Some Aspects of collective effects in neutrino oscillations

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Introduction: Non-Abelian Liouville Equation

For N_f flavours N_fxN_f density matrices are defined as (Wigner distributions)

$$\varrho_{ij}(\mathbf{r},\mathbf{p}) \equiv \int d^3\mathbf{r}'\, e^{-i\mathbf{p}\cdot\mathbf{r}'} \left\langle a_j^\dagger(\mathbf{r}-\mathbf{r}'/2) a_i(\mathbf{r}+\mathbf{r}'/2)\right\rangle = \int \frac{d^3\mathbf{\Delta}}{(2\pi)^3}\, e^{i\mathbf{\Delta}\cdot\mathbf{r}} \left\langle a_j^\dagger(\mathbf{p}-\mathbf{\Delta}/2) a_i(\mathbf{p}+\mathbf{\Delta}/2)\right\rangle ,$$

and analogously for anti-neutrinos, with $a_i(\mathbf{p})$ annihilator of neutrino with flavour i and momentum $\mathbf{p}$. Apart from the sources the equations of motion are Liouville equations with vacuum terms and refractive terms from a background medium and from self-interactions,

$$\partial_t \varrho(\mathbf{r},\mathbf{p}) + \mathbf{v}(\mathbf{r},\mathbf{p})\cdot\nabla_{\mathbf{r}}\varrho(\mathbf{r},\mathbf{p}) = -\,i\left[\Omega^0_{\mathbf{p}} + \Omega_m(\mathbf{r}) + \Omega^S(\mathbf{r},\mathbf{p}), \varrho(\mathbf{r},\mathbf{p})\right] ,$$

where $\Omega_p{}^0$ is the vacuum term, Ω_m is the matter term, and Ω^S the self-interaction,

$$\Omega^S(\mathbf{r},\mathbf{p}) = \mu(\mathbf{r}) \sum_{\mathbf{q}\neq\mathbf{p}} (1-\mathbf{v}_\mathbf{p}\cdot\mathbf{v}_\mathbf{q}) \Big\{ G_S[\rho(\mathbf{r},\mathbf{q}) - \bar\rho(\mathbf{r},\mathbf{q})]G_S + G_S \mathrm{Tr}\left[(\rho(\mathbf{r},\mathbf{q}) - \bar\rho(\mathbf{r},\mathbf{q}))G_S\right]\Big\} ,$$

where in general G_S=diag(1,…,1) for active neutrinos. For anti-neutrinos only the sign of $\Omega_p{}^0$ changes in the commutator in Eq.(1).

Generalisation to Non-Abelian Boltzmann Equation

For a Hamiltonian represented by the c-number flavour matrix $\mathsf{H}(t, \mathbf{r}, \mathbf{p})$ the non-Abelian Boltzmann equation for the space-time and momentum dependent c-number flavour density matrix $\varrho(t, \mathbf{r}, \mathbf{p})$ is generally written in the form

$$\partial_t \varrho + \frac{1}{2}\left\{\partial_{\mathbf{r}}\varrho, \partial_{\mathbf{p}}\mathsf{H}\right\} - \frac{1}{2}\left\{\partial_{\mathbf{p}}\varrho, \partial_{\mathbf{r}}\mathsf{H}\right\} = -i\left[\mathsf{H}, \varrho\right] + \mathscr{C}\left[\varrho\right] ,$$

where the left hand side is known as Liouville term and the right hand side consists of a commutator describing oscillations and the collision term $\mathscr{C}[\varrho]$ describes absorption, emission and scattering between different momentum modes.

If only the kinetic part in H depends on momentum then $\mathbf{V}_{\mathbf{p}} \equiv \partial_{\mathbf{p}}\mathsf{H}$ is a matrix of velocities with eigenvalues in the neutrino mass basis given by $\mathbf{v}_i = \mathbf{p}/(\mathbf{p}^2 + m_i^2)^{1/2}$ and $\frac{1}{2}\{\varrho, \mathbf{V}\}$ becomes a matrix of neutrino fluxes. Note that they contain group velocities, whereas the oscillation term contains phase velocities. The last term in the Liouville term is a force term which is usually neglected. In absence of mixing, forces and collisions the Liouville equation thus just expresses flux conservation. Note that in a stationary situation the flux is conserved, not the neutrino number.

We want to derive the non-Abelian Boltzmann equation from the Heisenberg equation $i\partial_t \hat{A} = [\hat{A}, \hat{H}]$ for operators $\hat{A}$ and Hamilton operator $\hat{H}$. To this end we introduce the annihilation and creation operators of a neutrino or antineutrino of momentum **p** and flavour i, $\hat{a}_i(\mathbf{p}, t), \hat{b}_i(\mathbf{p}, t), \hat{a}_i^\dagger(\mathbf{p}, t), \hat{b}_i^\dagger(\mathbf{p}, t)$ which correspond to the spatial operators

$$\hat{\psi}_i(\mathbf{r}) = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\, \hat{a}(\mathbf{p})\,,$$

and satisfy the anti-commutation relations

$$\{\hat{a}_i(\mathbf{p}, t), \hat{a}_j^\dagger(\mathbf{p}', t)\} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')\delta_{ij}\,,$$

and analogously for anti-neutrinos.

The following is based on G. Sigl and G. Raffelt, Nucl.Phys.B 406 (1993) 423
and Stirner, Sigl and Raffelt, JCAP 05 (2018) 016

Derivation through Wigner and Moyal Distributions

We define the operators

$$\hat{\mathsf{D}}_{ij}(\mathbf{p},\mathbf{p}',t) \equiv \hat{a}_j^\dagger(\mathbf{p}',t)\,\hat{a}_i(\mathbf{p},t)\,,$$

and relate them to a space- and momentum dependent density operator through a Wigner transformation,

$$\hat{\varrho}_{ij}(t,\mathbf{r},\mathbf{p}) = \int \frac{d^3\Delta}{(2\pi)^3}\, e^{i\Delta\cdot\mathbf{r}}\, \hat{D}_{ij}\left(\mathbf{p}+\frac{\Delta}{2},\mathbf{p}-\frac{\Delta}{2},t\right)\,.$$

For the Hamilton operator we make the mean-field ansatz

$$\hat{H} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\hat{a}_i^\dagger(\mathbf{p})\,\mathsf{H}_{ij}(\mathbf{p},\mathbf{p}')\,\hat{a}_j(\mathbf{p}')\,,$$

where H_{ij} is a c-number matrix.

Using Heisenberg's equation and after a few manipulations one arrives at

$$i\hbar\partial_t\hat{\varrho}(\mathbf{r},\mathbf{p}) = \mathrm{H}(\mathbf{r},\mathbf{p})\, e^{\frac{i}{2}\hbar(\overleftarrow{\partial}_{\mathbf{r}}\cdot\overrightarrow{\partial}_{\mathbf{p}}-\overleftarrow{\partial}_{\mathbf{p}}\cdot\overrightarrow{\partial}_{\mathbf{r}})}\,\hat{\varrho}(\mathbf{r},\mathbf{p}) - \hat{\varrho}(\mathbf{r},\mathbf{p})\, e^{\frac{i}{2}\hbar(\overleftarrow{\partial}_{\mathbf{r}}\cdot\overrightarrow{\partial}_{\mathbf{p}}-\overleftarrow{\partial}_{\mathbf{p}}\cdot\overrightarrow{\partial}_{\mathbf{r}})}\,\mathrm{H}(\mathbf{r},\mathbf{p})\,.$$

where the matrix $\mathrm{H}(\mathbf{r},\mathbf{p})$ is related to the matrix $\mathrm{H}_{ij}(\mathbf{p},\mathbf{p}')$ through the same kind of Wigner transformation relating $\hat{\varrho}(\mathbf{r},\mathbf{p})$ to $\hat{D}_{ij}(\mathbf{p},\mathbf{p}')$.

A similar equation for first quantised scalars was first derived by Moyal. This is an operator equation with so far no approximations.

Taking the expectation value $\varrho(\mathbf{r},\mathbf{p}) \equiv \langle\hat{\varrho}(\mathbf{r},\mathbf{p})\rangle$ (mean field theory) and expanding the exponential up to first order then immediately gives the Liouville equation with flavour mixing,

$$\partial_t\varrho + \frac{1}{2}\left\{\partial_{\mathbf{r}}\varrho, \partial_{\mathbf{p}}\mathrm{H}\right\} - \frac{1}{2}\left\{\partial_{\mathbf{p}}\varrho, \partial_{\mathbf{r}}\mathrm{H}\right\} = -\,i\left[\mathrm{H},\varrho\right]\,. \quad (3)$$

Higher orders would yield the collision terms and other quantum corrections. Planck's constant (here set to unity) appears both on the left hand side and in the exponent of the Moyal equation and thus cancels to lowest order, corresponding to the classicality of Liouville's equation.

Derivation through Husimi Distributions

We here simplify to one flavour $N_f = 1$. Smearing a Wigner distribution $f(\mathbf{r}, \mathbf{p})$ with Gaussians of width η in position and width σ in momentum space gives

$$F(\mathbf{r}, \mathbf{p}) \equiv \frac{1}{(2\pi\eta\sigma)^3} \int d^3\mathbf{r}' d^3\mathbf{p}' f(\mathbf{r}', \mathbf{p}') \exp\left[-\frac{(\mathbf{r} - \mathbf{r}')^2}{2\eta^2} - \frac{(\mathbf{p} - \mathbf{p}')^2}{2\sigma^2} \right] .$$

Choosing $\sigma = \hbar/(2\eta)$ the operator version can be put into the form

$$\hat{F}(\mathbf{r}, \mathbf{p}) = \frac{1}{(2\pi\eta^2)^{3/2}} \int \frac{d^3\mathbf{r}_1 d^3\mathbf{r}_2}{(2\pi\hbar)^3} \hat{\psi}^\dagger(\mathbf{r}_1) \hat{\psi}(\mathbf{r}_2) \exp\left[-\frac{(\mathbf{r} - \mathbf{r}_1)^2 + (\mathbf{r} - \mathbf{r}_2)^2}{4\eta^2} + \frac{i\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2)}{\hbar} \right] .$$

For the time evolution of the quantum field $\hat{\psi}$ we make the ansatz

$$i\hbar \partial_t \hat{\psi} = \mathbf{v} \cdot \hat{\mathbf{p}}\, \hat{\psi} + V(\mathbf{r}) \hat{\psi}, \qquad \text{or} \qquad \partial_t \hat{\psi} = -\mathbf{v} \cdot \partial_{\mathbf{r}} \hat{\psi} - iV(\mathbf{r}) \hat{\psi}/\hbar$$

which for $V(\mathbf{r}) = 0$ gives the relativistic dispersion relation $\hbar\omega = \mathbf{v} \cdot \mathbf{p}$.

A straightforward calculation and taking expectation values gives back the Liouville equation,

$$\partial_t F(\mathbf{r},\mathbf{p}) = -\,\mathbf{v}\cdot\partial_{\mathbf{v}} F + \partial_{\mathbf{r}} V\cdot\partial_{\mathbf{p}} F + \mathcal{O}(\hbar)\,,$$

to this order independent of η ! The higher order terms give rise to quantum corrections which become in particular relevant on scales comparable to the de Broglie scale $\hbar/p$.

Solutions of such Liouville/Vlasov equations crucially depend on boundary and initial conditions.

Collision Terms and Dynamical Decoherence

Assuming molecular chaos for neutral current interactions the collision (Boltzmann) term at a given location $\mathbf{r}$ has the form

$$\partial_t \rho_{\mathbf{p}}\Big|_{\text{coll,NC}} = \frac{1}{2}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\Big[W(q,p)(1-\rho_{\mathbf{p}})G\rho_{\mathbf{q}}G - W(p,q)\rho_{\mathbf{p}}G(1-\rho_{\mathbf{q}})G$$

$$+W(-q,p)(1-\rho_{\mathbf{p}})G(1-\overline{\rho}_{\mathbf{q}})G - W(p,-q)\rho_{\mathbf{p}}G\overline{\rho}_{\mathbf{q}}G + \text{h.c.}\Big],$$

where $W(q,p)$ is the (scalar) scattering rate from state of four-momentum $(E_{\mathbf{q}}, \mathbf{q})$ to a state of four-momentum $(E_{\mathbf{p}}, \mathbf{p})$, with $E_{\mathbf{p}}$ the energy corresponding to three-momentum $\mathbf{p}$, and G a dimensionless flavour matrix characterising the coupling strengths of the different neutrino flavours. The c-number flavour densities $\rho_{\mathbf{p}}$ and $\overline{\rho}_{\mathbf{p}}$ refer to neutrinos and anti-neutrinos, respectively. The first two terms describe scattering and the last two describe pair creation and annihilation. They are related by crossing particles and substituting $p \to -p \quad \rho_{\mathbf{p}} \to 1-\overline{\rho}_{\mathbf{p}}$, which also gives a similar equation for $\partial_t \overline{\rho}_{\mathbf{p}}\Big|_{\text{coll,NC}}$.

Charged current source terms at a given location $\mathbf{r}$ have the form

$$\partial_t \rho_{\mathbf{p}}\Big|_{\rm coll,CC} = \{\mathscr{P}(p), (1-\rho_{\mathbf{p}})\} - \{\mathscr{A}(p), \rho_{\mathbf{p}}\}\,,$$

with $\mathscr{P}(p)$ and $\mathscr{A}(p)$ flavour-diagonal $N_f \times N_f$ matrices with production and absorption rates of neutrinos of given flavour and four-momentum p on the diagonal. Assuming detailed balance for the background plasma this becomes

$$\partial_t \rho_{\mathbf{p}}\Big|_{\rm coll,CC} = \left\{ \mathscr{P}(\mathbf{p}), \left(1 - \frac{\rho_{\mathbf{p}}}{f_{0\mathbf{p}}}\right)\right\}\,,$$

with $f_{0\mathbf{p}}$ the equilibrium occupation numbers,

$$f_{0\mathbf{p}} \equiv f_{\rm eq}(E_{\mathbf{p}}) = \frac{1}{e^{(E_{\mathbf{p}}-\mu)/T}+1}\,.$$

Some Thermodynamical Aspects

If the neutrinos are coupled to a medium in thermal equilibrium characterized by a temperature T and a chemical potential μ for the lepton number, then one can show that the neutrino grand canonical potential

$$\Omega_\nu \equiv U_\nu - TS_\nu - \mu L_\nu \,,$$

can never increase, i.e. $\dot{\Omega}_\nu \leq 0$. Here, the internal energy of the neutrino ensemble U_ν, its total lepton number L_ν, and its entropy S_ν are given by

$$U_\nu = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \,|\,\mathbf{p}\,|\, \mathrm{Tr}(\rho_\mathbf{p} + \overline{\rho}_\mathbf{p}) \,,$$

$$L_\nu = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \,\mathrm{Tr}(\rho_\mathbf{p} - \overline{\rho}_\mathbf{p}) \,,$$

$$S_\nu = -\int \frac{d^3\mathbf{p}}{(2\pi)^3} \,\mathrm{Tr}\Big[\rho_\mathbf{p}\ln(\rho_\mathbf{p}) + (1-\rho_\mathbf{p})\ln(1-\rho_\mathbf{p}) + \overline{\rho}_\mathbf{p}\ln(\overline{\rho}_\mathbf{p}) + (1-\overline{\rho}_\mathbf{p})\ln(1-\overline{\rho}_\mathbf{p})\Big] \,.$$

Analogous expressions for bosons are obtained by substituting $1 - \rho_{\mathbf{p}} \to 1 + \rho_{\mathbf{p}}$, thus Pauli blocking turns into stimulated emission.

If there are only neutral current interactions, lepton number L_ν will be conserved and $\dot{\Omega}_\nu \leq 0$ implies that the neutrino free energy $F_\nu \equiv U_\nu - TS_\nu$ can never increase, $\dot{F}_\nu \leq 0$. If neutrinos interact only among themselves, the neutrino energy U_ν will be conserved in addition, and the neutrino entropy S_ν can never decrease, $\dot{S}_\nu \geq 0$. Derivation is similar to Boltzmann's H-theorem, but now with flavour matrices.

Real dynamical decoherence is tied to an increase of entropy, or a decrease of the grand canonical potential or the free energy and thus can only be caused by collision terms.

The entropy does not increase only if the occupation number matrices already equal their equilibrium values, $\partial_t \rho_{\mathbf{p}} = 0\,, \rho_{\mathbf{p}} = f_{0\mathbf{p}}$.

Missing Energy in inhomogeneous kinetic equation

The following is based on D.F.G. Fiorillo G. Raffelt and G. Sigl, Phys.Rev.Lett. 133 (2024) 2, 021002

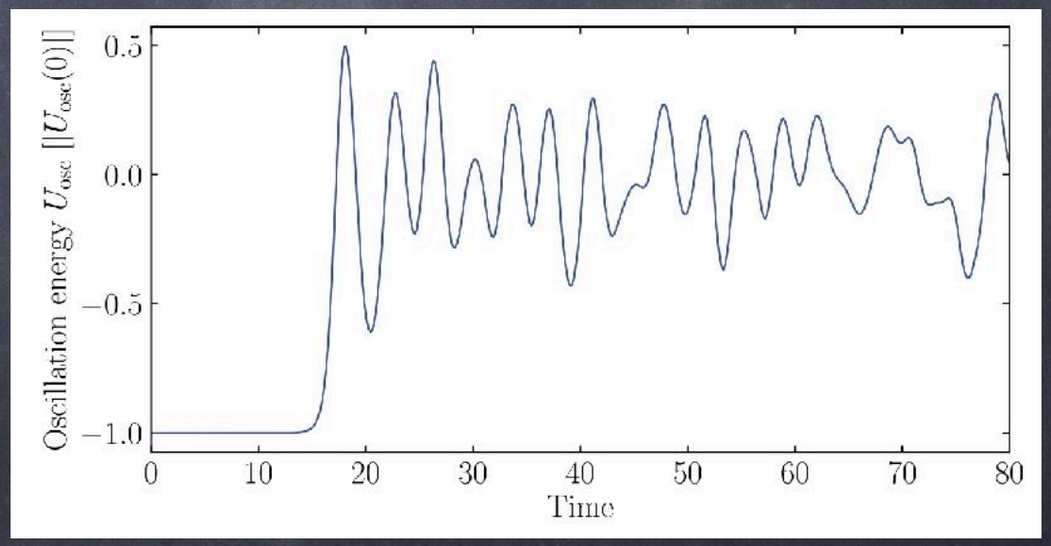
The oscillating interaction energy

$$U_{\rm osc} = \frac{\sqrt{2}G_{\rm F}}{4}\int d^3\mathbf{r}\sum_{\mathbf{p},\mathbf{p}'}(1-\cos\theta_{\mathbf{p},\mathbf{p}'})\,\mathbf{P}_{\mathbf{p}}\cdot\mathbf{P}_{\mathbf{p}'},$$

should be conserved, but in a setting with inhomogeneities in the z-direction one finds

$$\frac{dU_{\rm osc}}{dt} = -\frac{\sqrt{2}G_{\rm F}}{2}\int d^3\mathbf{r}\left(\mathbf{P}_0\cdot\partial_z\mathbf{P}_1 - \mathbf{P}_1\cdot\partial_z\mathbf{P}_2\right),$$

with $\mathbf{P}_k \equiv \sum_{\mathbf{p}} v_z^k\,\mathbf{P}_{\mathbf{p}}$.



In the massless limit this can be cured with a force term in the Liouville equation ($\mathbf{v}_\mathbf{p} \equiv \mathbf{p}/|\mathbf{p}|$):

$$\partial_t \rho_{\mathbf{p},\mathbf{r}} + \mathbf{v}_\mathbf{p} \cdot \nabla_\mathbf{r} \rho_{\mathbf{p},\mathbf{r}} = i\left[\rho_{\mathbf{p},\mathbf{r}}, \Omega_{\mathbf{p},\mathbf{r}}\right] + \nabla_\mathbf{r} \cdot \mathbf{\Phi}_{\mathbf{p},\mathbf{r}} - \nabla_\mathbf{p} \cdot \mathbf{F}_{\mathbf{p},\mathbf{r}},$$

of the form

$$\mathbf{F}_{\mathbf{p},\mathbf{r}} = \frac{\sqrt{2} G_\mathrm{F}}{2} \sum_{\mathbf{p}'} (1 - \cos\theta_{\mathbf{p},\mathbf{p}'}) \left[\left\{ \rho_{\mathbf{p}',\mathbf{r}}, \nabla_\mathbf{r} \rho_{\mathbf{p},\mathbf{r}} \right\} + 2\,\mathrm{Tr}(\rho_{\mathbf{p}',\mathbf{r}})\, \nabla_\mathbf{r} \rho_{\mathbf{p},\mathbf{r}} \right].$$

This pertains to an exchange of neutrino-neutrino refractive energy with the much larger reservoir of neutrino kinetic energy

$$K = \langle \mathscr{H}_0 \rangle = \int d^3\mathbf{r} \sum_\mathbf{p} \epsilon_\mathbf{p} \mathrm{Tr}(\rho_\mathbf{p}).$$

The role of matter refraction in toy models

The following is based on G. Sigl, Phys.Rev.D 105 (2022) 4, 043005

Notes:
all time- and length-scales are resolved
hierarchy of rates mimics hierarchy in real supernova:

$$\lambda(x) \gtrsim \mu(x) \gg f_s(x) \simeq f_{\rm NC}(x)$$

Numerical Setup of 1D Toy Model

Typically we use N_f =2 flavours, momentum modes with equal energy and one spatial dimension x with a one-dimensional array of N_p momentum modes whose velocity projections onto the x (radial) direction are isotropically distributed between -1 and +1,

$$v_x(i_p) = -1 + \frac{1}{N_p} + \frac{i_p - 1}{N_p - 1}\left(2 - \frac{2}{N_p}\right), \quad i_p = 1, \cdots, N_p, \text{ with } N_p \text{ even}.$$

The source term then is

$$\partial_t \rho(x, i_p)_{\text{coll,CC}} = f_s(x) f(x, i_p) \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \left(1 - \frac{\rho(x, i_p)}{f_0(x, i_p)}\right) \right\},$$

and analogously for anti-neutrinos. The vacuum term is

$$\Omega^0_{\nu_x} = \frac{\Delta m^2}{4} \begin{pmatrix} \cos 2\theta_0 & -\sin 2\theta_0 \\ -\sin 2\theta_0 & -\cos 2\theta_0 \end{pmatrix},$$

where $\Delta m^2 > 0$, $\cos 2\Theta_0 > 0$ corresponds to the inverted hierarchy.
The matter term is

$$\Omega_m(x) = \lambda(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\Omega^{\rm S}(x,i_p)=\mu(x)\sum_{i_q} g_{i_p,i_q}\left\{G_{\rm S}[\rho(x,i_q)-\bar{\rho}(x,i_q)]G_{\rm S}+G_{\rm S}{\rm Tr}\left[(\rho(x,i_q)-\bar{\rho}(x,i_q))G_{\rm S}\right]\right\}\,,$$

where in general G_S=diag(1,…,1) for active neutrinos. The self-interaction coefficients are normalised as

$$g_{i,j}=\frac{(1-\delta_{ij})(1-v_iv_j)}{\sum_{kl}(1-\delta_{kl})(1-v_kv_l)/N_p}\,,$$

which assures that the average coupling of one momentum mode summed over all other modes is unity and thus does not depend on N_p, before being multiplied with the characteristic self-coupling μ(x).

As initial conditions we typically put slightly unsure flavour eigenstates,

$$\rho(t=0,x,i_p)=f_i(x)f(x,i_p)M(x)\,,\quad \bar{\rho}(t=0,x,i_p)=f_i(x)\bar{f}(x,i_p)M(x)\,,$$

where

$$M(x)=\frac{1}{2}\begin{pmatrix}1+\cos\theta(x) & \exp[i\phi(x)]\sin\theta(x)\\ \exp[-i\phi(x)]\sin\theta(x) & 1-\cos\theta(x)\end{pmatrix}\,.$$

$f(x, i_p)$ and $\bar{f}(x, i_p)$ can contain a modulation in i_p and an asymmetry between neutrinos and anti-neutrinos, e.g. a crossing:

$$f(x, i_p) = \frac{1}{2}[1 - ah(x, i_p)][1 - bh(x, i_p)]\,, \quad \bar{f}(x, i_p) = \frac{1}{2}[1 - ah(x, i_p)][1 + bh(x, i_p)]\,,$$

with

$$h(x, i_p) = \left(2\frac{i_p - 1}{N_p - 1} - 1\right) g(x)\,,$$

where $g(x)$ vanishes at the boundaries.

At $x=L_x$ the boundary condition for the incoming modes, $v_x<0$, is given in terms of the initial condition (to make them consistent),

$$\rho(t, x = L_x, v_x < 0) = \rho(t = 0, x = L_x, v_x < 0) = f_i(L_x) f(L_x, i_p) M(x)\,,$$

and analogously for anti-neutrinos. Idea is to make them close to zero (no neutrinos coming from outside).
At $x=0$ the boundary conditions for the incoming modes, $v_x<0$, are given in terms of a reflective boundary,

$$\rho(t, x = 0, v_x) = \rho(t, x = 0, -v_x)\,, \quad \bar{\rho}(t, x = 0, v_x) = \bar{\rho}(t, x = 0, -v_x)\,.$$

Anti-neutrino initial and equilibrium densities and production rates are typically assumed to be equal to the ones of neutrinos. Flavor perturbations are initially driven by the vacuum frequency.

Partial differential equations are integrated within $0<x<L_x$ and $0<t<t_{max}$.
Mathematica 11.1 was used with the NDSolve routine.

Results

We show the following quantities: Total number of neutrinos

$$N(t,x) \equiv \sum_{v_x} \mathrm{Tr}\left[\rho(x,v_x) + \bar{\rho}(x,v_x)\right] ,$$

off diagonal terms

$$F_{\mathrm{off}}(t,x) \equiv \frac{\sum_{v_x} \left|\rho_{12}(x,v_x) + \bar{\rho}_{12}(x,v_x)\right|}{N(t,x)} ,$$

total flavour asymmetry of outgoing neutrinos

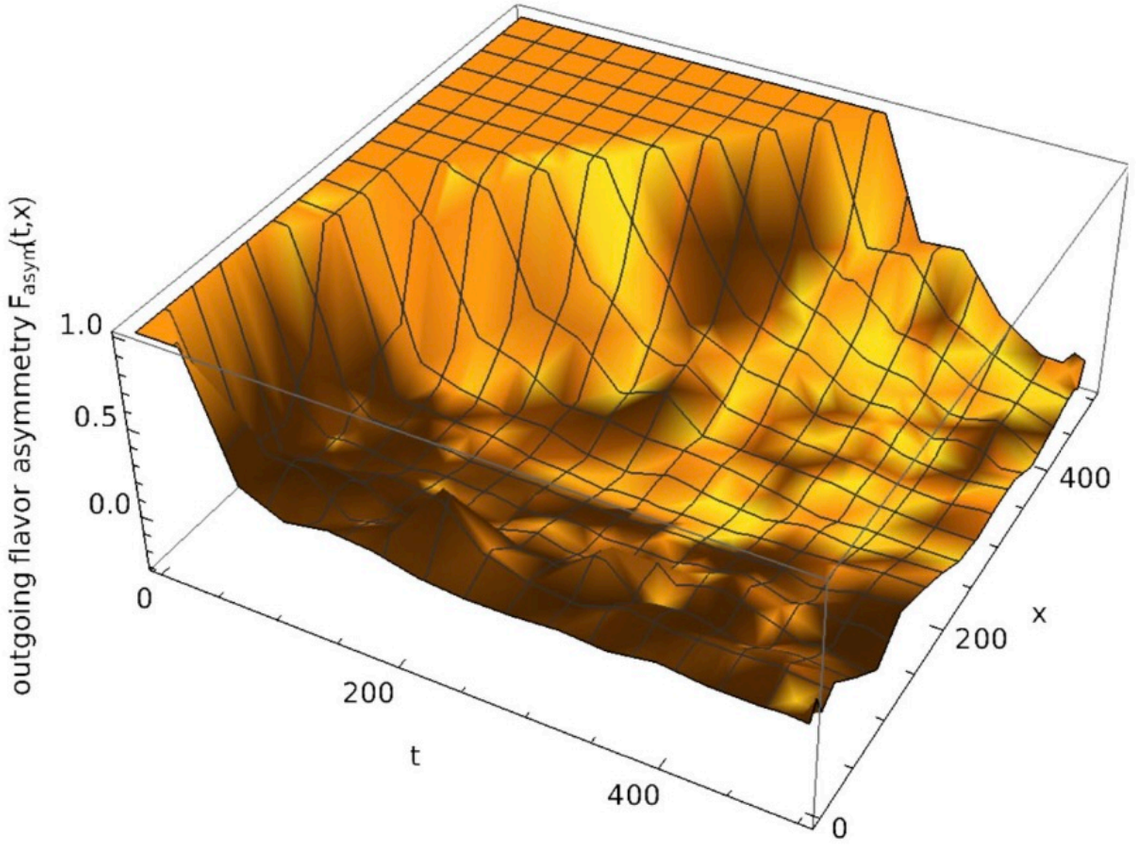
$$F_{\mathrm{asym}}(t,x) \equiv \frac{\sum_{v_x} > 0 \left[\rho_{11}(x,v_x) - \rho_{22}(x,v_x) + \bar{\rho}_{11}(x,v_x) - \bar{\rho}_{22}(x,v_x)\right]}{\sum_{v_x>0} \mathrm{Tr}\left[\rho(x,v_x) + \bar{\rho}(x,v_x)\right]} ,$$

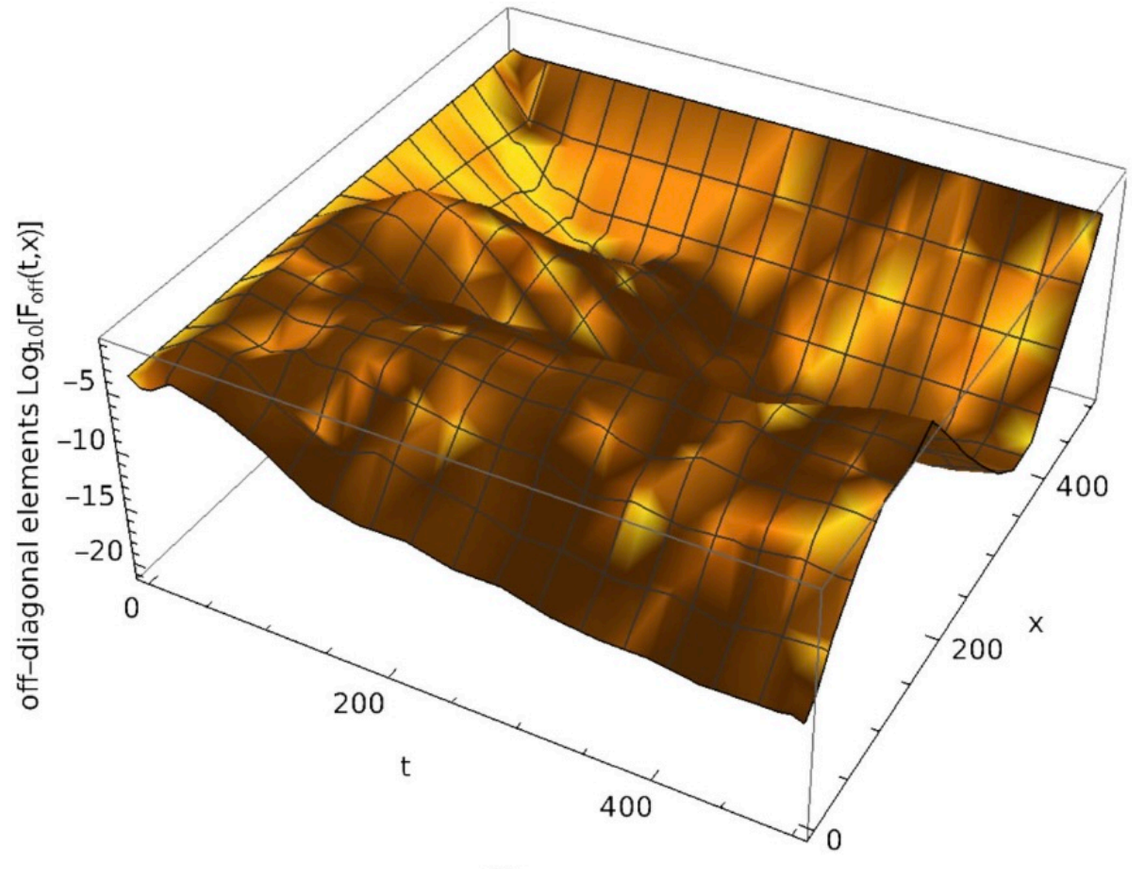
and the relative flavour-lepton number asymmetry

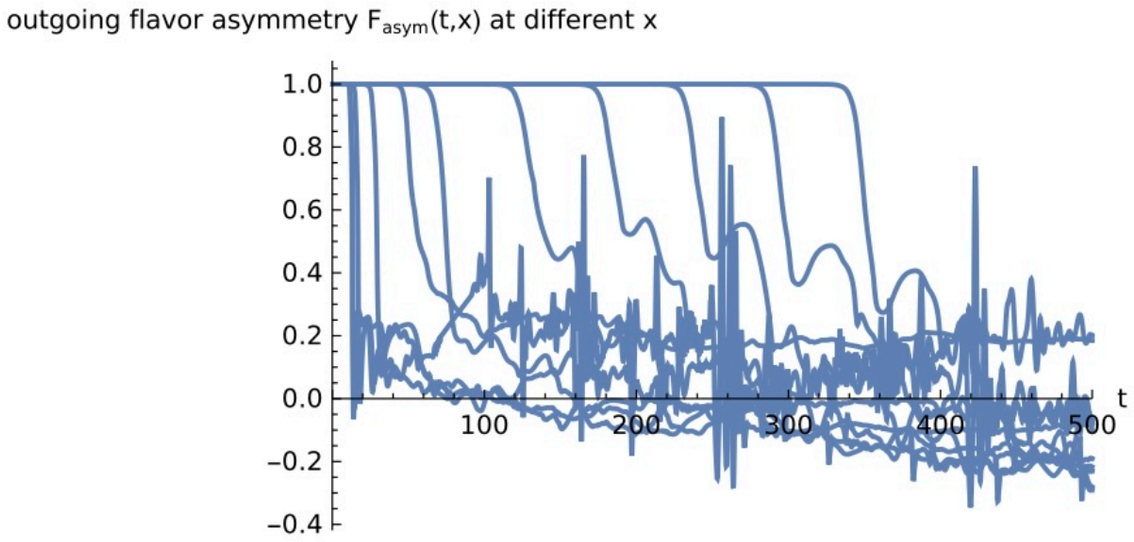
$$G(t,x,v_x) \equiv \frac{\rho_{11}(x,v_x) - \rho_{22}(x,v_x) - \bar{\rho}_{11}(x,v_x) + \bar{\rho}_{22}(x,v_x)}{N(t,x)} .$$

TABLE I. Summary of scenarios simulated. Here, L is the box size in the spatial coordinate, x_0 is the scale height of the exponential profiles, Δm^2 and θ_0 are the vacuum mixing parameters, $\mu(x)$ is the self-interaction strength, $\lambda(x)$ is the matter term and $f_{\rm NC}(x)$ is the normalization of the neutral current interactions. The initial conditions can have an electron lepton number crossing (ELNC) with $a = 0$, $b = 0.5$ in Eq. (18), where $g(x) = \sin(\pi x/L)\exp(-x/x_0)$ in Eq. (17) and random angles $\theta(x)$ and $\phi(x)$ as described in Eqs. (21) and (22) and shown in Fig. 1 with amplitude $A = 10^{-4}$, or they can represent an initially pure flavor 1 state in case (8). The boundary condition is usually reflective at $x = 0$, in which case at $x = L$ it is given by the initial condition for the incoming modes $v_x < 0$. For comparison, cases (9) and (10) represent the homogeneous case without profile and periodic boundary conditions. In this case the initial conditions are given by $g(x) = 1 + 0.1\sin(\pi x/L)$ in Eq. (17) and $\theta(x) = 10^{-5}$, $\phi(x) = 1.5$ in Eq. (22). The equilibrium and initial occupation numbers are $f_0(x) = \bar{f}_0(x) = f_i(x) = 0.8\exp(-x/x_0)$ and charged current interaction normalization is $f_s(x) = 0.1\exp(-x/x_0)$ for all simulations.

Fig.	L, x_0	Δm^2, θ_0	$\mu(x)$	$\lambda(x)$	$f_{\rm NC}(x)$	Initial	Boundary
1	2	200, 100	0,0	$50\exp(-x/100)$	0	0	ELNC random	Reflective at $x = 0$
2	3	200, 100	0,0	$50\exp(-x/100)$	$50\exp(-x/100)$	0	ELNC random	Reflective at $x = 0$
3	4	200, 100	0,0	$50\exp(-x/100)$	50	0	ELNC random	Reflective at $x = 0$
4	5	500, 250	0,0	$50\exp(-x/250)$	0	0	ELNC random	Reflective at $x = 0$
5	6	500, 250	0,0	$50\exp(-x/250)$	$50\exp(-x/250)$	0	ELNC random	Reflective at $x = 0$
6	7	500, 250	0,0	$50\exp(-x/250)$	0	$0.1\exp(-x/250)$	ELNC random	Reflective at $x = 0$
7	8	500, 250	0,0	$50\exp(-x/250)$	$50\exp(-x/250)$	$\exp(-x/250)$	ELNC random	Reflective at $x = 0$
8	10	500, 250	0.1, 10^{-4}	$50\exp(-x/250)$	0	0	Pure flavor 1	Reflective at $x = 0$
9	11	500, ∞	0,0	50	0	0	ELNC, nonrandom	Periodic
10	12	500, ∞	0,0	50	0	0.1	ELNC, nonrandom	Periodic







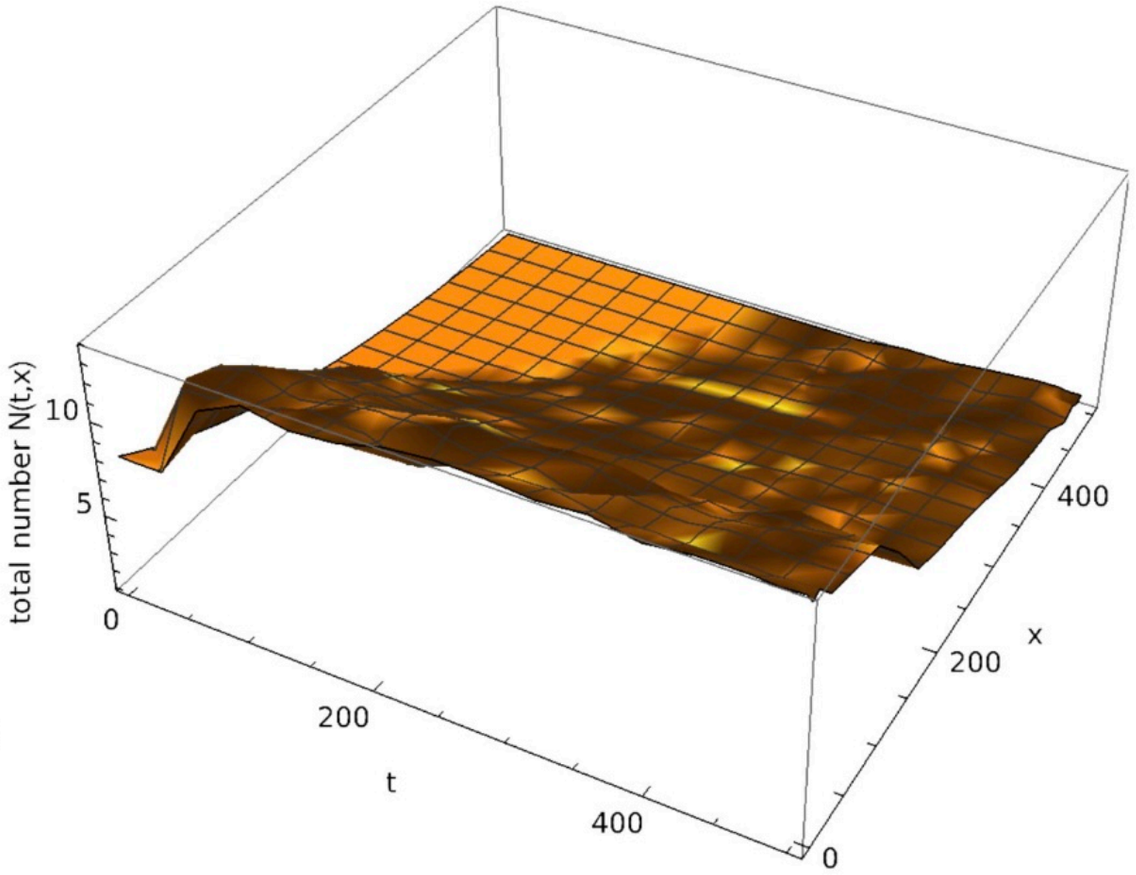
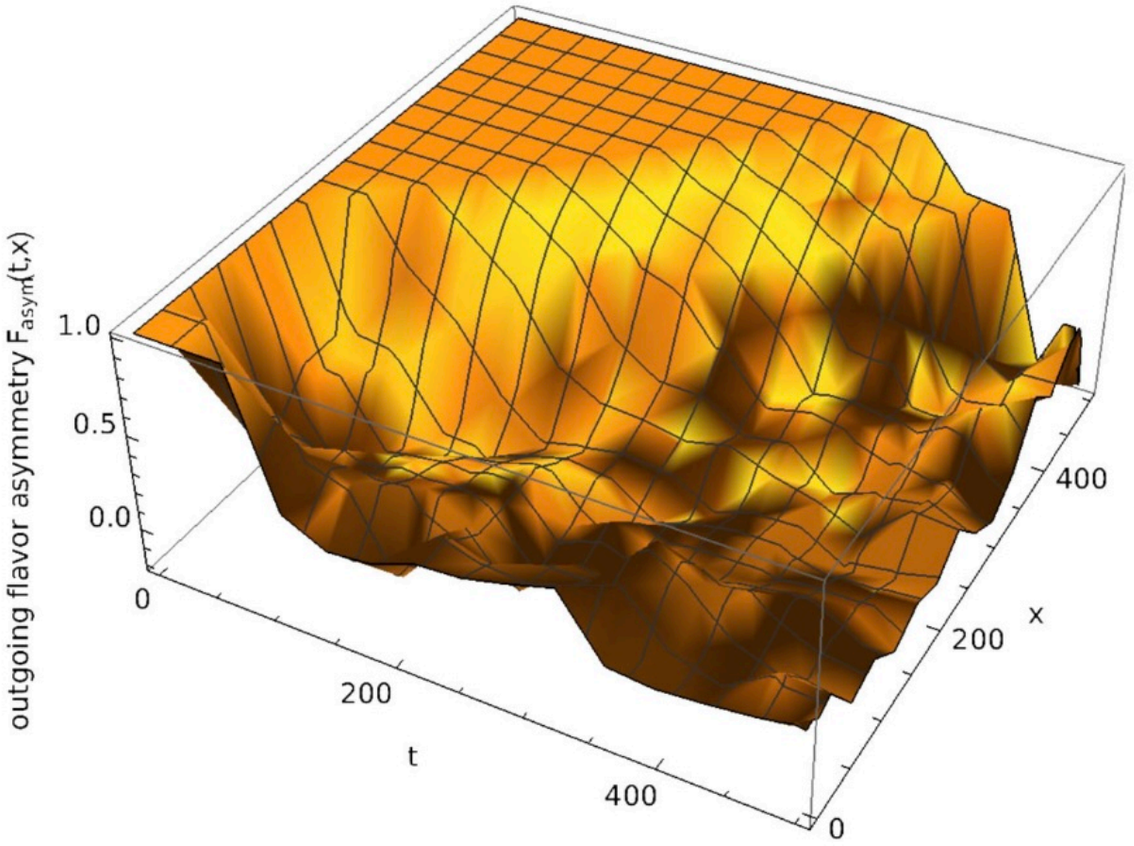
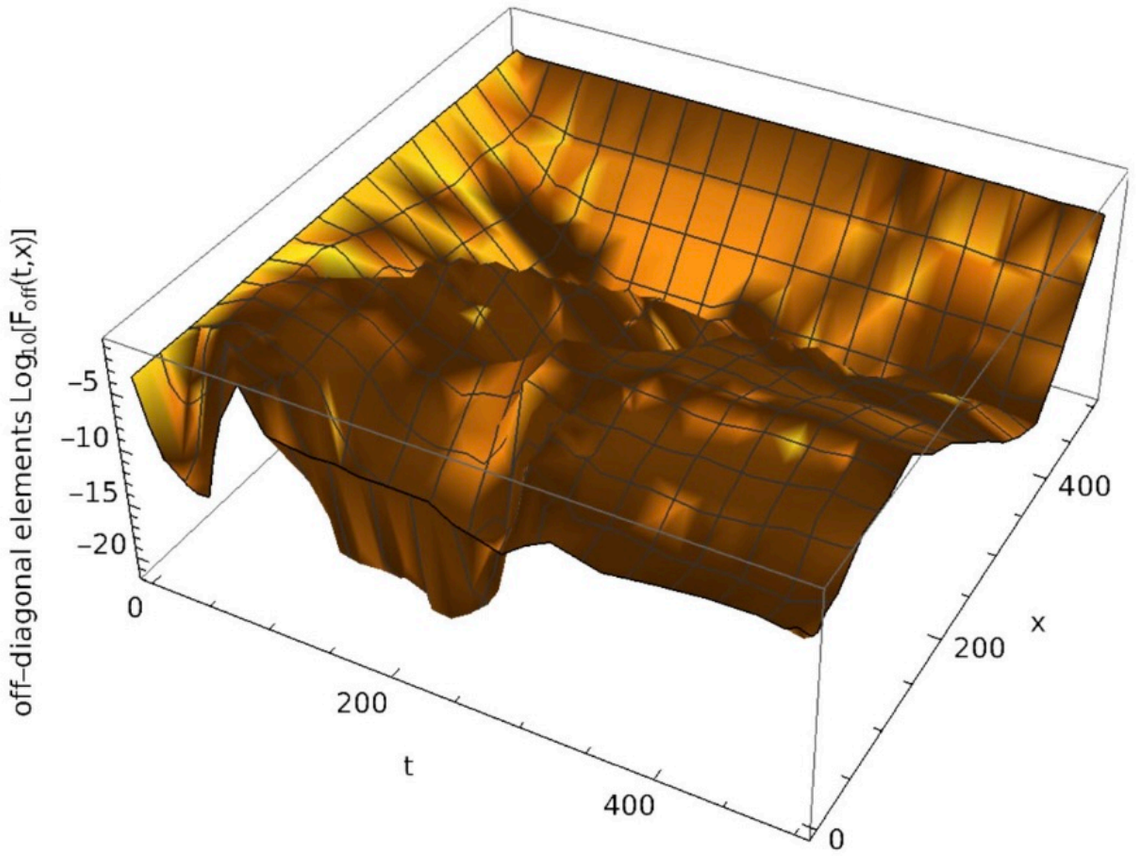
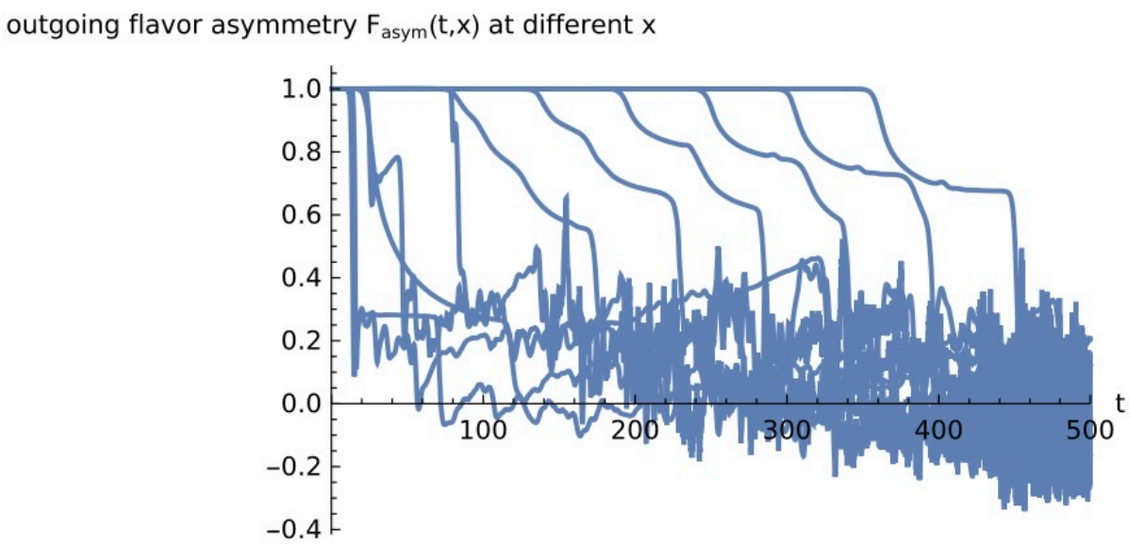


FIG. 5. Results for a case (4) simulation which is very similar to case (1) shown in Fig. 2, except for shallower profiles $\propto \exp(-x/250)$.







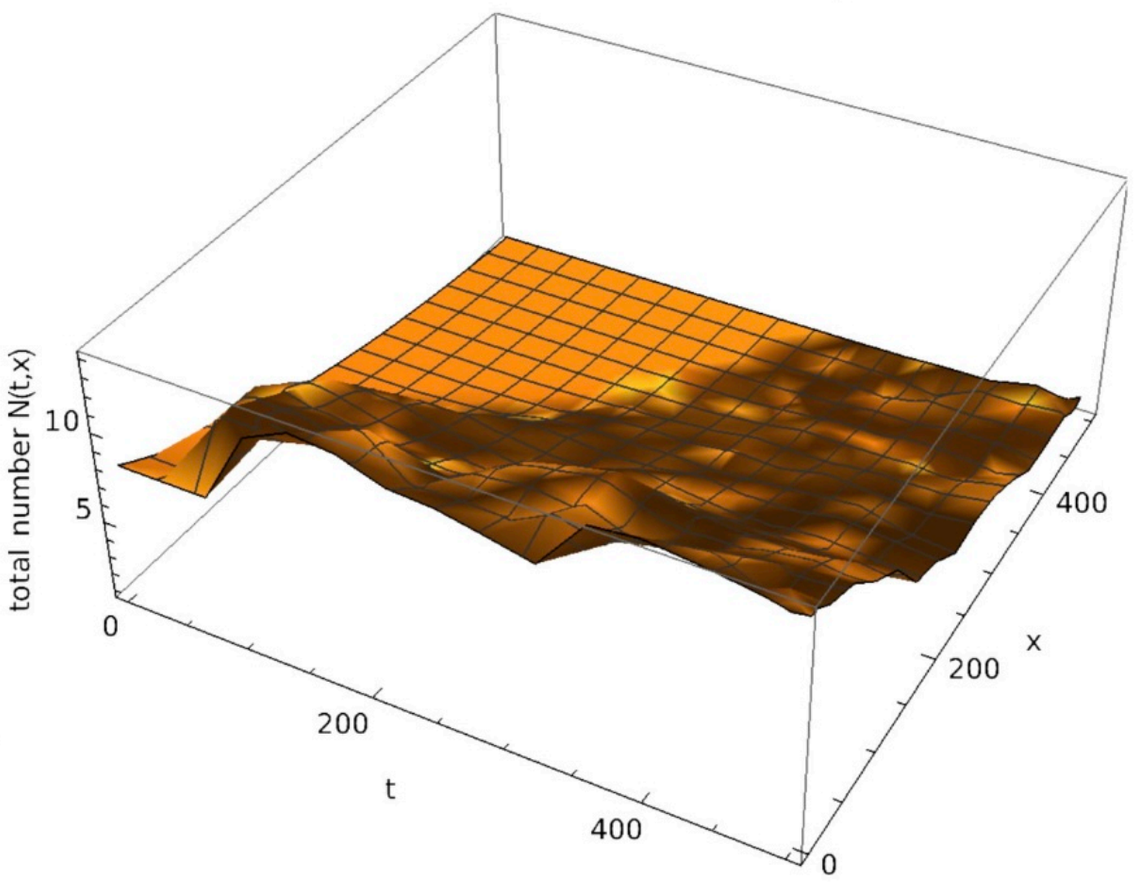
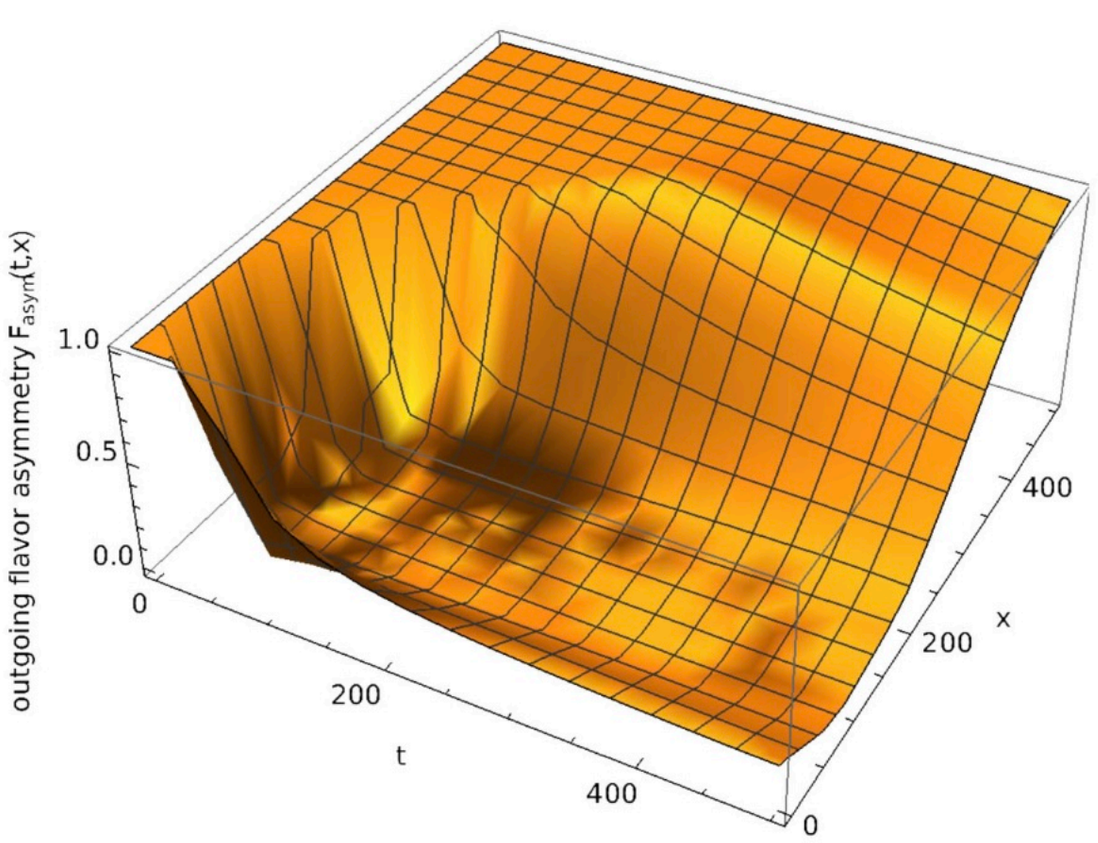
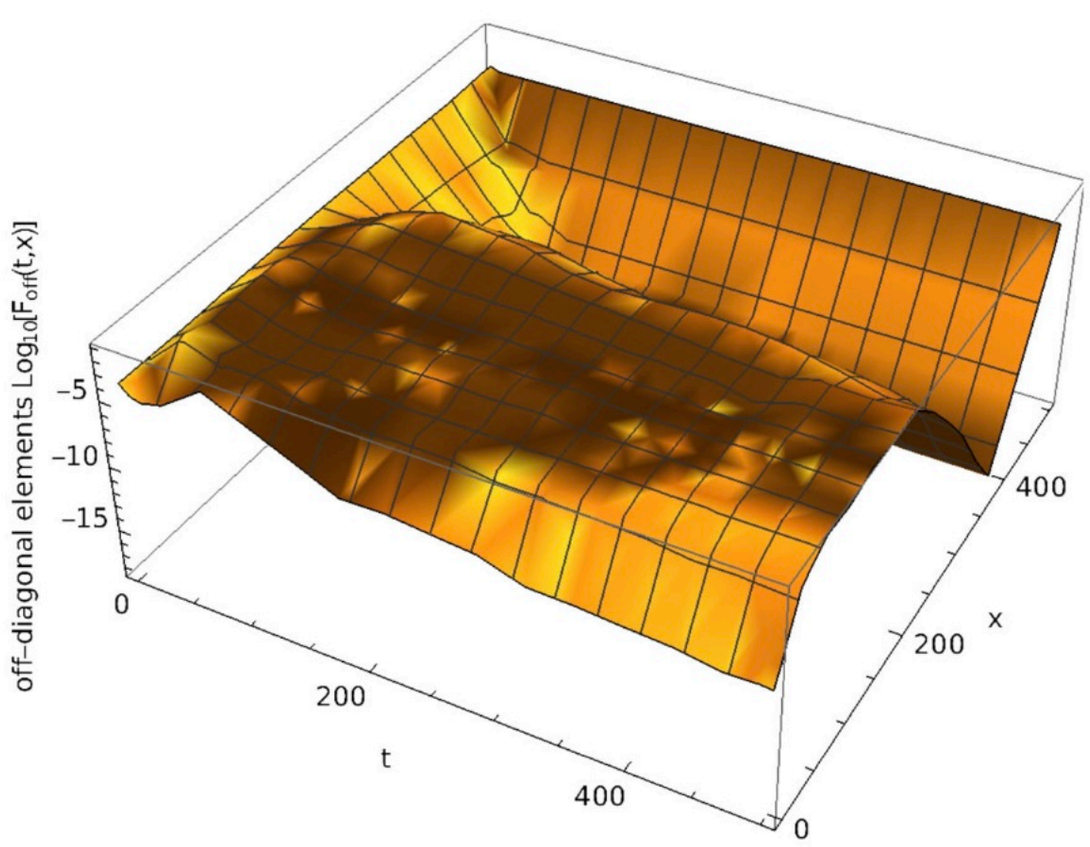


FIG. 6. Results for a case (5) simulation which is identical to case (4) but with an additional matter term $\lambda(x) = \mu(x) = 50\exp(-x/250)$. Note that the matter term partially suppresses and delays the flavor conversions.





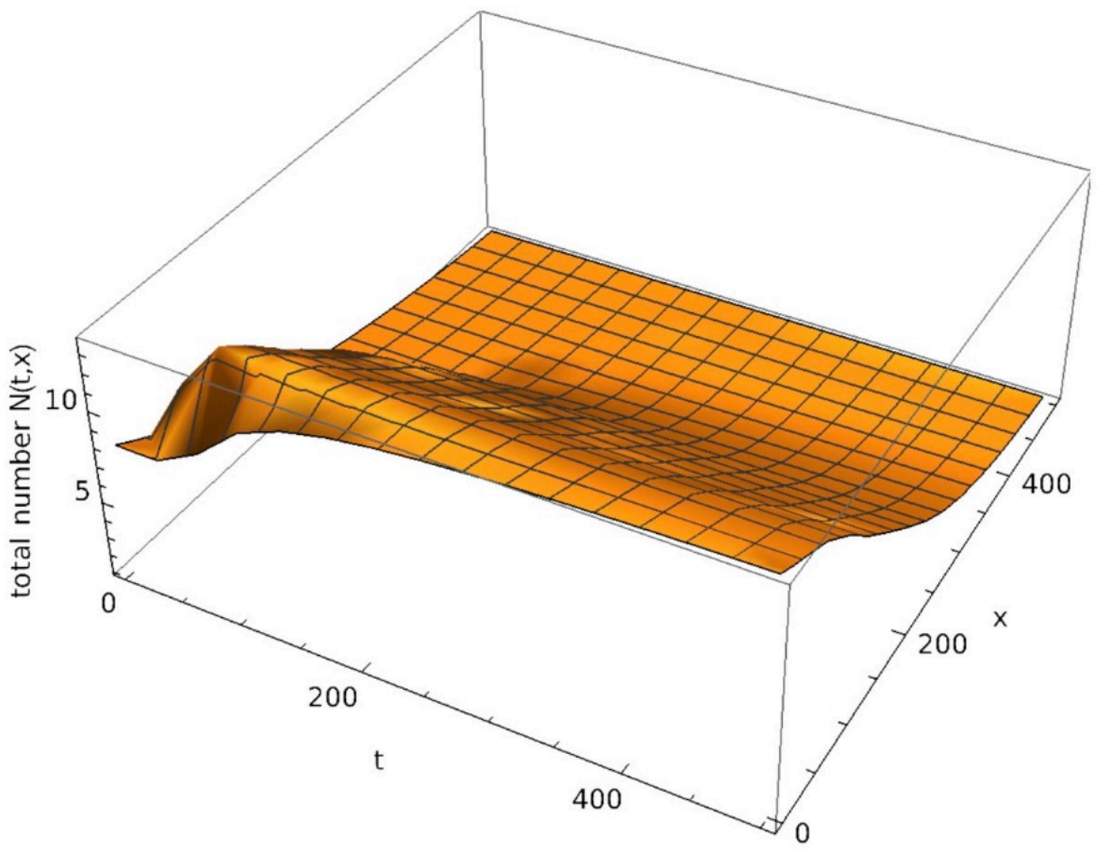
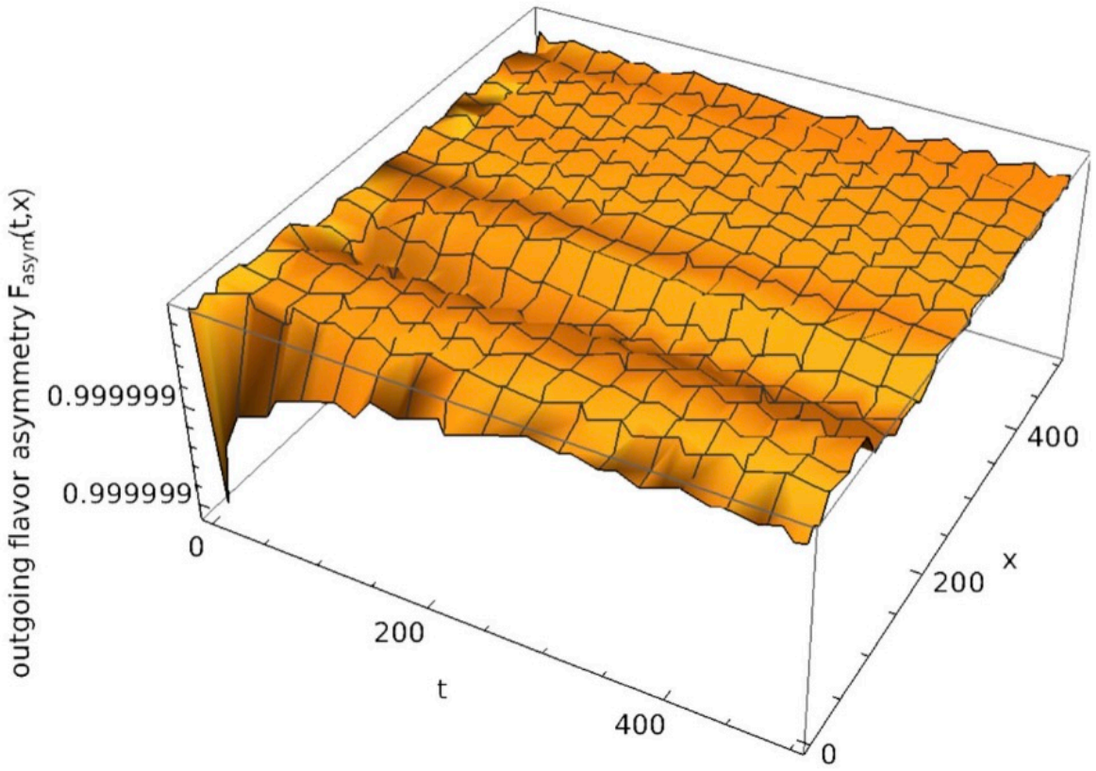
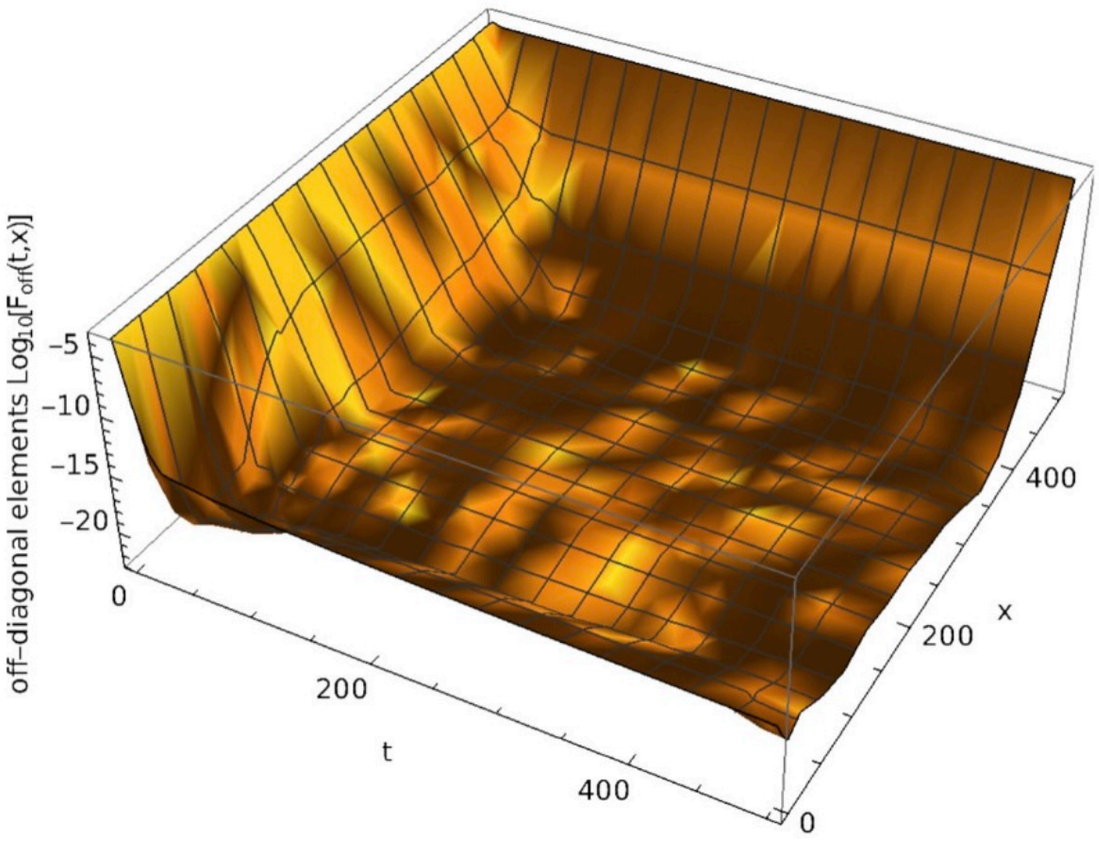
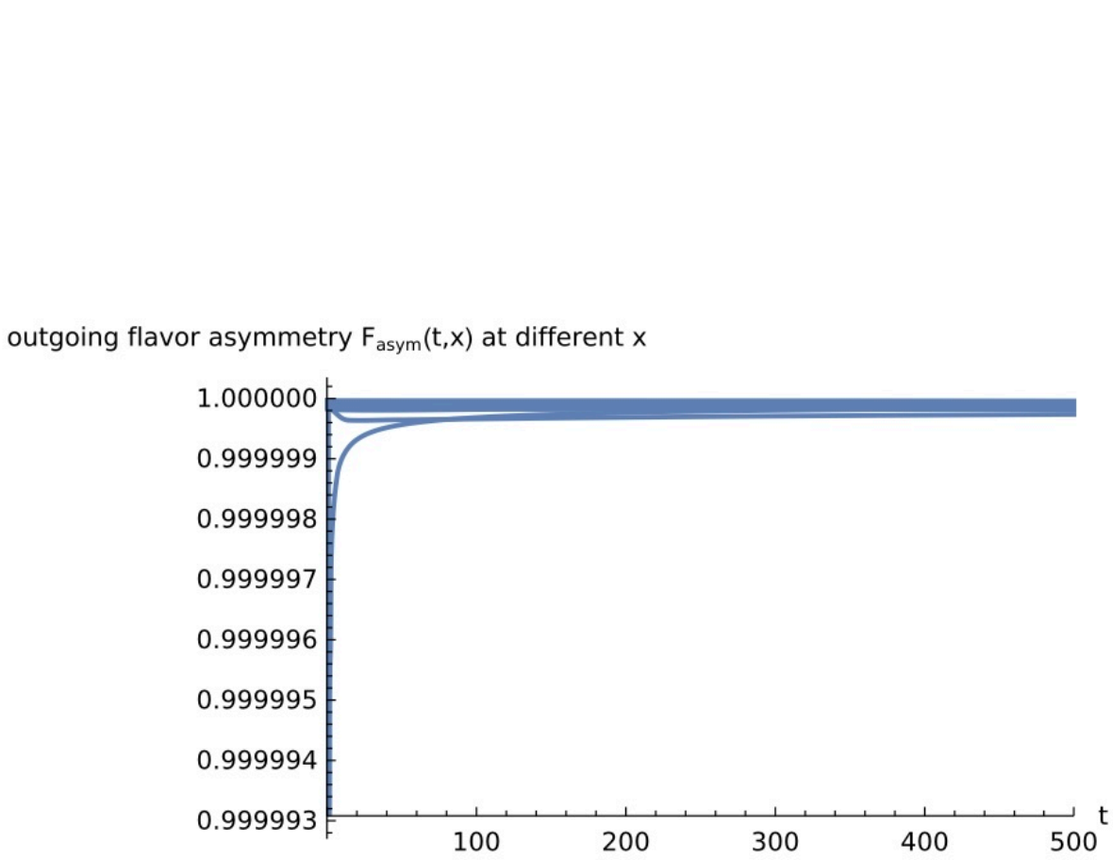


FIG. 7. Results for a case (6) simulation which is identical to case (4) but with an additional neutral current scattering term $\propto f_{\rm NC}(x) = 0.1\exp(-x/250)$. Note that the neutral current scattering term partially suppresses and delays the flavor conversions.







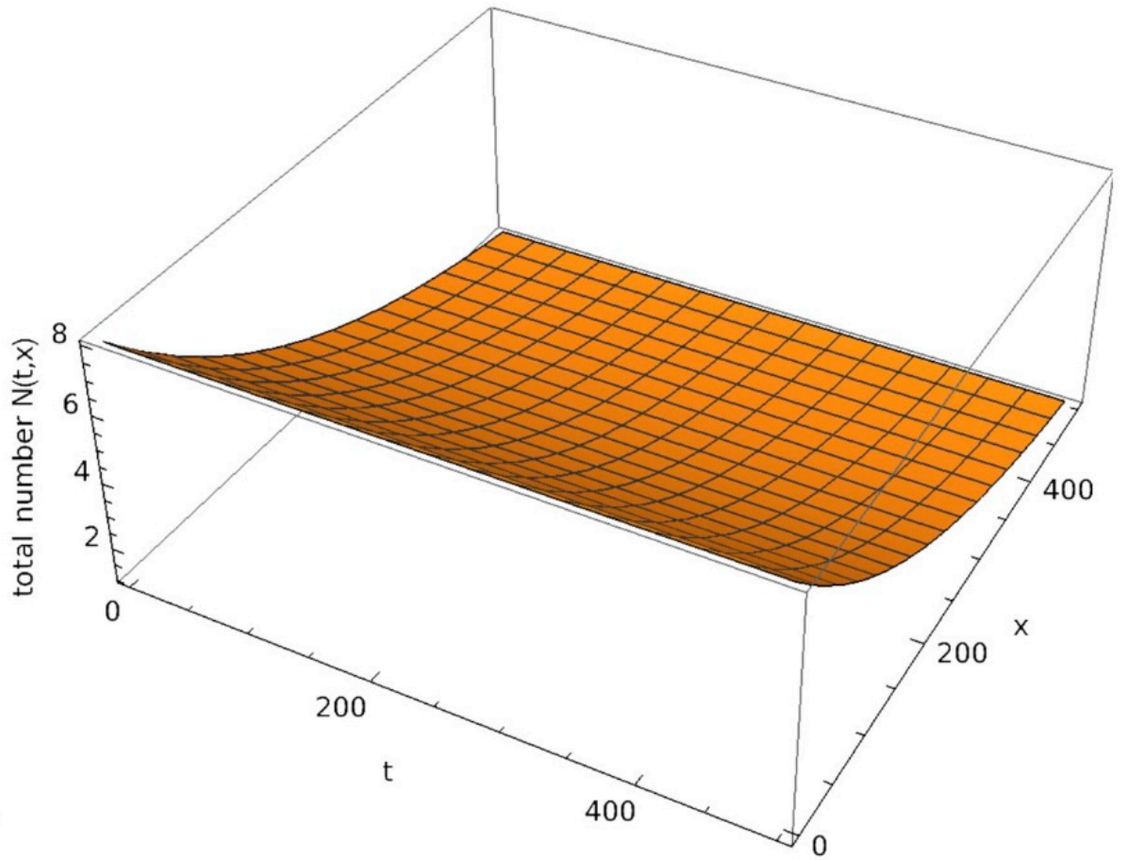


FIG. 8. Results for a case (7) simulation which is identical to case (4) but with an additional matter term $\lambda(x) = \mu(x) = 50\exp(-x/250)$ and an additional neutral current scattering term $\propto f_{\rm NC}(x) = \exp(-x/250)$. Note that the flavor conversions are completely suppressed in this case.

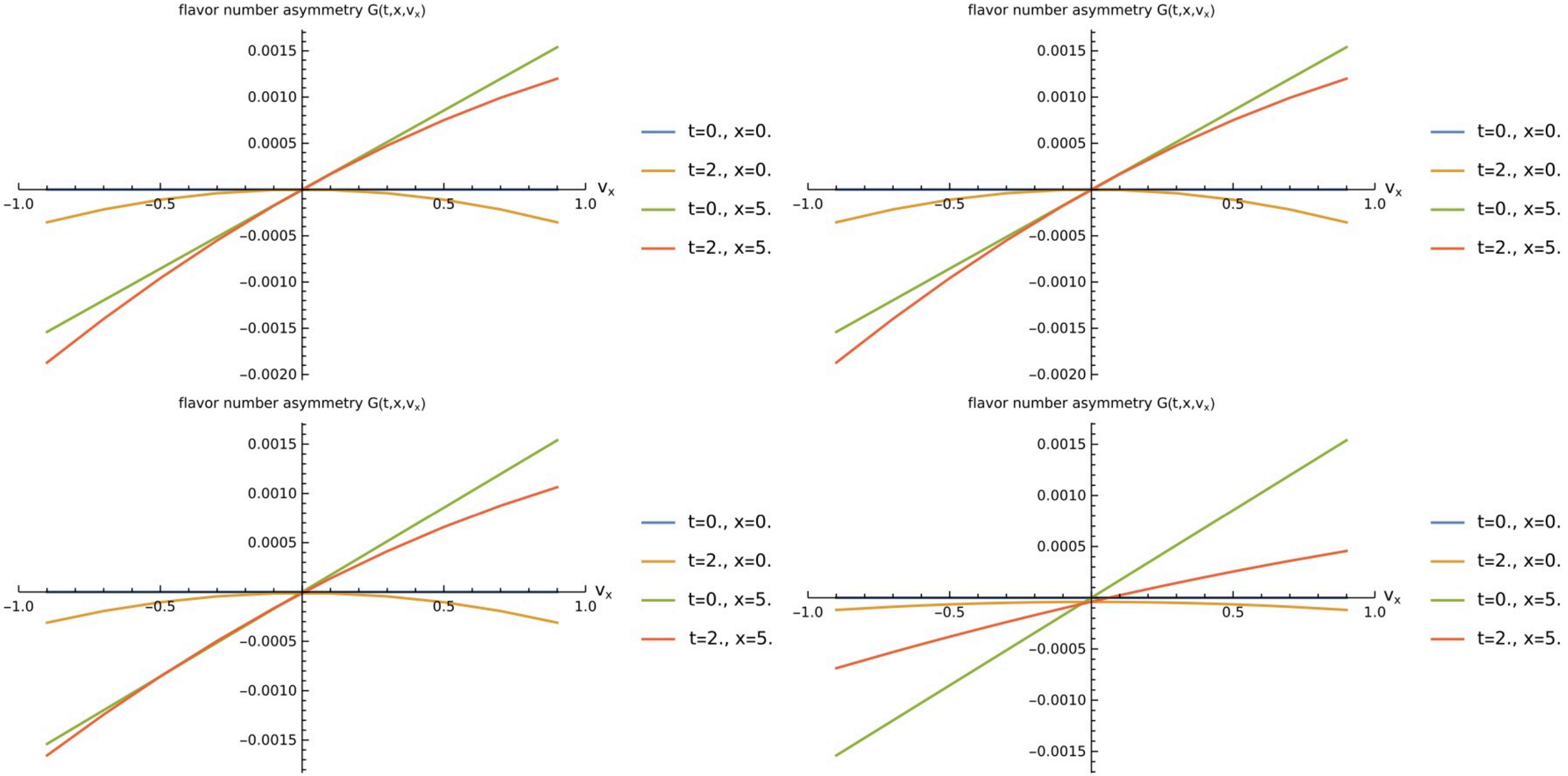


FIG. 9. The normalized flavor-lepton number asymmetry defined in Eq. (30) as a function of direction v_x for cases (4) (upper left), (5) (upper right), (6) (lower left) and (7) (lower right), at the values for t and x indicated. Note that at $t = 0$ the flavor-lepton number asymmetry is of order $bg(x)/N_p$ which for $x = 5$ gives $G(t = 0, x = 5, v_x) \simeq 0.00154 v_x$. At $x = L/2$ (not shown) the initial flavor asymmetry becomes maximal, $G(t = 0, x = L/2, v_x) \simeq 0.018 v_x$. Note that both the matter term and the neutral current scattering term tend to suppress the asymmetry already at early times.

Conclusions

1.) There is a deep connection between Schrödinger-like equations for the wave operators and the classical Liouville equation and its generalisation to collisional Vlasov/Boltzmann equations

2.) Classical behaviour emerges on length scales large compared to the de Broglie wavelength $\hbar/p$ to lowest order in an expansion in $\hbar$. To this order one has collisionless particle transport and kinematic decoherence

3.) Higher order terms would describe correlations between momentum modes and collision terms which can also lead to dynamic decoherence

4.) The forward scattering term then is the foundation for all the collective effects

5.) The self-induced exponential growth of small inhomogeneities violates conservation of neutrino-neutrino refractive energy. This is cured by a force term that leads to energy exchange with the large reservoir of neutrino kinetic energy

Conclusions on Matter Refraction in Toy Models

1.) Toy models can be used to understand the interplay between self-interaction, matter, vacuum, source terms and boundary conditions, although in general they are prohibitive for realistic conditions

2.) Consistent initial and boundary conditions are important and sometimes not completely straightforward

3.) Matter terms may not be trivially "rotated away" for profiles with significant slopes; probably depends on relation between profile scale height and oscillation lengths

4.) Fast flavor conversions tend to be suppressed or delayed in presence of inhomogeneous matter terms and neutral current non-forward scattering