$ or $\mathbf{3'}$, and carry weight $k_N = 0$;
- three charged lepton $SU(2)$ singlets E_1^c, E_2^c, E_3^c; transform as singlets of Γ_4, $\rho_{1,2,3} \sim \mathbf{1'}, \mathbf{1}, \mathbf{1'}$ and carry weights $k_{1,2.3} = 0, 2, 2$.

With these assumptions, the superpotential has the form:

$$W = \sum_{i=1}^{3} \alpha_i \left(E_i^c \, L \, f_{E_i}(Y) \right)_{\mathbf{1}} H_d + g \left(N^c \, L \, f_N(Y) \right)_{\mathbf{1}} H_u + \Lambda \left(N^c \, N^c \, f_M(Y) \right)_{\mathbf{1}}$$

$\alpha_{1,2,3}$, g, g', Λ are constants.

We work in a basis in which the S_4 generators S and T are represented by symmetric matrices for all irreducible representations $\mathbf{r}$. In this basis the triplet irreps of S and T to be used read:

$$S = \pm\frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \;\; T = \pm\frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix},$$

$\omega = e^{i2\pi\tau/3}$. **The plus (minus) corresponds to the irrep $\mathbf{3}$ ($\mathbf{3'}$) of S_4.**

In the employed basis we have:

$$ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

By specifying the weights of the matter fields one obtains the weights of the relevent modular forms.
After modular symmetry breaking, the matrices of charged lepton and neutrino Yukawa couplings, λ and $\mathcal{Y}$, as well as the Majorana mass matrix M for heavy neutrinos, are generated:

$$W = \lambda_{ij}\, E_i^c\, L_j\, H_d + \mathcal{Y}_{ij}\, N_i^c\, L_j\, H_u + \frac{1}{2}\, M_{ij}\, N_i^c\, N_j^c\,,$$

a sum over $i,j = 1,2,3$ is assumed. After integrating out N^c and after EWS breaking, the charged lepton mass matrix M_e and the light neutrino Majorana mass matrix M_ν are generated (we work in the L-R convention for the charged lepton mass term and the R-L convention for the light and heavy neutrino Majorana mass terms):

$$M_e = v_d\, \lambda^\dagger\,, \quad v_d \equiv \mathrm{vev}(\mathrm{H_d^0})\,,$$
$$M_\nu = -v_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y}\,, \quad v_u \equiv \mathrm{vev}(\mathrm{H_u^0})\,.$$

The Majorana mass term for heavy neutrinos

Assume $k_\Lambda = 0$, i.e., no non-trivial modular forms are present in $\Lambda\,(N^c\,N^c\,f_M\,(Y))_{\mathbf{1}}$, $k_N = 0$, and for both $\rho_N \sim \mathbf{3}$ or $\rho_N \sim \mathbf{3}'$

$$(N^c\,N^c)_{\mathbf{1}} = N_1^c\,N_1^c + N_2^c\,N_3^c + N_3^c\,N_2^c\,,$$

leading to the following mass matrix for heavy neutrinos:

$$M = 2\,\Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \textbf{for} \quad k_\Lambda = 0\,.$$

The spectrum of heavy neutrino masses is degenerate; the only free parameter is the overall scale Λ, which can be rendered real. The Majorana mass term conserves a "non-standard" lepton charge and two of the three heavy Majorana neutrinos with definite mass form a Dirac pair.

C.N. Leung, STP, 1983

The neutrino Yukawa couplings

The lowest non-trivial weight, $k_L = 2$, leads to

$$g \left(N^c \, L \, Y_{\mathbf{2}}^{(2)} \right)_{\mathbf{1}} H_u + g' \left(N^c \, L \, Y_{\mathbf{3'}}^{(2)} \right)_{\mathbf{1}} H_u \, .$$

There are 4 possible assignments of ρ_N and ρ_L we consider. Two of them, namely $\rho_N = \rho_L \sim \mathbf{3}$ and $\rho_N = \rho_L \sim \mathbf{3'}$ give the following form of $\mathcal{Y}$:

$$\mathcal{Y} = g \left[\begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & Y_2 & 0 \\ Y_2 & 0 & Y_1 \end{pmatrix} + \frac{g'}{g} \begin{pmatrix} 0 & Y_5 & -Y_4 \\ -Y_5 & 0 & Y_3 \\ Y_4 & -Y_3 & 0 \end{pmatrix} \right] , \quad \textbf{for} \quad k_L + K_N = 2 \quad \textbf{and} \quad \rho_N = \rho_L \, .$$

The two remaining combinations, $(\rho_N, \rho_L) \sim (\mathbf{3}, \mathbf{3'})$ and $(\mathbf{3'}, \mathbf{3})$, lead to:

$$\mathcal{Y} = g \left[\begin{pmatrix} 0 & -Y_1 & Y_2 \\ -Y_1 & Y_2 & 0 \\ Y_2 & 0 & -Y_1 \end{pmatrix} + \frac{g'}{g} \begin{pmatrix} 2Y_3 & -Y_5 & -Y_4 \\ -Y_5 & 2Y_4 & -Y_3 \\ -Y_4 & -Y_3 & 2Y_5 \end{pmatrix} \right] , \quad \textbf{for} \quad k_L + k_N = 2 \quad \textbf{and} \quad \rho_N \neq \rho_L \, .$$

In both cases, up to an overall factor, the matrix $\mathcal{Y}$ depends on one complex parameter g'/g and the VEV of τ, vev(τ).

$$Y_{\mathbf{2}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}} \left(\theta^4 + \varepsilon^4 \right) \\ -\sqrt{6} \, \varepsilon^2 \, \theta^2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} , \qquad Y_{\mathbf{3'}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}} \left(\theta^4 - \varepsilon^4 \right) \\ -2 \, \varepsilon \, \theta^3 \\ -2 \, \varepsilon^3 \, \theta \end{pmatrix} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} .$$

The charged lepton Yukawa couplings

In the minimal (in terms of weights) viable possibility for $L_{1,2,3}$ furnishing a 3-dim. irrep of S_4, i.e., $\rho_L \sim \mathbf{3}$ or $\mathbf{3}'$, and carrying a weight $k_L = 2$, and $E^c_{1,2,3}$ transforming as singlets of Γ_4, $\rho_{1,2,3} \sim 1', 1, 1'$ (up to permutations) and carrying weights $k_{1,2.3} = 0, 2, 2$, the relevant part of W, W_e, can take 6 different forms which lead to the same matrix U_e diagonalising $M_e M_e^\dagger = v_d^2\, \lambda^\dagger \lambda$, and thus do not lead to new results for the PMNS matrix. We give just one of these 6 forms corresponding to $\rho_L = \mathbf{3}$, $\rho_1 = 1'$, $\rho_2 = 1$, $\rho_3 = 1'$:

$$\alpha \left(E_1^c\, L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c\, L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c\, L\, Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}} H_d\,.$$

This leads to

$$\lambda = \begin{pmatrix} \alpha\, Y_3 & \alpha\, Y_5 & \alpha\, Y_4 \\ \beta\,(Y_1 Y_4 - Y_2 Y_5) & \beta\,(Y_1 Y_3 - Y_2 Y_4) & \beta\,(Y_1 Y_5 - Y_2 Y_3) \\ \gamma\,(Y_1 Y_4 + Y_2 Y_5) & \gamma\,(Y_1 Y_3 + Y_2 Y_4) & \gamma\,(Y_1 Y_5 + Y_2 Y_3) \end{pmatrix}\,,$$

In this “minimal” example the matrix λ depends on 3 free parameters, α, β and γ, which can be rendered real by re-phasing of the charged lepton fields.

We recall that

$$M_e = v_d\, \lambda^\dagger\,, \qquad v_d \equiv \mathrm{vev}(\mathrm{H_d^0})\,,$$
$$M_\nu = -v_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y}\,, \qquad v_u \equiv \mathrm{vev}(\mathrm{H_u^0})\,.$$

Parameters of the model: α, β, γ, g^2/Λ **– real;** g' **and VEV of** τ **– complex, i.e., 6 real parameters + 2 (1) phases for description of 12 observables (3 charged lepton masses, 3 neutrino masses, 3 mixing angles and 3 CPV phases). Excellent description of the data is obtained also for real** g' **(i.e., 6 real parameters + 1 phase, employing** gCP**).**

The 3 real parameters $v_d\alpha$, β/α, γ/α **– fixed by fitting** m_e, m_μ **and** m_τ**. The remaining 3 real parameters and 2 (1) phases –** $v_u^2 g^2/\Lambda$, $|g'/g|$, $|\tau|$ **and** $\arg(g'/g)$, $\arg\tau$ **(**$\arg\tau$**) – describe the 5** ν **measured observables - 3 mixing angles, 2** Δm_{ij}^2**.**

The model considered leads to testable predictions for $\min(m_j)$ **(**$\sum_i m_i$**), type of the** ν **mass spectrum (NO or IO), the 3 CPV Dirac and Majorana phases; predicted are also** $|\langle m \rangle|$**, the range of** θ_{23}**, as well as of correlations between different observables.**

Seven real parameters (5 real couplings + the complex VEV of τ**) – is the minimal number of parameters in the constructed so far phenomenologically viable lepton flavour models with massive Majorana neutrinos based on modular invariance.**

Numerical Analysis

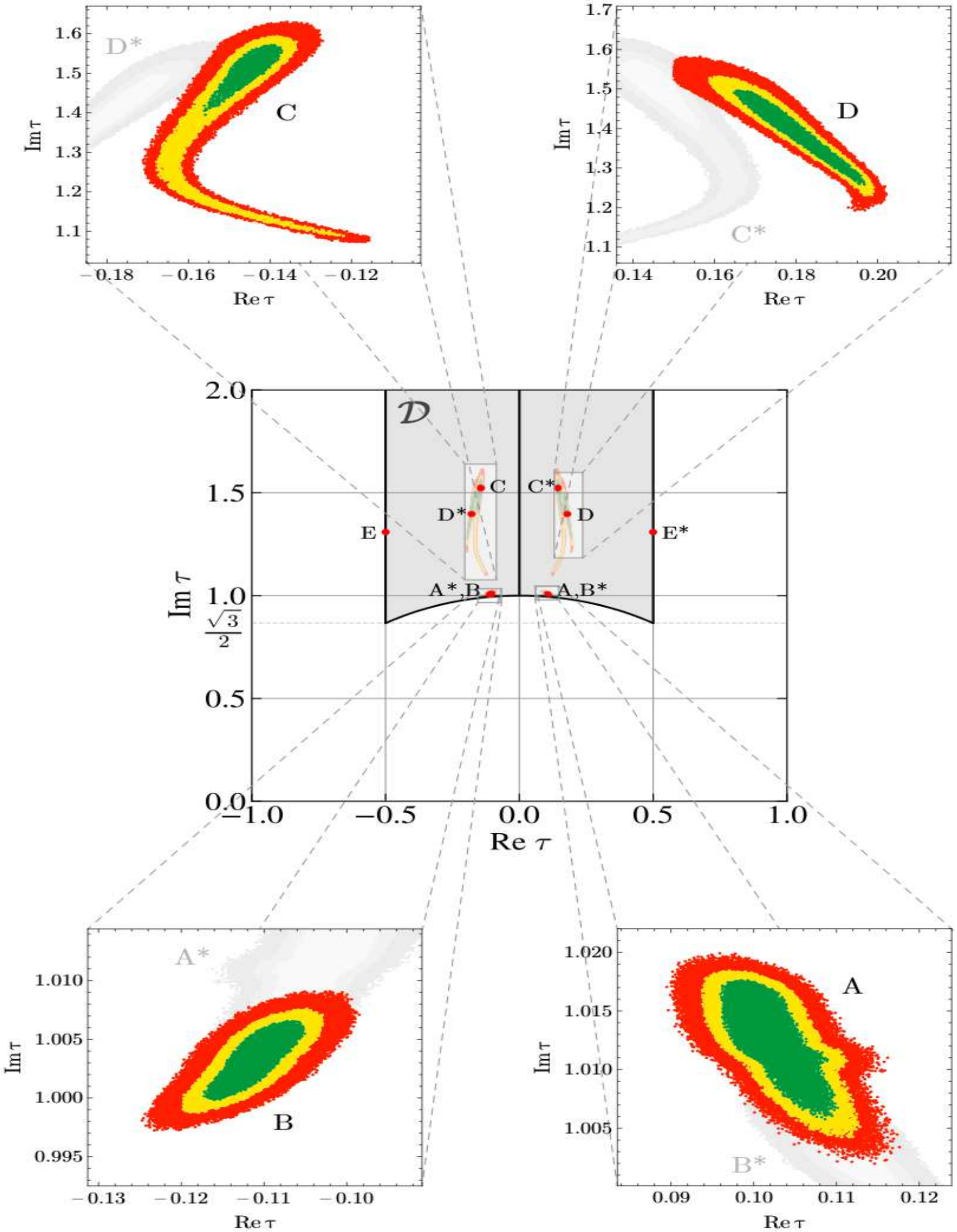
Each model depends on a set of dimensionless parameters

$$p_i = (\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots)\,,$$

which determine dimensionless observables (mass ratios, mixing angles and phases), and two overall mass scales: $v_d\,\alpha$ for M_e and $v_u^2\,g^2/\Lambda$ for M_ν. Phenomenologically viable models are those that lead to values of observables which are in close agreement with the experimental results summarized in the Table below. We assume also to be in a regime in which the running of neutrino parameters is negligible.

Observable	Best fit value and 1σ range	
m_e/m_μ	0.0048 ± 0.0002	
m_μ/m_τ	0.0565 ± 0.0045	
	NO	IO
$\delta m^2/(10^{-5}\ \mathrm{eV}^2)$	$7.34^{+0.17}_{-0.14}$	
$\lvert\Delta m^2\rvert/(10^{-3}\ \mathrm{eV}^2)$	$2.455^{+0.035}_{-0.032}$	$2.441^{+0.033}_{-0.035}$
$r \equiv \delta m^2/\lvert\Delta m^2\rvert$	0.0299 ± 0.0008	0.0301 ± 0.0008
$\sin^2\theta_{12}$	$0.304^{+0.014}_{-0.013}$	$0.303^{+0.014}_{-0.013}$
$\sin^2\theta_{13}$	$0.0214^{+0.0009}_{-0.0007}$	$0.0218^{+0.0008}_{-0.0007}$
$\sin^2\theta_{23}$	$0.551^{+0.019}_{-0.070}$	$0.557^{+0.017}_{-0.024}$
δ/π	$1.32^{+0.23}_{-0.18}$	$1.52^{+0.14}_{-0.15}$

Best fit values and 1σ ranges for neutrino oscillation parameters, obtained in the global analysis of F. Capozzi et al., arXiv:1804.09678, and for charged-lepton mass ratios, given at the scale 2×10^{16} GeV with the $\tan\beta$ averaging described in F. Feruglio, arXiv:1706.08749 obtained from G.G. Ross and M. Serna, arXiv:0704.1248. The parameters entering the definition of r are $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$. The best fit value and 1σ range of δ did not drive the numerical searches here reported.



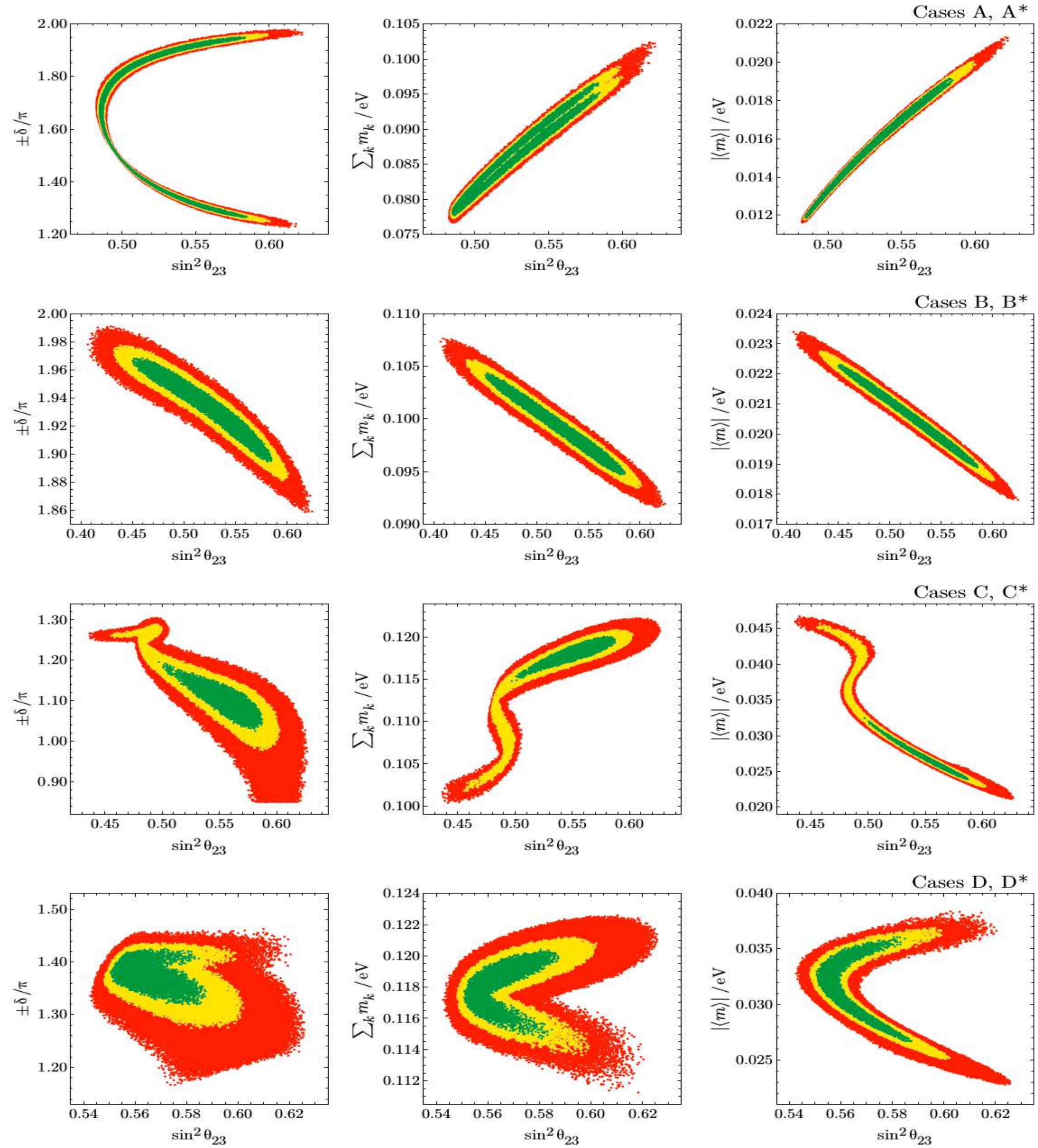
P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933

	Best fit value	2σ range	3σ range
$\mathrm{Re}\,\tau$	± 0.1045	$\pm(0.09597 - 0.1101)$	$\pm(0.09378 - 0.1128)$
$\mathrm{Im}\,\tau$	1.01	1.006 – 1.018	1.004 – 1.018
β/α	9.465	8.247 – 11.14	7.693 – 12.39
γ/α	0.002205	0.002032 – 0.002382	0.001941 – 0.002472
$\mathrm{Re}\,g'/g$	0.233	−0.02383 – 0.387	−0.02544 – 0.4417
$\mathrm{Im}\,g'/g$	± 0.4924	$\pm(-0.592 - 0.5587)$	$\pm(-0.6046 - 0.5751)$
$v_d\,\alpha$ [MeV]	53.19		
$v_u^2\,g^2/\Lambda$ [eV]	0.00933		
m_e/m_μ	0.004802	0.004418 – 0.005178	0.00422 – 0.005383
m_μ/m_τ	0.0565	0.048 – 0.06494	0.04317 – 0.06961
r	0.02989	0.02836 – 0.03148	0.02759 – 0.03224
δm^2 [10^{-5} eV^2]	7.339	7.074 – 7.596	6.935 – 7.712
$\|\Delta m^2\|$ [10^{-3} eV^2]	2.455	2.413 – 2.494	2.392 – 2.513
$\sin^2\theta_{12}$	0.305	0.2795 – 0.3313	0.2656 – 0.3449
$\sin^2\theta_{13}$	0.02125	0.01988 – 0.02298	0.01912 – 0.02383
$\sin^2\theta_{23}$	0.551	0.4846 – 0.5846	0.4838 – 0.5999
Ordering	NO		
m_1 [eV]	0.01746	0.01196 – 0.02045	0.01185 – 0.02143
m_2 [eV]	0.01945	0.01477 – 0.02216	0.01473 – 0.02307
m_3 [eV]	0.05288	0.05099 – 0.05405	0.05075 – 0.05452
$\sum_i m_i$ [eV]	0.0898	0.07774 – 0.09661	0.07735 – 0.09887
$\|\langle m\rangle\|$ [eV]	0.01699	0.01188 – 0.01917	0.01177 – 0.02002
δ/π	± 1.314	$\pm(1.266 - 1.95)$	$\pm(1.249 - 1.961)$
α_{21}/π	± 0.302	$\pm(0.2821 - 0.3612)$	$\pm(0.2748 - 0.3708)$
α_{31}/π	± 0.8716	$\pm(0.8162 - 1.617)$	$\pm(0.7973 - 1.635)$
$N\sigma$	0.02005		

Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases A and A*, (which refer to $(k_\Lambda, k_g) = (0,2)$ and $\tau = \pm 0.1045 + i\,1.01$).

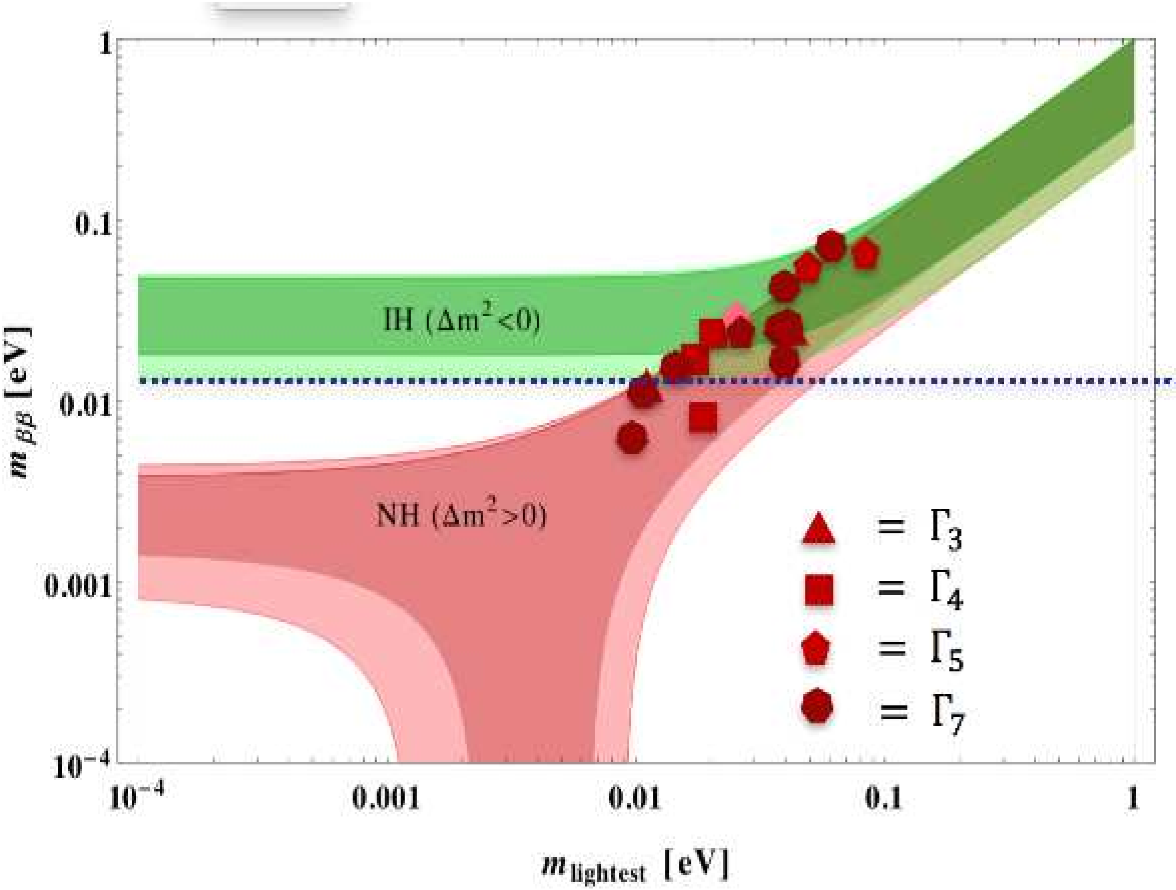
	Best fit value	2σ range	3σ range
Re τ	∓ 0.109	$\mp(0.1051 - 0.1172)$	$\mp(0.103 - 0.1197)$
Im τ	1.005	0.9998 – 1.007	0.9988 – 1.008
β/α	0.03306	0.02799 – 0.03811	0.02529 – 0.04074
γ/α	0.0001307	0.0001091 – 0.0001538	0.0000982 – 0.0001663
Re g'/g	0.4097	0.3513 – 0.5714	0.3241 – 0.5989
Im g'/g	∓ 0.5745	$\mp(0.5557 - 0.5932)$	$\mp(0.5436 - 0.5944)$
$v_d\,\alpha$ [MeV]	893.2		
$v_u^2\,g^2/\Lambda$ [eV]	0.008028		
m_e/m_μ	0.004802	0.004425 – 0.005175	0.004211 – 0.005384
m_μ/m_τ	0.05649	0.04785 – 0.06506	0.04318 – 0.06962
r	0.0299	0.02838 – 0.03144	0.02757 – 0.03223
δm^2 [10^{-5} eV^2]	7.34	7.078 – 7.59	6.932 – 7.71
$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.455	2.414 – 2.494	2.393 – 2.514
$\sin^2\theta_{12}$	0.305	0.2795 – 0.3314	0.2662 – 0.3455
$\sin^2\theta_{13}$	0.02125	0.0199 – 0.02302	0.01914 – 0.02383
$\sin^2\theta_{23}$	0.551	0.4503 – 0.5852	0.4322 – 0.601
Ordering	NO		
m_1 [eV]	0.02074	0.01969 – 0.02374	0.01918 – 0.02428
m_2 [eV]	0.02244	0.02148 – 0.02522	0.02101 – 0.02574
m_3 [eV]	0.05406	0.05345 – 0.05541	0.05314 – 0.05577
$\sum_i m_i$ [eV]	0.09724	0.09473 – 0.1043	0.0935 – 0.1056
$\lvert\langle m\rangle\rvert$ [eV]	0.01983	0.01889 – 0.02229	0.01847 – 0.02275
δ/π	± 1.919	$\pm(1.895 - 1.968)$	$\pm(1.882 - 1.977)$
α_{21}/π	± 1.704	$\pm(1.689 - 1.716)$	$\pm(1.681 - 1.722)$
α_{31}/π	± 1.539	$\pm(1.502 - 1.605)$	$\pm(1.484 - 1.618)$
$N\sigma$	0.02435		

Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases B and B*, (which refer to $(k_\Lambda, k_g) = (0,2)$ and $\tau = \pm 0.109 + i\,1.005$).



P.P. Novichkov et al., arXiv:1811.04933

Predictions for the neutrinoless double beta decay effective Majorana mass.



F. Feruglio, talk at Bethe Colloquium, 18/06/2020

Predictions of modular invariant models of lepton flavour for the neutrinoless double beta decay effective Majorana mass. The predictions are in the range of sensitivity of some of the current and upcoming neutrinoless double beta decay experiments (LEGEND, nEXO, KamLAND-Zen II, NEXT).

Success led to Ambitious Program

The charged lepton mass hierarchy is decsribed correctly by the model due to a fine-tuning of the constants β/α and γ/α.

This is a common problem of the numerous proposed lepton and quark flavour models based on modular invariance and constructed prior 2021.

Idea: the fermion mass hierarchies should arise as a cosequence of the properties of the modular forms rather than by fine-tuning the constants present in the fermion mass matrices.

The problem of having the charged lepton (and quark) mass hierarchies determined by the properties of the modular forms (without fine tuning) was studied in H. Okada, M. Tanimoto, 2009.14242, 2012.0188; F. Feruglio et al., 2101.08718.

A solution was proposed in P.P. Novichkov, J.T. Penedo, STP, arXiv:2102.07488, where also a non-fine-tuned model of lepton flavour was constructed. Makes use of the existence of values of the VEV of τ (fixed points) which break the modular symmetry only partially to certain residual discrete symmetries.

Interesting results were obtained in studies of the modulus stabilisation, i.e., finding the VEV of the modulus as minima of modulus potential derived from "first principles" (see, e.g., P. Novichkov et al., 2201.02020).

Recent developments in modular invaraince approach to the flavour problem:

Solution to the strong CP problem without axions: F. Feruglio et al., 2305.08908; M. Tanimoto, S.T.P., 2404.00858; J.T. Penedo, S.T.P., 2404.08032.

The modulus τ as inflaton: G.-J. Ding et al., 2405.06497; S.F. King, Xin Wang 2405.08924.

JUNO

20 kt LS detector of reactor $\bar{\nu}_e$ via IBD
$\bar{\nu}_e + p \rightarrow n + e^+$; $E_{res} = 3\%/\sqrt{E}$; $L \cong 53$ **km;**
thermal power of the used reactors: 26.6 GW;
Sphere with a diameter of 38 m.
Cost: 300×10^6 **US Dollars.**
Built in China by international collaboration of more than 700 scientists from 74 Institutions in 17 countries/regions. Expected to start data-taking at the beginning of 2025.
After 6 years of operation: NMO at 3σ **(using reactor** ν **data only). Adding** ν_{atm} **data can improve the sensitivity by** $(0.8 - 1.4)\sigma$**.**

The idea put forward in S.T.P., M. Piai, PLB 553 (2002) 94 (hep-ph/0112074).

Based on: $P_{NO}(\bar{\nu}_e \rightarrow \bar{\nu}_e) \neq P_{IO}(\bar{\nu}_e \rightarrow \bar{\nu}_e)$

$$P^{NO}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - \frac{1}{2}\sin^2 2\theta_{13}\left(1 - \cos\frac{\Delta m^2_{atm}\,L}{2\,E_\nu}\right) - \frac{1}{2}\cos^4\theta_{13}\sin^2 2\theta_{12}\left(1 - \cos\frac{\Delta m^2_{\odot}\,L}{2\,E_\nu}\right)$$

$$+ \frac{1}{2}\sin^2 2\theta_{13}\sin^2\theta_{12}\left(\cos\left(\frac{\Delta m^2_{atm}\,L}{2\,E_\nu} - \frac{\Delta m^2_{\odot}\,L}{2\,E_\nu}\right) - \cos\frac{\Delta m^2_{atm}\,L}{2\,E_\nu}\right), \quad \Delta m^2_{\odot} \equiv \Delta m^2_{21},$$

$$P^{IO}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - \frac{1}{2}\sin^2 2\theta_{13}\left(1 - \cos\frac{\Delta m^2_{atm}\,L}{2\,E_\nu}\right) - \frac{1}{2}\cos^4\theta_{13}\sin^2 2\theta_{12}\left(1 - \cos\frac{\Delta m^2_{\odot}\,L}{2\,E_\nu}\right)$$

$$+ \frac{1}{2}\sin^2 2\theta_{13}\cos^2\theta_{12}\left(\cos\left(\frac{\Delta m^2_{atm}\,L}{2\,E_\nu} - \frac{\Delta m^2_{\odot}\,L}{2\,E_\nu}\right) - \cos\frac{\Delta m^2_{atm}\,L}{2\,E_\nu}\right).$$

$\Delta m^2_{\rm atm} = \Delta m^2_{31(32)}(NO)$, $\Delta m^2_{\rm atm} = \Delta m^2_{32(31)}(IO)$,

$$\bar{\nu}_e + p \rightarrow e^+ + n$$

Spectrum of e^+ - sensitive to the difference between $P^{NO}(\bar{\nu}_e \rightarrow \bar{\nu}_e)$ and $P^{IO}(\bar{\nu}_e \rightarrow \bar{\nu}_e)$ - can be used to determine neutrino mass ordering. Optimal L exists.

S.T.P., M. Piai, 2001

JUNO (China, International collaboration)

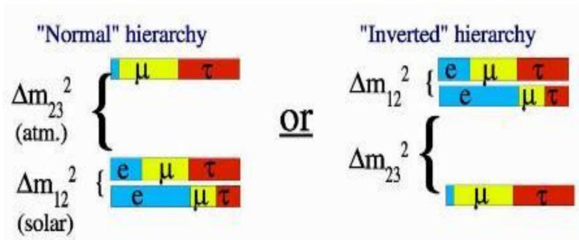
S. Choubey, S.T.P., M. Piai, PRD 68 (2003) 113006 ((hep-ph/0306017): can measure $\sin^2\theta_{12}$, Δm^2_{21} and Δm^2_{31} with exceptionally high precision.

After 6 years of datataking:
$\sin^2\theta_{12}$**: 0.5%;** Δm^2_{21}**: 0.3%;** Δm^2_{31}**: 0.2% (**1σ**)**
(Y. Wang, talk given at CERN on March 20, 2024).

Wide program of research: atmospheric ν oscillations, solar neutrinos, SN neutrinos, geo-neutrinos, nucleon decay; distant future: $(\beta\beta)_{0\nu}$ decay.

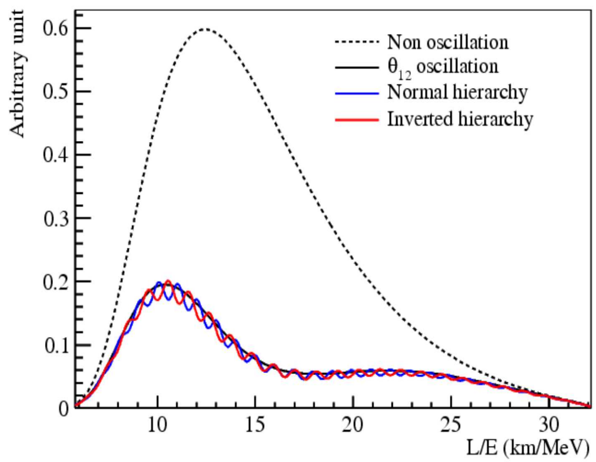

Guangzhou
Guang Zhou
Dongguan
Huizhou
Zhaoqing
CNS
2.5 h drive
Shen Zhen
Huizhou NPP
Lufeng NPP
Daya Bay
NPP
Zhongshan
Zhu Hai
Zhujiang River Estuary
Hong Kong
Hong Kong
Macau
Macau
JUNO site
53 km
53 km
Taishan NPP
Yangjiang NPP

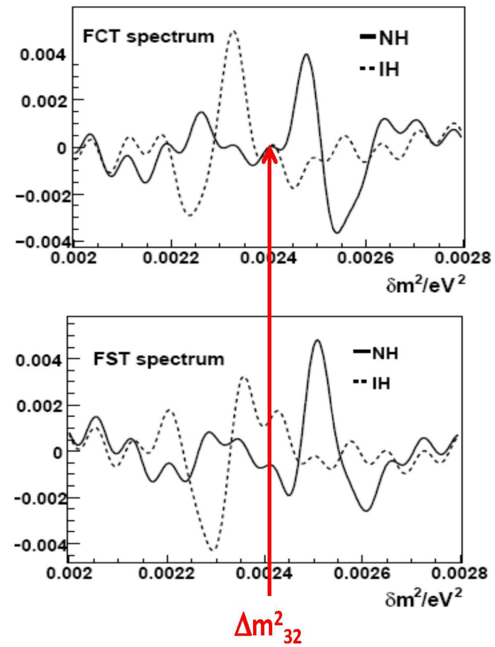
Mass Ordering by Reactor Neutrinos



$$
\begin{aligned}
\Delta m_{31}^2 &= \Delta m_{32}^2 + \Delta m_{21}^2 \\
\mathrm{NH}: \ |\Delta m_{31}^2| &= |\Delta m_{32}^2| + |\Delta m_{21}^2| \\
\mathrm{IH}: \ |\Delta m_{31}^2| &= |\Delta m_{32}^2| - |\Delta m_{21}^2|
\end{aligned}
$$

$$\boxed{\frac{\Delta m_{21}^2}{|\Delta m_{32}^2|} \sim 3\%}$$

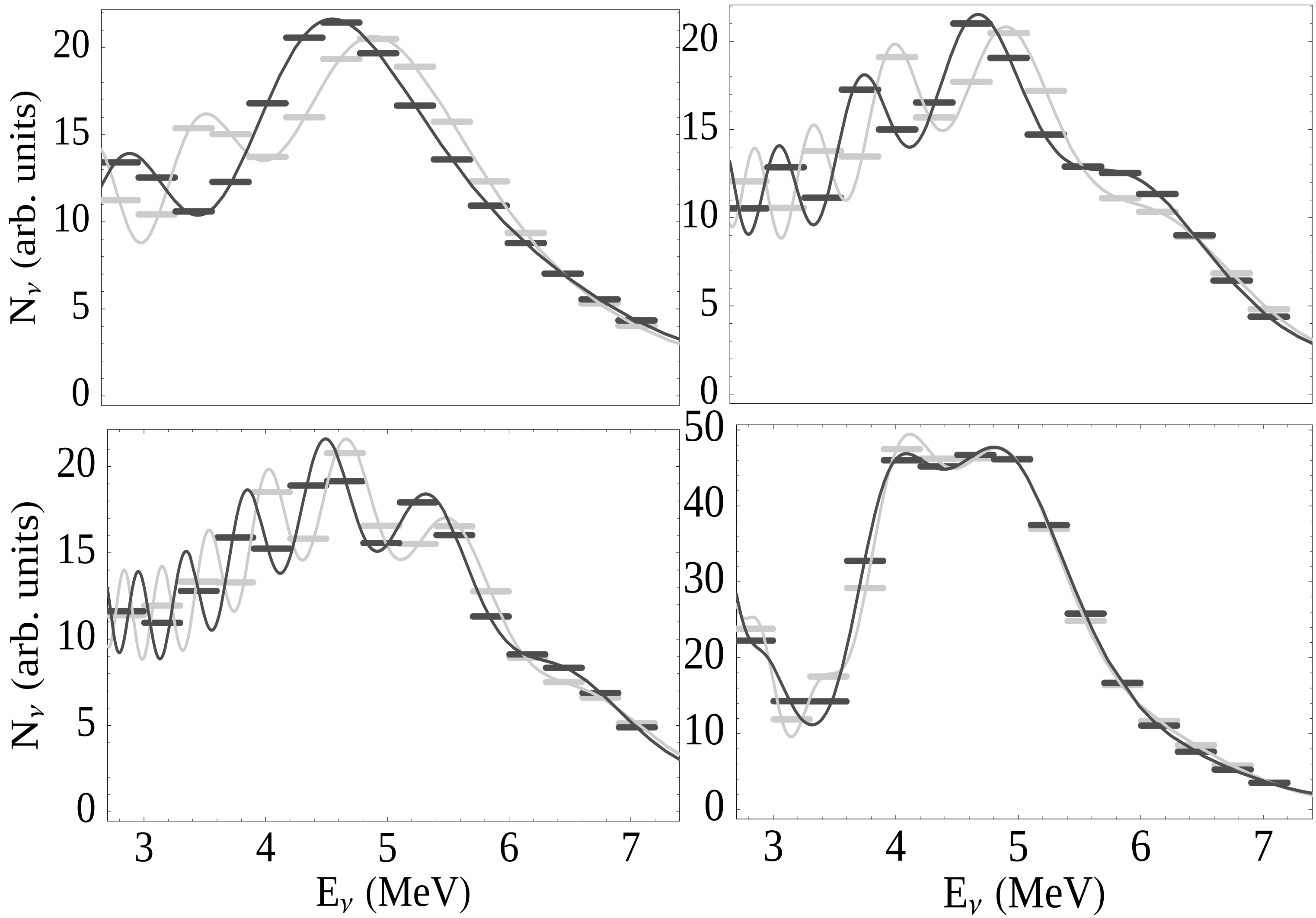




$$
\begin{aligned}
P_{ee}(L/E) &= 1 - P_{21} - P_{31} - P_{32} \\
P_{21} &= \cos^4(\theta_{13}) \sin^2(2\theta_{12}) \sin^2(\Delta_{21}) \\
P_{31} &= \cos^2(\theta_{12}) \sin^2(2\theta_{13}) \sin^2(\Delta_{31}) \\
P_{32} &= \sin^2(\theta_{12}) \sin^2(2\theta_{13}) \sin^2(\Delta_{32})
\end{aligned}
$$

S. Petcov and Piai, Phys. Lett. B 553, 94-106(2002)
J. Learned et al., PRD 78(2008)071302
L. Zhan, YFW et al., PRD 78(2008)111103

Y. Wang, talk given at CERN on March 20, 2024

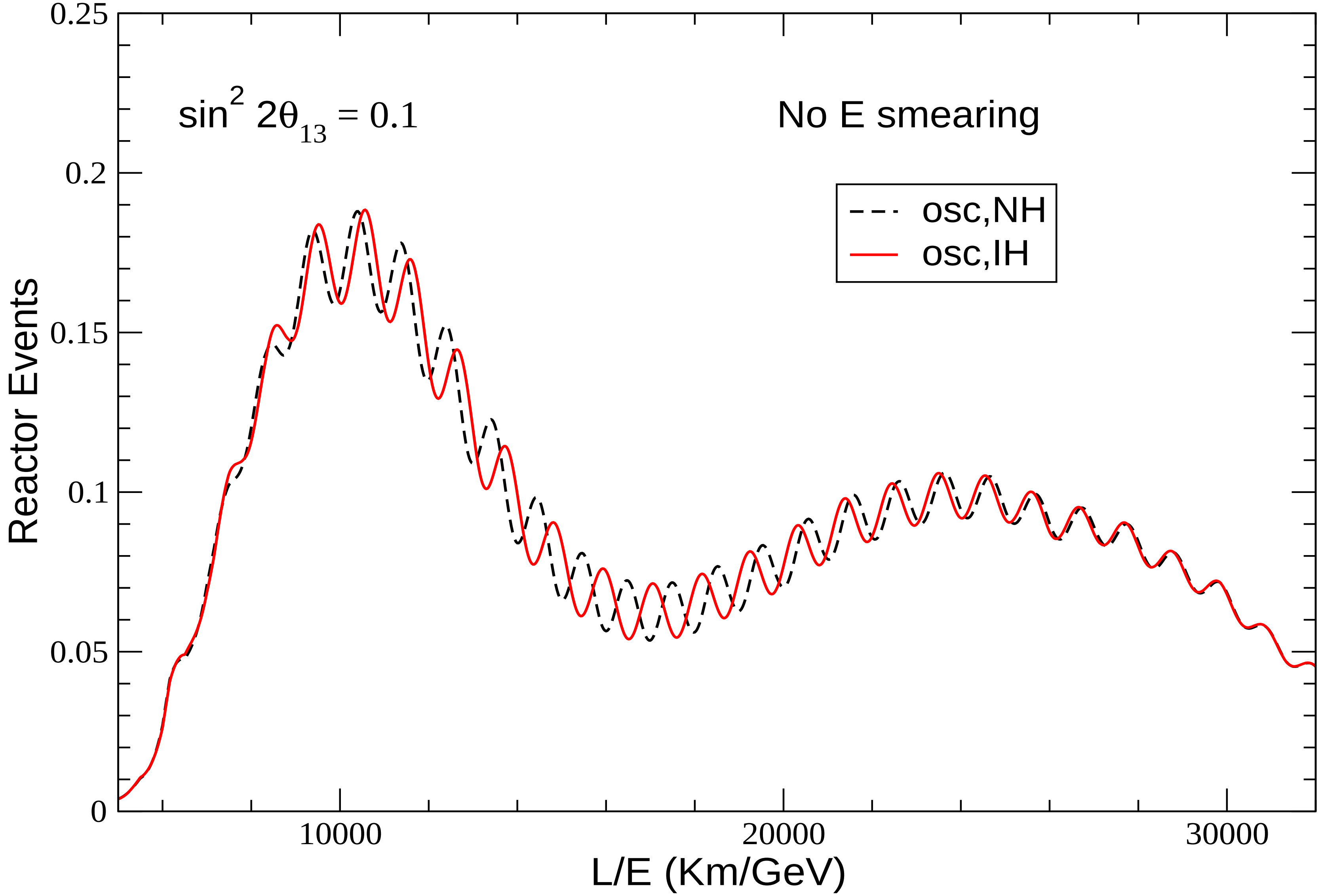


S.T.P., M. Piai, PLB 553 (2002) 94 (hep-ph/0112074)

$\Delta m^2_{21} = 2 \times 10^{-4}$ eV^2, $\Delta m^2_{31} = 1.3; 2.5; 3.5 \times 10^{-3}$ eV^2;

$\Delta m^2_{21} = 6 \times 10^{-4}$ eV^2, $\Delta m^2_{31} = 2.5 \times 10^{-3}$ eV^2; $\sin^2\theta_{12} = 0.8$;$\sin^2\theta_{13} = 0.05$.

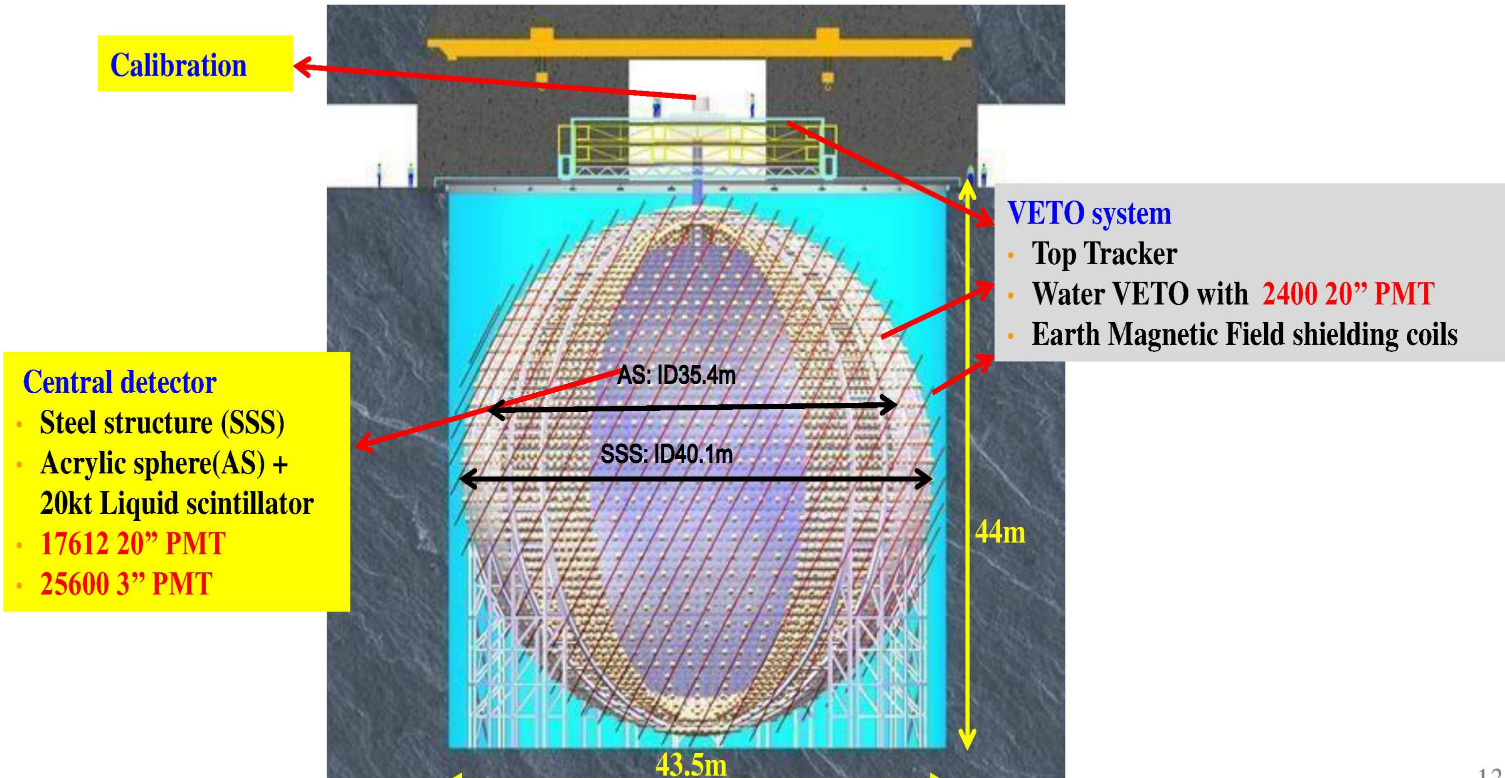
"However, as it is well known, "Only those who wager can win." (W. Pauli, Letter to Participants of a Physics Meeting in Tubingen, Germany, December 4, 1930."

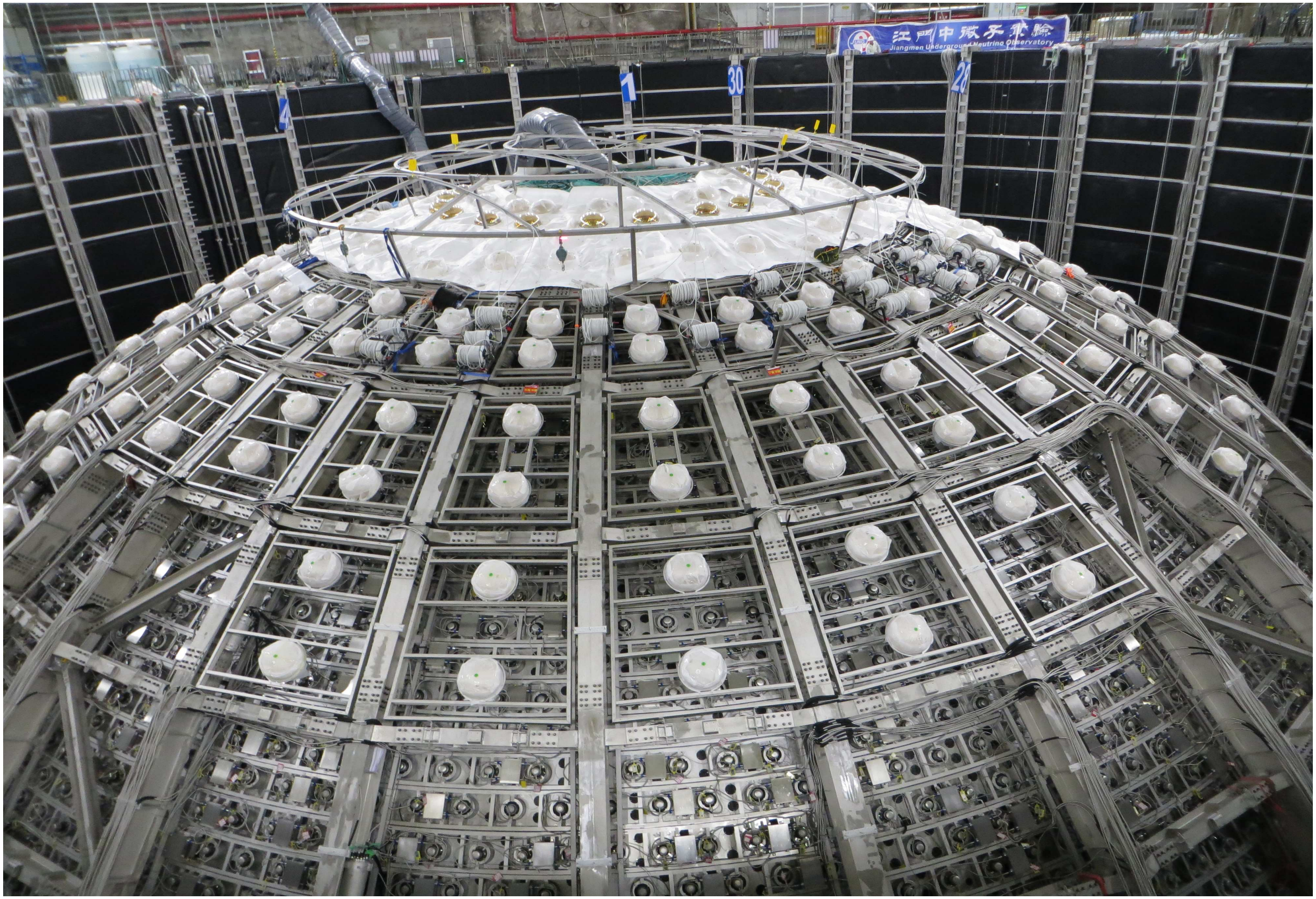


P. Ghoshal, S.T.P., JHEP 03 (2011) 058 (arXiv:1011.1646)

Concept of JUNO for Mass Ordering

- Two-layer structure for simplicity and cost: stainless steel frame + Acrylic tank
- Water as VETO and Buffer ➔ radiopurity control of water




JUNO
江門中微子實驗
Jiangmen Underground Neutrino Observatory


江門中微子實驗
Jiangmen Underground Neutrino Observatory

DUNE

DUNE (LBNE): Fermilab-DUSEL, $L = 1290$ km, 1.2 MW (2.3 MW) proton beam, wide band ν beam (first and second osc. maxima at $E = 2.4$ GeV and 0.8 GeV); 34 kt fiducial volume LAr detecors; plans to run 5 years with ν_μ and 5 years with $\bar{\nu}_\mu$; 2028-2029

DUNE could have very good sensitivity to CP-violation with a 60% coverage at 3σ in the allowed range of values of $\sin^2 2\theta_{13}$ (assuming it will run for 5 years in neutrinos and 5 years in antineutrinos).

DUNE could achieve the determination of the NMO at 3σ in less then a year.

Hyper-Kamiokande and T2HK

Hyper-Kamiokande: water-Cherenkov, $\sim$0.25 Mton, fiducial $\sim$0.2 Mton; 2027; T2HK.

Instead of Conclusions

We have presented two approaches to the lepton flavour problem based respectively on non-Abelian discrete symmetries and modular invariance. Both approaches lead to specific testable predictions. Only with the additional data from the current and upcoming experiments (T2K, NOνA, JUNO, HK, T2HK, DUNE,..., LEGEND, KamLAND-Zen II, nEXO, SNO+, CUORE, CUPID, NEXT,...) it will be possible to perform thorough tests of these predictions. The planned high precision measurements of θ_{12} (JUNO), θ_{23} (T2HK, DUNE) and especially of the Dirac CPV phase δ (T2HK, DUNE, ESSνSB), if successful, will allow us to perform such tests. Experimental confirmation of some of the discussed specific predictions will imply the exitence of a new basic symmetry in particle physics. This will have profound implications.

I personally am looking very much forward to the upcoming new data on neutrino mixing, leptonic CP violation, neutrino mass ordering, absolute neutrino mass scale and the nature of massive neutrinos and the better understanding of neutrinos and the associated development of the theory of neutrino masses, mixing and leptonic CP vilation these data will bring.

Supporting Slides

CP Symmetry in Modular Invariant Flavour Models

The formalism of combined finite modular and generalised CP (gCP) symmetries for theories of flavour was developed in P.P. Novichkov et al., arXiv:1905.11970.

gCP invariance was shown to imply that the constants g, which accompany each invariant singlet in the superpotential, must be real (in a symmetric basis of S and T and at least for $\Gamma^{(\prime)}_N$, $N \leq 5$). Thus, the number of free parameters in modular-invariant models which also enjoy a gCP symmetry gets reduced, leading to "minimal" models which have higher predictive power.

In these models, the only source of both modular symmetry breaking and CP violation is the VEV of the modulus τ.

The "minimal" phenomenologically viable modular-invariant flavour models with gCP symmetry constructed so far
– of the lepton sector with massive Majorana neutrinos (12 observables) contain $\geq$ 7 real parameters – 5 real couplings + the complex τ (6 real constants + 1 phase);
– of the quark sector contain $\geq$ 9 real parameters – 7 real coulplings + the complex τ;
– while the models of lepton and quark flavours (22 observables) have $\geq$ 15 real parameters - 13 real couplings + the complex τ.

See, e.g., B.-Y. Qu et al., arXiv:2106.11659

Under the CP transformatoion,

$$\tau \xrightarrow{\text{CP}} -\tau^* \,.$$

P.P. Novichkov et al., 1905.11970; A. Baur et al., 1901.03251 and 1908.00805

It was further demonstrated that CP is conserved for

$$\text{Re}\tau = \pm 1/2\,;\;\; \tau = e^{i\theta}\,,\; \theta = [\pi/3, 2\pi/3]\,; \text{Re}\tau = 0\,,\; \text{Im}\tau \geq 1\,.$$

i.e., for the values of τ's VEV at the boundary of the fundamental domain and on the imaginary axis.

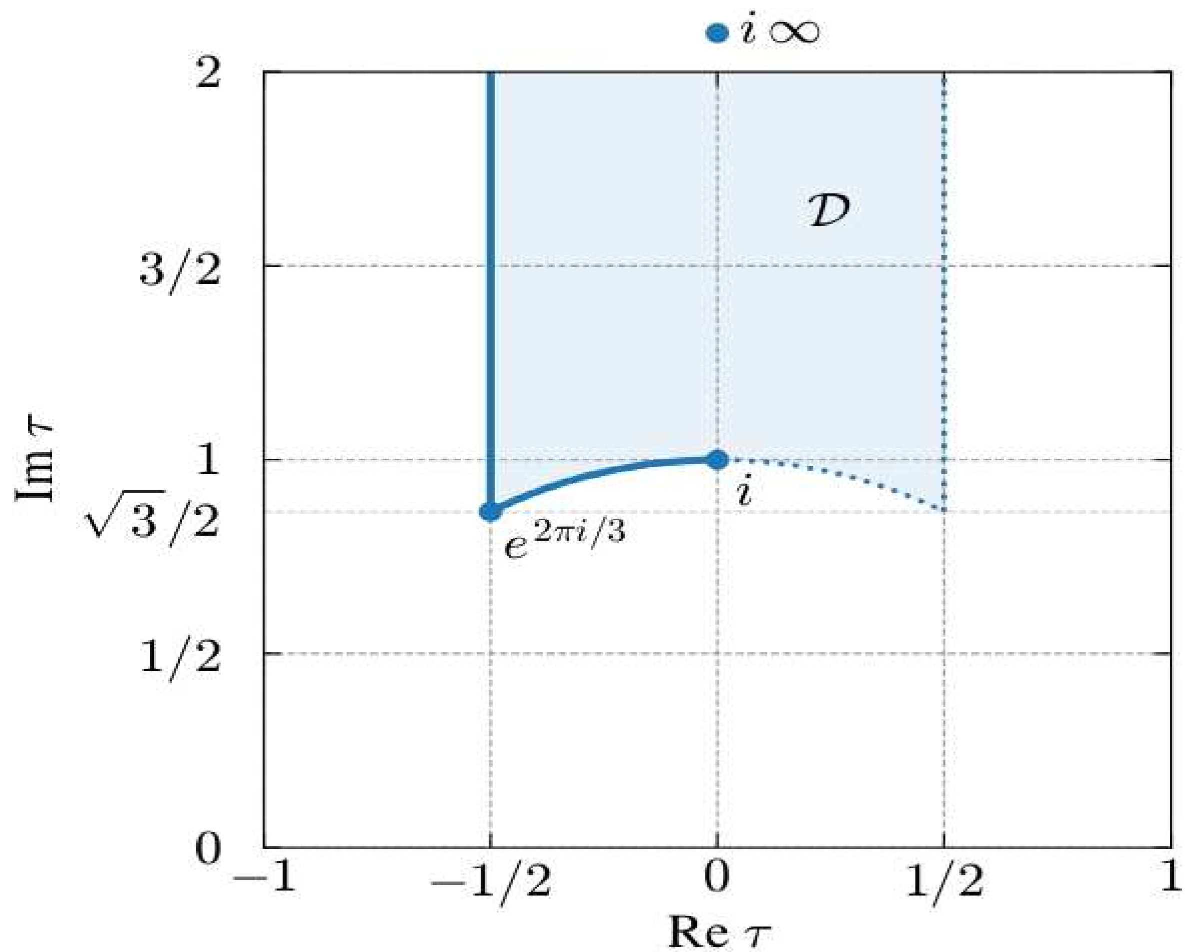


Figure from P.P. Novichkov et al., arXiv:2006.03058

Residual Symmetries

The breakdown of modular symmetry is parameterised by the VEV of τ.
There is no value of τ's VEV which preserves the full symmetry $\Gamma^{(\prime)}$ ($\Gamma_N^{(\prime)}$).
At certain "symmetric points" $\tau = \tau_{\text{sym}}$, $\Gamma^{(\prime)}$ ($\Gamma_N^{(\prime)}$) is only partially broken, with the unbroken generators giving rise to residual symmetries.
The $R = S^2$ generator ($\Gamma_N^{(\prime)}$) is unbroken for any value of τ, thus a $\mathbb{Z}_2^R$ symmetry is always preserved.

There are only 3 inequivalent symmetric points in $\mathcal{D}$:

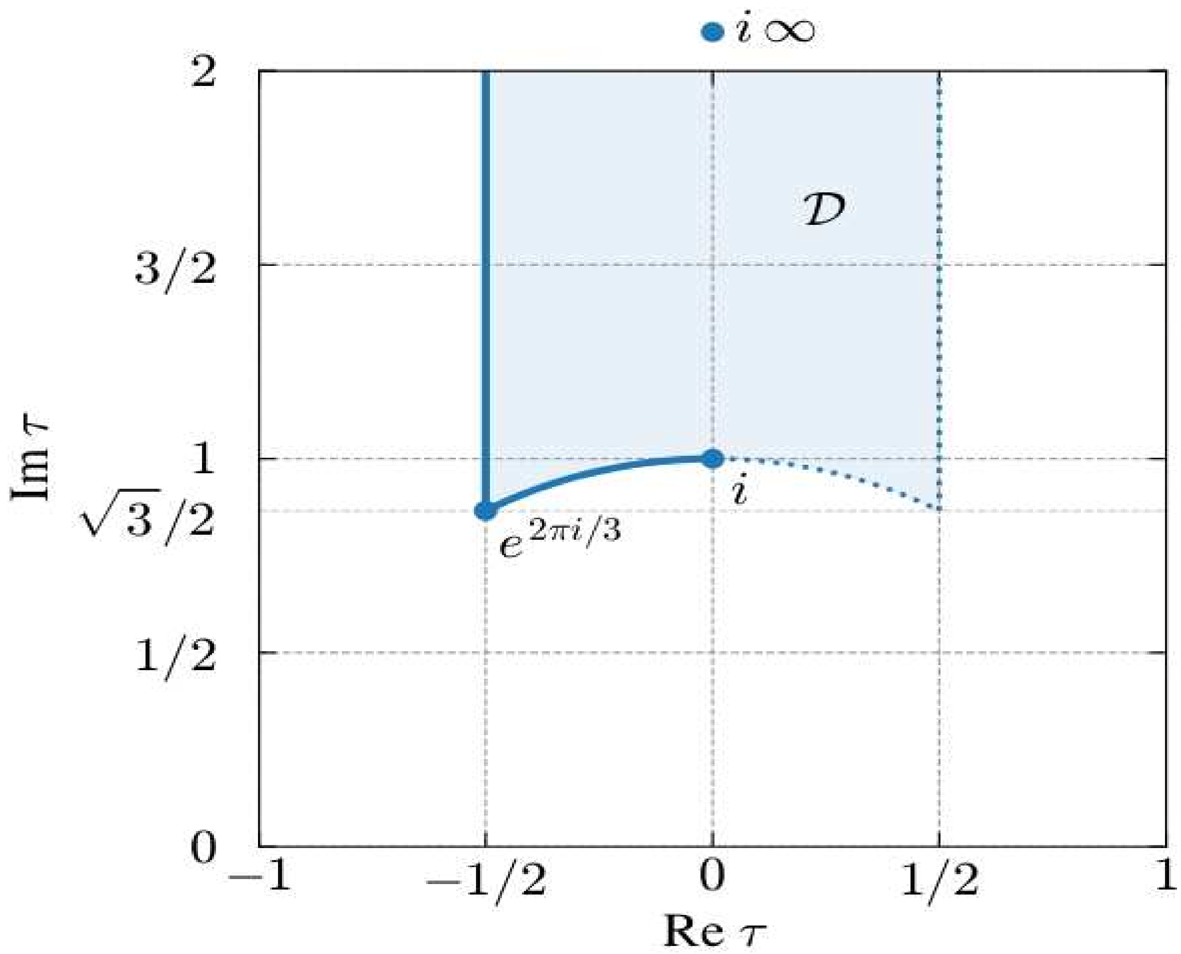
- $\tau_{\text{sym}} = i\infty$, **invariant under T, preserving $\mathbb{Z}_N^T$;**
- $\tau_{\text{sym}} = i$, **invariant under S, preserving $\mathbb{Z}_2^S$ ($\mathbb{Z}_4^S$, $S^2 = R$);**
- $\tau_{\text{sym}} = \omega \equiv \exp(2\pi i/3)$, **invariant under ST, preserving $\mathbb{Z}_3^{ST}$.**

P.P. Novichkov et al., arXiv:1811.04933 and arXiv:2006.03058

These symmetric values of τ preserve the CP ($\mathbb{Z}_2^{CP}$) symmetry of a CP- and modular-invariant theory (e.g. a modular theory where the couplings satisfy a reality condition).

P.P. Novichkov et al., arXiv:1911.04933 and arXiv:2006.03058

The CP ($\mathbb{Z}_2^{\text{CP}}$) symmetry is preserved for $\text{Re}\,\tau = 0$ or for τ lying on the border of the fundamental domain $\mathcal{D}$, but is broken at generic values of τ.



The fundamental domain $\mathcal{D}$ of the modular group Γ and its three symmetric points $\tau_{\text{sym}} = i\,\infty, i, \omega$. At the solid and dotted lines (which include the three points) CP is also preserved. The value of τ can always be restricted to $\mathcal{D}$ by a suitable modular transformation.

Figure from P.P. Novichkov et al., arXiv:2006.03058

Fermion Mass Hierarchies without Fine-Tuning

The $l-$ and $q-$ mass hierarchies in practically all modular flavour models proposed in the literarture before arXiv:2102.07488 – obtained with fine-tuning.

Fine-tuning:
i) unjustified hierarchies between model's parameters, and/or
ii) high sensitivity of observables to model parameters.

The flavour structure of the fermion mass matrices M_F can be severely constrained by the residual symmetries present at each of the 3 symmetry points,
$\tau_{\text{sym}} = i$,
$\tau_{\text{sym}} = \omega \equiv \exp(i\,2\pi/3) = -\,1/2 + i\sqrt{3}/2$, and
$\tau_{\text{sym}} = i\infty$:
residual symmetries may enforce the presence of multiple zeros in M_F.
The posibility to build viable flavour models with observed charged lepton (quark) mass hirarchies in the vicinity of the symmetry points was studied in H. Okada, M. Tanimoto, 2009.14242, 2012.0188; F. Feruglio et al., 2101.08718 (see also G-J. Ding et al., 1910.03460).

As τ moves away from τ_{sym}, the zero entries in M_F will become non-zero. Their magnitude will be controlled by the size of the departure ϵ

from τ_{sym} and by the field transformation properties under the residual symmetry group.

Thus, fine-tuning might be avoided in the vicinity of τ_{sym} as $l-$ and $q-$ mass hierarchies would follow from the properties of the modular forms present in the corresponding M_F rather than being determined by the values of the accompanying constants also present in M_F.

P.P. Novichkov, J.T. Penedo, STP, arXiv:2102.07488.

Mass Matrices

Consider the bilinear (i.e., mass term)

$$\psi_i^c \, M(\tau)_{ij} \, \psi_j \,,$$

where the superfields ψ and ψ^c transform as

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho_r(\gamma) \, \psi \quad (\rho(\gamma)\,,\ \Gamma_N^{(\prime)}\,,\ N = 2, 3, 4, 5)\,,$$

$$\psi^c \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c_{r^c}(\gamma) \, \psi^c\,, \quad (\rho^c(\gamma)\,,\ \Gamma_N^{(\prime)})\,.$$

Modular invariance: $M(\tau)_{ij}$ must be modular form of level N and weight $K \equiv k + k^c$,

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger \,.$$

$\tau_{\text{sym}} = i\infty$

At $\tau_{\text{sym}} = i\infty$ we have $\mathbb{Z}_N^T$ symmetry ($\tau_{\text{sym}} = i\infty$ is invariant under T).

Consider T-diagonal basis for the group generators S and T.
In this basis $\rho^{(c)}(T) = \text{diag}(\rho_i^{(c)})$.

By setting $\gamma = T$ in the equation for $M(\gamma\tau)$ one finds

$$M_{ij}(T\tau) = (\rho_i^c \rho_j)^* \, M_{ij}(\tau)\,.$$

M_{ij} is a function of $q \equiv \exp(2\pi i\tau/N)$ (recall the $q-$expansions) and

$$\epsilon \equiv |q| = e^{-2\pi \, \text{Im}\, \tau/N}$$

parameterises the deviation of τ from the symmetric point.

The entries $M_{ij}(q)$ depend analytically on q. Further,
$q \xrightarrow{T} \zeta q$ ($T\tau = \tau + 1$), with $\zeta \equiv \exp(2\pi i/N)$. Thus, in terms of q,

$$M_{ij}(\zeta q) = (\rho_i^c \rho_j)^* M_{ij}(q)\,.$$

Expanding both sides in powers of q, one finds

$$\zeta^n M_{ij}^{(n)}(0) = (\rho_i^c \rho_j)^* M_{ij}^{(n)}(0)\,, \qquad (1)$$

$M_{ij}^{(n)}$ is the n-th derivative of M_{ij} with respect to q. This means that $M_{ij}^{(n)}(0)$ can only be non-zero for values of n such that $(\rho_i^c \rho_j)^* = \zeta^n$.

In the symmetric limit $q \to 0$, e.g., $M_{ij} = M_{ij}^{(0)}(0) \neq 0$ only if $\rho_i^c \rho_j = 1$.

More generally, if $(\rho_i^c \rho_j)^* = \zeta^l$ with $0 \leq l < N$,

$$M_{ij}(q) = a_0\, q^l + a_1\, q^{N+l} + a_2\, q^{2N+l} + \ldots$$

in the vicinity of the symmetric point.

Thus, the entry M_{ij} is expected to be $\mathcal{O}(\epsilon^l)$ whenever $\mathrm{Im}\,\tau$ is large. The power l only depends on how the representations of ψ and ψ^c decompose under the residual symmetry group $\mathbb{Z}_N^T$.

Summary

τsym $= i\infty$, $\mathbb{Z}_N^T$ symmetry: for $(\rho_i^c \rho_j)^* = \zeta^l$ with $0 \leq l < N$, $\zeta \equiv \exp(2\pi i/N)$

$$M_{ij}(q) = a_0\, q^l + a_1\, q^{N+l} + a_2\, q^{2N+l} + \ldots\,,\ \ |q_N| = e^{-2\pi\, \mathrm{Im}\,\tau/N} \equiv \epsilon\,, e.g.\ |q_4| \leq 0.26\,,$$

in the vicinity of the symmetric point.
The entry $M_{ij} \sim \mathcal{O}(\epsilon^l)$ whenever $\mathrm{Im}\,\tau$ is large; $l = 0, 1, 2;\ 3;\ 4$ for $A_4^{(\prime)}$; $S_4^{(\prime)}$; $A_5^{(\prime)}$.
The power l only depends on how the representations of ψ and ψ^c decompose under the residual symmetry group $\mathbb{Z}_N^T$. Thus, we can have, for example:
$m_\tau : m_\mu : m_e \sim (1, \epsilon, \epsilon^2)$ **for $A_4^{(\prime)}$;** $\quad m_\tau : m_\mu : m_e \sim (1, \epsilon, \epsilon^3)$ **for $S_4^{(\prime)}$.**

τsym $= i$, $\mathbb{Z}_4^S$ symmetry: for $(i^{k^c} i^k \rho_i^c \rho_j)^* = (-1)^n$, $n = 0, 1, 2, \ldots$,
$M_{ij}^n(0) \neq 0$, $M_{ij} \sim \mathcal{O}(\epsilon^m)$, $m = 0, 1$, $\epsilon \equiv |s|$, $s \equiv (\tau - i)/(\tau + i)$. Not sufficient to reproduce the $l-$ and $q-$ mass hierarchies!
The power $m = 0, 1$ depends on how the representations of ψ and ψ^c decompose under $\mathbb{Z}_4^S$ and on their respective weights k^c and k^c.

τsym $= \omega$, $\omega \equiv exp(i2\pi/3)$, $\mathbb{Z}_3^{ST}$ symmetry: for $(\omega^{k^c} \rho_i^c \omega^k \rho_j)^* = \omega^{2n}$, $\omega^3 = 1$,
$M_{ij}^n(0) \neq 0$, $M_{ij} \sim \mathcal{O}(\epsilon^m)$, $m = 0, 1, 2$, $\epsilon \equiv |u|$, $u \equiv (\tau - \omega)/(\tau - \omega^2)$.
The power $m = 0, 1, 2$ depends on how the representations of ψ and ψ^c decompose under $\mathbb{Z}_3^{ST}$ and on their respective weights k^c and k^c. In this case we can have:
$m_\tau : m_\mu : m_e \sim (1, \epsilon, \epsilon^2)$ **for $A_4^{(\prime)}$, $S_4^{(\prime)}$ and $A_5^{(\prime)}$.**

Decomposition under Residual Symmetries

As τ departs from τ_{sym}, the entries M_{ij} of M_F are of $\mathcal{O}(\epsilon^l)$, where ϵ parameterises the deviation of τ from τ_{sym}.
The powers l are extracted from products of factors which, correspond to representations of the residual symmetry group.
One can systematically identify these residual symmetry representations for the different possible choices of Γ'_N representations of matter fields. This knowledge can be exploited to construct hierarchical M_F via controlled corrections to entries which are zero in the symmetric limit.

The matter fields ψ furnish 'weighted' representations (r, k) of Γ'_N.
When a residual symmetry is preserved by the value of τ,
ψ decompose into unitary representations of the residual symmetry group.

Modulo a possible $\mathbb{Z}_2^R$ factor, these groups are $\mathbb{Z}_N^T$, $\mathbb{Z}_4^S$, and $\mathbb{Z}_3^{ST}$.
A cyclic group $\mathbb{Z}_n \equiv \langle a \,|\, a^n = 1 \rangle$ has n inequivalent 1-dimensional irreps 1_k, $k = 0, \ldots, n-1$ is sometimes referred to as a "charge". The group generator a is represented by one of the n-th roots of unity,

$$1_k \; : \qquad \rho(a) = \exp\left(2\pi i \frac{k}{n}\right) .$$

For odd n, the only real irrep of $\mathbb{Z}_n$ is the trivial one, 1_0; for even n, there is one more real irrep, $1_{n/2}$. All other irreps are complex, and split into pairs of conjugated irreps: $(1_k)^* = 1_{n-k}$.

Consider as an example a $(3, k)$ triplet ψ of S'_4.
It transforms under the unbroken $\gamma = ST$ at $\tau = \omega$ as

$$\psi_i \xrightarrow{ST} (-\omega - 1)^{-k} \rho_3(ST)_{ij} \psi_j = \omega^k \rho_3(ST)_{ij} \psi_j \,.$$

The eigenvalues of $\rho_3(ST)$ are 1, ω and ω^2.
So, in a ST-diagonal basis the transformation rule explicitly reads

$$\psi \xrightarrow{ST} \omega^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \psi = \begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^{k+1} & 0 \\ 0 & 0 & \omega^{k+2} \end{pmatrix} \psi \,,$$

Thus, ψ decomposes as $\psi \rightsquigarrow \mathbf{1}_k \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+2}$ under $\mathbb{Z}_3^{ST}$.

One can find the residual symmetry representations for any other multiplet of a finite modular group in a similar way. For a given level N, the decompositions of fields under a certain residual symmetry group only depend on the pair (r, k).

The decompositions of the weighted representations of Γ'_N $(N \leq 5)$ under the three residual symmetry groups, i.e. the residual decompositions of the irreps of $\Gamma'_2 \simeq S_3$, $\Gamma'_3 \simeq A'_4 = T'$, $\Gamma'_4 \simeq S'_4 = SL(2, \mathbb{Z}_4)$, and $\Gamma'_5 \simeq A'_5 = SL(2, \mathbb{Z}_5)$ are listed in Tables 6–9 of Appendix A in P.P. Novichkov et al., arXiv:2102.07488.

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$
			$\tau \simeq i\infty$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$ with $\mathbf{1}_a \neq (\mathbf{1}_b)^*$
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes [\mathbf{2} \oplus \mathbf{1}$, or $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}']$, $\mathbf{3}' \otimes [\mathbf{2} \oplus \mathbf{1}'$, or $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}']$, $\hat{\mathbf{3}}' \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}']$, $\hat{\mathbf{3}} \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}']$
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes \mathbf{3}'$

Hierarchical mass patterns which can be realised in the vicinity of symmetric points. These patterns are unaffected by the exchange $\mathrm{r} \leftrightarrow \mathrm{r}^c$ and may only be viable for certain weights. Subscripts run over irreps of a certain dimension. Primes in parenthesis are uncorrelated.

Leading-order mass spectra patterns of bilinears $\psi^c\psi$ in the vicinity of the symmetric points ω and $i\infty$, for 3d multiplets $\psi \sim (\mathrm{r}, k)$ and $\psi^c \sim (\mathrm{r}^c, k^c)$ of the finite modular groups Γ'_N, $N = 2, 3, 4, 5$, i.e., for S_3, A'_4, S'_4 and A'_5 is given in Tables 10 - 13 of Appendix B in P.P. Novichkov et al., arXiv:2102.07488.

The number of cases which can lead to viable hierechical charged lepton or quark mass mass patterns is extremely limitted.

A_5' Model with $L \sim \mathbf{3}$, $E^c \sim \mathbf{3}'$, $N^c \sim \hat{\mathbf{2}}'$

$L \sim (\mathbf{3}, k_L = 3)$, $E^c \sim (\mathbf{3}', k_E = 1)$, $N^c \sim (\hat{\mathbf{2}}', k_N = 2)$; vicinity of $\tau = i\infty$.

We consider first the most 'structured' series of hierarchical models, i.e. the case with both fields L, E^c furnishing complete irreps of the finite modular group.

At level $N = 5$ the only such possibility arises in the vicinity of $\tau = i\infty$ when L and E^c are different triplets of A_5'.
For neutrino masses generated via a type I seesaw, we have considered gauge-singlets N^c furnishing a complete irrep of dimension 2 or 3.

We performed a detailed search for a model which
i) is phenomenologically viable in the regime of interest,
ii) produces a charged-lepton spectrum which is not fine-tuned,
iii) involvs at most 8 effective parameters (including τ).

An observable O is typically considered fine-tuned with respect to some parameter p if BG $\equiv |\partial \ln O / \partial \ln p| \gtrsim 10$.

G. Giudice and R. Barbieri, 1987

Found one model satisfying these requirements:
$L \sim (\mathbf{3}, k_L = 3)$, $E^c \sim (\mathbf{3}', k_E = 1)$, $N^c \sim (\hat{\mathbf{2}}', k_N = 2)$.
The charged-lepton mass matrix has the following structure:

$$M_e^\dagger \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix}, \quad \epsilon \simeq q_5\,, \quad q_5 = \exp(i2\pi\tau/5)\,.$$

The predicted charged-lepton mass pattern is $(m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^4)$**.**

S'_4 Model with $L \sim \hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$, $E^c \sim \hat{\mathbf{3}}'$, $N^c \sim \mathbf{3}$

$L \sim (\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}, k_L = 2)$, $E^c \sim (\hat{\mathbf{3}}', k_E = 2)$, $N^c \sim (\mathbf{3}, k_N = 1)$; vicinity of $\tau = i\infty$.

In the second most 'structured' case, one of the fields L, E^c is an irreducible triplet, while the other decomposes into a doublet and a singlet of the finite modular group.
This possibility is realised at level $N = 4$ in the vicinity of $\tau = i\infty$.
For definiteness, we take $L = L_{12} \oplus L_3$ with $L_{12} \sim (\hat{\mathbf{2}}, k_L)$, $L_3 \sim (\hat{\mathbf{1}}, k_L)$, and $E^c \sim (\hat{\mathbf{3}}', k_E)$.

We have performed a systematic scan restricting ourselves to models involving at most 8 effective parameters (including τ) with no no limit on modular form weights.
Models predicting $m_e = 0$ are rejected.
N^c (when present) furnish a complete irrep of dimension 2 or 3.

Out of the 60 models thus identified, we have selected the only one which
i) is viable in the regime of interest and
ii) produces a charged-lepton spectrum which is not fine-tuned.
This model turns out to be consistent with the experimental bound on the Dirac CPV phase. It corresponds to $k_L = k_E = 2$ and $N^c \sim (\mathbf{3}, 1)$.

Using as expansion parameter $\epsilon \equiv \varepsilon/\theta \simeq 2q$, $\mathbf{q} =$ exp$(i\pi\tau/2)$, $M_e^\dagger$ is approximately given by:

$$M_e^\dagger \sim v_d \begin{pmatrix} \epsilon^2 & \epsilon & \epsilon^3 \\ 1 & \epsilon^3 & \epsilon \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix} \; ; \; M_e^\dagger \simeq \frac{\sqrt{3}}{2} v_d \alpha_1 \theta^8 \begin{pmatrix} \epsilon^2 & \frac{(\tilde{\alpha}_2+\sqrt{3})}{2\sqrt{6}}\epsilon & \frac{(7\tilde{\alpha}_2-\sqrt{3})}{2\sqrt{6}}\epsilon^3 \\ -\frac{\tilde{\alpha}_2}{6} & \frac{(7\sqrt{3}\tilde{\alpha}_2+9)}{6\sqrt{6}}\epsilon^3 & \frac{(\sqrt{3}\tilde{\alpha}_2-9)}{6\sqrt{6}}\epsilon \\ \tilde{\alpha}_3\epsilon^2 & -\frac{\tilde{\alpha}_3}{\sqrt{2}}\epsilon & \frac{\tilde{\alpha}_3}{\sqrt{2}}\epsilon^3 \end{pmatrix} , \; \tilde{\alpha}_{2(3)} \equiv \alpha_{2(3)}/\alpha_1$$

The charged-lepton mass pattern is predicted to be $(m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^3)$.